Hyperplane sections in arithmetic hyperbolic manifolds

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Abstract

In this paper, we prove that the homology groups of immersed totally geodesic hypersurfaces of compact arithmetic hyperbolic manifolds virtually inject in the homology group of the ambient manifold.

1 Introduction

Let $\mathbb{H}^n$ be the $n$-dimensional hyperbolic space. A hyperplane is a codimension one totally geodesic subspace of $\mathbb{H}^n$; note that a hyperplane is isometric to $\mathbb{H}^{n-1}$. Let $M = \Gamma \backslash \mathbb{H}^n$ be a compact hyperbolic manifold. A $\Gamma$-hyperplane is a hyperplane $H$ of $\mathbb{H}^n$ such that $\text{Stab}_\Gamma(H) \backslash H$ is compact. A $\Gamma$-hyperplane thus projects onto a compact immersed totally geodesic submanifold in $M$.

Recall the following celebrated theorem of Millson \[1\].

**Theorem 1.1 (Millson)** Let $M = \Gamma \backslash \mathbb{H}^n$ be an arithmetic hyperbolic manifold and $H \subset \mathbb{H}^n$ be a $\Gamma$-hyperplane. Then, there exists a finite cover $\widetilde{M}$ of $M$ such that $H$ projects onto an embedded submanifold $F \hookrightarrow M$ and

$$[F] \neq 0 \text{ in } H_{n-1}(M).$$

In this paper we prove the following generalization of Millson’s theorem.

**Theorem 1.2** Let $M = \Gamma \backslash \mathbb{H}^n$ be a compact arithmetic manifold and $H \subset \mathbb{H}^n$ be a $\Gamma$-hyperplane. Then, there exists a finite cover $\widetilde{M}$ of $M$ such that $H$ projects onto an embedded submanifold $F \hookrightarrow M$ and

$$H_k(F, \mathbb{Z}) \rightarrow H_k(\widetilde{M}, \mathbb{Z})$$

is injective for every integer $k \geq 0$.

This theorem partly confirms a conjecture the first author stated in \[2\]. The proof may be deduced from recent work \[3\] of the last two authors (and from lemmas of \[4\]). In this paper we give a direct and self-contained proof of theorem 1.2.

We now give a more precise description of the content of this paper.
First recall the general definition of an arithmetic hyperbolic manifold. Let \( G \) be a \( \mathbb{Q} \)-algebraic group such that its group of real points, \( G(\mathbb{R}) \), is the product (with finite intersection) of a compact group by \( G^\text{nc} = SO(n,1) \).

A congruence subgroup \( \Gamma \) of \( G(\mathbb{Q}) \) is the intersection \( G(\mathbb{Q}) \cap K \), where \( K \) is a compact-open subgroup of \( G(A_f) \) the group of finite adèles points of \( G \). According to a classical theorem of Borel and Harish-Chandra, it is a lattice in \( G^\text{nc} = SO(n,1) \). It is a cocompact lattice if and only if \( G \) is anisotropic over \( \mathbb{Q} \). We will always assume it is a cocompact lattice. If \( \Gamma \) is sufficiently deep, i.e. \( K \) is a sufficiently small compact-open subgroup of \( G(A_f) \), then \( \Gamma \) is moreover torsion-free.

Let \( K_\infty \) be a maximal compact subgroup of \( G(\mathbb{R}) \), then \( G(\mathbb{R})/K_\infty \) - the associated symmetric space - is isometric to the \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \). If \( \Gamma \subset G(\mathbb{Q}) \) is a torsion-free congruence subgroup, \( \Gamma \backslash \mathbb{H}^n \) is a \( n \)-dimensional congruence hyperbolic manifold. In general, a hyperbolic manifold is called arithmetic if it shares a common finite (Riemannian) cover with a congruence hyperbolic manifold.

**Statement of the main results.** Let \( M = \Gamma \backslash \mathbb{H}^n \) be as above. It is well known, and follows from Tits’s classification of algebraic groups, that there exists \( \Gamma \)-hyperplanes in \( \mathbb{H}^n \) if and only if the group \( G \) comes, by restriction of scalars, from an orthogonal group over a totally real number field. Following [?], we call standard the particular arithmetic manifolds thus obtained. Theorem 1.2 immediately follows from the following more general theorem. All homology groups we consider are with \( \mathbb{Z} \) coefficients.

**Theorem 1.3** Let \( M = \Gamma \backslash \mathbb{H}^n \) be a compact standard arithmetic hyperbolic manifold. If \( \Gamma \) is sufficiently deep then any totally geodesic (convex cocompact) immersed submanifold \( F \) in \( M \) lifts to a finite cover \( \widetilde{M} \) of \( M \) such that \( F \) embeds in \( \widetilde{M} \) and

\[
H_k(F) \to H_k(\widetilde{M})
\]

is injective for every integer \( k \geq 0 \).

We also prove the following theorem which improves on a result of Millson and Raghunathan [?] and, in the standard arithmetic case, on a result of the first author [?].

**Theorem 1.4** Let \( M = \Gamma \backslash \mathbb{H}^n \) be a compact standard arithmetic hyperbolic manifold and \( F_1 \) and \( F_2 \) be two totally geodesic immersed submanifolds in \( M \). Assume \( F_1 \) and \( F_2 \) transversally intersects in at least one point. Then there exists a finite cover \( \widetilde{M} \) of \( M \) and two connected components \( \widetilde{F}_1 \) and \( \widetilde{F}_2 \) of the preimages of \( F_1 \) and \( F_2 \) in \( \widetilde{M} \) such that \( \widetilde{F}_1 \) and \( \widetilde{F}_2 \) are both embedded in \( \widetilde{M} \) and their intersection \( \widetilde{F}_1 \cap \widetilde{F}_2 \) is connected and non trivial in \( H_1(\widetilde{M}) \).

Extensions of theorems 1.3 and 1.4 to non-compact or non-arithmetic manifolds are discussed in the last section.

**Plan of the proof.** Let \( M = \Gamma \backslash \mathbb{H}^n \) be a compact standard arithmetic hyperbolic manifold. Since the pioneering work of Milson, a key point of the subject has been the separability of (stabilizers of) hyperplanes in \( \Gamma \), namely:
Lemma 1.5 Let $H \subset \mathbb{H}^n$ be a hyperplane. Then $\Gamma_H := \text{Stab}_\Gamma(H)$ is a separable subset of $\Gamma$, i.e. it is closed in the profinite topology of $\Gamma$.

Recall the profinite topology on a group $\Gamma$ is the topology generated by the basis consisting of cosets of finite index subgroups of $\Gamma$.

Separability of stabilizers of geodesics (and in fact of any finitely generated subgroups) in surface groups was first proved by Scott [?] by a nice geometric argument to which we shall come back. Separability of stabilizers of hyperplanes in hyperbolic 3-manifolds was then proved by Long [?]. More generally the first author proved in [?] the separability of stabilizers of totally geodesic subspaces (not necessarily of codimension one) in hyperbolic (or more generally, locally symmetric) manifolds of any dimensions. In the last two cases, both proofs are algebraic; they nevertheless apply to any (not necessarily arithmetic) hyperbolic manifold. They all follow from the following general lemma.

Lemma 1.6 Let $V$ be a real algebraic subset of some $GL(N, \mathbb{R})$ and $\Gamma$ be a finitely generated subgroup of $GL(N, \mathbb{R})$. Then $\Gamma \cap V$ is a separable subset of $\Gamma$.

Sketch of proof. Let $\gamma \notin \Gamma \cap V$. Then, there exists a polynomial $P$ on the $N \times N$ matrices such that $P(\gamma) \neq 0$. Malcev’s proof of residual finiteness of finitely generated linear groups, provides a congruence quotient $\Gamma \to \tilde{\Gamma}$ such that $P(\tilde{\gamma}) \neq 0$. But $P(v) = 0$ for each $v \in V$, so we have separated $\gamma$ from $\Gamma \cap V$. \hfill $\square$

Separability of hyperplanes can be used (see [?] for example) to get an easy proof of Millson’s theorem, here codimension one is crucial: a codimension-1 submanifold gives a non trivial class in homology if and only if it is non-separating.

Scott’s geometric method and the algebraic method of lemma 1.6 appear to be the only known tools to prove separability of hyperplanes. Scott’s method does not work in general, it applies when the group $\Gamma$ embeds into a right-angled hyperbolic Coxeter subgroup of the isometries of hyperbolic space. Although there are quite a lot of such groups in low dimensions this imposes strong restrictions, e.g. on the dimension. Nevertheless Scott’s method not only gives the separability of hyperplanes, it gives a retraction. This seems not to have been emphasized until the recent papers of Wise [?] and Long and Reid [?]; it seems however to be more useful than the separability statement when applied to the topological study of hyperbolic manifolds. A careful examination of Scott’s proof indeed gives the following theorem ([?, Theorem 2.6]).

Theorem 1.7 Let $M = \Gamma \backslash \mathbb{H}^n$ be a compact hyperbolic manifold such that $\Gamma$ embeds into a right-angled hyperbolic Coxeter subgroup of the isometries of $\mathbb{H}^n$. Let $F$ be a compact totally geodesic (convex cocompact) immersed submanifold of $M$. Then, there exists a finite cover $\tilde{M}$ of $M$ such that $F$ lifts to $\tilde{M}$ and there exists a continuous retraction

$$r: \tilde{M} \to F.$$ 

In particular, $H_\ast(F)$ injects into $H_\ast(\tilde{M})$.

Proof. Let $P$ be a right-angled polyhedron in $\mathbb{H}^n$, $C(P)$ the Coxeter group generated by reflections in the faces of $P$ such that $\Gamma \subset C(P)$. The group
Λ := π₁(F) is then a convex cocompact subgroup of C(P). The geometric convex hull of Λ can be expanded to a tiling hull T(Λ) coming from the right-angled Coxeter group: if N(Λ) denotes the convex hull of the limit set of Λ, then T(Λ) is defined to be the smallest convex union of copies of P which contains N(Λ) in its interior. It is easy to see that T(Λ) is Λ-invariant, and the key step is to show that the quotient Λ\T(Λ) is compact. Note then that Λ\T(Λ) clearly retracts onto F.

The group generated by reflections in the sides of T(Λ), denoted by K(Λ), is discrete with fundamental polyhedron T(Λ). By the Λ-invariance, it follows that K(Λ) is normalized by Λ and K(Λ) ∩ Λ = 1. Define V(Λ) = K(Λ) ⋊ Λ. Then V(Λ)\Hⁿ, and thus ˜M := (V(Λ) ∩ Γ)\Hⁿ, clearly retracts on Λ\T(Λ). It only remains to prove ˜M is a finite cover of M or that V(Λ) is of finite index in C(P), but this easily follows from the fact that Λ\T(Λ) is compact.

Recently, in [?], the second author has extended this result to the case of quasi-convex subgroups of abstract right-angled Coxeter groups, i.e. finitely generated groups given by a presentation by elements of order two and where the only other relators are commutators of the generators. In particular, theorem 1.7 still holds when Γ only embeds in an abstract right-angled Coxeter group. Quite surprisingly many groups Γ embed in abstract right-angled Coxeter group; in [?] the last two authors make a systematic study of such groups. In a coffee break discussion we realized that the methods of [?] easily yield that standard cocompact arithmetic lattices in SO(n, 1) virtually embed in abstract right-angled Coxeter groups. From these considerations one easily gets theorem 1.3. Here we give a direct proof of the following result which illustrates the tools developed in the above mentioned papers.

Theorem 1.8 Let Γ be an arithmetic (congruence) subgroup of SO(n, 1). Then there exists a finite index (congruence) subgroup Γ′ ⊂ Γ such that Γ′ embeds in a right-angled Coxeter group.

Theorem 1.8 is proved in three steps:

1. We construct a CAT(0) (simply connected non positively curved) cube complex C on which Γ acts properly and cocompactly. This occupies section 2 and 3 and relies on the fact that Γ\Hⁿ contains lots of Γ-hyperplanes (take the Hecke translates of a fixed Γ-hyperplane). The preimages of a well chosen finite number of these hyperplanes give a locally finite family of hyperplanes in Hⁿ whose dual graph is quasi-isometric to Hⁿ. Following Sageev, by “filling in” the skeletons of cubes in this graph we get the requested CAT(0) cube complex. Its hyperplanes are in one-to-one correspondence with our chosen locally finite family of hyperplanes in Hⁿ.

2. We then consider the abstract right-angled Coxeter group generated by the Γ-equivalence classes of these hyperplanes. Let DM(Γ) be its associated CAT(0) cube complex. To conclude we just have to prove C isometrically embeds into DM(Γ) in a Γ-equivariant way provided that Γ is sufficiently deep. We first construct the requested map and note that, by results of Gromov on CAT(0) cube complexes, it is sufficient to check that this map is a local isometry on the 2-skeleton.
3. We then prove that when $\Gamma$ is sufficiently deep the constructed map is indeed a local isometry on the 2-skeleton. This step involves the separability of hyperplanes (lemma 1.5) as well as the following refinement of it.

**Lemma 1.9** Let $H_1, H_2 \subset \mathbb{H}^n$ be two hyperplanes of $\mathbb{H}^n$. Then $\Gamma \cap \text{Stab}_{SO(n,1)}(H_1)\text{Stab}_{SO(n,1)}(H_2)$ is a separable subset of $\Gamma$.

This lemma follows from lemma 1.6 and the fact that $\text{Stab}_{SO(n,1)}(H_1)\text{Stab}_{SO(n,1)}(H_2)$ is a real algebraic subset of $SO(n,1)$ as can easily be checked by following the proof of [?], Proposition 10. Note that a related lemma is the key point in the proof of the main theorem in [?] which is strongly related to theorem 1.4.

### 2. Arithmetic hyperbolic manifolds coming from orthogonal groups

Let $F$ be a totally real number field and $(V, q)$ be a quadratic space over the field $F$. Let $G'$ be the special orthogonal group of $(V, q)$; it is an algebraic group over $F$. Let $G = \text{Res}_{F/\mathbb{Q}}(G')$ be the $\mathbb{Q}$-algebraic group obtained by restricting the scalars from $F$ to $\mathbb{Q}$. The group of real points of $G$ is isomorphic to a product of special orthogonal groups; assume its non-compact part, $G^{nc}$, is isomorphic to $SO(n, 1)$. Fix a maximal open compact subgroup $K_{\text{max}}$ in $G(A_f)$. Any congruence subgroup $\Gamma \subset \Gamma_{\text{max}} := G(\mathbb{Q}) \cap K_{\text{max}}$ is a lattice in $SO(n, 1)$; if chosen sufficiently deep, $\Gamma$ is torsion free and the quotient $M(\Gamma) = \Gamma \backslash \mathbb{H}^n$ is a finite volume hyperbolic manifold. We will always assume $G$ anisotropic, equivalently $q$ does not represent 0 over $F$, and so $M(\Gamma)$ is compact. It is well known that this construction gives examples of compact hyperbolic manifolds in any dimension. A *standard arithmetic manifolds* is a hyperbolic manifold commensurable with such an $M(\Gamma)$. These are the only arithmetic hyperbolic manifolds of even dimension but there are other examples in odd dimensions to which the methods of this paper are awkward to apply since these examples never contain codimension-1 totally geodesic compact submanifolds.

Any totally positive 1-dimensional subspace of $V$ yields a $\Gamma$-hyperplane. Let $H \subset \mathbb{H}^n$ be such a $\Gamma$-hyperplane. Define $\mathcal{H}(H)$ to be the following subset of hyperplanes in $\mathbb{H}^n$:

$$\mathcal{H}(H) = \{g \cdot H : g \in G(\mathbb{Q})\}.$$  

For any finite subset $S \subset G(\mathbb{Q})$ we define

$$\mathcal{H}^S(H) = \{g \cdot H : g \in (\Gamma_{\text{max}} \cdot S) \subset G(\mathbb{Q})\}.$$  

Note that $\Gamma_{g \cdot H} = g\Gamma_{H}g^{-1}$. When $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup, each hyperplane $H' \in \mathcal{H}(H)$ projects in $M(\Gamma)$ onto a totally geodesic compact immersed submanifold. In fact $H$ projects onto $M_H(\Gamma_H) := \Gamma_H \backslash H$ which immerses inside $M(\Gamma)$ and $g \cdot H$ projects onto its Hecke translate $T_g(M_H(\Gamma_H))$. Here $T_g$ is the Hecke operator associated to $g \in G(\mathbb{Q})$ and $T_g(M_H(\Gamma_H))$ is equal to the image of $\Gamma_{g \cdot H} \backslash g \cdot H$ inside $M(\Gamma)$.  

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The set $\mathcal{H}(H)$ thus gives a preferred collection of hyperplanes in $\mathbb{H}^n$, and this collection is dense in the set of all hyperplanes as $G(Q)$ is dense in $G_{\text{nc}}$. Recall the notion of a \textit{wallspace structure}, due to Haglund and Paulin [?]: a collection $\mathcal{W}$ of hyperplanes in $\mathbb{H}^n$ defines a wallspace structure if

1. $\mathcal{W}$ is locally finite in $\mathbb{H}^n$, and
2. there exists a constant $K > 0$ such that for any two points $x, y$ of $\mathbb{H}^n$ not contained in some hyperplane of $\mathcal{W}$, we have:

   \[ \frac{1}{K} (n_{\mathcal{W}}(x, y) - 1) \leq d(x, y) \leq K (n_{\mathcal{W}}(x, y) + 1), \]

where $n_{\mathcal{W}}(x, y)$ denotes the number of hyperplanes of $\mathcal{W}$ that separates $x$ and $y$ and $d$ is the hyperbolic metric on $\mathbb{H}^n$.

\textbf{Proposition 2.1} There exists a finite set $S \subset G(Q)$ such that $\mathcal{H}^S(H)$ defines a wallspace structure on $\mathbb{H}^n$.

\textit{Proof.} As any hyperplane $H' \in \mathcal{H}(H)$ projects onto a compact subset of $\Gamma_{\text{max}} \backslash \mathbb{H}^n$, any $\Gamma_{\text{max}}$-orbit in $\mathcal{H}(H)$ gives a locally finite family of hyperplanes in $\mathbb{H}^n$. The collection $\mathcal{H}^S(H)$, which is a finite union of $\Gamma_{\text{max}}$-orbit, is thus always locally finite.

In particular, there exists a constant $K = K(G, S)$ such that any ball of radius one intersects at most $K$ hyperplanes. It then follows that

\[ n_{\mathcal{H}^S(H)}(x, y) \leq K(d(x, y) + 1). \]

Since $\Gamma_{\text{max}}$ is cocompact in $SO(n, 1)$, there exists a closed ball $\overline{B}(x_0, r)$ such that

\[ \bigcup_{\gamma \in \Gamma_{\text{max}}} \gamma \cdot \overline{B}(x_0, r) = \mathbb{H}^n. \] \hspace{1cm} (2.1)

As a closed convex subset of $\mathbb{H}^n$, the closed ball $\overline{B}(x_0, r+1)$ is the intersection of the closed half-spaces containing it. By compactness of $S(x_0, r+2)$, there is a finite subset of these half-spaces whose intersection is a polyhedron $P_0$ such that $B(x_0, r+1) \subset P_0 \subset B(x_0, r+2)$.

By density of $\mathcal{H}(H)$ in the set of all hyperplanes of $\mathbb{H}^n$, there exists some finite set $S \subset G(Q)$ such that we may approximate the polyhedron $P_0$ by a polyhedron $P$ whose faces span hyperplanes $H_1, \ldots, H_k$ in $\mathcal{H}^S(H)$ and such that $\overline{B}(x_0, r) \subset P \subset B(x_0, r+2)$.

As $B(x_0, r+2)$ is of diameter $2(r+2)$, any geodesic $\gamma$ with initial point in $\overline{B}(x_0, r) \subset P \subset B(x_0, r+2)$ and length $> 2(r+2)$ has its terminal point outside $B(x_0, r+2)$. Thus $\gamma$ crosses $\partial P$, which means that $\gamma$ intersects some bounding hyperplane $\in \mathcal{H}^S(H)$. Now, by (2.1), the initial point of any geodesic of length $> 2(r+2)$ may be translated into $\overline{B}(x_0, r)$ by an element of $\Gamma_{\text{max}}$, it thus must intersect some hyperplane in $\mathcal{H}^S(H)$. The proposition now follows since if $x, y$ are two points of $\mathbb{H}^n$ not contained in some hyperplane of $\mathcal{H}^S(H)$,

\[ d(x, y) \leq 2(r+2) \times (n_{\mathcal{H}^S(H)}(x, y) + 1). \] \hfill $\square$
From now on fix $S$ as in the proposition. Note that any arithmetic subgroup $\Gamma \subset \Gamma_{\text{max}}$ preserves the wallspace structure $\mathcal{H}^S(H)$.

Given an arithmetic subgroup $\Gamma \subset \Gamma_{\text{max}}$, let $\mathcal{H}_\Gamma^S(H)$ be the subset of $\mathcal{H}^S(H)$ consisting of the hyperplanes which project onto $M(\Gamma)$ as embedded submanifolds. Recall (see e.g. [?]) that lemma 1.5 immediately implies that for any finite subset $S \subset G(\mathbb{Q})$, we have $\mathcal{H}_\Gamma^S(H) = \mathcal{H}^S(H)$ for $\Gamma$ sufficiently deep.

### 3 Cubulation

**Generalities.** A space with walls $(V, \mathcal{W})$ is naturally associated to any wallspace structure on $\mathbb{H}^n$: in our case let $V$ be the set of connected components of $\mathbb{H}^n - \cup_{H \in \mathcal{H}(H)} H'$ and let $\mathcal{W}$ be the set of all the partitions, called walls, of $V$ into 2 subsets induced by the hyperplanes $H' \in \mathcal{H}^S(H)$. Note that any two distinct components are separated by a finite, non-zero number of walls. This number defines the wall metric on $V$; as a metric space, $V$ is quasi-isometric to $\mathbb{H}^n$.

Recall a cube complex $C$ is a CW-complex such that the attaching map of each $k$-cell is defined on the boundary of $I^k$ ($I = [-1, 1]$) and its restriction to each $(k-1)$-face of $\partial I^k$ into $C^{k-1}$ is an isometry onto $I^{k-1}$ postcomposed with some $(k-1)$-cell of $C^{k-1}$. The image of a $k$-cell is a $k$-cube. A map between cube complexes is combinatorial if it maps each $k$-cell onto a $k$-cell isomorphically on the interior.

By a theorem of Gromov a cube complex is $CAT(0)$ if and only if it is nonpositively curved and simply connected, see [?] for more about $CAT(0)$ cube complexes. Recall the following lemma (see [?]) which easily follows from a classical theorem of Gromov characterizing non positively curved cube complex. Let $C_k$ be the subcomplex of $[-1, 1]^k$ consisting of the union of all 2-cubes containing $v_k = (1, \ldots, 1)$.

**Lemma 3.1** A cube complex $C$ is nonpositively curved if and only if its 2-skeleton is simple, i.e. the link of each vertex is a simplicial complex, and for each integer $k \geq 3$ any $k$-corner, i.e. combinatorial map $C_k \to C$, extends to exactly one $k$-cube $I^k \to C$.

The property in lemma 3.1 characterizes non positively curved cube complexes and easily yields the following:

**Lemma 3.2** Let $C$ and $C'$ be two simple cube complexes. Assume $C'$ is non positively curved. Then any combinatorial map $C^2 \to C'$ extends to a unique combinatorial map $C \to C'$.

Note that a $CAT(0)$ cube complex admits natural totally geodesic hyperplanes, whose intersections with a $k$-cube are either empty, or consist in a coordinate hyperplane of $[-1, 1]^k$ (see [?])). It follows that the set of vertices of a $CAT(0)$ cube complex has a natural structure of space with walls.

Conversely, the cubulation of space with walls embeds it in a $CAT(0)$ cube complex; it is an abstract version of a construction due to Sageev [?]. In [?] Nica shows that any space with walls admits such a cubulation, see [?] for an

\[ \text{Note that any arithmetic subgroup } \Gamma \subset G(\mathbb{Q}) \text{ has an open compact closure in } G(k_f); \text{ changing } K_{\text{max}}, \text{ we are thus dealing with all arithmetic subgroups.} \]
alternate construction. As a consequence he obtains that a proper group action on a space with walls extends naturally to a proper group action on a CAT(0) cube complex.

Cubulation of $\mathcal{H}^S(H)$. Our wallspace structure $\mathcal{H}^S(H)$ on $\mathbb{H}^n$ is a locally finite collection of hyperplanes in $\mathbb{H}^n$ and the dual graph to $\mathcal{H}^S(H)$ in $\mathbb{H}^n$ then includes into the 1-skeleton $\mathcal{C}_{H,S}$ of a cube complex. This cube complex $\mathcal{C}_{H,S}$ is obtained by “filling in” the dual graph with isometric copies of euclidean cubes by inductively adding a $k$-dimensional cube whenever its $(k-1)$-skeleton or a $k$-corner is present.

Here are some more details: A $S$-half-space of $\mathbb{H}^n$ is an open half-space bounded by a hyperplane of $\mathcal{H}^S(H)$, its complementary half-space is the complement of its closure. We can encode each vertex $v$ of the dual graph by a family of $S$-half-spaces defined as follows. Let $P_v$ be the connected component corresponding to $v$, so that the closure of $P_v$ is a compact (simple) polyhedron. We consider the family $\sigma_v$ of $S$-half-spaces containing $P_v$. Obviously $v$ is determined by $\sigma_v$ and we have the following properties:

1. For any $S$-half-spaces $A, B$ such that $A \subset B$ and $A \in \sigma_v$ we have $B \in \sigma_v$.
2. For any $S$-half-spaces $A$, either $A \in \sigma_v$ or its complementary half-space belongs to $\sigma_v$.

More generally the vertices of $\overline{\mathcal{C}_{H,S}}$ consist by definition in families $\sigma$ of $S$-half-spaces satisfying the two above properties, and furthermore is finitely generated, in other words there is a finite family $\{A_1, \cdots, A_k\} \subset \sigma$ such that any $S$-half-space $A$ of $\sigma$ contains some $A_i$. For example all families $\sigma_v$ are finitely generated. Indeed when a hyperplane $H'$ is too far away from $P_v$ there is a hyperplane $H''$ of $\mathcal{H}^S(H)$ which separates $P_v$ from $H'$. So $\overline{\mathcal{C}_{H,S}}$ already contains the set of vertices of the dual graph. Note also that for any geodesic ray $\xi : \mathbb{R}^+ \to \mathbb{H}^n$ the set $\sigma_\xi$ of $S$-half-spaces which contain some half-line $\xi([M, +\infty[)$ indeed satisfies properties 1. and 2. above, but it is not finitely generated.

When there is an edge of the dual graph from $v$ to $w$ then $P_v, P_w$ are separated by a unique hyperplane $H'$ of $\mathcal{H}^S(H)$, and the family $\sigma_w$ is the same as $\sigma_v$, except that the $S$-half-space bounded by $H'$ and containing $P_w$ is replaced by the complementary $S$-half-space. More generally one puts an edge in $\overline{\mathcal{C}_{H,S}}$ between two families $\sigma, \sigma'$ provided the symmetric difference $\sigma \triangle \sigma'$ consists in two complementary $S$-half-spaces.

Then for $k \geq 1$ the $(k+1)$-skeleton of $\overline{\mathcal{C}_{H,S}}$ is defined out of the inductively defined $k$-skeleton by adding a $(k+1)$-cube filling each copy of $\partial [-1, 1]^k$. By construction the 1-skeleton of the cube complex $\overline{\mathcal{C}_{H,S}}$ contains the dual graph, which is connected, and we denote by $\mathcal{C}_{H,S}$ the connected component of the dual graph. Using lemma 3.1, one can finally show that $\mathcal{C}_{H,S}$ is a CAT(0) cube complex (see [2] for details).

Since the wall metric is quasi-isometric to the hyperbolic metric on $\mathbb{H}^n$, any group which acts properly on $\mathbb{H}^n$ and preserves the wallspace structure, acts properly on $\mathcal{C}_{H,S}$.

Claim: there exists a constant $\rho$ (depending only on the family $\mathcal{H}^S(H)$ of hyperplanes of $\mathbb{H}^n$) such that for any vertex $\sigma \in \mathcal{C}_{H,S}$ there is a finite generating family $\{A_1, \cdots, A_k\} \subset \sigma$ and a ball $B = B_\sigma$ of radius $\rho$ which intersects non-trivially the boundary hyperplane of each half-space $A_i$. 

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Indeed consider first an arbitrary vertex $v$ (identified with $\sigma_v$) and assume that $\sigma$ contains a half-space $A$ far away from $P_v$. Let $q$ be the projection of some base point $p_v \in P_v$ onto $A$. By hyperbolicity there is a universal constant $\eta$ such that any hyperplane $H'$ that both cuts $A$ and separates $p_v$ from its projection $q$ indeed enters the $\eta$-ball around $q$. Thus by (uniform) local finiteness of $\mathcal{H}^S(H)$, in the set of hyperplanes of $\mathcal{H}^S(H)$ separating $P_v$ from $q$ there are at most $K$ and cutting $A$; and all the other ones separate $P_v$ from $A$. Observe that there is a path $\gamma$ in the dual graph which starts at $v$ and ends at some vertex $w$ such that $q \in P_w \subseteq A$, and the hyperplanes crossed by $\gamma$ are precisely the hyperplanes separating $P_v$ from $q$. Since the wall metric is quasi-isometric to the hyperbolic metric on $\mathbb{H}^n$, there is a constant $R$ such that if $d_{\mathbb{H}^n}(A, P_v) > R$ then the length of $\gamma$ is $> 2K + 1$, and it then follows that the vertex $v'$ on $\gamma$ at distance $2K + 1$ of $v$ satisfies $d_{\text{wall}}(v', \sigma) < d_{\text{wall}}(v, \sigma)$. Considering a vertex $v$ that minimizes $d_{\text{wall}}(v, \sigma)$ we reach the conclusion that no half-space $A$ of $\sigma$ can be $R$-far away from $P_v$.

Now for $r$ large enough any hyperplane which is $r$ far away from the $R$-neighborhood of $P_v$ can be separated from this neighborhood by some hyperplane of $\mathcal{H}^S(H)$. The claim follows with $\rho = r + R$.

Since $\mathcal{H}^S(H)$ is locally finite and $\Gamma$ acts cocompactly on $\mathbb{H}^n$ it acts cofinitely on the set of hyperplanes of $\mathcal{H}^S(H)$ intersecting some $\rho$-ball. It then follows from the claim that $\Gamma$ is cofinite on the set of vertices of $C_{H,S}$. Since the claim also implies that $C_{H,S}$ is locally compact, it follows that $\Gamma$ is cocompact on $C_{H,S}$. This would also result from the more general lemma of Hruska and Wise [?]:

**Lemma 3.3** If a Gromov-hyperbolic group acts properly and cofinitely on a space with walls, then its induced proper action on the associated $\text{CAT}(0)$ cube complex is cocompact.

We thus get the following proposition.

**Proposition 3.4** Any arithmetic subgroup $\Gamma \subset \Gamma_{\max}$ acts properly and cocompactly on a $\text{CAT}(0)$ cube complex $C$. Moreover, the hyperplanes of $C$ are in one-to-one correspondence with the hyperplanes $\in \mathcal{H}^S(H)$, with equal stabilizers in $\Gamma$.

Here, recall a hyperplane of $C$ is a component of the cube complex whose cubes are the midcubes of cubes of $C$. Where a midcube in $I^n$ is the subset obtained by restricting one of the coordinates to 0, so the midcube is parallel to two $(n-1)$-faces of $I^n$. The edges of $I^n$ dual to this midcube are the edges perpendicular to it. An edge of $C$ is dual to some hyperplane $H$ if its midpoint is a vertex of $H$.

### 4 Right-angled Coxeter groups

From now on, we fix the $\text{CAT}(0)$ cube complex $C$. Let $\Gamma \subset \Gamma_{\max}$ be a torsion free arithmetic subgroup. For $\Gamma$ sufficiently deep, we want to isometrically embed $C(\Gamma) := \Gamma \backslash C$ into some quotient of a $\text{CAT}(0)$ cube complex by a subgroup of a right-angled Coxeter group.

We shall now construct a candidate map:
Call hyperplane of $C(\Gamma)$ the projection of any hyperplane of $C$. Let $G(\Gamma)$ be the simplicial graph whose vertices are hyperplanes of $C(\Gamma)$ and whose edges connect distinct intersecting hyperplanes.

We then associate to $\Gamma$ the right-angled Coxeter group $C(\Gamma)$ presented by:

$$\langle \text{vertices of } G(\Gamma) | s^2 = 1, [s, t] = 1 \text{ where } s, t \text{ are joined by an edge of } G(\Gamma) \rangle.$$ (4.2)

Recall the Davis-Moussong realization of the Coxeter group $C(\Gamma)$ (see for example [? or ?>]). This is the unique CAT(0) cube complex $\text{DM}(\Gamma)$ whose 2-skeleton is as follows. The vertices of $\text{DM}(\Gamma)$ are the elements of $C(\Gamma)$. There is an edge between $c_1$ and $c_2$ if and only if $c_1 = c_2 s$ for some generator $s$. And there is a square with vertices $c_1, c_2, c_3, c_4$ whenever there are generators $s$ and $t$ adjacent in $G(\Gamma)$ such that $c_2 = c_1 s, c_3 = c_2 t, c_4 = c_3 s$.

For any choice of a base vertex $v \in C$ there is a canonical map $f_v^0$ from the 0-skeleton $C^0$ of $C$ to $\text{DM}(\Gamma)^0 = C(\Gamma)$ sending $v$ to 1: if $w$ is a vertex of $C$ any edge path in $C^1$ between $v$ and $w$ is a sequence of hyperplanes and thus defines an element in $C(\Gamma)$; since any two edge-paths between $v$ and $w$ are homotopic by a sequence of square-homotopies, the associated element in $C(\Gamma)$ is independent of the choice of the edge path: we denote it by $c(v, w) \in C(\Gamma)$.

The map $f_v^0 : C^0 \to C(\Gamma)$ is $\Gamma$-equivariant, where $\Gamma$ acts naturally on $C$ and through the representation $c_v : \Gamma \to C(\Gamma)$

$$c_v : \Gamma \to C(\Gamma)$$

on $C(\Gamma)$. Note that the group $C(\Gamma)$ acts on $\text{DM}(\Gamma)$. We first extend the map $f_v^0$ to a $\Gamma$-equivariant map

$$f_v^1 : C^1 \to \text{DM}(\Gamma)$$

on the 1-skeleton piecewise linearly. The map $f_v^1$ maps the boundary of any square $Q$ onto either the boundary of a square $Q'$ in $\text{DM}(\Gamma)$ or an edge $E$. We extend $f_v^1$ at $Q$ by either an isomorphism $Q \to Q'$ or by any piecewise linear map $Q \to E$. Doing this in a $\Gamma$-equivariant manner leads to a map

$$f_v^2 : C^2 \to \text{DM}(\Gamma).$$

The map $f_v^2$ may not be injective nor combinatorial; note however that $C$ and $\text{DM}(\Gamma)$ are both simply connected, [?, Proposition 4.14] 2 and lemma 3.2 hence implies that we may extend $f_v^2$ to an isometric embedding

$$f_v : C \to \text{DM}(\Gamma)$$

if and only if $f_v^2 : C^2 \to \text{DM}(\Gamma)^2$ is a combinatorial local isometry. Recall a combinatorial map $\phi$ between simple cube complexes is a local isometry if

1. it is an immersion, i.e. $\phi : \text{link}(v) \to \text{link}(\phi(v))$ is injective for each vertex $v$ of the domain, and
2. $\phi(\text{link}(v))$ is a full subcomplex 3 of $\text{link}(\phi(v))$.

---

2Here we use that a finite dimensional nonpositively curved cube complex is a complete length space of nonpositive curvature.

3Recall that a subcomplex $K \subset L$ of a simplicial complex is full if any simplex of $L$ whose vertices are in $K$ is in fact entirely contained in $K$.  

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5 Proof of theorem 1.8

According to the last paragraph of section 4 above, theorem 1.8 follows directly from the following proposition.

Proposition 5.1 For $\Gamma$ sufficiently deep, $f_2^2$ is a combinatorial local isometry.

Proof. Note that the only way $f_2^2$ may not be combinatorial is that it may be degenerate on some square. In such case $f_2^2$ is not an immersion. In order to prove proposition 5.1 we thus only have to check conditions 1. and 2. in the definition of a local isometry for the map $f_2^2$.

We first check condition 1. that the map $f_2^2$ is an immersion. Let $w$ be a vertex of $C^2$ and $a$ and $b$ be two distinct points of link$(w)$ such that $f_2^2(a) = f_2^2(b)$. Think of $a$ and $b$ as edges of $C$ with initial point $w$. There are two cases: either $C$ contains a square with edges $a$ and $b$ (in this case there is an edge between $a$ and $b$ in link$(w)$) or not.

In the first case, the hyperplanes dual to the edges $a$ and $b$ of $C$ intersects. As $f_2^2(a) = f_2^2(b)$ those distinct hyperplanes should belong to the same $\Gamma$-orbit. There should thus exist $H' \in \mathcal{H}^S(H)$ and $\gamma \in \Gamma$ such that $\gamma \cdot H' \neq H'$ and $\gamma \cdot H' \cap H' \neq \emptyset$.

In the second case, the hyperplanes dual to $a$ and $b$ are distinct hyperplanes $H'$ and $\gamma \cdot H'$ in the same $\Gamma$-orbit; in this case they don’t intersect but no hyperplane of $\mathcal{H}^S(H)$ separates $H'$ from $\gamma \cdot H'$.

In the first case, the projection of $H'$ in $M(\Gamma)$ is not embedded and we already know this does not happen when $\Gamma$ is sufficiently deep. The two cases may, in fact, be handled in the same way: in both cases, the set

$$\text{Bad}(\Gamma, H') = \{ \gamma \in \Gamma : \gamma \cdot N(H') \cap N(H') \neq \emptyset \} - \Gamma_{H'}$$

is nonempty, here $N(H')$ denotes the regular neighbourhood of $H'$ in $C$ and $\Gamma_{H'}$ the stabilizer of $H'$ in $\Gamma$. The following lemma is then a simple corollary of lemma 1.5.

Lemma 5.2 There exists a finite index (congruence) subgroup $\Gamma'$ of $\Gamma$ such that

$$\text{Bad}(\Gamma', H') = \emptyset.$$  

Proof. As the group $\Gamma_{H'}$ acts cocompactly on the set of edges dual to $H'$, there are only finitely many $\Gamma_{H'}$-orbits of edges in $N(H')$. Since $C$ is locally finite, there are finitely many edges meeting at a given vertex. There are thus finitely many elements $b_1, \ldots, b_l \in \Gamma - \Gamma_{H'}$ such that

$$\text{Bad}(\Gamma, H') = \bigcup_{i=1}^l \Gamma_{H'} b_i \Gamma_{H'}.$$  

According to lemma 1.5, $\Gamma_{H'}$ is separable in $\Gamma$ and there exists a finite index (congruence) subgroup $\Gamma'' \subset \Gamma$ containing $\Gamma_{H'}$ and disjoint from $\{b_1, \ldots, b_l\}$. We then clearly have $\text{Bad}(\Gamma'', H') = \text{Bad}(\Gamma, H') \cap \Gamma'' = \emptyset$ and lemma 5.2 is proved.
There are only a finite number of hyperplanes \( H_1, \ldots, H_m \) such that each hyperplane of \( \mathcal{H}^S(H) \) is in the \( \Gamma \)-orbit of one of the \( H_i \)'s. Lemma 5.2 thus implies that there exists a finite index (congruence) subgroup \( \Gamma' \subset \Gamma \) such that \( \text{Bad}(\Gamma', H_i) = \emptyset \). Then let \( \Gamma' \subset \Gamma \) denote a finite index (congruence) normal subgroup contained in \( \cap_i \Gamma_i' \). We have \( \text{Bad}(\Gamma', H_i) = \emptyset \) for each \( i \) and thus \( \text{Bad}(\Gamma', H') = \emptyset \) for each hyperplane \( H' \in \mathcal{H}^S(H) \). If we replace \( \Gamma \) by \( \Gamma' \) the map \( f^2_v \) becomes an immersion. For \( \Gamma \) sufficiently deep the map \( f^2_v \) is thus an immersion.

We now check condition 2. that for each vertex \( w \) of \( C^2 \), the image \( f^2_v(\text{link}(w)) \) is a full subcomplex of \( \text{link}(f^2_v(w)) \). As \( \text{link}(w) \) is a one dimensional complex, we just have to check that for \( \Gamma \) sufficiently deep, if \( H_a \) and \( H_b \) are non-intersecting hyperplanes in \( C \) dual to two distinct points \( a \) and \( b \) in \( \text{link}(w) \) then their images in \( M(\Gamma) \) don’t intersect. (In other words, if \( DM(\Gamma) \) contains a square with edges \( j^2_v(a) \) and \( j^2_v(b) \), then \( C^2 \) contains a square with edges \( a \) and \( b \).)

Assume by contradiction that \( H_a \) and \( H_b \) don’t intersect and that there exists \( \gamma \in \Gamma \) such that \( H_b \) intersects \( \gamma \cdot H_a \). Then \( \gamma \) belongs to the set

\[
\text{Bad}(\Gamma, H_a, H_b) = \{ \gamma \in \Gamma : \gamma \cdot H_a \cap H_b \neq \emptyset \}.
\]

In particular \( \gamma \) does not belong to \( \text{Stab}_{SO(n,1)}(H_a) \text{Stab}_{SO(n,1)}(H_b) \) and lemma 1.9 implies the following lemma.

**Lemma 5.3** There exists a finite index (congruence) subgroup \( \Gamma' \) of \( \Gamma \) such that

\[
\text{Bad}(\Gamma', H_a, H_b) = \emptyset.
\]

**Proof.** There exists finitely many \( b_1, \ldots, b_i \in \Gamma \) such that

1. \( \text{Bad}(\Gamma, H_a, H_b) = \bigcup_i \Gamma_{H_i} b_i \Gamma_{H_i} \), and

2. \( \{b_1, \ldots, b_m\} \cap \text{Stab}_{SO(n,1)}(H_a) \text{Stab}_{SO(n,1)}(H_b) = \emptyset. \)

Lemma 1.9 then implies that there exists a finite index (congruence) normal subgroup \( \Gamma' \subset \Gamma \) such that \( (b_1 \Gamma' \cup \ldots \cup b_i \Gamma') \cap \text{Stab}_{SO(n,1)}(H_a) \text{Stab}_{SO(n,1)}(H_b) = \emptyset. \) We then have \( \text{Bad}(\Gamma', H_a, H_b) = \text{Bad}(\Gamma, H_a, H_b) \cap \Gamma' = \emptyset \) and lemma 5.3 is proved. □

There are only finitely many pairs of hyperplanes \( (H_{a_1}, H_{b_1}), \ldots, (H_{a_i}, H_{b_i}) \) such that any pair of non-intersecting hyperplanes dual to two distinct points in the link of some vertex of \( C \) is in the \( \Gamma \)-orbit of one of the \( (H_{a_i}, H_{b_i}) \)'s. Consider finite index (congruence) subgroups \( \Gamma_i' \subset \Gamma \) such that \( \text{Bad}(\Gamma_i', H_{a_i}, H_{b_i}) = \emptyset. \) Let \( \Gamma' \subset \Gamma \) denote a finite index (congruence) normal subgroup contained in \( \cap_i \Gamma_i' \). We have \( \text{Bad}(\Gamma', H_{a_i}, H_{b_i}) = \emptyset \) for each \( i \) and thus \( \text{Bad}(\Gamma', H_a, H_b) = \emptyset \) for each pair \( (H_a, H_b) \) of non-intersecting hyperplanes dual to two distinct points in the link of some vertex of \( C \). If we replace \( \Gamma \) by \( \Gamma' \) the image \( f^2_v(\text{link}(w)) \) is a full subcomplex of \( \text{link}(f^2_v(w)) \) for each vertex \( w \) of \( C^2 \). For \( \Gamma \) sufficiently deep, the map \( f^2_v \) is thus a local isometry and proposition 5.1 is proved. □
6 Retractions and proof theorem 1.3

We first prove theorem 1.2.

Proof of theorem 1.2. Let $\Gamma$ be as in theorem 1.2. The existence of a $\Gamma$-hyperplane forces $\Gamma$ to be a standard arithmetic lattice as in §2. Let $H$ be the $\Gamma$-hyperplane considered in theorem 1.2. According to proposition 2.1 $\Gamma$ preserves a wallspace structure on $\mathbb{H}^n$ which contains $H$ as a wall. Let $C$ be the associated $CAT(0)$ cube complex on which $\Gamma$ acts properly and cocompactly (proposition 3.4). The hyperplane $H$ defines a hyperplane of $C$. Passing to a finite index subgroup of $\Gamma$ we may assume $\Gamma$ is sufficiently deep so that proposition 5.1 applies. According to it, the cube complex $C$ then isometrically embeds into the right-angled Coxeter complex $DM(\Gamma)$. Note that this embedding is $\Gamma$-equivariant, where $\Gamma$ acts naturally on $C$ and through the representation $c_v$ followed by the usual action of $C(\Gamma)$ on $DM(\Gamma)$. To $H$ is associated a hyperplane of $C$ which extends to a hyperplane of $DM(\Gamma)$. We denote by $H_{DM}$ the union of cubes of $C$ which intersect the hyperplane corresponding to $H$. Then $H_{DM}$ is a convex \(^4\) subcomplex of $C$ and hence of $DM(\Gamma)$. The subcomplex $H_{DM}$ is invariant and cocompact under $\Gamma_H$.

An abstract right-angled Coxeter group $C(\Gamma)$ retracts onto the stabilizer of any of the hyperplanes of $DM(\Gamma)$. More generally, Scott’s method implies that a finite index subgroup of $\Gamma$ retracts onto $DM(\Gamma)$. This finally implies that $\Gamma_H \setminus H$ homologically embeds into $M = M(\Gamma')$ and theorem 1.2 is proved. □

Now recall a subset $Y$ of a geodesic metric space is said to be quasiconvex if there exists a constant $k > 0$ such that for all $x, y \in Y$ each geodesic joining $x$ to $y$ is contained in the $k$-neighborhood of $Y$. Let $\Gamma$ be a group with finite generating set $S$. A subgroup $\Lambda$ of $\Gamma$ is said to be quasiconvex in $(\Gamma, S)$ if $\Lambda$ is quasiconvex in the Cayley graph of $\Gamma$ with respect to the set of generators $S$. Recall that if $\Gamma$ is Gromov-hyperbolic and $\Lambda \subset \Gamma$ is a subgroup which is quasiconvex in $(\Gamma, S)$ for some finite generating set $S$ then it is quasiconvex in $(\Gamma, S')$ for any finite generating set $S'$. If moreover $\Gamma$ acts properly cocompactly on any proper geodesic metric space $X$ then any orbit of $\Lambda$ in $X$ is quasiconvex in $X$.

Let $G$ a group and $H$ a subgroup. We say $G$ virtually retracts on $H$ if there is a finite index subgroup $V$ of $G$ such that

- $H \subset V$, and
- there is a homomorphism $V \rightarrow H$ which is the identity when restricted to $H$.

\(^4\)A subcomplex $Y$ of a $CAT(0)$ cube complex $X$ is said to be convex provided $Y$ is connected and the link of $Y$ at each vertex $v \in Y$ is a full subcomplex of the link of $X$ at $v$. 

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The proof of theorem 1.7 ultimately yields the following, see [?] for more details.

**Theorem 6.1** Let \( \mathcal{G} \) be some finite graph with vertex set \( S \),

\[
C = \{ s \in S \mid s^2 = 1, \ [s, t] = 1 \text{ where } s, t \in S \text{ are joined by an edge of } \mathcal{G} \}
\]

be an abstract right-angled and \( \Lambda \) be a quasiconvex subgroup in \( (C, S) \). Then \( C \) virtually retracts on \( \Lambda \).

*Proof of theorem 1.3.* Let \( \Gamma \) be as in theorem 1.3. Let \( C \) and \( DM(\Gamma) \) be as in §3 and 4. Passing to a finite index subgroup of \( \Gamma \) we may assume \( \Gamma \) is sufficiently deep so that proposition 5.1 applies. According to it, the cube complex \( C \) then isometrically embeds into the right-angled Coxeter complex \( DM(\Gamma) \). Note that this embedding is \( \Gamma \)-equivariant, where \( \Gamma \) acts naturally on \( C \) and through the representation \( c_n \) followed by the usual action of \( C(\Gamma) \) on \( DM(\Gamma) \).

Let \( \Lambda \) be a quasiconvex subgroup of \( \Gamma \). As \( \Gamma \) is Gromov-hyperbolic and acts properly and cocompactly on \( C \), any orbit of \( \Lambda \) in \( C \) is quasiconvex. Since \( C \) is convex in \( DM(\Gamma) \), \( \Lambda \) is a quasiconvex subgroup of \( C(\Gamma) \) (with respect to the generating set \{ vertices of \( \mathcal{G}(\Gamma) \} \)). It then follows from Theorem 6.1 that \( C(\Gamma) \) virtually retracts onto \( \Lambda \). This gives, by restriction to \( \Gamma \), a virtual retraction of \( \Gamma \) onto \( \Lambda \).

Consider now a compact totally geodesic immersed submanifold \( F \) in \( M = M(\Gamma) \). Let \( \Lambda \) be its fundamental group; it is clearly a quasiconvex subgroup of \( \Gamma \) which thus virtually retracts onto it. Denote \( \Gamma' \) the finite index subgroup of \( \Gamma \) which retracts \( \Gamma' \rightarrow \Lambda \) onto \( \Lambda \). Theorem 1.3 follows with \( \hat{M} = \Gamma' \backslash \mathbb{H}^n \).

\[ \square \]

## 7 Cup-products of geodesic class

The above proof shows that if \( \Gamma' \) is as in theorem 1.3, it virtually retracts onto any of its quasiconvex subgroups. Theorem 1.4 will thus easily follow from the following (well known) lemma.

**Lemma 7.1** Let \( F_1 \) and \( F_2 \) as in theorem 1.4 with respective fundamental groups \( \Lambda_1 \) and \( \Lambda_2 \). Then, there exists a finite index subgroup \( \Gamma' \subset \Gamma \) such that \( \Lambda_1 \subset \Gamma' \) and the group

\[
\Lambda_1 \ast_{\Lambda_1 \cap \Lambda_2} \Lambda_2',
\]

where \( \Lambda_2' = \Gamma' \cap \Lambda_2 \), injects as a quasiconvex subgroup in \( \Gamma' \).

*Proof.* We prove that there exists \( \Gamma' \) such that the group generated by \( \Lambda_1 \) and \( \Lambda_2' \) in the isometry group of \( \mathbb{H}^n \) is convex-cocompact and isomorphic to

\[
\Lambda_1 \ast_{\Lambda_1 \cap \Lambda_2} \Lambda_2'.
\]

We first prove that the group \( \langle \Lambda_1, \Lambda_2' \rangle \) is convex-cocompact. As it is a subgroup of a cocompact subgroup \( \Gamma' \) of the group of isometries of the hyperbolic \( n \)-space, it suffices to prove that it is geometrically finite, i.e. that some (any) Dirichlet polyhedron of \( \langle \Lambda_1, \Lambda_2' \rangle \) is finite-sided. This is easily checked using the
separability of $\Lambda_1$ (equal to the stabilizer of a totally geodesic subspace of $\mathbb{H}^n$) in $\Gamma$. For any (large) real $R > 0$, this indeed implies the existence of a finite index subgroup $\Gamma' \subset \Gamma$ such that $\Lambda_1 \subset \Gamma'$ and every element in $\Gamma' - \Lambda_1$ has translation length bigger than $R$.

Take a point $x$ belonging to the totally geodesic subspace fixed by $\Lambda_1 \cap \Lambda_2$ and form the Dirichlet polyhedron $D$ associated to $x$ and the group $\langle \Lambda_1, \Lambda_2' \rangle$. Denote by $H_1$ and $H_2$ the (finite) sets of hyperplanes bounding the Dirichlet polyhedron associated to $x$ and the (convex-cocompact) groups $\Lambda_1$ and $\Lambda_2'$ respectively. If $R$ is chosen large enough, a hyperplane $H_1$ of $H_1$ intersects a hyperplane $H_2$ of $H_2$ only if $H_2$ is a hyperplane bounding the Dirichlet polyhedron associated to $x$ and the (convex-cocompact) group $\Lambda_1 \cap \Lambda_2$. The hyperplanes bounding $D$ are then all in $H_1 \cup H_2$ and the convex-cocompactness follows.

It moreover follows from the above proof that the only relations in the group $\langle \Lambda_1, \Lambda_2' \rangle$, the so-called Poincaré relations corresponding to the intersections of two bounding hyperplanes of $D$, are the relations inside $\Lambda_1$, the relations inside $\Lambda_2'$ and the amalgamation along $\Lambda_1 \cap \Lambda_2$. The group $\langle \Lambda_1, \Lambda_2' \rangle$ is thus isomorphic to $\Lambda_1 \ast_{\Lambda_1 \cap \Lambda_2} \Lambda_2'$.

This proves lemma 7.1. \qed

Proof of theorem 1.4. Let $\Gamma$ be as in theorem 1.4. Passing to a finite index subgroup we may assume $\Gamma$ is as in the proof of theorem 1.3. Let $\Lambda_1$ and $\Lambda_2$ be the fundamental groups of $F_1$ and $F_2$ in the statement of theorem 1.4. According to lemma 7.1, passing to a finite index subgroup of $\Gamma$ we may assume that the group generated by $\Lambda_1$ and $\Lambda_2$ in $\Gamma$ is a quasiconvex subgroup isomorphic to $\Lambda_1 \ast_{\Lambda_1 \cap \Lambda_2} \Lambda_2$.

Theorem 1.3’s proof implies $\Gamma$ virtually retracts onto $\Lambda_1 \ast_{\Lambda_1 \cap \Lambda_2} \Lambda_2$ and thus that $\Lambda_1 \ast_{\Lambda_1 \cap \Lambda_2} \Lambda_2$ is separable inside $\Gamma$. There exists in particular a finite cover $\tilde{M}$ of $M$ such that the convex core of $\Lambda_1 \ast_{\Lambda_1 \cap \Lambda_2} \Lambda_2$ injects in $\tilde{M}$. But $F_1$ and $F_2$ both inject into this convex core and intersect along a connected totally geodesic manifold with fundamental group $\Lambda_1 \cap \Lambda_2$. That this totally geodesic submanifold is non trivial in $H_4(\tilde{M})$ follows from (the proof of) theorem 1.3. \qed

Note that theorem 1.4’s proof (especially lemma 7.1) furthermore implies that one can have $\tilde{M}$ such that $\tilde{F}_1 = F_1$.

By duality between homology and cohomology, theorem 1.4 implies that many geodesic class, i.e. harmonic forms dual to totally geodesic (embedded) submanifolds, have non zero cup-product. This supports a conjecture made in [?].

8 Relation to the Lefschetz’ properties of [?]

\footnote{Recall that this follows from lemma 1.6.}

\footnote{Recall that proofs of separability theorems was the first goal of Scott’s method.}
We now explicit the relations between theorems 1.2, 1.3 and 1.4 and the conjectured Lefschetz’ properties of \[ \ldots \]. These properties are stated in \[ \ldots \] for congruence manifolds and at the level of projective limits (to avoid the sentence “up to a finite cover”), this require some notations.

Let \( G \) be an anisotropic \( \mathbb{Q} \)-algebraic group such that its group of real points, \( G(\mathbb{R}) \), is the product (with finite intersection) of a compact group by \( G^{\text{nc}} = SO(n,1) \). If \( \Gamma' \subset \Gamma \) are two congruence subgroups of \( G(\mathbb{Q}) \), we get a finite covering of compact manifolds:

\[
\Gamma' \backslash \mathbb{H}^n \to \Gamma \backslash \mathbb{H}^n. \tag{8.3}
\]

This induces a surjective map

\[
H_*(\Gamma' \backslash \mathbb{H}^n) \to H_*(\Gamma \backslash \mathbb{H}^n) \tag{8.4}
\]

and an injective map

\[
H^*(\Gamma \backslash \mathbb{H}^n) \to H^*(\Gamma' \backslash \mathbb{H}^n). \tag{8.5}
\]

Here (co-)homology groups are with complex coefficients. We thus get a direct (resp. inverse) system of homology (resp. cohomology) groups indexed by congruence subgroups of \( G(\mathbb{Q}) \). By taking the direct (resp. inverse) limit, we define \(^7\)

\[
H_*(\text{Sh}^0G) = \lim_\leftarrow H_*(\Gamma \backslash \mathbb{H}^n), \tag{8.6}
\]

\[
H^*(\text{Sh}^0G) = \lim_\rightarrow H^*(\Gamma \backslash \mathbb{H}^n). \tag{8.7}
\]

Consider now a \( \mathbb{Q} \)-subgroup \( H \) of \( G \), of the same type, with \( H^{\text{nc}} = SO(k,1) \) \((1 \leq k \leq n-1)\). The symmetric space associated to \( H \) then embeds in \( \mathbb{H}^n \) - the symmetric space associated to \( G \) - as a totally geodesic \( k \)-dimensional subspace \( \Sigma \subset \mathbb{H}^n \). The immersion

\[
\Gamma \backslash \Sigma \to M(\Gamma)
\]

induces the natural map

\[
H_*(\Gamma \backslash \Sigma) \to H_*(M(\Gamma)). \tag{8.8}
\]

Now if \( g \in G(\mathbb{Q}) \) consider more generally the immersion

\[
(g^{-1}\Gamma g \cap \text{H}) \backslash \Sigma \to M(\Gamma)
\]

induced by the totally geodesic embedding \( \Sigma \to \mathbb{H}^n \); \( x \mapsto g \cdot x \). The virtual restriction map is the natural map

\[
H^*(M(\Gamma)) \to \prod_{g \in G(\mathbb{Q})} H^*((g^{-1}\Gamma g \cap \text{H}) \backslash \Sigma), \tag{8.10}
\]

induced from the family of maps (8.9) for varying \( g \in G(\mathbb{Q}) \). Taking the limit as \( \Gamma \) varies over all the congruence subgroup of \( G(\mathbb{Q}) \), we get the natural maps

\[
H_*(\text{Sh}^0H) \to H_*(\text{Sh}^0G) \quad \text{and} \quad H^*(\text{Sh}^0G) \to \prod_{g \in G(\mathbb{Q})} H^*(\text{Sh}^0H). \tag{8.11}
\]

We can now state the main conjecture of \[ \ldots \].

\(^7\)Note that these are only notations; we won’t deal with such a space as \( \text{Sh}^0G \) and thus won’t try to give a meaning to its (co)homological groups.
Conjecture 8.1 Let $H$ and $G$ be as above. Then,

1. for any integer $i \geq n/2$, the natural map
   \[ H_i(\text{Sh}^0 H) \to H_i(\text{Sh}^0 G) \]
   is injective;

2. for any nonnegative integer $i \leq k/2$, the natural map
   \[ H^i(\text{Sh}^0 G) \to \prod_{g \in G(\mathbb{Q})} H^i(\text{Sh}^0 H) \]
   is injective.

Theorem 1.2 implies that part 1. of conjecture 8.1 is “almost” true in case $k = n - 1$, in the sense that it is true if we consider all (maybe noncongruence) arithmetic subgroups $\Gamma \subset G(\mathbb{Q})$. In counterpart in that case ($k = n - 1$) and, more generally, in the standard case where $G$ comes, by restriction of scalars, from an orthogonal group over a totally real field, part 1. of conjecture 8.1 then holds with $\mathbb{Z}$ coefficients and for any $i$. In the case $k < n - 1$, this last statement follows from the proof of theorem 1.2 by adding a “fake hyperplane” corresponding to $\Sigma$ to the cube complex $C$.

Part 2. of conjecture 8.1 is known only for $i = 1$, see however [?] for some evidences. Theorem 1.4 may be related to it: if the cup-product of a cohomology class $\omega$ with some geodesic class dual to $F$ is non zero, then the restriction of $\omega$ to $F$ is certainly non trivial in cohomology.

9 Generalisations

Compact Coxeter (hyperbolic) groups also have lots of hyperplanes and the proofs of theorems 1.3 and 1.4 still works in this case. Note that hybrid (non arithmetic) cocompact lattices of $SO(n,1)$, constructed by Gromov and Piatetski-Shapiro [?] also have lots of hyperplanes and thus also embed in abstract right-angled Coxeter groups. More simply note that the hybrid cocompact lattices of Gromov and Piatetski-Shapiro are still “arithmetic” in the following weak sense.

Proposition 9.1 Let $\Gamma$ be a hybrid cocompact lattice in $SO(n,1)$ as constructed by Gromov and Piatetski-Shapiro in [?]. Then $\Gamma$ virtually embeds as a quasiconvex subgroup of a standard cocompact arithmetic lattice in $SO(n+1,1)$.

Proof. First recall the construction of the hybrid hyperbolic manifolds. Let $F$ be a totally real field and $(V_0, q_0)$ be a quadratic space over the field $F$ such that the group of real points of the $\mathbb{Q}$-algebraic group $G_0 = \text{Res}_{F/\mathbb{Q}}(O(V_0))$ is isomorphic to $SO(n - 1,1)$ modulo a compact factor. Let $a_1, a_2$ be two totally positive numbers in $F$. The $F$-quadratic space

$$(V_1, q_1) = (F \oplus V_0, a_1 x^2 \oplus q_0), \quad (V_2, q_2) = (F \oplus V_0, a_2 x^2 \oplus q_0)$$

and

$$(V_3, q_3) = (F \oplus F \oplus V_0, a_1 x^2 \oplus a_2 y^2 \oplus q_0)$$
yield cocompact arithmetic lattices in $SO(n, 1)$, for $V_1$ and $V_2$, and in $SO(n + 1, 1)$, for $V_3$. Let $G_i$, $i = 1, 2, 3$, be the corresponding $\mathbb{Q}$-algebraic groups. Choose a torsion free congruence subgroup $\Gamma_3 \subset G_3(\mathbb{Q})$ and let $\Gamma_i = G_i(\mathbb{Q}) \cap \Gamma_3$ for $i = 0, 1, 2$. Passing to a deeper congruence subgroup $\Gamma_3$ we may further assume that $\Gamma_0 \backslash \mathbb{H}^{n-1}$ embeds inside each $\Gamma_i \backslash \mathbb{H}^n$, $i = 1, 2$, as an hyperplane. A hybrid manifold is then obtained by a cut and paste procedure of $\Gamma_i \backslash \mathbb{H}^n$, $i = 1, 2$, along the hyperplane $\Gamma_0 \backslash \mathbb{H}^{n-1}$.

Note that the quadratic subspaces $V_1, V_2 \subset V_3$ yield hyperplanes in $\mathbb{H}^{n+1}$ which intersect along a codimension two totally geodesic subspace corresponding to $V_0$. The proof of lemma 7.1 then easily implies that, passing to a finite index subgroup of $\Gamma_3$ we may assume that the universal cover of our hybrid manifold embeds in $\mathbb{H}^{n+1}$ as a quasi-convex, $\Gamma_3$-finite union of pieces of totally geodesic hyperplanes. This implies proposition 9.1. □

By restricting the virtual retractions already constructed for the standard cocompact arithmetic lattice in $SO(n + 1, 1)$ (in the hybrid case), or simply by copying the “special” cubulation of standard cocompact arithmetic lattices in $SO(n, 1)$ (in the Coxeter case) we get:

**Theorem 9.2** Let $M$ be a compact hyperbolic manifold whose fundamental group is commensurable either with a Coxeter group or with an hybrid lattice constructed by Gromov and Piatetski-Shapiro. Then the fundamental group of $M$ virtually retracts onto any of its quasiconvex subgroups.

Without changes the proofs of theorem 1.3 and 1.4 give the following two results.

**Theorem 9.3** Let $M$ be a compact hyperbolic manifold as in theorem 9.2. Then for any compact totally geodesic immersed submanifold $F$ in $M$ there exists a finite cover $\hat{M}$ of $M$ and a connected component $\hat{F}$ of the preimage of $F$ in $\hat{M}$ such that $\hat{F}$ is embedded and

$$H_k(\hat{F}) \to H_k(\hat{M})$$

is injective for every integer $k \geq 0$.

**Theorem 9.4** Let $M$ be a compact hyperbolic manifold as in theorem 9.2 and let $F_1$ and $F_2$ be two totally geodesic immersed submanifolds in $M$. Assume $F_1$ and $F_2$ transversally intersects in at least one point. Then there exists a finite cover $\hat{M}$ of $M$ and two connected components $\hat{F}_1$ and $\hat{F}_2$ of the preimage of $F$ in $\hat{M}$ such that $\hat{F}_1$ and $\hat{F}_2$ are both embedded in $\hat{M}$ and their intersection $\hat{F}_1 \cap \hat{F}_2$ is connected and non trivial in $H_*(\hat{M})$.

**Remark.** Ian Agol has recently found a new way to construct non-arithmetic hyperbolic manifolds, see [?]. He works only with hyperbolic manifolds of dimension four but as he points out the method is general and our results are exactly what he needs to extend it to higher dimensions. He constructed compact hyperbolic manifolds with arbitrary small geodesics. These manifolds have also a lot of hyperplanes and theorem 1.3 and 1.4 still apply. Note however that it is not clear if the lattices he construct embed as quasiconvex subgroups in
higher dimensional arithmetic lattices; such embeddings should not be obtained by “bending” along a hypersurface as in the proof of proposition 9.1 because of the existence of small geodesics \footnote{We would like to thank Ian Agol for pointing out a mistake in a first version of this remark.}.

Finally note that it is not necessary to assume $M$ compact to get a cubulation (proper action on a $CAT(0)$ cube complex), it is still true when $M$ is of finite volume. It then follows easily from above that theorem 1.2 still hold when $M$ is only of finite volume. The separation of quasiconvex subgroup in non cocompact arithmetic lattices will be the subject of another paper of the last two authors.

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