(II) THE BRAUER GROUP OF A VARIETY

II.1. Brauer group of a field

$k$ perfect field (usually of char $0$), $\bar{k}$ alg. closure,

$$\Gamma = \text{Gal} (\bar{k}/k) = \varprojlim \text{Gal} (L/k)$$

$L$ is finite Galois

An abelian group equipped with a continuous action of $\Gamma = \Gamma^\ell_k$ - module.
Examples:

1. $A = \mathbb{Z}/m$ (trivial action)

2. $A = \mu_m(\bar{k})$

3. $A = G(\bar{k})$, $G$ commutative object in $k$-group.

For a $\Gamma'$-module $A$, define the Galois cohomology groups $H^i(\Gamma', A) = H^i(k, A)$ for $i \geq 0$.

e.g.: $H^0(\Gamma', A) = A^{\Gamma'}$ ("invariants").
Some properties:

1. If the action is trivial, then $H^1(\Gamma', A) = \text{Hom}_c(\Gamma', A)$.

2. $H^c(\Gamma', A)$ is covariant in $A$, contravariant in $\Gamma'$.

3. If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence of $\Gamma'$-modules, then there is a long exact sequence

$$0 \rightarrow H^1(\Gamma', A) \rightarrow H^1(\Gamma', B) \rightarrow H^1(\Gamma', C) \rightarrow H^2(\Gamma', A) \rightarrow \cdots$$
Examples:

\[ H^1(\Gamma, \mathbb{Q}_\ell(*)) = 0 \] (Hilbert's 90).

If (for \( k \neq n \)), then

\[ H^1(\Gamma, \mu_n(k)) = \mathbb{Q}^* / k^* \]

(Kummer theory; use the exact sequence

\[ 0 \rightarrow \mu_n(k) \rightarrow \mathbb{Q}^* / k^* \rightarrow 0 \]

and Hilbert's 90).

Rem.: For \( i > 1 \), the group

\[ H^i(\Gamma, A) \] is torsion.

\[ (H^i(\Gamma, A) = \text{lim}_{\rightarrow} H^i(\text{Gal}(L/k), A)) \]
Def: The Brauer group of $k$ is $H^2(\Gamma, \mathbb{P}^*_k) = H^2(k, G_m)$.

Exc.: Br $C = 0$, similarly for $\text{IF}_q$, $C(t)$.

Br $\mathbb{R} = \mathbb{Z}/2$.

Br $\mathbb{Q}_p = \mathbb{Q}/\mathbb{Z}$ (local class field theory).

For a number field $k$, there is an exact sequence:

$$0 \rightarrow \text{Br } k \rightarrow \mathbb{Q} \rightarrow \text{Br } k_0 \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

(global class field theory).
Other definition of Br $k$:

$Br_k$ can be defined as equivalence classes of central simple algebras over $k$, with the law $\otimes$.

$A \sim B$ iff $A \cong M_n(D)$, $B \cong M_m(D)$ for the same division algebra $D$.

Thus $Br_k = 0 \iff$ every division algebra with center $k$ is $\cong k$. 
Let $a, b$ in $k^*$. Then the Hilbert symbol $(a, b)$ is an element of $Br_k[2]$. It corresponds to the quaternion algebra $k$-algebra with basis $(1, i, j, k)$, relations $i^2 = -ab$, $j^2 = a$, $k^2 = b$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

For $k = \mathbb{R}$ or $\mathbb{Q}$, $Br_k[2] = \mathbb{Z}/2$, recover the def. in the 1. talk. $(a, b)$ can be defined as a cup product in group cohomology.
II.2. Generalization to arbitrary schemes

Let $X$ be an algebraic variety over a field $k$, $G$ a commutative $\mathbb{G}_a$ or $\mathbb{G}_m$ group (e.g. $G = \mathbb{G}_a(k)$, $G = \mathbb{G}_m(k)$).

Then one can define étale cohomology groups $H^i_c(X, G)$ for $i \geq 0$.

If $X = \text{Spec } k$, then $H^i_c(X, G) = H^i(k, G(k))$ (Borel cohomology group).
Properties: \( H^0(X, G) = G(X) = \text{Mor}_k(X, G) \)

\( H^1(X, G) \) is covariant in \( G \), contravariant in \( X \).

- long exact sequence associated to an exact sequence \( 0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \) of \( k \)-groups.

Examples: \( H^1(X, G_m) = \text{Pic} X \)

- If \( X \) is projective over an alg. closed field \( k \), then
  \[ H^1(X, \mu_m) = \text{Pic} X [m] \quad (m, \text{rank} k \neq 1) \]
  (use \( 0 \rightarrow \mu_m \rightarrow \mu_n \rightarrow \mu_{mn} \rightarrow 0 \)).
**Def:** The (cohomological)
Brauer group of $X$ is
\[ H^2(X, \mathbb{G}_m) = \text{Br}(X) \]

**Prop:** $\text{Br} \left( \text{Spec } k \right) = \text{Br}(k)$. (can be defined similarly for
any scheme $X$).

**Th.** (Grothendieck) For $X$ smooth
over $k$ and integral,
\[ \text{Br}(X) \longrightarrow \text{Br} \left( \mathbb{A}^1(k) \right) \]
function field of $X$.

Thus $\text{Br}(X)$ is torsion.
II.3. The Brauer-Manin obstruction

Let $X$ be a projective and smooth variety over a number field $k$. Define $X(A_k)$ as the set of $k$-rational points. Assume $X(A_k) \neq \emptyset$.

Let $d \in Br X$. For $P_0 \in X(k_v)$, evaluate $d(P_0) \in Br k_v$, then take the local invariant $j_\nu (d(P_0)) \in \mathbb{Q}/\mathbb{Z}$. 
Then by reciprocity law for Br \( k \), if \((P_v)\) \(v \in k\) comes from a rational point \( P \in X(k) \), then:

\[
\sum_{v \in k} j_v (d(P_v)) = 0
\]

for any \( d \in \text{Br} X \)

"Monin conditions" (Monin, ICM 70)

\( X \) is smooth and proj. model of \( y^2 + z^2 = (x^2 - 2)(3 - x^2) \)

\( d = (-1, x^2 - 2) \in \text{Br} (k(X)) \) is in fact in \( \text{Br} X \)

and for any \((P_v) \in X(k_v)\), \( \sum j_v (d(P_v)) \neq 0 \)

\(-\) Monin obstruction to the Hasse principle.
Similarly, if \((P_v) \in X((A/k))\) is such that \(\sum_v j_{v,d}(P_v) \neq 0\) for some \(d \in Br X\), then \((P_v)\) is not in the closure of \(X(f)\) for the product of \(v\)-adic topologies.

\(\Rightarrow\) Brauer-Manin obstruction to weak approximation (Colliot-Thélène/Sansuc).

Remark: "Constant" elements of \(Br X\) do not give any obstruction.

If \(X \times \mathbb{A}^1 \) is rational, \(Br X/Br k\) is finite \(\Rightarrow\) BM obs. in "comme à Lille"
II.4. Some results and conjectures

Conjecture [Colliot-Thélène/Sansuc]

For a (geometrically) rational surface $X$, Brower-Morin obstruction to the Hasse principle and weak approximation is the only one.

-known-for: Intersection of two quadrics in $\mathbb{P}^n$, $n > 8$.

(CT/Sansuc, Swinnerton-Dyer, 1987)

- Diagonal cubic surfaces over $\mathbb{Q}$, assuming finiteness of III (Swinnerton-Dyer, 2000).

+ Some additional condition.
Th [Serre, 1981]: The orologue holds for principal homogeneous spaces of linear algebraic groups.


Th [Korobogatov] There exists a bielliptic surface $X/\mathcal{G}$ such that:

i) $X(\mathcal{G}) = \emptyset$

ii) There exists an adelic point on $X$ satisfying Mori's conditions!