Compactness of the solution operator to $\overline{\partial}$ in weighted L^2 -spaces.

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Abstract

In this paper we discuss compactness of the canonical solution operator to $\overline{\partial}$ on weigthed L^2 spaces on $\mathbb{C}^n$. For this purpose we apply ideas which were used for the Witten Laplacian in the real case and various methods of spectral theory of these operators. We also point out connections to the theory of Dirac and Pauli operators.

Introduction.

Background for bounded pseudoconvex domains

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. We consider the $\overline{\partial}$-complex

$$L^2(\Omega) \stackrel{\overline{\partial}^{(0,0)}}{\longrightarrow} L^2_{(0,1)}(\Omega) \stackrel{\overline{\partial}^{(0,1)}}{\longrightarrow} \dots \stackrel{\overline{\partial}^{(0,n-1)}}{\longrightarrow} L^2_{(0,n)}(\Omega) \stackrel{\overline{\partial}^{(0,n)}}{\longrightarrow} 0 \,,$$

where $L^2_{(0,q)}(\Omega)$ is the space of $(0,q)$-forms on Ω with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial}\left(\sum_J{}' a_J \, d\overline{z}_J\right) = \sum_{j=1}^n \sum_J{}' \frac{\partial a_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J \,,$$

where $\sum{}'$ means the sum over increasing multi-indices J.

The complex Laplacian $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}$ acts as an unbounded selfadjoint operator on $L^2_{(0,q)}(\Omega),\ 1 \leq q \leq n,$ it is surjective and therefore has a continuous inverse, the $\overline{\partial}$-Neumann operator $N_q.$ If v is a closed $(0, q+1)$-form, then $\overline{\partial}^*\, N_{q+1}v$ provides the canonical solution to $\overline{\partial}u = v,$ which is orthogonal to the kernel of $\overline{\partial}$ and so has minimal norm (see for instance [CheSha]).

The case of unbounded domains

In this talk, we discuss the compactness of the canonical solution operator to $\overline{\partial}$ on weighted L^2-spaces over $\mathbb{C}^n$:

$$L^2(\mathbb{C}^n, \varphi) = \{f \ : \int_{\mathbb{C}^n} |f(z)|^2 \exp(-2\varphi(z))\, d\lambda(z) < \infty\},$$

where φ is a suitable weight-function.

Main point :

Continue to explore the connection of $\overline{\partial}$ with the theory of Schrödinger operators with magnetic fields, see for example [Chr], [Bernd], [FuStr3] and [ChrFu]. Give necessary or sufficient conditions in terms of the weight function φ for the solution operator to be compact on $L^2(\mathbb{C}^n, \varphi)$ continuing the work from [Has3] and using results from [AusBen], [HelMoh], [Iwa], [She], and [Stein].

The complex one-dimensional case

Let φ be a subharmonic $\mathcal{C}^2$-function. We want to solve

$$\overline{\partial} u = f \tag{1}$$

for $f \in L^2(\mathbb{C}, \varphi)$. The canonical solution operator (if it exists) to $\overline{\partial}$ gives a solution with minimal $L^2(\mathbb{C}, \varphi)$-norm. With $v = u\, e^{-\varphi}$ and $g = f\, e^{-\varphi}$, the equation becomes

$$\overline{\partial}_\varphi v = g \ , \tag{2}$$

with

$$\overline{\partial}_\varphi = e^{-\varphi} \frac{\partial}{\partial \overline{z}} e^{\varphi}.$$

u is the minimal solution to the $\overline{\partial}$-equation in $L^2(\mathbb{C}, \varphi)$ iff v is the solution to $\overline{\partial}_\varphi v = g$ which is minimal in $L^2(\mathbb{C})$.

The formal adjoint of $\overline{\partial}_\varphi$ is

$$\overline{\partial}^*_\varphi = -e^{\varphi} \frac{\partial}{\partial z} e^{-\varphi}. \tag{3}$$

Let us introduce

$$\square_{\varphi}^{(0,0)} = \overline{\partial}_{\varphi}^{*}\, \overline{\partial}_{\varphi}\ ,\ \square_{\varphi}^{(0,1)} = \overline{\partial}_{\varphi}\, \overline{\partial}_{\varphi}^{*}\ . \tag{4}$$

(We write $\square_{\varphi}^{(0,0)}$ and $\square_{\varphi}^{(0,1)}$ because they are actually the "Witten"- Laplacians on respectively $(0,0)-$ and $(0,1)-$ forms).
Note that :

$$\square_{\varphi}^{(0,0)} = \frac{1}{4}(-\Delta_A - B)\ ,\ \square_{\varphi}^{(0,1)} = \frac{1}{4}(-\Delta_A + B), \tag{5}$$

where

$$A_1 = -\partial_y \varphi\, ,\ A_2 = \partial_x \varphi\, , \tag{6}$$

$$\Delta_A = \left(\frac{\partial}{\partial x} - iA_2\right)^2 + \left(\frac{\partial}{\partial y} + iA_1\right)^2\ , \tag{7}$$

and the magnetic field B satisfies

$$B(x,y) = \Delta\varphi(x,y)\ . \tag{8}$$

Hence $\Box_{\varphi}^{(0,\cdot)}$ is (up to a multiplicative constant) a Schrödinger operator with magnetic field and an electric potential $\pm B$. These operators are ([Sima] essentially self-adjoint on $C_0^\infty(\mathbb{C})$.

The canonical Solution Operator

When $\square_\varphi^{(0,1)}$ is invertible, it is easy to see that the solution to the problem (2) is given by the so called canonical operator

$$S_\varphi := \overline{\partial}_\varphi^* \left(\square_\varphi^{(0,1)}\right)^{-1} . \tag{9}$$

The existence of S_φ was established by M. Christ [Chr]) under very weak assumptions on φ (class $\mathcal{W}$) and Haslinger [Has3] observed that then S_φ was compact iff $\square_\varphi^{(0,1)}$ has compact resolvent.

Now we prove a criterion[1] of compactness, in terms of φ only.
Previous results : Molchanov, Helffer and Morame ([HelMor]), Iwatsuka ([Iwa]), Shen ([She]), Kondratiev-Shubin....

[1] We warmly thank P. Auscher for discussions on this topic.

We assume that φ is a subharmonic $\mathcal{C}^2$ function and that $\Delta\varphi$ belongs to the reverse Hölder class $B_2(\mathbb{R}^2)$ consisting of L^2 positive (strictly positive a.e.) functions V for which there exists a constant $C > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q V^2\,dx\right)^{\frac{1}{2}} \leq C\left(\frac{1}{|Q|}\int_Q V\,dx\right),$$

for any cube Q in $\mathbb{R}^2$.

It is known that if V is in B_q for some $q > 1$ then V is in the Muckenhoupt class A_∞ and the corresponding measure $V(x)dx$ is doubling (condition appearing in the definition of $\mathcal{W}$). More precisely it is known from [Stein] that

$$A_\infty = \cup_{q>1} B_q\ .$$

Note that any positive polynomial is in B_q for any $q > 1$.

Theorem 1.
Let φ be a subharmonic $\mathcal{C}^2$ function on $\mathbb{R}^2$ such that

$$\Delta\varphi \in B_2(\mathbb{R}^2)\,. \tag{10}$$

Then S_φ is compact if and only if

$$\lim_{|z|\to\infty} \int_{B(z,1)} \Delta\varphi(y)^2\, dy = +\infty\,. \tag{11}$$

Step 1
It's enough to show that $-\Delta_A + \Delta\varphi$ has compact resolvent. Using the comparison between selfadjoint operators :

$$-2\Delta_A \geq -\Delta_A + \Delta\varphi \geq -\Delta_A \tag{12}$$

we see that $-\Delta_A + \Delta\varphi$ has compact resolvent iff $-\Delta_A$ has compact resolvent. Note that the invertibility results of the strict positivity of $\Delta\varphi$ a. e. (so the proof is independent of Christ's result).

Step 2: necessary condition

We can apply a result of Iwatsuka ([Iwa])

Proposition 2.

Suppose that $A \in H^1_{loc}$ and that $-\Delta_A$ has compact resolvent. Then

$$\lim_{|z|\to\infty} \int_{B(z,1)} B(y)^2 \, dy = +\infty \,, \tag{13}$$

with $B = curlA$.

Step 3: sufficient condition

By the diamagnetic property, we observe that, if $-\Delta + \Delta\varphi$ has compact resolvent, then $-\Delta_A + \Delta\varphi$ has compact resolvent. So it is enough to prove that $-\Delta + V$ has compact resolvent with $V = \Delta\varphi$.

By the definition of $B_2(\mathbb{R}^2)$, Assumption (11) implies that

$$\lim_{|z|\to\infty} \int_{B(z,1)} \Delta\varphi(y) \, dy = +\infty \,. \tag{14}$$

By Iwatsuka's criterion, it suffices to show that

$$\lim_{|z|\to\infty} \lambda_{0,V}(B(z,1)) = +\infty \,, \qquad (15)$$

where $\lambda_{0,V}(B(z,1))$ is the lowest Dirichlet eigenvalue of $-\Delta+V$ in $B(z,1)$. We use the following improved version of the Fefferman-Phong Lemma as given in [AusBen].

Some Fefferman-Phong Lemma due to P. Auscher and Ben Ali

Lemma 3.
If $V \in A_\infty$, *then there exists* $C_V > 0$ *and* $\beta_V \in]0,1[$ *such that, for all cubes* Q *(with sidelength* R*), for all* $u \in C_0^\infty(Q)$,

$$C_V \frac{m_\beta(R^2\Theta_Q)}{R^2} \int |u(y)|^2 \, dy \leq \int (|\nabla u(y)|^2 + V(y)|u(y)|^2) \, dy \qquad (16)$$

where

$$\Theta_Q = \frac{1}{|Q|} \int_Q V(y) dy \; ,$$

and

$m_\beta(t) = t$ *for* $t \leq 1$, *and* $m_\beta(t) = t^{\beta_V}$ *for* $t \geq 1$.

End of the proof

We apply Lemma 3 with $R = 1$ and $V = \Delta\varphi$.

Remark 4.

1. *We actually only need assumptions at* ∞.

2. *We can restrict the assumptions to cubes of size* $\leq R_0$.

The $\overline{\partial}$-equation in weighted L^2 - spaces of several complex variables: preliminaries

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-weight function and define the spaces :

- $$L^2(\mathbb{C}^n, \varphi) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \;:\; \int_{\mathbb{C}^n} |f|^2\, e^{-2\varphi}\, d\lambda < \infty\},$$

- $L^2_{(0,1)}(\mathbb{C}^n, \varphi) = (0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$,

- $L^2_{(0,2)}(\mathbb{C}^n, \varphi) = (0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$,

- $A^2(\mathbb{C}^n, \varphi)$ = entire functions belonging to $L^2(\mathbb{C}^n, \varphi)$.

We consider the $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n,\varphi) \overset{\overline{\partial}^{(0,0)}}{\longrightarrow} L^2_{(0,1)}(\mathbb{C}^n,\varphi) \overset{\overline{\partial}^{(0,1)}}{\longrightarrow} L^2_{(0,2)}(\mathbb{C}^n,\varphi) \dots .$$

The $\overline{\partial}$-problem consists now in solving $\overline{\partial} u = f$ in $L^2(\mathbb{C}^n,\varphi) \cap \ker \overline{\partial}^{\perp}$ for some $f \in L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ satisfying $\overline{\partial} f = 0$.

As before we consider the "Witten" distorted $\overline{\partial}$-complex

$$\overline{\partial}_\varphi = \exp\varphi\, \overline{\partial}\, \exp -\varphi \ ,$$

on the usual L^2-forms.

The Witten $\square$- Laplacians $\square_\varphi^{(0,0)}$ and $\square_\varphi^{(0,1)}$ are defined by

$$\begin{aligned} \square_\varphi^{(0,0)} &= (\overline{\partial}_\varphi^{(0,0)})^* \quad \overline{\partial}_\varphi^{(0,0)} \quad , \\ \square_\varphi^{(0,1)} &= (\overline{\partial}_\varphi^{(0,1)})^* \quad \overline{\partial}_\varphi^{(0,1)} + (\overline{\partial}_\varphi^{(0,0)}) \quad (\overline{\partial}_\varphi^{(0,0)})^* \ . \end{aligned} \tag{17}$$

By computation, we verify

$$\square_\varphi^{(0,1)} = \square_\varphi^{(0,0)} \otimes I + 2M_\varphi, \tag{18}$$

where

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{jk}, \tag{19}$$

is the Levi-matrix.

For φ in $\mathcal{C}^2$, it can be shown (Simader [Sima]) that $\Box_\varphi^{(0,1)}$ can be extended to a densely defined self-adjoint operator on $L^2_{(0,1)}(\mathbb{C}^n)$, still denoted by $\Box_\varphi^{(0,1)}$.

Remember also :

$$4\,\Box_\varphi^{(0,0)} = \Delta_\varphi^{(0)} - \Delta\varphi, \tag{20}$$

where

$$\Delta_\varphi^{(0)} = -\sum_{j=1}^{n}\left(\left(\frac{\partial}{\partial x_j} + i\frac{\partial\varphi}{\partial y_j}\right)^2 + \left(\frac{\partial}{\partial y_j} - i\frac{\partial\varphi}{\partial x_j}\right)^2\right)\ .$$

About general criteria of compact resolvent

Before we start with the analysis of the canonical solution operator to $\overline{\partial}$ we recall a theorem due to Helffer-Mohamed ([HelMoh]) on compact resolvents of Schrödinger operators with magnetic fields :

$$P_A = \sum_{j=1}^{n} (D_{x_j} - A_j(x))^2 . \qquad (21)$$

Here $D_{x_j} = -i\frac{\partial}{\partial x_j}$ and the magnetic potential $A(x) = (A_1(x), A_2(x), \cdots, A_n(x))$ is supposed to be C^∞. Under these conditions, the operator is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. We introduce :

$$B_{jk} = \frac{1}{i}[X_j, X_k] = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} , \quad \text{for } j, k = 1, \ldots, n ,$$

the components of the magnetic fields and for $q \geq 1$ the quantities :

$$m_q(x) = \sum_{j<k} \sum_{|\alpha|=q-1} |\partial_x^\alpha B_{jk}(x)| \; . \qquad (22)$$

Then the criterion is

Theorem 5. *([HelMoh])*
Let us assume that there exists r and a constant C such that

$$m_{r+1}(x) \leq C \, \left(1 + \sum_{q=1}^{r} m_q(x) \right) \; , \; \forall x \in \mathbb{R}^n \; , \qquad (23)$$

and

$$\sum_{q=1}^{r} m_q(x) \rightarrow +\infty \; , \quad \textit{as } |x| \rightarrow +\infty \; . \qquad (24)$$

Then P_A has a compact resolvent.

(see also [She] and [KS] for further results in this direction.)

We will mainly apply this result for the case of real dimension $2n,$ where we will write the elements of $\mathbb{R}^{2n}$ in the form $(x_1, y_1, \ldots, x_n, y_n)$ and for the magnetic potential

$$A = \left(-\frac{\partial \varphi}{\partial y_1}, \frac{\partial \varphi}{\partial x_1}, \ldots, -\frac{\partial \varphi}{\partial y_n}, \frac{\partial \varphi}{\partial x_n} \right) . \qquad (25)$$

The analysis of the canonical solution operator

Now suppose that $\overline{\partial} f = 0$, for $f \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. It is easy to see that the canonical solution operator S to $\overline{\partial}$ is compact in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ iff the canonical solution operator S^{can}_{φ} to $\overline{\partial}_{\varphi} v = g$ is compact $L^2_{(0,1)}(\mathbb{C}^n)$. More precisely, assuming that $\Box^{(0,1)}_{\varphi}$ is invertible, we define

$$S_{\varphi} := (\overline{\partial}^{(0,0)}_{\varphi})^* \, (\Box^{(0,1)}_{\varphi})^{-1} \,, \qquad (26)$$

and observe that in this case the restriction S^{can}_{φ} of S_{φ} to $\mathrm{Ker}\, \overline{\partial}^{(0,1)}$,

$$S^{can}_{\varphi} = (S_{\varphi})_{\ \mathrm{Ker}\ \overline{\partial}^{(0,1)}} \,,$$

permits to solve the $\overline{\partial}_{\varphi}$ problem (or equivalently, after reverse conjugation, the $\overline{\partial}$-problem).

Theorem 6.
Let φ *be plurisubharmonic* $\mathcal{C}^2$ *such that for the lowest eigenvalue* λ_φ *of the Levi matrix* M_φ *the condition*

$$\liminf_{|z|\to\infty} \lambda_\varphi(z) > 0\,, \tag{27}$$

is satisfied. Then $\Box_\varphi^{(0,1)}$ *has a bounded inverse* N_φ *on* $L^2_{(0,1)}$ *and* S_φ *is well defined and continuous on* $L^2_{(0,1)}$.

Proof

We have by (18),

$$\langle \Box_\varphi^{(0,1)} v, v\rangle \geq 2\langle M_\varphi v, v\rangle \ , \forall v \ . \tag{28}$$

Using Persson's Theorem, Assumption (27) implies that the bottom of the essential spectrum of $\Box_\varphi^{(0,1)}$ is strictly positive. Hence it remains to show that $\Box_\varphi^{(0,1)}$ is injective. This is easy using Kazdan's uniqueness theorem, inequality

$$\langle \Box_\varphi^{(0,1)} v, v\rangle \geq \int_{\mathbb{C}^n} \lambda_\varphi(z)|v(z)|^2 \, d\lambda(z). \tag{29}$$

and the positivity of λ_φ.

Theorem 7.
Let φ be plurisubharmonic $\mathcal{C}^2$ such that

$$\lim_{|z|\to\infty} \lambda_\varphi(z) = +\infty \,. \tag{30}$$

Then S_φ is compact.

Proof
Using (29) and (30), it follows that $\Box_\varphi^{(0,1)}$ has compact resolvent. It is then easy to show that S_φ is compact.

Example :

$$\varphi(z) = \left(\sum_{j=1}^{n} |z_j|^2\right)^m ,$$

for some integer $m > 1$.

Remark 8.
If 0 *is not in the spectrum of* $\Box_\varphi^{(0,1)}$*, then we have*

$$\int_{\mathbb{C}^n} |u(z)|^2 \, e^{-2\varphi(z)} \, d\lambda(z) \leq \frac{1}{2} \langle M_\varphi^{-1} \overline{\partial} u \, , \, \overline{\partial} u \rangle_{L^2_{(0,1)}(\mathbb{C}^n, \varphi)}, \tag{31}$$

for u *orthogonal to* *Ker* $\overline{\partial}$*.*
Very short proof inspired by a proof of Brascamp-Lieb inequality by Witten Laplacian techniques. By Ruelle's Lemma [Ru] and (28)*, we get*

$$N_\varphi \leq \frac{1}{2} M_\varphi^{-1} \, .$$

This implies in particular Hörmander's statement (in his book in complex analysis), that for u *orthogonal to* *Ker* $\overline{\partial}$*,*

$$\int_{\mathbb{C}^n} |u(z)|^2 \, e^{-2\varphi(z)} \, d\lambda(z) \leq \int_{\mathbb{C}^n} |\overline{\partial} u(z)|^2 \, \frac{e^{-2\varphi(z)}}{\lambda_\varphi(z)} \, d\lambda(z).$$

The constant is improved but Hörmander established the result in greater generality (for pseudo-convex open sets).

Finally we mention a variant of Theorem 7 using the results from [HelMoh], together with ideas of M. Derridj (see [HelNou] and references therein).

Theorem 9.
Let φ be plurisubharmonic $\mathcal{C}^2$ and suppose that there exists $t \in (0, 1/4)$ and a compact set K such that

$$M_\varphi \geq t\Delta\varphi \otimes I\,, \text{ in } \mathbb{C}^n \setminus K.$$

Suppose λ_φ does not vanish identically. Assume that $\Delta_\varphi^{(0)}$ has compact resolvent. Then S_φ is compact.

Proof
Using (18), we have :

$$\Box_\varphi^{(0,1)} \geq (\Box_\varphi^{(0,0)} + 2t\Delta\varphi) \otimes I\,, \text{ in } \mathbb{C}^n \setminus K\,. \quad (32)$$

By formula (20), we are then reduced to the analysis of the compactness of the resolvent of

$$\frac{1}{4}\Delta_\varphi^{(0)} + (2t - 1/4)\Delta\varphi\,.$$

The case of decoupled weights

Here we consider weights φ of the form

$$\varphi(z_1, \ldots, z_n) = \sum_{j=1}^{n} \varphi_j(z_j),$$

where the functions φ_j are C^∞ functions on $\mathbb{C}$

About Dirac and Pauli operators

In this case an interesting connection to Dirac and Pauli operators is of importance (see [CFKS], [Erd], [HelNouWa], [Roz], [Tha]). Let us first consider the real two dimensional case.

The Dirac operator $\mathbb{D}$ is defined by

$$\mathbb{D} = \sigma_1 \left(\frac{1}{i} \partial_x - A_1(x,y) \right) + \sigma_2 \left(\frac{1}{i} \partial_y - A_2(x,y) \right),$$

where the σ_j are the Pauli matrices.
It turns out that the square of $\mathbb{D}$ is diagonal with the Pauli operators $P_\pm$ on the diagonal:

$$\mathbb{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix},$$

where

$$P_\pm = -\Delta_A \pm B(x,y) .$$

We get

$$4\Box_\varphi^{(0,0)} = P_- .$$

It is proved in [HelNouWa] that at least one of the operators $P_\pm$ has non compact resolvent if φ satisfies in $\mathbb{C}$ the following condition (H_r) :
There exists a sequence of disjoint balls B_n of radius ≥ 1 such that (23) is satisfied in the union of these balls.

This is in particular the case when the magnetic potentials are polynomials.

Note also the interesting independent result (cf [CFKS]) that the spectra of P_+ and P_- coincide except at 0. So if P_+ has compact resolvent then P_- has its essential spectrum reduced to $\{0\}$.

Main results and proofs

Our main theorem is the following

Theorem 10.
Let $n \geq 2$ and let φ be a decoupled weight such that one of the φ_j satisfies for some $r_j > 0$ the condition (H_{r_j}), then $\Box_\varphi^{(0,1)}$ has a non compact resolvent.

Proof
A simple computation shows that the operator $\Box_\varphi^{(0,1)}$ becomes diagonal, each component on the diagonal being

$$\mathcal{S}_k = \Box_\varphi^{(0,0)} + 2\,\frac{\partial^2 \varphi_k}{\partial z_k \partial \overline{z}_k}. \qquad (33)$$

Then

Proposition 11.
Let $n \geq 2$. Under the assumptions of the theorem on the weight function φ, there always exists a k such that $\mathcal{S}_k$ is not with compact resolvent.

We observe that $\mathcal{S}_k$ can be rewritten in the form

$$4\mathcal{S}_k = \sum_{j \neq k} P_-^{(j)} + P_+^{(k)}\,,$$

where each operator $P_\pm^{(\ell)}$ is the previously analyzed Pauli operator in variables the (x_ℓ, y_ℓ). The result is then obtained from the results by Helffer-Nourrigat-Wang.

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