**NOTATION:**

$k$ number field, $k_v$ completion of $k$ at the place $v$.

$\Omega_k$ set of places of $k$, $\bar{k}$ algebraic closure of $k$.

$X$ smooth, proper, and geometrically integral $k$-variety.

$X(\mathbb{A}_k) = \prod_{v \in \Omega_k} X(k_v)$ set of adelic points of $X$.

**DEFINITION:**

i) A class $\mathcal{C}$ of (smooth, proper, and geometrically integral) $k$-varieties is said to satisfy the *Hasse principle* (HP) if, for any $X \in \mathcal{C}$,

$$X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$$

ii) A smooth and proper $k$-variety $X$ satisfies the *weak approximation* (WA) if $X(k)$ is dense in $X(\mathbb{A}_k)$.

These happen to be $k$-**birational properties**.

For any (integral) $k$-variety $V$, we define :

$V$ satisfies HP (resp. WA) := a smooth and proper model $X$ of $V$ satisfies HP (resp. WA).
EXAMPLES:

- $\mathbb{P}_k^n$ satisfies WA.
- Quadrics satisfy HP and WA (Hasse-Minkowski)
- $K/k$ cyclic extension, $a \in k^*$, $\omega_1, \ldots, \omega_r$ basis of $K/k$.

\[ V : N_{K/k}(x_1\omega_1 + \ldots + x_r\omega_r) = a \]

satisfies HP and WA.

COUNTER-EXAMPLES TO HP:

- (Hasse) $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{13}, \sqrt{17})$.

\[ V : N_{K/k}(x_1\omega_1 + \ldots + x_r\omega_r) = -1 \]

- (Reichardt, Lindt) Some curves of genus one:

\[ V : y^2 = 2(x^4 - 17) \]

- Some cubic surfaces (Swinnerton-Dyer, Cassels and Guy)
- Some principal homogeneous spaces of semi-simple linear algebraic groups (Serre).
THE BRAUER-MANIN OBSTRUCTION:

\( X/k \) proper, smooth, geometrically integral.

\( \text{Br} \, X := H^2_{\text{et}}(X, \mathbb{G}_m) \) : Brauer group of \( X \)

\( \text{Br} \, X \subset \text{Br} \, k(X) \).

\( j_v : \text{Br} \, k_v \leftrightarrow \mathbb{Q}/\mathbb{Z} \) local invariant.

Let \( M_v \in X(k_v), \ A \in \text{Br} \, X \); then \( A(M_v) \in \text{Br} \, k_v \).

Actually \( A(M_v) \equiv 0 \) outside a finite (independent of \( M_v \)) set \( S \) of places of \( k \).

Therefore, for any adelic point \( (M_v) \in X(A_k) \), the sum:

\[
\sum_{v \in \Omega_k} j_v(A(M_v)) \in \mathbb{Q}/\mathbb{Z}
\]

is well-defined. Recall the reciprocity law in global class field theory:

\[
0 \to \text{Br} \, k \to \bigoplus_{v \in \Omega_k} \text{Br} \, k_v \xrightarrow{\sum j_v} \mathbb{Q}/\mathbb{Z} \to 0
\]

If for any adelic point \( (M_v) \), there exists an element \( A \) of \( \text{Br} \, X \) such that \( \sum j_v(A(M_v)) \neq 0 \), then \( X(k) \) is empty: this is the Brauer-Manin obstruction (BM) to the HP.
$X(A_k)^{Br} := \{(M_v)_{v \in \Omega_k}, \forall A \in \text{Br}X, \sum_{v \in \Omega_k} j_v(A(M_v)) = 0\}$

BM obstruction to HP : $X(A_k)^{Br} = \emptyset$

BM obstruction to WA : $X(A_k)^{Br} \neq X(A_k)$.

**REMARK** : ”Constant elements” of Br $X$ don’t give any BM obstruction $\Rightarrow$ it makes sense to replace Br $X$ with Br $X$/Br $k$ in the definition of the BM obstruction.

**Theorem[Manin, 1970]**. Assume the finiteness of the Tate-Shafarevic group of elliptic curves. Then the BM obstruction to the HP is the only one for curves of genus one.
AN EXPLICIT EXAMPLE:

Let $V$ be the smooth affine surface:

$$y^2 + z^2 = (x^2 - 2)(3 - x^2) \neq 0$$

Theorem[Iskovskih, 1970]. Let $X$ be a smooth and proper model of $V$. Then $X$ is a counter-example to the HP.

Proof: Let $f := (x^2 - 2)$, $K := \mathbb{Q}(\sqrt{-1})$, $Q_{\infty} := \mathbb{R}$. Let $X$ be a smooth and proper model of $V$; $k(V) = k(X)$. The element $A := (-1, f)$ of $Br_k(V)$ belongs to $Br X$.

$M_p := (y_p, z_p, x_p) \in V(Q_p)$, $K_p := K \otimes \mathbb{Q} Q_p$.

$$A(M_p) = (-1, f(x_p)) \in \mathbb{Z}/2 \text{ (local Hilbert symbol)}.$$

$$(-1, f(x_p)) = 0 \Leftrightarrow f(x_p) \in N_{K_p/Q_p}(K_p^*).$$

- For $p \neq 2$, $f(x_p)$ is a local norm of $K_p/Q_p$ (any $M_p$).
- For $p = 2$, $f(x_p)$ cannot be a local norm of $K_p/Q_p$. 
On the other hand $V(Q_p) \neq \emptyset$ for all $p$ (easy). Because of the implicit function theorem, we have:

i) For all $p$, there exists $M_p \in X(Q_p)$.

ii) For $p \neq 2$, $A(M_p) = 0$ for any $M_p \in X(Q_p)$.

iii) For $p = 2$, $A(M_p) \neq 0$ for any $M_p \in X(Q_p)$.

Finally $X(A_k)$ is not empty but for any adelic point $(M_p)$:

$$\sum_p j_p(A(M_p)) \neq 0$$

This is a Brauer-Manin obstruction to the HP.
**DESCENT ON RATIONAL VARIETIES:**

**Definition:** A (geometrically integral) $k$-variety $X$ is *rational* if $\overline{X} := X \times_k \bar{k}$ is birational to $\mathbb{P}^n$. 

$(\Leftrightarrow \bar{k}(X)$ purely transcendental over $\bar{k}(T_1, \ldots, T_n))$.

If $X$ is a (proper and smooth) rational variety, then Pic $\overline{X}$ is of finite type and torsion-free, and Br $\overline{X} = 0$, hence Br $X$/Br $k = H^1(k, \text{Pic} \overline{X})$ is finite.

The descent method was developped by Colliot-Thélène and Sansuc. Their fundamental idea is to use principal homogeneous spaces (under certain algebraic tori) over $X$ as auxiliary varieties, for which there is no Brauer-Manin obstruction.
Let $M$ be a finite type and torsion free $G$-module
$(G := \text{Gal}(\bar{k}/k) : \text{absolute Galois group of } k)$.
Then $S := \text{Hom}(M, \mathbf{G}_m)$ ($k$-group dual to $M$) is a $k$-torus.
There is an exact sequence of abelian groups:

$$0 \rightarrow H^1(k, S) \rightarrow H^1_\text{ét}(X, S) \rightarrow \text{Hom}_G(M, \text{Pic } \overline{X})$$

**Definition:** Let $X$ be a proper and smooth $k$-variety such that $\text{Pic } \overline{X}$ is of finite type and torsion-free (e.g. $X$ rational). Let $S_0$ be the $k$-torus dual to $\text{Pic } \overline{X}$.

A *universal $X$-torsor* is a principal homogeneous space $\mathcal{T}$ (over $X$) under $S_0$ such that the image of the class $[\mathcal{T}]$ of $\mathcal{T}$ under the map $H^1_\text{ét}(X, S_0) \rightarrow \text{Hom}_G(\text{Pic } \overline{X}, \text{Pic } \overline{X})$ is the identity map.
**Theorem [Colliot-Thélène, Sansuc].** Let $X$ be a smooth, projective, and geometrically integral $k$-variety.

Assume that $X$ is rational and that $X(\mathbb{A}_k)^{\text{Br}}$ is not empty. Then:

i) There exists a universal $X$-torsor $\mathcal{T}$ such that $\mathcal{T}(\mathbb{A}_k) \neq \emptyset$.

ii) Let $\mathcal{T}$ be any universal torsor and let $\mathcal{T}^c$ be a smooth compactification of $\mathcal{T}$. Then $\text{Br } \mathcal{T}^c / \text{Br } k = 0$.

iii) One can find a finite number of universal $X$-torsors $\mathcal{T}_1, \ldots, \mathcal{T}_n$ (equipped with $k$-morphisms $p_1, \ldots, p_n$ to $X$) such that:

$$X(k) = \bigcup_{i=1}^{n} p_i(\mathcal{T}_i(k)) \quad X(\mathbb{A}_k)^{\text{Br}} = \bigcup_{i=1}^{n} p_i^c(\mathcal{T}_i^c(\mathbb{A}_k))$$

→ If universal torsors satisfy HP (resp. WA), then the BM obstruction to the HP (resp. WA) is the only one for $X$. 
**Hope**: universal torsors satisfy HP and WA.

But it is an **open question** even if \( X \) is a rational surface.

**KNOWN RESULTS**:

- **Châtelet surfaces**: \( y^2 - az^2 = P(x) \), \( \text{deg} P = 4 \), \( a \in k^* \) (Colliot-Thélène, Sansuc, Swinnerton-Dyer).

- **Conic bundles** over the projective line with at most 4 degenerate fibers (Salberger, Colliot-Thélène).

- **Smooth intersections of two quadrics** in \( \mathbb{P}^4_k \) containing a \( k \)-rational point (Salberger, Skorobogatov).

In some very special cases, the universal torsors are \( k \)-birational to the projective space, but the approach which has given most of the results is to use a **fibration method** to prove that they satisfy HP and WA.
THE FIBRATION METHOD :

Idea : To prove the HP (resp. WA) for a variety $V$, try to find a surjective morphism $f$ from $V$ to a $k$-variety $B$ for which the HP and the WA hold, and such that the fibers of $f$ satisfy HP (resp. WA) as well.

Theorem. Let $B$ be a smooth and projective $k$-variety that satisfies HP and WA. Let $V$ be a geometrically integral $k$-variety equipped with a projective and surjective morphism $f$ to $B$. Assume :

a) All geometric fibers of $f$ are geometrically integral.

b) HP (resp. WA) holds for almost all $k$-fibers of $f$.

Then the HP (resp. the WA) holds for a smooth and proper model of $V$.

almost all: on a dense Zariski open subset

Note that $V$ itself is not assumed to be smooth.
The proof uses the **implicit function theorem** and **weak approximation on** $\mathbf{B}$ to get $k_v$-points in some $k$-fiber $V_m$ for $v$ in $S$, $S$ finite set of places.

Then the condition that all geometric fibers are geometrically integral ensures, thanks to the **Lang-Weil theorem**, that for $v \not\in S$, $X_m$ has $k_v$-points as well.

It remains to apply the HP (or the WA) for the $k$-fiber $V_m$.

**Question**: Is it possible to replace the assumption that the $k$-fibers satisfy the HP with the weaker assumption that the BM obstruction to the HP is the only one for those fibers? That is, prove directly by a fibration method that the Brauer-Manin obstruction to the HP (or to WA) is the only one?
MORE FIBRATION METHODS :

Theorem. Let $V$ be a geometrically integral $k$-variety equipped with a projective and surjective morphism $f$ to $\mathbf{P}^1_k$. Let $V_\eta$ be the generic fiber of $f$ and suppose that it is smooth. Let $K$ be the function field of $\mathbf{P}^1_k$ and put $\overline{V}_\eta = V \times_K \overline{K}$. Assume :

i) $\text{Br} \overline{V}_\eta = 0$ and $\text{Pic} \overline{V}_\eta$ is torsion-free (e.g. $V_\eta$ is a rational variety).

ii) All geometric fibers of $f$ are geometrically integral.

iii) For almost all $k$-fibers of $f$, the Brauer-Manin obstruction to the HP (resp. to WA) is the only one.

Then the Brauer-Manin obstruction to the HP (resp. to WA) is the only one for a smooth and proper model of $V$.

The framework of the proof is the same as in the previous fibration theorem but it is not sufficient to get a $k$-fibre $V_m$ with points in each completion of $k$: indeed, not only $V_m(\mathbf{A}_k)$, but also $V_m(\mathbf{A}_k)^{\text{Br}}$, should be non-empty.
Two supplementary ingredients:

• Control the Brauer group of the $k$-fibers. This may be done, using the geometric assumptions on the generic fiber and Hilbert’s irreducibility theorem.

• Let $A$ be an element of $\text{Br} V_\eta$ ($A \not\in \text{Br} K$); take a smooth and proper model $X$ of $V$.

Problem: $A$ does not necessary belong to $\text{Br} X$!

The key is to use the following result:

**Theorem.** Let $X$ be a proper, smooth, and geometrically integral $k$-variety. Let $U$ be a non-empty open subset of $X$ and $A$ be an element of $\text{Br} U$ which does not belong to $\text{Br} X$. Then there exists infinitely many places $\nu$ of $k$ such that the evaluation map: $U(k_\nu) \to \text{Br} k_\nu$ associated to $A$ is not constant.
Example of application of the fibration theorem:
If the BM obstruction to the HP (resp. WA) is the only one for smooth cubic surfaces, then HP and WA hold for smooth cubic hypersurfaces of dimension at least three.

It is worth noting that even if the fibers (in the theorem) are rational varieties, $V$ itself is not necessary rational.
Actually, the following situation may occur: there is an obstruction to the HP for $V$ which does not come from the elements of $[\ker \Br X \to \Br \overline{X}] = H^1(k, \Pic \overline{X})$, but from a “transcendental element”.
TRANSCENDENTAL BM OBSTRUCTIONS:

Take $p$ a prime with $p \equiv -1 \mod 4$;

$$V : y^2 - g(t)z^2 = [f(x)^2 + \frac{2^7}{p}] [1 + g(t)^2 - g(t)(f(x)^2 + \frac{2^7}{p} + 2)]$$

$$f(x) = \frac{1}{(x^2 + 1)} \quad g(t) = (-2^6 p/t^2 + 1) - 1$$

$$W : y^2 - tz^2 = (x^2 + 1/p) [1 + t^2 - t(x^2 + 1/p + 2)]$$

Transcendental BM obstruction to HP for $V$.

Transcendental BM obstruction to WA for $W$ ($W(k) \neq \emptyset$).
WHAT SHOULD WE EXPECT IN GENERAL?

One may hope that the BM obstruction to the HP is the only one for rational varieties.

For smooth complete intersections of dimension at least three, it is not expected.

Very recently, Skorobogatov constructed a bielliptic (proper and smooth) surface over \( \mathbb{Q} \) for which:

The BM obstruction to the HP is not the only one.

Another direction is the following: it is possible to define a similar BM obstruction for the zero-cycles of degree one (instead of the rational points). Colliot-Thélène has conjectured that this obstruction to the existence of a zero-cycle of degree 1 is the only one for all varieties over a number field \( k \). For example, this conjecture is true for conic bundles over the projective line with an arbitrary number of degenerate fibers (Colliot-Thélène, Swinnerton-Dyer).