SCATTERING THEORY ON GEOMETRICALLY FINITE QUOTIENTS WITH RATIONAL CUSPS

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Abstract. We study Eisenstein functions and scattering operator on geometrically finite hyperbolic manifolds with infinite volume and rational non-maximal rank cusps. For both we prove the meromorphic extension and we show that the scattering operator belongs to a certain class of pseudo-differential operators on the conformal infinity which is a manifold with fibred boundaries. Then we obtain results relating Q-curvature of the boundary, scattering operator at energy n and renormalized volume.

On montre le prolongement méromorphe des fonctions d'Eisenstein et de l'opérateur de diffusion sur les variétés hyperboliques géométriquement finie, de volume infini, dont les cusps de rang non-maximal sont rationnels. Dans ce cas, l'opérateur de diffusion appartient à une certaine classe d'opérateur pseudo-differentiel sur l'infini conforme, qui est une variété avec bord fibré. Enfin on obtient quelques résultats reliant Q-courbure de l'infini conforme, operateur de diffusion à energie n et volume renormalisé.

1. Introduction and results

The purpose of this work is to study the Eisenstein functions and scattering operator on a class of geometrically finite hyperbolic quotients $\Gamma\backslash\mathbb{H}^{n+1}$ with non-maximal rank cusps. As a consequence, we investigate relations between the conformal geometry of the boundary (which is non-compact) and the scattering operator, in the spirit of Graham-Zworski's work on asymptotically Einstein manifolds [7].

Such problems involving spectral and scattering theory on geometrically finite hyperbolic quotients have been studied probably since Selberg and lead to many important results. However, most of the results known are obtained when the group has no parabolic elements of non-maximal rank, in other words when the quotient $X = \Gamma\backslash\mathbb{H}^{n+1}$ has no cusps of non-maximal rank. As far as we know, the only results concerning meromorphic extension of the resolvent or scattering operator for this cases were due, until recently, to Froese-Hislop-Perry [3] in dimension 3. However, Bunke and Olbrich [1] deal in a preprint with the meromorphic extension of the scattering operator in all generality using a very different approach; in particular they do not study the (pseudo-differential) structure of this operator. We lead the reader to the introduction of [8] for a more detailed review of works touching meromorphic extension of the resolvent for the Laplacian through the essential spectrum, resonances (i.e. the poles of this extension), meromorphic continuation of Eisenstein functions and scattering operator for geometrically finite hyperbolic manifolds, though we do not claim to be complete about references therein.

We consider an infinite volume hyperbolic quotient $X := \Gamma\backslash\mathbb{H}^{n+1}$ where Γ is a discrete group of isometries of $\mathbb{H}^{n+1}$ which admits a fundamental domain with finitely many sides, X is said geometrically finite, and such that each parabolic subgroup of Γ does not contain irrational rotation. For exemple, this last condition is always satisfied in dimension $n+1 = 3$ and, in general, can be reduced to the case where each parabolic subgroup is conjugate to a lattice of translations in $\mathbb{R}^n$ (in the model $\mathbb{H}^{n+1} = (0,\infty)\times\mathbb{R}^n$), possibly by passing to a finite cover, thus resolvent, scattering operator and Eisenberg functions are obtained as a finite sum on the cover. Similarly, elliptic elements of Γ can also be excluded by passing to a finite cover, X is then a

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smooth manifold, and since the presence of maximal-rank cusps do not add difficulties, we will avoid them for simplicity of exposition. The Laplacian on such manifolds have been studied by Froese-Hislop-Perry [3] in dimension 3 and by Perry [23] in higher dimension. The manifold X equipped with the hyperbolic metric is complete and the spectrum of the Laplacian Δ_X splits into continuous spectrum $[\frac{n^2}{4},\infty)$ and a finite number of L^2 eigenvalues included in $(0,\frac{n^2}{4})$ which form the point spectrum $\sigma_{pp}(\Delta_X)$ (see Lax-Phillips [14]). In [8] we proved that the modified resolvent

$$R(\lambda) := (\Delta_X - \lambda(n-\lambda))^{-1}$$

extends from $\{\Re(\lambda) > \frac{n}{2}\}$ to $\mathbb{C}$ meromorphically with poles of finite multiplicity (i.e. the rank of the polar part in the Laurent expansion at each pole is finite) from $L^2_{comp}(X)$ to $L^2_{loc}(X)$, these poles are called resonances.

In the present work, we define a Poisson operator, Eisenstein functions, a scattering operator and we show that they extend meromorphically to $\mathbb{C}$. To explain the main Theorems, we recall briefly the structure at infinity of the manifold X but in any case, we lead to reader to Section 2 of Mazzeo-Phillips [19] for a comprehensive description of geometrically finite quotients $\Gamma\backslash\mathbb{H}^{n+1}$ (see also [2, 23, 8]). The first approach is to see X as the interior of a smooth compact manifold with boundary $\bar{X}$. If ρ is a boundary defining function of the boundary $\partial\bar{X}$ and if g is the hyperbolic metric on X, then $\rho^2 g$ extends as a smooth non-negative tensor on $\bar{X}$ which is a metric outside some submanifolds of the boundary $\partial\bar{X}$ where it becomes degenerate. Each one of these submanifolds arises from a cusp point of X, i.e. a fixed point at infinity of $\mathbb{H}^{n+1}$ for a parabolic subgroup of Γ, and is diffeomorphic to a k-dimensional torus T^k if the parabolic subgroup has rank k. If we note c the union of these submanifolds, $B = \partial\bar{X}\setminus c$ is a non-compact manifold which can be thought as the infinity of X; actually $B = \Gamma\backslash\Omega$ where $\Omega\subset S^n$ is the domain of discontinuity of Γ. By blowing-up these submanifolds in $\bar{X}$, this gives a manifold $\bar{X}_c$ with corners of codimension 2 which is the compactification of X defined by Mazzeo-Phillips [19] in the general case. The topological boundary of $\bar{X}_c$ splits into two kind of smooth hypersurfaces with boundaries, the regular ones whose union is a compactification $\bar{B}$ of B and the cusp ones which are diffeomorphic to $S^{n-k}_+\times T^k$, S^{n-k}_+ being an $n-k$ dimensional half-sphere with boundary. It turns out that B has ends diffeomorphic to $(\mathbb{R}^{n-k}_y\setminus\{|y|<1\})\times T^k$, each end arising from a rank-k parabolic subgroup of Γ fixing a point at infinity of $\mathbb{H}^{n+1}$. The compactification $\bar{B}$ of B corresponds to the radial compactification in the y variable in each end thus $\bar{B}$ is a fibred boundary manifold in the sense of Mazzeo-Melrose [18], the fibrations being the projections

$$S^{n-k-1}\times T^k \to S^{n-k-1}.$$

When equipped with the metric $h_0 := \rho^2 g|_B$, (B,h_0) is conformal to an 'exact Φ-type metric' near its infinity as defined in [18], the conformal factor decreasing enough to make the volume of B finite - the vanishing rate is even stronger than the fibred cusp metrics (see Figure 1 for illustration).

We construct Poisson and scattering operators $\mathcal{P}(\lambda), S(\lambda)$ by solving a Poisson problem in a way similar to that introduced on Euclidean manifolds by Melrose and on many other settings by various authors (see [21] for review). However, in view of the sensible structure of the metric near the cusps c, it appears that $\mathcal{P}(\lambda), S(\lambda)$ do not act naturally on $C^\infty(\partial\bar{X})$ but much on subspaces related to this structure. We then define the subalgebra $C^\infty_{\rm acc}(\bar{X})$ of $C^\infty(\bar{X})$ of functions which are asymptotically constant in the cusps, these are the $f\in C^\infty(\bar{X})$ such that

$$Z(f|_c) = 0, \quad Z((X_1\dots X_N f)|_c) = 0$$

for all smooth vector fields $X_1,\dots,X_N$ on $\bar{X}$ ($\forall N\in\mathbb{N}$) and all smooth vector fields Z on c. In other words, these are the functions whose restrictions at the cusp submanifolds are locally constant and similarly for all derivatives. It is actually possible to find a boundary defining function ρ in this subalgebra. Then the volume form ${\rm dvol}_g$ of g can be expressed by $\rho^{-n-1}R_c^2\mu_{\bar{X}}$

for a function R_c which is smooth positive in $\bar{X}\setminus c$ with $R_c^2\in C^\infty_{\rm acc}(\bar{X})$ vanishing at order $2k$ at each k-dimensional component of c and where $\mu_{\bar{X}}$ is a smooth volume density on $\bar{X}$. The functions R_c and ρ are not uniquely determined but we show that the set $R_c^{-1}C^\infty_{\rm acc}(\bar{X})$ is independent of the choice of R_c^2,ρ in $C^\infty_{\rm acc}(\bar{X})$ (but it certainly depends on the metric). Then we define $C^\infty_{\rm acc}(\partial\bar{X})$ and $R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X})$ by restriction of $C^\infty_{\rm acc}(\bar{X})$ and $R_c^{-1}C^\infty_{\rm acc}(\bar{X})$ at $\partial\bar{X}$ and $B=\partial\bar{X}\setminus c$ (here we use the same notation for R_c and its restriction $R_c|_{\partial\bar{X}}$). For any boundary defining function $\rho\in C^\infty_{\rm acc}(\bar{X})$, one can define the Poisson operator $\mathcal{P}(\lambda)$ by showing that if $\Re(\lambda)\geq\frac{n}{2}$ and $\lambda\notin\frac{n}{2}+\mathbb{N}$, then for all $f\in R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X})$ there exists a unique solution $\mathcal{P}(\lambda)f$ of the following Poisson problem

$$\left\{\begin{array}{l}(\Delta_X-\lambda(n-\lambda))\mathcal{P}(\lambda)f=0\\ \mathcal{P}(\lambda)f=\rho^{n-\lambda}F(\lambda,f)+\rho^\lambda G(\lambda,f)\\ F(\lambda,f),G(\lambda,f)\in R_c^{-1}C^\infty_{\rm acc}(\bar{X})\\ F(\lambda,f)|_{\rho=0}=f\end{array}\right. .$$

The construction of the solution is a consequence of an indicial equation for Δ_X and the precise mapping property of the extended resolvent

$$R(\lambda):\dot{C}^\infty(\bar{X})\to\rho^\lambda R_c^{-1}C^\infty_{\rm acc}(\bar{X}).$$

where $\dot{C}^\infty(\bar{X})$ is the set of functions in $C^\infty(\bar{X})$ vanishing at all order at $\partial\bar{X}$.

Next we analyze Eisenstein functions. The metric h_0 induces an $L^2(B)$ Hilbert space on B and we prove

Theorem 1.1. *If $R(\lambda;w;w')$ denotes the Schwartz kernel of the modified resolvent, the Eisenstein function*

$$E(\lambda;b;w'):=\lim_{w\to b}[\rho(w)^{-\lambda}R(\lambda;w;w')],\quad b\in B,w'\in X$$

is a smooth function on $B\times X$ if λ is not a resonance. There exists $C>1$ such that for all $N>0$ it is the Schwartz kernel of a meromorphic operator

$$E(\lambda):\rho^NL^2(X)\to L^2(B)$$

in $\Re(\lambda)>\frac{n}{2}-C^{-1}N$ with poles of finite multiplicity, satisfying $\mathcal{P}(\lambda)=(2\lambda-n)^tE(\lambda)$ on $R_c^{-1}C^\infty_{acc}(\partial\bar{X})$. Except possibly at $\{\lambda;\Re(\lambda)<\frac{n}{2},\lambda(n-\lambda)\in\sigma_{pp}(\Delta_X)\}$, the set of poles of $E(\lambda)$ coincide with the set of resonances.

Using the asymptotic expression of $\mathcal{P}(\lambda)f$, the scattering operator is then defined (with the same notations) by

$$S(\lambda):\left\{\begin{array}{ccc}R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X})&\to&R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X})\\ f&\to&F(\lambda,f)|_{\rho=0}\end{array}\right. .$$

For $\Re(\lambda)=\frac{n}{2}$, $S(\lambda)$ can be extended to $L^2(B)$ as a unitary operator and it gives, in a sense, a parametrization of the absolutely continuous spectrum of Δ_X. Then, we prove the following result which is expressed in more details in Theorem 6.5, Lemma 6.1, Corollary 6.3 and Proposition 7.1:

Theorem 1.2. *The scattering operator $S(\lambda)$ extends meromorphically to $\mathbb{C}$ as a family of pseudo-differential operators in the full Φ-calculus on the manifold with fibred boundary $\bar{B}$ in the sense of Mazzeo-Melrose* [18]. *In $\{\Re(\lambda)\leq\frac{n}{2},\lambda(n-\lambda)\notin\sigma_{pp}(\Delta_X)\}$, λ_0 is a pole of $S(\lambda)$ if and only if λ_0 is a resonance and it has finite multiplicity. In $\{\Re(\lambda)>\frac{n}{2}\}$, $S(\lambda)$ has only first order poles whose residue is*

$$Res_{\lambda_0}S(\lambda)=\left\{\begin{array}{ll}-\frac{(-1)^{j+1}2^{-2j}}{j!(j-1)!}P_j+\Pi_{\lambda_0}&\text{if }\lambda_0=\frac{n}{2}+j,j\in\mathbb{N}\\ \Pi_{\lambda_0}&\text{if }\lambda_0\notin\frac{n}{2}+\mathbb{N}\end{array}\right.$$

where P_j is the j-th GJMS conformal Laplacian of [6] *on (B,h_0) and Π_{λ_0} is an operator with rank* $\dim\ker_{L^2}(\Delta_X-\lambda_0(n-\lambda_0))$.

Note that the GJMS conformal Laplacians P_j in [6] are well-defined for all j if $n \geq 3$ (resp. for $j \leq 1$ if $n = 2$) if the manifold is locally conformally flat (it is actually done in the compact setting but they can be extended for non-compact manifolds by using the same local expression in the curvature tensor), which is the case for B.

In last part, we prove some results similar to Graham-Zworski theorems in [7] for this class of manifolds. By changing the boundary defining function $\hat{\rho} = e^{\omega}\rho \in C^{\infty}_{\rm acc}(\bar{X})$ (with $\omega \in C^{\infty}_{\rm acc}(\bar{X})$) we obtain a metric $\hat{h}_0 := \rho^2 g|_B = e^{2\omega_0}h_0$ on B conformal to h_0 (where $\omega_0 = \omega|_B$), this induces a subconformal class $[h_0]_{\rm acc}$ of h_0 on the boundary B. If we replace ρ by $\hat{\rho}$ in Poisson problem, this defines different Poisson and scattering operators $\hat{\mathcal{P}}(\lambda), \hat{S}(\lambda)$ and by uniqueness, $\hat{S}(\lambda)$ is related to $S(\lambda)$ by the covariant rule $\hat{S}(\lambda) = e^{-\lambda\omega_0}S(\lambda)e^{(n-\lambda)\omega_0}$, thus $\hat{S}(\lambda)$ depends only on the conformal representative $\hat{h}_0$ and $S(\lambda)$, this makes the scattering operator a conformally covariant operator with respect to the subconformal class $[h_0]_{\rm acc}$. Similarly $\hat{P}_j$ is related to P_j by the covariant rule $\hat{P}_j = e^{-(\frac{n}{2}+j)\omega_0}P_j e^{(\frac{n}{2}-j)\omega_0}$.

If n is even, one can use the operators P_j to define Branson's Q-curvature of h_0 on B and we show

Theorem 1.3. *Let n be even, then for any choice of ρ, the Q-curvature of $h_0 = \rho^2 g|_B$ on the boundary B satisfies*

$$Q = \frac{(-1)^{\frac{n}{2}}2^{-n}}{\frac{n}{2}!(\frac{n}{2}-1)!}S(n)1.$$

Moreover it has a conormal behaviour of order $-n$ at $\partial\bar{B}$, is in $L^1(B, dvol_{h_0})$ and

$$\frac{(-1)^{\frac{n}{2}}2^{1-n}}{\frac{n}{2}!(\frac{n}{2}-1)!}\int_B Q \ dvol_{h_0} = L$$

where L is the log term, independent of ρ, appearing in the expansion of the volume

$$vol_X(\{\rho > \epsilon\}) \sim c_0\epsilon^{-n} + \cdots + c_{n-2}\epsilon^{-2} + L\log(\epsilon^{-1}) + V + o(1).$$

To conclude, we deduce from Theorem 1.3 and the fact that every geometrically finite 3-manifolds satisfy our assumptions (since there is no rotational part in this case by lack of dimension),

Corollary 1.4. *If $X = \Gamma\backslash\mathbb{H}^3$ is a geometrically finite hyperbolic manifold, its renormalized volume is*

$$L = -\pi\chi(\bar{B}) = -\pi\chi(\partial\bar{X}) = -2\pi\chi(\bar{X})$$

where $\chi(\bullet)$ means Euler characteristic.

This gives a generalization in dimension 3 of Epstein's formula [22] for the renormalized volume of a convex co-compact hyperbolic manifold. These results show a certain continuity when a convex co-compact group degenerates to a cusp case.

The case of irrational cusps is more technically involved and it is not clear if such precise results can be obtained, at least the meromorphic extension of the resolvent will probably be carried out in a following paper. It is also important to add that this analysis could be used to study the divisors of Selberg's zeta function as Patterson-Perry [22] did for convex co-compact hyperbolic manifolds.

The paper is organized as follows: we first introduce in section 2 the geometric setting, discuss the compactification $\bar{X}$ of the manifold X and analyze its infinity B; then in section 3 we define the class of pseudo-differential operator on B which contains the scattering operator and in section 4 we study the mapping properties and the structure of the resolvent for the Laplacian. In section 5, we construct the Poisson operator and Eisenstein functions using section 4 and in section 6 we define and describe the scattering operator. To conclude we investigate the relation between the conformal geometry of B and the scattering theory on X.

Along the paper, we will identify operators with their Schwartz kernel and we consider operators acting on functions for simplicity of exposition though the correct approach would be to use half-densities. Consequently the kernels of pseudo-differential operators have to be understood as tensorized by appropriate half-densities.

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2. Geometry of the manifold

2.1. **Assumptions on the group.** We describe here with more details the assumptions about the cusps discussed roughly in the introduction; we strongly use Section 2 of Mazzeo-Phillips [19]. Let Γ a discrete subgroup of orientation preserving isometries of the hyperbolic space $\mathbb{H}^{n+1}$. Recall that Γ acts also on the natural compactification $\bar{\mathbb{H}}^{n+1} = \{m \in \mathbb{R}^{n+1}; ||m|| \leq 1\}$ of $\mathbb{H}^{n+1}$ and on its boundary S^n; an element γ is called hyperbolic if it fixes exactly two points on S^n and no point in $\mathbb{H}^{n+1}$, parabolic if it fixes one point on S^n and no point in $\mathbb{H}^{n+1}$, then γ is elliptic if it fixes a point of $\mathbb{H}^{n+1}$. If Γ contains elliptic elements, there exists a subgroup Γ_0 of finite index of Γ without elliptic elements thus X is finitely covered by $\Gamma_0\backslash\mathbb{H}^{n+1}$, the latter being a smooth manifold. Since we study resolvent of the Laplacian and other related objects, we can always pass to a finite cover without difficulties: objects on X can indeed be obtained by summing on a finite set objects on the finite cover. Thus we exclude elliptic elements in Γ. We suppose that Γ is geometrically finite, which means here that it admits a fundamental domain F with finitely many sides. Each fixed point $p \in S^n$ of a parabolic element of Γ is called a cusp point, and for each cusp point p, let Γ_p be the subgoup of Γ fixing p. Actually Γ_p contains only parabolic elements and it can be shown that there is a Γ_p invariant neighbourhood U_p of p such that $\Gamma\backslash(F \cap U_p)$ is isometric to a neighbourhood of p in $\Gamma_p\backslash(F \cap U_p)$. The subgroup Γ_p has a maximal free abelian subgoup Γ_a with rank k, the rank of the cusp p is defined to be the integer k. We suppose that $k \leq n-1$ for each p since this case is well known in term of scattering theory. Using now conjugation, it suffices to look at the case where $p = \infty$ in the upper half model $\mathbb{H}^{n+1} = \mathbb{R}^+ \times \mathbb{R}^n$. Section 2 of [19] (the arguments come from Thurston's lecture notes) shows that there is an affine subspace $\mathbb{R}^k \subset \mathbb{R}^n$ globally fixed by Γ_∞ on which Γ_a acts as a group of k translations. This allows to see that every $\gamma \in \Gamma_\infty$ acts as

$$\gamma(y, z) = (Ry, Az + b) \text{ on } \mathbb{R}_y^{n-k-1} \oplus \mathbb{R}_z^k$$

for some $A \in O(k), R \in O(n-k-1)$ and $b \in \mathbb{R}^k$; elements in Γ_a have $A = \text{Id}$. There is a flat compact manifold $N = \Gamma_\infty\backslash\mathbb{R}^k$ such that $\Gamma_\infty\backslash\mathbb{R}^n$ is a flat vector bundle with basis N and $T^k := \Gamma_a\backslash\mathbb{R}^k$ such that $\Gamma_a\backslash\mathbb{R}^n$ is a flat bundle over T^k. We assume that the holonomy representation of these bundles $\Gamma \to O(n-k-1)$ has finite image, so that all rotations R have rational angle $p\pi/q$ for some $p, q \in \mathbb{N}$. Then there is a finite cover of this bundle which is $T^k \times \mathbb{R}^{n-k}$, T^k being a flat torus, and it suffices to study the case where each rotation R is the identity.

2.2. **Neighbourhoods of infinity, models.** From previous discussions and assumptions on the cusps and using [2, 23, 8] we obtain a covering of the manifold X by model charts. There exists a compact K of X such that $X \setminus K$ is covered by a finite number of charts isometric to either a regular neighbourhood (M_r, g_r) or a rank-k cusp neighbourhood (M_k, g_k) where

$$M_r := \{(x, y) \in (0, \infty) \times \mathbb{R}^n; x^2 + |y|^2 < 1, \}, \quad g_r = x^{-2}(dx^2 + dy^2),$$

$$M_k := \{(x, y, z) \in (0, \infty) \times \mathbb{R}^{n-k} \times T^k; x^2 + |y|^2 > 1\}, \quad g_k = x^{-2}(dx^2 + dy^2 + dz^2)$$

for $k = 1, \ldots, n-1$ with (T^k, dz^2) a k-dimensional flat torus.

Note that we could allow maximal rank cusps as in [8] without difficulties but since these cases are well-known, we restrict ourselves to the non-maximal rank cusps cases for simplicity of exposition. We will make as if there was only one neighbourhood of each type to simplify the notations, we then note I_r, $(I_k)_k$ the corresponding chart isometries. One can also choose the covering such that $I_k^{-1}(M_k)\cap I_j^{-1}(M_j)=\emptyset$ for $k\neq j$, possibly by adding regular neighbourhoods.

The model M_k can be considered as a subset of the quotient $X_k=\Gamma_k\backslash\mathbb{H}^{n+1}$ of $\mathbb{H}^{n+1}$ by a rank-k parabolic subgroup Γ_k of Γ which fixes a single point at infinity of $\mathbb{H}^{n+1}$. Indeed, modulo conjugation by an isometry, one can suppose that the fixed point is the point at infinity of $\mathbb{H}^{n+1}$ in the half-space model $(0,\infty)\times\mathbb{R}^n$. Γ_k can then be considered as a lattice of k independent translations acting on $\mathbb{R}^n$, therefore it is the image of the lattice $\mathbb{Z}^k$ by a map $A_k\in GL_k(\mathbb{R})$ and the flat torus $T^k:=\Gamma_k\backslash\mathbb{R}^k$ is well defined. Then X_k is isometric to $\mathbb{R}^+_x\times\mathbb{R}^{n-k}_y\times T^k_z$ equipped with the metric

$$g_k=\frac{dx^2+dy^2+dz^2}{x^2}$$

dz^2 being the flat metric on a k-dimensional torus T^k. Therefore M_k is the subset of X_k with $x^2+|y|^2>1$. As a matter of fact it will be often useful to consider $\mathbb{R}^+\times\mathbb{R}^{n-k}$ as the $n-k+1$-dimensional hyperbolic space $\mathbb{H}^{n-k+1}$. Hence X_k can be compactified into the compact manifold with boundary $\bar{X}_k=\bar{\mathbb{H}}^{n-k+1}\times T^k$ where $\bar{\mathbb{H}}^{n-k+1}$ is the ball $\{|w|\leq 1\}$ in $\mathbb{R}^{n-k+1}$. Then

$$\rho_k(x,y,z):=\frac{x}{|y|^2+x^2+1}=(2\cosh(d_{\mathbb{H}^{n-k+1}}(x,y;1,0)))^{-1}$$

is a natural boundary defining function in $\bar{X}_k$ ($\partial\bar{X}_k=\{\rho_k=0\}$ and $d\rho_k\neq 0$ on $\partial\bar{X}_k$). Let us define the new coordinates

$$t:=\frac{x}{x^2+|y|^2},\quad u:=\frac{-y}{x^2+|y|^2}\tag{2.1}$$

which induce an isometry from (M_k,g_k) to

$$\{(t,u,z)\in(0,\infty)\times\mathbb{R}^{n-k}\times T^k;t^2+|u|^2<1\}$$

equipped with the metric

$$\frac{dt^2+du^2+(t^2+|u|^2)^2dz^2}{t^2}\tag{2.2}$$

and $\rho_k(t,u)=\rho_k(x,y)$. These coordinates can be thought as compactification coordinates for M_k, since t and u extend smoothly to $\bar{X}_k\setminus\{x=y=0\}$. The infinity of X in the chart M_k is then given by $\{\rho_k=0\}$ or equivalently $\{t=0\}$. Also we will call cusp submanifold the submanifold $\{t=u=0\}$ of $\bar{X}_k$ it will be noted c_k and we remark that $c_k\simeq\infty\times T^k\simeq T^k$ in $\bar{X}_k$ where ∞ is the point at infinity in the half-space model of $\mathbb{H}^{n-k+1}$. We also have $M_k=\{w\in X_k;t(w)^2+|u(w)|^2<1\}$ which is a subset of $\bar{X}_k$ and we will denote

$$\bar{M}_k:=\{w\in\bar{X}_k;t^2(w)+|u(w)|^2<1\}.$$

At last we define the manifold

$$Y_k:=\mathbb{R}^{n-k}\times T^k$$

which can be viewed as $(\bar{X}_k\setminus c_k)\cap\{x=0\}$.

The model M_r is simpler and can be considered as a subset of $\mathbb{H}^{n+1}$. We define as before $\bar{M}_r:=\{(x,y)\in[0,\infty)\times\mathbb{R}^n;x^2+|y|^2<1\}$.

There exist some smooth functions $\chi,\chi^r,\chi^1,\dots,\chi^{n-1}$ on respectively $X,M_r,M_1,\dots,M_{n-1}$ which, through the isometric charts $I_r,I_1,\dots,I_n$, satisfy

$$I_r^*\chi^r+\sum_{k=1}^{n-1}I_k^*\chi^k+\chi=1$$

with χ having compact support in X. Note that it is possible to choose χ^k which does not depend on the variable $z \in T^k$.

For what follows we will consider $M_k, M_r, \bar{M}_k, \bar{M}_r$ as neighbourhoods in $\bar{X}$ instead of using the notations $I_k^{-1}(M_k), I_r^{-1}(M_r)$...

2.3. **Compactification, volume densities.** Using the previous discussion, one obtains a compactification of X as a smooth compact manifold with boundary $\bar{X}$. Moreover, with no loss of generality one can choose a boundary defining function ρ which is equal to the function t in each neighbourhood $\bar{M}_k$. The boundary $\partial\bar{X}$ is covered by some charts $B_1, \dots, B_{n-1}, B_r$ induced by $M_1, \dots, M_{n-1}, M_r$ by taking

$$B_k := \bar{M}_k \cap \partial\bar{X} \simeq \{(u, z) \in \mathbb{R}^{n-k} \times T^k; |u|^2 < 1\}$$

$$B_r := \bar{M}_r \cap \partial\bar{X} \simeq \{y \in \mathbb{R}^n; |y|^2 < 1\}.$$

From the discussion above, we see that the metric on X can be expressed by

$$g = \frac{H}{\rho^2}$$

with H a smooth non-negative symmetric 2-tensor on $\bar{X}$ which degenerates at the cusps submanifolds $(c_k)_{k=1,\dots,n-1}$. Let us define $c := (\cup_k c_k) \subset \partial\bar{X} \subset \bar{X}$, and $B := \partial\bar{X} \setminus c$, then the restriction

$$h_0 := H|_B = (\rho^2 g)|_B \tag{2.3}$$

is a smooth metric on the non-compact manifold B.

We will also need to use functions representing the distance to the cusps submanifolds as follows: for $k = 1, \dots, n-1$, let r_{c_k} be a continuous non-negative function in $\bar{X}$, smooth and positive in $\bar{X} \setminus c_k$ which satisfies

$$I_{k*}(r_{c_k}) = \sqrt{t^2 + |u|^2}$$

in $\bar{M}_k$ and is equal to 1 in M_j when $j \neq k$. Then we define the functions

$$r_c := \prod_{k=1}^{n-1} r_{c_k}, \quad R_c := \prod_{k=1}^{n-1} (r_{c_k})^k \tag{2.4}$$

on $\bar{X}$ and we will also denote by r_{c_k}, r_c and R_c their restriction to $\partial\bar{X}$. It can easily be checked that B equipped with the metric h_0 of (2.3) has a volume density dvol_{h_0} which is of the form

$$\mathrm{dvol}_{h_0} = R_c^2 \mu_{\partial\bar{X}} \tag{2.5}$$

with $\mu_{\partial\bar{X}}$ a smooth non-vanishing density (volume density) on $\partial\bar{X}$. Similarly the volume density dvol_g on X can be expressed by

$$\mathrm{dvol}_g = \rho^{-n-1} R_c^2 \mu_{\bar{X}} \tag{2.6}$$

for a smooth volume density $\mu_{\bar{X}}$ on $\bar{X}$. In what follows, we will write $L^2(X)$ and $L^2(B)$ for the Hilbert spaces of square integrable functions on X and B with respect to the volume densities dvol_g and dvol_{h_0}.

2.4. **Class of functions.** For a compact manifold $\bar{M}$ with boundary $\partial\bar{M}$, we denote by $\dot{C}^\infty(\bar{M})$ the set of smooth functions on $\bar{M}$ which vanish at all orders at $\partial\bar{M}$. Its topological dual is the set of extendible distribution on $\bar{M}$, denoted $C^{-\infty}(\bar{M})$ (note that a correct definition would include density bundles).

There will be a special set of smooth functions on $\bar{X}, \partial\bar{X}$ which will play an important role for what follows, these are the functions which are “asymptotically constant in the cusp variables”.

To give a precise definition we begin by introducing the sets $\mathcal{C}(T\bar{X})$, $\mathcal{C}(T\partial\bar{X})$ and $\mathcal{C}(Tc)$ of smooth vector fields on $\bar{X}, \partial\bar{X}, c$. Then we set

$$C^\infty_{\rm acc}(\bar{X}) := \{f \in C^\infty(\bar{X}); \forall X_1, \dots, X_N \in \mathcal{C}(T\bar{X}), \forall Z \in \mathcal{C}(Tc), Z(f|_c) = 0, Z(X_1 \dots X_N f|_c) = 0\}$$

and $C^\infty_{\rm acc}(\partial\bar{X}), C^\infty_{\rm acc}(\bar{X}_k), C^\infty_{\rm acc}(\partial\bar{X}_k)$ are defined similarly by replacing $\bar{X}$ by $\partial\bar{X}, \bar{X}_k, \partial\bar{X}_k$. These functions are constant on each cusp submanifold c_k and their derivatives too. In local coordinates (t, u, z) near the cusp $c_k = \{t = u = 0\}$, one can check by a Taylor expansion at $(0, 0, z) \in c_k$ and Borel Lemma that a function $f \in C^\infty_{\rm acc}(\bar{X})$ can be decomposed locally as a sum

$$f(t, u, z) = f_0(t, u) + O((t^2 + |u|^2)^\infty) = f_0(t, u) + O(r_c^\infty) \tag{2.7}$$

for some f_0 smooth. We remark the following properties, the proofs of which are straightforward:

Lemma 2.1. *The set $C^\infty_{acc}(\bar{X})$ is a subalgebra of $C^\infty(\bar{X})$ which is stable under the action of $\mathcal{C}(T\bar{X})$, and stable by composition with smooth real functions.*

Observe also that r_c^2 and R_c^2 defined by (2.4) are in $C^\infty_{\rm acc}(\bar{X})$. Actually this implies that if $\hat{\rho} \in C^\infty_{\rm acc}(\bar{X})$ is a boundary defining function of $\partial\bar{X}$ and $\hat{R}_c^2 \in C^\infty_{\rm acc}(\bar{X})$ is a non-negative function vanihing at order $2k$ at each c_k such that ${\rm dvol}_g = \hat{\rho}^{-n-1}\hat{R}_c^2\hat{\mu}_{\bar{X}}$ for a smooth volume form on $\bar{X}$, then

$$\hat{\rho} = F_1\rho, \quad \hat{R}_c^2 = F_2 R_c^2, \quad \hat{\mu}_{\bar{X}} = F_3\mu_{\bar{X}}$$

for some functions $F_1, F_2 \in C^\infty_{\rm acc}(\bar{X})$ and $F_3 \in C^\infty(\bar{X})$ satisfying $F_1^{-n-1}F_2F_3 = 1$ and $F_1 > 0$, $F_3 > 0$. Then necessarily $F_3 \in C^\infty_{\rm acc}(\bar{X})$ and $F_2 > 0$ which shows that $R_c^{-1}C^\infty_{\rm acc}(\bar{X}) = \hat{R}_c^{-1}C^\infty_{\rm acc}(\bar{X})$ and this space does not depend on the choices of ρ, R_c^2 in $C^\infty_{\rm acc}(\bar{X})$. Actually the map $f \to f|{\rm dvol}_g|^{\frac{1}{2}}$ naturally identifies $R_c^{-1}C^\infty(\bar{X})$ with the space of smooth half-densities $C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}})$ defined in the 0-calculus of Mazzeo-Melrose [17] (depending only on the C^∞ structure of $\bar{X}$) and the space $R_c^{-1}C^\infty_{\rm acc}(\bar{X})$ could then be considered as a subspace of $C^\infty(\bar{X}, \Gamma_0^{\frac{1}{2}})$ (depending on the metric) if we worked with densities.

We also define the set of smooth functions on $\bar{X}_k$ (resp. $\bar{X}$) vanishing at all order at the cusps

$$\dot{C}^\infty_c(\bar{X}) := \{f \in C^\infty(\bar{X}); \forall X_1, \dots, X_N \in \mathcal{C}(T\bar{X}), f|_c = 0, (X_1 \dots X_N f)|_c = 0\}$$

and $\dot{C}^\infty_c(\partial\bar{X}), \dot{C}^\infty_c(\partial\bar{X}_k), \dot{C}^\infty_c(\partial\bar{X}_k)$ similarly. Remark that there is a natural identification

$$\dot{C}^\infty_c(\partial\bar{X}) \longleftrightarrow \dot{C}^\infty(\bar{B})$$

if $\bar{B}$ is defined as the blow-up of $\partial\bar{X}$ around c. By similar arguments, the spaces $C^\infty_{\rm acc}(\partial\bar{X})$, $\dot{C}^\infty_c(\partial\bar{X})$, $R_c^{-1}C^\infty_{\rm acc}(\partial\bar{X})$ can be defined (here we note again R_c instead of $R_c|_B$) an they coincide with the restriction of $C^\infty_{\rm acc}(\bar{X})$, $\dot{C}^\infty_c(\bar{X})$, and $R_c^{-1}C^\infty_{\rm acc}(\bar{X})$ at $B = \partial\bar{X} \setminus c$.

To conclude this part, remark the following inclusions

$$\dot{C}^\infty(\bar{X}) \subset \dot{C}^\infty_c(\bar{X}) \subset C^\infty_{\rm acc}(\bar{X}).$$

and the same for their restriction at B.

2.5. **Model form for the metric.** To use the same ideas than for asymptotically hyperbolic manifolds, we need to choose boundary defining functions of $\partial\bar{X}$ in $\bar{X}$ which induce product decompositions of the metric near infinity. The different choices of boundary defining functions induce a conformal class of smooth tensors on $\partial\bar{X}$ which are metrics on B, this is the conformal class $[h_0]$ of $h_0 := \rho^2 g|_{\partial\bar{X}}$. However, in view of the presence of the cusps, we need to consider the following smaller class of conformal metrics on B

$$[h_0]_{\rm acc} := \{fh_0; f > 0 \in C^\infty_{\rm acc}(\partial\bar{X})\}.$$

Lemma 2.2. *For all $\hat{h}_0 \in [h_0]_{acc}$, there exists a boundary defining function $\hat{\rho} \in C^\infty_{acc}(\bar{X})$ of $\partial\bar{X}$ in $\bar{X}$ such that $|d\hat{\rho}|_{\hat{\rho}^2 g} - 1 \in \dot{C}^\infty(\bar{X})$ in a collar neighbourhood of $\partial\bar{X}$ and $\hat{\rho}^2 g|_B = \hat{h}_0$. Moreover, $\hat{\rho}$ is uniquely determined modulo $\dot{C}^\infty(\bar{X})$ by $\hat{h}_0$.*

Proof: for $\hat{h}_0 \in [h_0]$, the construction of a boundary defining function $\hat{\rho} = \rho e^{\omega}$ which satisfies $|d\hat{\rho}|_{\hat{\rho}^2 g} = 1$ and $\hat{\rho}^2 g|_B = h_0$ is equivalent to solving the PDE

$$2(\nabla_{\rho^2 g}\rho)(\omega) + \rho|d\omega|^2_{\rho^2 g} = \frac{1 - |d\rho|^2_{\rho^2 g}}{\rho} \tag{2.8}$$

with initial condition $\omega|_{\partial\bar{X}} = \omega_0$ where $\hat{h}_0 = e^{2\omega_0}h_0$ (see [4, Lem. 2.1]). The construction of a solution is possible in regular neighbourhoods $\bar{M}_r$ and is unique since the equation is non-characteristic there. In $\bar{M}_k$, we write the equation in coordinates and this gives

$$2\partial_t\omega + t\left((\partial_t\omega)^2 + |\partial_u\omega|^2 + (t^2 + |u|^2)^{-2}|\partial_z\omega|^2\right) = 0$$

in view of the form of the metric (2.2) there (recall that $\rho = t$ in $\bar{M}_k$). Taking this equation at $t = 0$, we see that $\partial_t\omega|_{t=0} = 0$ and by differentiating it N times with respect to t and setting $t = 0$ we see by induction that all the values $\partial_t^j\omega|_{t=0}$ in $\{u \neq 0\}$ are determined by $\omega|_{t=0}$ for $j \leq N + 1$. In particular when j is odd this is 0 (see again [4] for a similar study). Since $w_0 \in C^\infty_{\rm acc}(\partial\bar{X})$, we can write it locally under the form (2.7) which shows by induction that $\partial_t^j\omega|_{t=0} \in C^\infty_{\rm acc}(\partial\bar{X})$; the essential arguments to use are that the singular term in the equation is killed by $|\partial_z\omega| = O((t^2 + |u|^2)^\infty)$ and the properties of $C^\infty_{\rm acc}(\partial\bar{X})$ discussed previously. By using Borel lemma, we can construct a smooth function ω in a neighbourhood of $\partial\bar{X}$ in X with those derivatives, thus ω satisfies (2.8) modulo $O(\rho^\infty)$ and this proves that there exists a function $\hat{\rho}$ which satisfies the Lemma, the uniqueness of its Taylor expansion with respect to ρ at $\partial\bar{X}$ is clear from the construction. □

We will now use this function to obtain a certain model form of the metric near $\partial\bar{X}$. Using again the same arguments than [4, 9], it suffices to consider the collar neighbourhood $[0, \epsilon)_s \times \partial\bar{X}$ of $\partial\bar{X}$ induced by the flow $\varphi_s(m)$ of the gradient $\nabla_{\hat{\rho}^2 g}\hat{\rho}$ with initial condition $\varphi_0(m) = m$ for $m \in \partial\bar{X}$, that is the diffeomorphism

$$\varphi : (s, m) \to \varphi_s(m)$$

from $[0, \epsilon) \times \partial\bar{X}$ to its image. We consider the function ω constructed in the proof of previous Lemma (thus $\hat{\rho} = \rho e^{\omega}$) and since $\partial_s\hat{\rho}(\varphi_s(m)) = 1 + O(\rho^\infty) = 1 + O(s^\infty)$, we deduce

$$\rho = se^{-\omega} + O(s^\infty).$$

Now, we remark that the identity $|\nabla_{\hat{\rho}^2 g}\hat{\rho}|_{\hat{\rho}^2 g} = 1 + O(s^\infty)$ implies that $s^2 g$ can be expressed by

$$s^2\varphi^* g = ds^2 + \hat{h}(s) + O(s^\infty)$$

in $[0, \epsilon) \times \partial\bar{X}$ where $\hat{h}(s)$ is a smooth family of tensors on $\partial\bar{X}$ which are positive for $s > 0$, with $\hat{h}(0) = \hat{h}_0$ positive on B. We have seen in the proof of last Lemma that, in $\bar{M}_k$, ω is an even function of $\rho = t$, thus s is an odd function of t and t is an odd function of s. Let $(v, \zeta) \in \mathbb{R}^{n-k} \times T^k$ some coordinates on $\partial\bar{X}$ near c_k. We have $\varphi_0(v, \zeta) = (v, \zeta)$ and using the form (2.2) of g

$$\partial_s\varphi_s(v, \zeta) = \nabla_{\hat{\rho}^2 g}\hat{\rho} = e^{-\omega}(1 + t\partial_t\omega)\partial_t + te^{-\omega}\partial_u\omega.\partial_u + \frac{te^{-\omega}}{(t^2 + |u|^2)^2}\partial_z\omega.\partial_z$$

then the function $\varphi(s, v, \zeta) = \varphi_s(v, \zeta)$ can be locally written near c_k (in coordinates (t, u, z))

$$\varphi(s, v, \zeta) = \Big(t = se^{-\omega} + t_1, u = v + su_1, z = \zeta + sz_1\Big) \tag{2.9}$$

$$t_1 \in \dot{C}^\infty(\bar{X}), \quad u_1 \in C^\infty_{\rm acc}(\bar{X}), \quad z_1 \in \dot{C}^\infty_c(\bar{X}).$$

Using that ω is even in s and t odd in s, it is straightforward to verify that u, z are even in s. We deduce that locally

$$dt = l_1(s, v, ds, dv) + O(r_c^\infty), \quad du = l_2(s, v, ds, dv) + O(r_c^\infty), \quad dz = d\zeta + O(r_c^\infty). \tag{2.10}$$

for some smooth tensors l_1, l_2, even in s. We want now to write the metric g in these coordinates (s, v, ζ). By looking at the expression (2.2) and using (2.9), (2.10) with the properties of $C^\infty_{\rm acc}(\bar{X})$ discussed in previous section, we obtain that

$$\hat{h}(s) = h_1(s, v, dv) + h_2(s, v, z, dv, d\zeta) + e^{2\omega} r_c^4 d\zeta^2 + O(s^\infty) \tag{2.11}$$

where h_1, h_2 are smooth tensors, even in s, such that $h_2 = O(r_c^\infty)$. Since $\hat{\rho} - s = O(\hat{\rho}^\infty)$, we can replace s by $\hat{\rho}$ in (2.11) and we have the same expression for the metric. Now in a regular neighbourhood M_r, there exists coordinates $(x, y) \in (0, \epsilon) \times \mathbb{R}^n$ such that $g = x^{-2}(dx^2 + dy^2)$, thus by writing $\hat{\rho} = xe^\theta$ for some θ smooth, we have by mimicking last Lemma that (from (2.8))

$$2\partial_x\theta + x((\partial_x\theta)^2 + |\partial_y\theta|^2) = O(x^\infty)$$

with $\theta|_{x=0} = \theta_0$ satisfying $\hat{h}_0 = e^{2\theta_0}dy^2$. Exactly as before for M_k, this gives that $\hat{\rho}$ is odd in x, thus x is odd in s and y even in s, which easily implies that $\hat{h}(s)$ has an even Taylor expansion in s at $s = 0$.

This discussion proves that there exists a collar neighbourhood $(0, \epsilon)_{\hat{\rho}} \times \partial\bar{X}$ of $\partial\bar{X}$ in $\bar{X}$ such that

$$g = \frac{d\hat{\rho}^2 + \hat{h}(\hat{\rho})}{\hat{\rho}^2} + O(\hat{\rho}^\infty) \tag{2.12}$$

for a smooth family of symmetric tensors $\hat{h}(\hat{\rho})$ on $\partial\bar{X}$ with an even Taylor expansion in $\hat{\rho}$ at $\hat{\rho} = 0$, positive for $\hat{\rho} > 0$, $\hat{h}(0) = \hat{h}_0$ being positive on B and with the local expression (2.11) near the cusps c_k. Actually, the evenness of the metric in $\hat{\rho}$ is a consequence of the constant curvature of X and is studied in detail in [9] for asymptotically hyperbolic manifolds.

Is is quite direct and similar to a result of Graham [4] to check that for two functions $\hat{\rho}_1, \hat{\rho}_2$ satisfying Lemma 2.2, then for all $j \in \mathbb{N}$

$$\partial^{2j}_{\hat{\rho}_1}\hat{\rho}_2|_{\partial\bar{X}} = 0, \quad \partial^{2j}_{\hat{\rho}_2}\hat{\rho}_1|_{\partial\bar{X}} = 0$$

which will be useful to define renormalized volume in an invariant way.

There is however a very special case of boundary defining function $\hat{\rho}$ which can be chosen to put the metric into a simpler form. It is obtained by taking $\hat{\rho} = t$ in the neighbourhood $\bar{M}_k$ of the cusp c_k and extending it to a neighbourhood of $\partial\bar{X}$ so that it satisfies $|d\hat{\rho}|_{\hat{\rho}^2 g} = 1$ in this neighbourhood and $\hat{\rho}^2 g|_{\partial\bar{X}} = h_0$. To prove the existence of such an extension, it suffices to go back to the proof of Lemma 2.2 and we see that this amounts to solve the PDE (2.8) without the error term $O(\rho^\infty)$ and with initial condition $\omega|_{\partial\bar{X}} = 0$. Since the equation is non-characteristic out of the cusp c, there exists a unique solution ω in some neighbourhood $\{\rho < \epsilon, \delta < r_c\}$ (for some $\delta, \epsilon > 0$) of the boundary $\partial\bar{X}$ avoiding the cusp c, and it is clear that $\omega = 0$ satisfies the equation in $\bar{M}_k$.

For what follows, we will often work with this boundary defining functions $\hat{\rho}$ and by convention we will note it ρ, forgetting the previous choice of function ρ. Then we have in some collar neighbourhood $(0, \epsilon)_\rho \times \partial\bar{X}$ of $\partial\bar{X}$

$$g = \frac{d\rho^2 + h(\rho)}{\rho^2} \tag{2.13}$$

for some smooth family of symmetric tensors $h(\rho)$ on $\partial\bar{X}$, depending smoothly on ρ, positive for $\rho > 0$, with $h(0) = h_0$ positive on B and satisfying

$$h(\rho) = du^2 + (\rho^2 + |u|^2)^2 dz^2$$

in each $\bar{M}_k$.

2.6. **Geometry of B.** To study the scattering operator and to define the class of pseudo-differential operators which contains it, we can consider the manifold B as the union of a compact manifold $\mathcal{E}_r$ (covered by the charts B_r) and $n-1$ ends $\mathcal{E}_1, \dots, \mathcal{E}_k$ with $\mathcal{E}_k$ diffeomorphic to

$$\{(y,z) \in \mathbb{R}^{n-k} \times T^k; |y| > 1\} \subset Y_k = \mathbb{R}^{n-k} \times T^k.$$

For simplicity, we will consider $\mathcal{E}_k$ as this last subset of Y_k. By using the radial compactification in the y variable in each end $\mathcal{E}_k$ we see that the manifold B compactifies in a smooth compact manifold with boundary $\bar{B}$, the boundary $\partial\bar{B}$ being a disjoint union on $k = 1, \dots, n-1$ of products $\partial\mathcal{E}_k := S^{n-k-1} \times T^k$. A boundary defining function of $\partial\mathcal{E}_k$ is given by $v = r_{c_k} = r_c = |y|^{-1}$ and r_c is a boundary defining function of $\partial\bar{B}$. Note that $\partial\bar{B} \neq \partial\bar{X}$ but $\partial\bar{B}$ is actually the blow-up of $\partial\bar{X}$ around the cusps submanifolds $c_1, \dots, c_{n-1}$. The structure of the compactified manifold $\bar{B}$ near $\partial\mathcal{E}_k$ is $[0,1)_v \times \partial\mathcal{E}_k$ and $\partial\mathcal{E}_k$ is a fibred boundary in the sense that there is a fibration (this is the projection here)

$$\phi_k : S^{n-k-1} \times T_k \to S^{n-k-1}. \tag{2.14}$$

The metric h_0 on B is not exactly a fibred cusp metric since too much decreasing at infinity

$$h_0 = dv^2 + v^2 d\omega^2 + v^4 dz^2.$$

For following purposes, it is also quite natural to consider B with the metric $\widetilde{h}_0 := r_c^{-4} h_0$ conformal to h_0 since this is the flat metric $dy^2 + dz^2$ on each end $\mathcal{E}_k$. Note that $\widetilde{h}_0$ in $(0,1)_v \times S_\omega^{n-k-1} \times T_z^k$ is

$$\widetilde{h}_0 = \frac{dv^2}{v^4} + \frac{d\omega^2}{v^2} + dz^2$$

which is an "exact Φ-metric" in the sense of Mazzeo-Melrose [18]. The volume induced by the metric h_0 on B is finite whereas the volume of B with the metric $\widetilde{h}_0$ is not finite.

3. Pseudo-differential operators at infinity

There is a natural way to define pseudo-differential operators on B using the euclidean structure of each end $\mathcal{E}_k$. Recall first from Schwartz theorem that for any continuous linear operator $A : \dot{C}^\infty(\bar{B}) \to C^{-\infty}(\bar{B})$ there exists a unique extendible distribution $a \in C^\infty(\bar{B} \times \bar{B})$ (we dropped the density factor for simplicty), called Schwartz kernel, such that

$$\langle A\phi, \psi\rangle = \langle a, \psi \otimes \phi\rangle, \quad \forall \phi, \psi \in \dot{C}^\infty(\bar{B}).$$

Thus we will identify Schwartz kernel with its associated operator. We can define the space $\Psi^{m,l}(B)$ of pseudo-differential operators of order $(m,l) \in \mathbb{R}^2$ as the set of linear operators

$$A : \dot{C}^\infty(\bar{B}) \to C^{-\infty}(\bar{B}) \tag{3.1}$$

such that in each compact coordinate patch on B (those are the B_r of previous section), A has a distributional Schwartz kernel of the type

$$A(w; w') = \int_{\mathbb{R}^n} e^{i\xi.(w-w')} a(w,\xi) d\xi \tag{3.2}$$

with $a(w,\xi)$ a symbol in the coordinate patch, i.e. $a(w,\xi)$ is smooth and

$$|\partial_w^\alpha \partial_\xi^\beta a(w,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|},$$

whereas on the end $\mathcal{E}_k$ with coordinates $w = (y,z) \in \mathbb{R}^{n-k} \times T^k$, the distributional kernel of A is of the form (3.2) but with $a(w;\xi)$ smooth and satisfying

$$|\partial_y^\alpha \partial_z^\beta \partial_\xi^\gamma a(y,z,\xi)| \leq C_{\alpha,\beta,\gamma}(1+|y|)^{-l-|\alpha|}(1+|\xi|)^{m-|\gamma|}.$$

It is not hard to check the mapping property (3.1). One can also define classical (or polyhomogeneous) pseudo-differential operators of order $m, l \in \mathbb{C}$ as operators in $\Psi^{\Re(m),\Re(l)}(B)$ with the symbol in (3.2) satisfying (for all k)

$$a(y,z,\xi) = |y|^{-l}|\xi|^m \widetilde{a}(|y|^{-1}, y/|y|, z, |\xi|^{-1}, \xi/|\xi|) \quad \text{for } |\xi| > 1$$

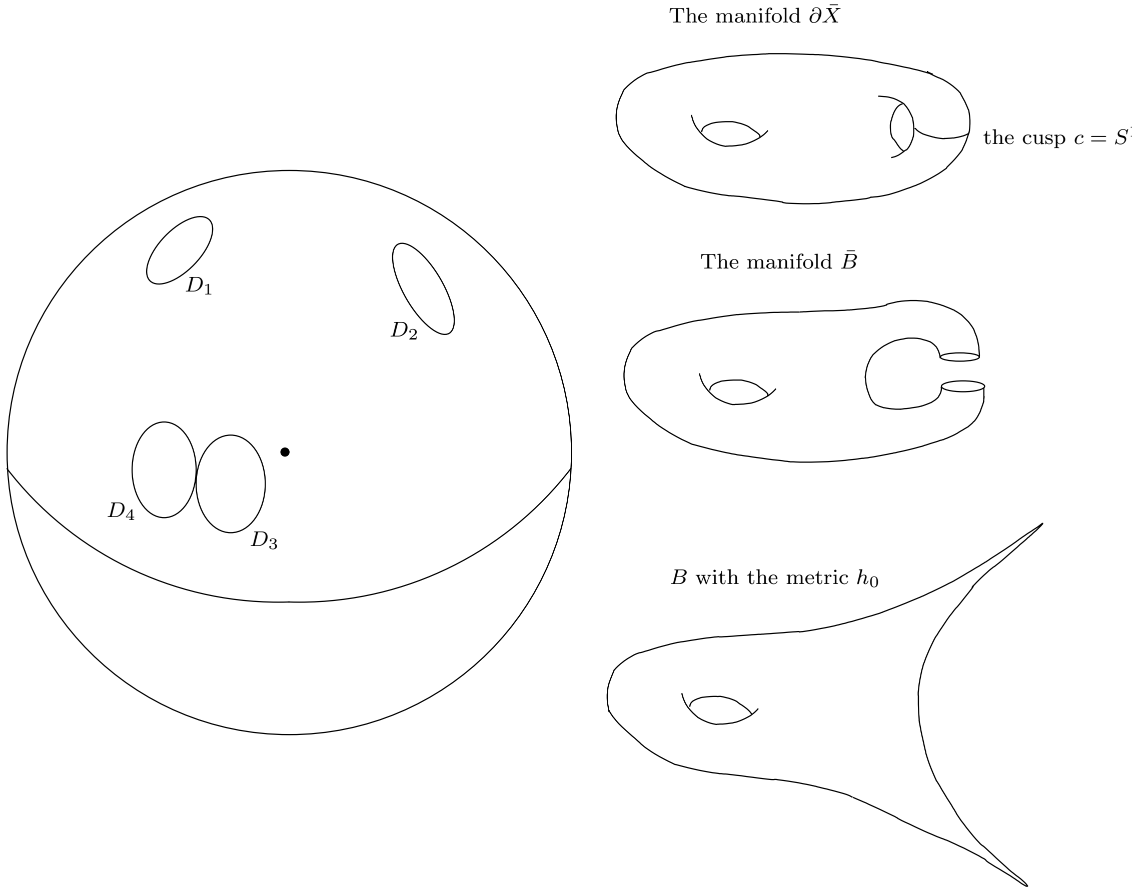

FIGURE 1. The infinity B of the quotient $X = \Gamma\backslash\mathbb{H}^3$ where Γ is a Schottky group gluing $D_3 \longleftrightarrow D_4$ and $D_1 \longleftrightarrow D_2$; $\bar{B}$ is a manifold with fibred boundary.

for some $\widetilde{a} \in C^\infty([0,1) \times S^{n-k-1} \times T^k \times [0,1) \times S^{n-k-1})$, we will use the notation $\Psi^{m,l}_{cl}(B)$. In each end $\mathcal{E}_k$, this corresponds in a sense to the class of pseudo-differential treated by Hörmander in the $y \in \mathbb{R}^{n-k}$ variable (or the Scattering Calculus of Melrose [21]) but with the additional compact variable $z \in T^k$. In particular, an operator $A \in \Psi^{m,l}(B)$ can be defined in term of its distributional kernel lifted from $\bar{B} \times \bar{B}$ to a blown-up version of this product. This is a standard way due to Melrose to describe in details the various singularities of the kernel: we always have the usual conormal singularity at the diagonal of $\bar{X} \times \bar{X}$ (like in the compact setting) but for non-compact manifolds, it is important to include informations in the symbol about the behaviour at infinity, these can be interpreted as conormal singularities for the kernel on the boundaries of the compactification $\bar{X} \times \bar{X}$ (boundary of the compactification = infinity of the manifold). Since singularities with different nature intesects at the diagonal of the corner $\partial\bar{X} \times \partial\bar{X}$, it is convenient to define a bigger manifold, the blow-up, where the kernel is more readable.

The blow-up here is slightly different from that of Scattering Calculus, it is in a sense the scattering blow-up defined in [21] but only in y variable. This blow-up corresponding to manifolds with fibred boundaries is explained in generality by Mazzeo-Melrose in [18], it is achieved in two essential steps. The principle is to start with the manifold with corners $\bar{X} \times \bar{X}$ and to construct a larger manifold with corners where the phase of (3.2) defines a smooth submanifold ("the diagonal") intersecting transversally the boundary of this larger manifold at only one hypersurface.

For what follows, we will use part of the notations of [18]. The manifold $\bar{B}\times\bar{B}$ has $2n-2$ boundary hypersurfaces $\mathcal{L}_k := \partial\mathcal{E}_k\times\bar{B}$, $\mathcal{R}_k = \bar{B}\times\partial\mathcal{E}_k$ for $k=1,\dots,n-1$ and we have $\mathcal{L}_k\cap\mathcal{L}_j=\emptyset$ if $j\neq k$, the same with $\mathcal{R}_k$ and finally $\mathcal{L}_k\cap\mathcal{R}_j=\partial\mathcal{E}_k\times\partial\mathcal{E}_j$ is a corner of codimension 2. We need to define the first blow-up of $\bar{B}\times\bar{B}$ by taking the "b"blow-up

$$\bar{B}\times_b\bar{B} := [\bar{B}\times\bar{B};\partial\mathcal{E}_1\times\partial\mathcal{E}_1;\dots;\partial\mathcal{E}_{n-1}\times\partial\mathcal{E}_{n-1}]$$

which means that we blow-up successively each corner $\partial\mathcal{E}_k\times\partial\mathcal{E}_k$ of $\mathcal{E}_k\times\mathcal{E}_k\subset\bar{B}\times\bar{B}$. This is done by replacing in $\bar{B}\times\bar{B}$ the submanifold $\partial\mathcal{E}_k\times\partial\mathcal{E}_k$ by its spherical normal interior pointing bundle in $\bar{B}\times\bar{B}$. The blow-down map is denoted

$$\beta_b:\bar{B}\times_b\bar{B}\to\bar{B}\times\bar{B}.$$

The manifold $\bar{B}\times_b\bar{B}$ has $3n-3$ boundary hypersurfaces, the first $2n-2$ are the top and bottom faces

$$\mathcal{B}'_k := \overline{\beta_b^{-1}(B\times\partial\mathcal{E}_k)},\quad \mathcal{T}'_k := \overline{\beta_b^{-1}(\partial\mathcal{E}_k\times B)},\quad k=1,\dots,n-1.$$

The new ones are called front faces $(\mathcal{F}'_k)_{k=1,\dots,n-1}$ for the b blow-up and $\mathcal{F}'_k$ is the spherical normal interior pointing bundle of $\partial\mathcal{E}_k\times\partial\mathcal{E}_k$ in $\bar{B}\times\bar{B}$ and is mapped by β_b on $\partial\mathcal{E}_k\times\partial\mathcal{E}_k$. Note that $\mathcal{F}'_k$ is diffeomorphic to $[-1,1]_\tau\times\partial\mathcal{E}_k\times\partial\mathcal{E}_k$ using the function $\tau=\frac{v-v'}{v+v'}$ (see Melrose [20]), thus we will identify them.

The closure $D_b := \overline{\beta_b^{-1}(D_B)}$ of the diagonal D_B of $B\times B$ meets the boundary of $\bar{B}\times_b\bar{B}$ only at the (interior of the) hypersurfaces $\mathcal{F}'_k$ and it does transversally at a submanifold denoted ∂D_b. The blow-up of $\bar{B}\times_b\bar{B}$ along ∂D_b would give the blow-up associated to the Scattering Calculus but it turns out that the second kind of blow-up we need for our purpose are the successive blow-ups of $\bar{B}\times_b\bar{B}$ along the submanifolds

$$\Phi_k=\{(0,m,m')\in\mathcal{F}'_k=[-1,1]_\tau\times\partial\mathcal{E}_k\times\partial\mathcal{E}_k;\phi_k(m)=\phi_k(m')\},$$

with ϕ_k the fibration of (2.14), this gives the manifold with corners

$$\bar{B}\times_\Phi\bar{B} := [\bar{B}\times_b\bar{B};\Phi_1;\dots;\Phi_{n-1}].$$

The blow-down maps are

$$\bar{B}\times_\Phi\bar{B}\overset{\beta_{\Phi-b}}{\longrightarrow}\bar{B}\times_b\bar{B}\overset{\beta_b}{\longrightarrow}\bar{B}\times\bar{B},\quad \beta_\Phi:=\beta_b\circ\beta_{\Phi-b}.$$

The boundaries of $\bar{B}\times_\Phi\bar{B}$ are the top and bottom faces

$$\mathcal{B}_k=\overline{\beta_\Phi^{-1}(B\times\partial\mathcal{B}'_k)},\quad \mathcal{T}_k=\overline{\beta_\Phi^{-1}(\partial\mathcal{B}'_k\times B)}$$

the front faces of the b blow-up

$$\mathcal{F}_k := \overline{\beta_{\Phi-b}^{-1}(\mathcal{F}'_k\setminus\Phi_k)}$$

and the front face of the Φ blow-up is the normal spherical interior pointing bundle of Φ_k in $\bar{B}\times_b\bar{B}$

$$\mathcal{I}_k := SN_+(\Phi_k;\bar{B}\times_b\bar{B}).$$

We will denote by $\rho_{\mathcal{T}_k},\rho_{\mathcal{B}_k},\rho_{\mathcal{F}_k},\rho_{\mathcal{I}_k}$ some functions which define the respective hypersurfaces:

$$\{\rho_{\mathcal{T}_k}=0\}=\mathcal{T}_k,\quad \{\rho_{\mathcal{B}_k}=0\}=\mathcal{B}_k,\quad \{\rho_{\mathcal{F}_k}=0\}=\mathcal{F}_k,\quad \{\rho_{\mathcal{I}_k}=0\}=\mathcal{I}_k.$$

The closure $D_\Phi := \overline{\beta_\Phi^{-1}(D_B)}$ meets the topological boundary of $\bar{B}\times_\Phi\bar{B}$ only at (the interior of) the hypersurfaces $\mathcal{I}_k$ and it does transversally. One can thus define (using extension through the boundary hypersurface) the set $I^m(\bar{B}\times_\Phi\bar{B};D_\Phi)$ of distributions classically conormal of order m to the submanifold D_Φ.

The important point is that β_Φ^* is a one-to-one map between $\dot{C}^\infty(\bar{B}\times\bar{B})$ and $\dot{C}^\infty(\bar{B}\times_\Phi\bar{B})$, this induces a one-to-one map between their respective duals, which allows to indentify continuous operators (3.1) with their Schwartz kernel lifted to $\bar{B}\times_\Phi\bar{B}$. With this identification, we define the space

$$\Psi_\Phi^{m,l}(\bar{B}) := \{K\in\rho_{\mathcal{I}_k}^l I^m(\bar{B}\times_\Phi\bar{B};D_\Phi);\forall k, K\equiv 0 \text{ at } \mathcal{F}_k,\mathcal{T}_k,\mathcal{B}_k\}$$

for $m,l\in\mathbb{C}$, where $\equiv$ means equality of Taylor series. This forms the (classical) "small Φ-calculus" and it is not difficult to check that $\Psi_{cl}^{m,l}(B)=\Psi_\Phi^{m,l}(\bar{B})$ with the notations introduced

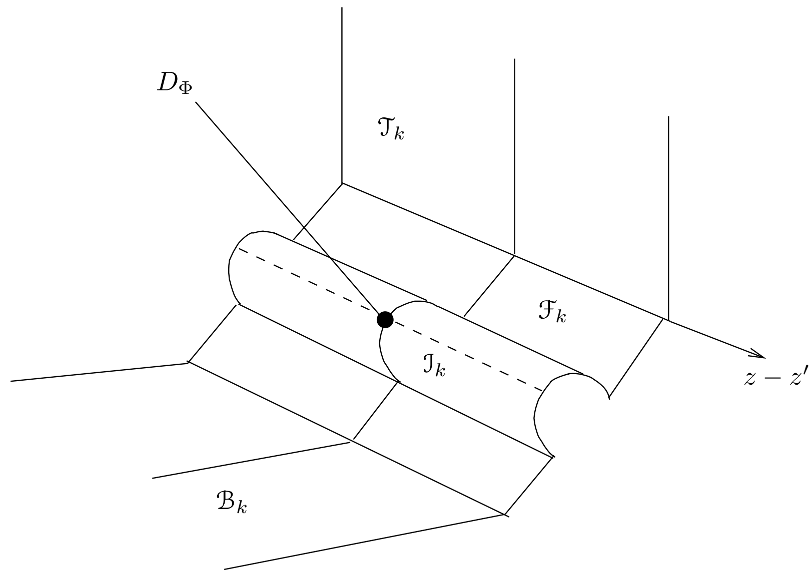

FIGURE 2. The blow-up of Φ_k in $\bar{B} \times_b \bar{B}$

before for the standard pseudo-differential operators on B. We sketch the proof of the sense $\Psi^{m,l}_{cl}(B) \subset \Psi^{m,l}(\bar{B})$. Recall that

$$v = |y|^{-1}, \omega = \frac{y}{|y|}, v' = |y|', \omega' = \frac{y'}{|y'|}, z, z'$$

give some local coordinates near the corner $\partial\mathcal{E}_k \times \partial\mathcal{E}_k$ on $\bar{B} \times \bar{B}$ and

$$s = \frac{v}{v'}, v', \omega, \omega', z, z' \text{ with } |\omega| = |\omega'| = 1$$

give some coordinates on $\bar{B} \times_b \bar{B}$ near the front face $\mathcal{F}'_k$ (valid out of $\mathcal{B}'_k$), in particular $\Phi_k = \{v' = 0; s = 1; \omega = \omega'\}$. If $A \in \Psi^{m,l}_{cl}(B)$, the expression (3.2) with $w = (y, z), w' = (y', z')$ can be put in these coordinates

$$A(w; w') = \int e^{i(\frac{1}{v'}(\frac{\omega}{s} - \omega').\xi_1 + (z-z').\xi_2)} a\Big(\frac{\omega}{v's}, z; \xi_1, \xi_2\Big) d\xi_1 d\xi_2. \tag{3.3}$$

It can be checked that $\frac{\omega_i}{s} - \omega'_i, \omega'_i, v', z, z'$ for $i = 1, \dots, n-k$ give some coordinates near $\mathcal{F}'_k \cap \Phi_k$ and $\Phi_k = \{\frac{\omega}{s} - \omega' = 0\}$. The functions $(\omega_i - s\omega'_i)/(sv')$ lift under $\beta_{\Phi-b}$ to some functions W_i which are smooth near $\mathcal{I}_k \setminus (\mathcal{I}_k \cap \mathcal{F}_k)$ and we have near $D_\Phi \cap \mathcal{I}_k$

$$D_\Phi = \{W_1 = \dots = W_{n-k} = 0; z = z'\}, \quad \mathcal{I}_k = \{v' = 0\}$$

in coordinates $W := (W_1, \dots, W_{n-k}), \omega', v', z, z'$ with $\sum_i {\omega'_i}^2 = 1$. This gives in (3.3)

$$A(w; w') = \int e^{i(W.\xi_1 + (z-z').\xi_2)} a\Big(W + \frac{\omega'}{v'}, z; \xi_1, \xi_2\Big) d\xi_1 d\xi_2$$

with $\{W = 0\} = D_\Phi$. This last expression shows that $A(w; w')$ has a classical conormal singularity at D_Φ of order m. Near the front face $\mathcal{I}_k$, that is when $v' \to 0$, then ${v'}^{-l} a(\frac{W+\omega'}{v'}, z; \xi)$ is a smooth function near $D_\Phi \cap \mathcal{I}_k$. Using other systems of coordinates covering $\mathcal{I}_k \cap \mathcal{F}_k$ one easily see that $\beta^*_\Phi(A)$ vanishes at all order at $\mathcal{F}_k$ (using integration by parts in oscillating integrals and the "polynomial growth" of $a(w, \xi)$ in $|w|$) and that $\rho^{-l}_{\mathcal{I}_k} \beta^*_\Phi(A) \in I^m(\bar{B} \times_\Phi \bar{B}; D_\Phi)$. The vanishing of (3.3) at $\{v' = 0; |\omega - s\omega'| > \epsilon; 1 > s\}$ comes by integration by parts and shows the vanishing of $\beta^*_\Phi(A)$ at all order at the boundaries near $\mathcal{F}_k \cap \mathcal{T}_k$ and the behaviour near $\mathcal{F}_k \cap \mathcal{B}_k$ is similar.

Finally the vanishing at $\mathcal{T}_k$ and $\mathcal{B}_k$ far from $\mathcal{F}_k$ is again a consequence of non-stationary phase (3.2).

The converse $\Psi_{\Phi}^{m,l}(\bar{B}) \subset \Psi_{cl}^{m,l}(B)$ is essentially similar.

Now one can define the "full Φ-calculus" by considering the set of operators (identifying lifted kernels and operators)

$$\Psi_{\Phi}^{m,l,E}(\bar{B}) := \Psi_{\Phi}^{m,l}(\bar{B}) + \prod_{\substack{F=\mathcal{F},\mathcal{I},\mathcal{T},\mathcal{B}\\ k=1,\dots,n-1}} (\rho_{F_k})^{E(F_k)} C^{\infty}(\bar{B} \times_{\Phi} \bar{B}) \tag{3.4}$$

$$E = \{E(\mathcal{T}_1), E(\mathcal{B}_1), E(\mathcal{F}_1), E(\mathcal{I}_1), \dots, E(\mathcal{T}_{n-1}), E(\mathcal{B}_{n-1}), E(\mathcal{F}_{n-1}), E(\mathcal{I}_{n-1})\}, \quad E(F_k) \in \mathbb{C}$$

i.e. we allow some classically conormal singularities at all faces. For operators we deal with, the conormal singularity at the front faces $\mathcal{I}_k$ will be of the same order for both terms, that is $l = E(\mathcal{I}_1) = \dots = E(\mathcal{I}_{n-1})$, hence we will write $\Psi_{\Phi}^{m,E}(\bar{B})$ instead of $\Psi_{\Phi}^{m,l,E}(\bar{B})$. Finally, a subclass with much more regularity will appear as error terms in the expression of the scattering operator, those are operators with kernels of the form

$$\prod_k (r_{c_k})^{a_k} (r'_{c_k})^{b_k} C^{\infty}(\partial\bar{X} \times \partial\bar{X}).$$

where $a_k, b_k \in \mathbb{C}$ and $r_{c_k}(w, w') := r_{c_k}(w)$, $r'_{c_k}(w, w') := r_{c_k}(w')$. Recall again that $\partial\bar{X}$ can be viewed as the smooth compact manifold without boundary obtained from $\bar{B}$ by collapsing each $\partial\mathcal{E}_k \simeq S^{n-k-1} \times T^k$ to $\phi_k(\partial\mathcal{E}_k) = c_k \simeq T^k$.

Actually, since we forgot the density factors for the kernels, the orders of such pseudo-differential operators depend on the density we use to pair two fonctions in $\dot{C}^{\infty}(\bar{B})$, thus it will be necessary to precise it.

4. Resolvent

In this section we analyze the meromorphic extension of the modified resolvent

$$R(\lambda) := (\Delta_X - \lambda(n-\lambda))^{-1}$$

and more precisely the necessary informations we shall need to define Eisenstein functions, Poisson operator and scattering operator. The meromorphic extension of the resolvent is proved in [8] by parametrix construction. Using also spectral theorem, this can be summarized as follows:

Theorem 4.1. *There exists $C > 1$ such that for all $N > 0$, the modified resolvent $R(\lambda)$ on X extends meromorphically with poles of finite multiplicity from $\{\Re(\lambda) > \frac{n}{2}\}$ to $\{\Re(\lambda) > \frac{n}{2} - CN\}$ with values in the bounded operators from $\rho^N L^2(X)$ to $\rho^{-N} L^2(X)$. The only poles of $R(\lambda)$ in $\{\Re(\lambda) > \frac{n}{2}\}$ are first order poles at each λ_0 such that $\lambda_0(n-\lambda_0) \in \sigma_{pp}(\Delta_X)$ and with residue*

$$Res_{\lambda_0} R(\lambda) = (2\lambda_0 - n)^{-1} \sum_{j=1}^{r} \phi_j \otimes \phi_j, \quad \phi_j \in \rho^{\lambda_0} R_c^{-1} C_{acc}^{\infty}(\bar{X}) \subset L^2(X)$$

where $(\phi_j)_{j=1,\dots,r}$ is an orthonormal basis of $\ker_{L^2}(\Delta_X - \lambda_0(n-\lambda_0))$.

Actually the form of ϕ_j is a consequence of (4.20) which will be proved in this section.

To construct the Poisson operator, we need more precise information about the mapping properties of $R(\lambda)$ and about its Schwartz kernel structure near infinity. One of the main points is to analyze the Schwartz kernel of the meromorphic extension of the resolvent

$$R_{X_k}(\lambda) = (\Delta_{X_k} - \lambda(n-\lambda))^{-1}$$

for the Laplacian Δ_{X_k} on the model spaces $X_k = \Gamma_k \backslash \mathbb{H}^{n+1}$, and its mapping properties.

Recall that $\bar{X}$ is a compact manifold with boundary $\partial\bar{X}$, hence $\bar{X}\times\bar{X}$ is a manifold with corners on which we define the functions

$$\rho(w,w'):=\rho(w),\quad \rho'(w,w'):=\rho(w'),\quad R_c(w,w'):=R_c(w),\quad R_c'(w,w'):=R_c(w'). \tag{4.1}$$

Since ρ, R_c are well defined on $\bar{M}_k$ via I_k, the functions (4.1) can also be defined on $\bar{M}_k\times\bar{M}_k$.

Lemma 4.2. *Let $\theta,\theta'\in C^\infty(\bar{X}_k)$ be functions with support in $\bar{M}_k$ and constant near c_k, then the extended resolvent $R_{X_k}(\lambda)$ satisfies*

$$\theta R_{X_k}(\lambda)\theta' : \dot{C}^\infty(\bar{X}_k)\to \rho^\lambda R_c^{-1}C^\infty_{acc}(\bar{X}_k) \tag{4.2}$$

for $\lambda\notin(\frac{k}{2}-\mathbb{N}_0)$ if $n-k+1$ is odd and for $\lambda\in\mathbb{C}$ otherwise. If moreover θ,θ' are chosen satisfying supp$(\theta)\cap c_k=\emptyset$ and $\theta\theta'=0$ then

$$\theta' R_{X_k}(\lambda)\theta\in\rho^\lambda{\rho'}^\lambda R_c^{-1}C^\infty(\bar{X}_k\times\bar{X}_k),\quad \theta R_{X_k}(\lambda)\theta'\in\rho^\lambda{\rho'}^\lambda {R_c'}^{-1}C^\infty(\bar{X}_k\times\bar{X}_k) \tag{4.3}$$

Proof: clearly, it is enough to show the lemma with θ,θ' which are independent of the variable $z\in T^k$. We recall from [8] that the explicit formula for the resolvent on X_k can be obtained by Fourier analysis on the $z\in T^k$ variable, $R_{X_k}(\lambda)$ admits a meromorphic continuation to $\mathbb{C}$ and its Schwartz kernel can be written

$$R_{X_k}(\lambda)=\sum_{m\in\mathbb{Z}^k}e^{i\omega_m.(z-z')}R_m(\lambda) \tag{4.4}$$

for $\lambda\notin(\frac{k}{2}-\mathbb{N}_0)$ if $n-k+1$ is odd and for $\lambda\in\mathbb{C}$ otherwise, with

$$R_m(\lambda;x,y;x',y'):=C_k\int_{\mathbb{R}^k}e^{i\omega_m.z}R_{\mathbb{H}^{n+1}}(\lambda;x,y,z;x',y',0)dz \tag{4.5}$$

where C_k is a constant, $R_{\mathbb{H}^{n+1}}(\lambda)$ is the kernel of the resolvent of the Laplacian on $\mathbb{H}^{n+1}$ and $\omega_m:=2\pi{}^t(A_k^{-1})m$. Note that $R_m(\lambda)$ can be considered as an operator -a resolvent- on $\mathbb{H}^{n-k+1}$. We have seen in [8] that if

$$\tau:=\frac{xx'}{r^2+|z|^2},\quad r^2:=|y-y'|^2+x^2+{x'}^2,\quad d:=\frac{xx'}{r^2}$$

then for all $N\in\mathbb{N}\cup\infty$ there exists a function $F_N(\lambda,\tau)$ smooth in $\tau\in[0,\frac12)$ with a conormal singularity at $\tau=\frac12$ such that

$$R_{\mathbb{H}^{n+1}}(\lambda;x,y,z;x',y',0)=\tau^\lambda\sum_{j=0}^{N-1}\alpha_j(\lambda)\tau^{2j}+\tau^{\lambda+2N}F_N(\lambda,\tau)$$

for some $\alpha_j(\lambda)$ meromorphic in λ (with only poles at $-\mathbb{N}_0$ if $n+1$ is even) and if $N=\infty$, $F_\infty(\lambda,\tau)=0$ and the sum converges locally uniformly if $\tau\neq\frac12$ (see also [12] and [23, Appendix A]). Thus by a change of variable $w=z/r$ in (4.5), one has as in [8, Sect. 3.1]

$$R_m(\lambda)=d^\lambda r^k\sum_{j=0}^{N-1}d^{2j}F_{j,\lambda}(r|\omega_m|)+d^{\lambda+2N}r^k\int_{\mathbb{R}^k}e^{-ir\omega_m.z}\frac{F_N(\lambda,d(1+|z|^2)^{-1})}{(1+|z|^2)^{\lambda+2N}}dz \tag{4.6}$$

with

$$F_{j,\lambda}(u):=C_{k,j}(\lambda)|u|^{\lambda-\frac{k}{2}+2j}K_{-\lambda+\frac{k}{2}-2j}(|u|),\quad F_{j,\lambda}(0):=D_{k,j}(\lambda)$$

$K_s(z)=\int_0^\infty\cosh(st)e^{-z\cosh(t)}dt$ being modified Bessel function, $C_{k,j}(\lambda)$ some holomorphic functions and $D_{k,j}(\lambda)$ some meromorphic functions in $\mathbb{C}$ with only first order poles at $\frac{k}{2}-\mathbb{N}_0$ if $n-k+1$ is even (in fact we have $R_0(\lambda)=(xx')^{\frac{k}{2}}R_{\mathbb{H}^{n-k+1}}(\lambda-\frac{k}{2})$). The sum (4.6) with $N=\infty$ is locally uniformly convergent in $\{d<\frac12, 0<r\}$.

We first show (4.3) using these explicit formulae. We will better use the compactification coordinates (t,u) on M_k, the functions r and d become

$$d=\frac{tt'}{|u-u'|^2+t^2+t'^2},\quad r^2=\frac{t^2+t'^2+|u-u'|^2}{(t^2+|u|^2)(t'^2+|u'|^2)}. \tag{4.7}$$

On the support of $\theta R_{X_k}(\lambda)\theta'$ we have $t^2+t'^2+|u-u'|^2>\epsilon$ and $d\leq\frac{1}{2}-\epsilon$ for some $\epsilon>0$ since $\theta\theta'=0$, thus (4.6) with $N=\infty$ is absolutely convergent there and $r\to+\infty$ when $t^2+|u|^2\to 0$, that is when we approach the cusp submanifold c_k with respect to variables (t,u). Since Bessel's function $K_s(x)=K_{-s}(x)$ and all its derivatives with respect to x vanish exponentially when $x\to\infty$, the kernel

$$\sum_{m\neq 0}\theta R_m(\lambda)e^{i\omega_m.(z-z')}\theta'$$

is in $\rho^\lambda\rho'^\lambda {R_c}^{-1}C^\infty(\{\bar{X}_k\setminus c_k\}\times\bar{X}_k)$ and can be extended to $\bar{X}_k\times\bar{X}_k$ with

$$\sum_{m\neq 0}\theta R_m(\lambda)\theta' e^{i\omega_m.(z-z')}\in\rho^\lambda\rho'^\lambda C^\infty(\bar{X}_k\times\bar{X}_k)$$

vanishing at all order at $(c_k\times\bar{X}_k)\cup(\bar{X}_k\times c_k)$. Note that we have used that $\rho=t$ in M_k. For the term $R_0(\lambda)$, it is clear, using (4.6) and (4.7) that

$$\theta R_0(\lambda)\theta'\in\rho^\lambda\rho'^\lambda {R'_c}^{-1}C^\infty(\bar{X}_k\times\bar{X}_k)$$

which concludes the proof of (4.3) using the symmetry of the resolvent kernel.

The property (4.2) is more technical since it involves the singularity of $R_{X_k}(\lambda)$ near the diagonal. Let $f\in\dot{C}^\infty(\bar{X}_k)$, with support in $\bar{M}_k$. We first study for $m\neq 0$ the function $\theta R_m(\lambda)\theta' f_m$ in $\bar{M}_k$ where $f_m=\langle f,e^{i\omega_m.z}\rangle_{T_k}$ is the m-th Fourier mode on T^k of f. We clearly have $f_m\in\dot{C}^\infty(\bar{\mathbb{H}}^{n-k+1})$ with

$$\forall l\in\mathbb{N}, |\partial^\alpha f_m|\leq C_{\alpha,l}|\omega_m|^{-l}$$

with $C_{\alpha,l}$ uniform in m. For simplicity, we consider (4.6) with $N=0$ and decompose

$$F_0(\lambda,\tau)=\chi(\tau)F_0(\lambda,\tau)+(1-\chi(\tau))F_0(\lambda,\tau)=:F_{0,1}(\lambda,\tau)+F_{0,2}(\lambda,\tau)$$

with χ a $C_0^\infty([0,1/4))$ which is equal to 1 near $\tau=0$. The integral

$$\theta(t,u)\theta'(t',u')r^k d^\lambda\int_{\mathbb{R}^{n-k}}e^{-ir\omega_m.z}(1+|z|^2)^{-\lambda}F_{0,1}(\lambda,d(1+|z|^2)^{-1})dz$$

is well defined for $\Re(\lambda)>\frac{k}{2}$ and is equal by integration by parts to

$$\kappa_1:=\theta(t,u)\theta'(t',u')(r|\omega_m|)^{-2N}r^k d^\lambda\int_{\mathbb{R}^{n-k}}e^{-ir\omega_m.z}\Delta_z^N\left(\frac{F_{0,1}(\lambda,d(1+|z|^2)^{-1})}{(1+|z|^2)^\lambda}\right)dz \tag{4.8}$$

for all $N>0$. In view of the smoothness of $F_{0,1}(\lambda,\tau)$ for $\tau\in\mathbb{R}^+$, it is straightforward to see that the integrand in (4.8) satisfies

$$\left|\Delta_z^N\left(\frac{F_{0,1}(\lambda,d(1+|z|^2)^{-1})}{(1+|z|^2)^\lambda}\right)\right|\leq C_N(1+|z|^2)^{-\Re(\lambda)-N}$$

and is a smooth function of d for $\lambda\in\mathbb{C}\setminus-\mathbb{N}_0$, now integrable with respect to $z\in\mathbb{R}^k$ if $\Re(\lambda)+N>\frac{k}{2}$. Now since $f_m(t',u')=O(t'^\infty)$, we have in $\bar{\mathbb{H}}^{n-k+1}\times\bar{\mathbb{H}}^{n-k+1}$

$$|\partial^\alpha_{t,u}(d/t)\partial^\beta f_m|\leq C_{\alpha,\beta,l}|\omega_m|^{-l},\quad |\partial^\alpha_{t,u}d\partial^\beta f_m|\leq C_{\alpha,\beta,l}|\omega_m|^{-l}$$

$$|\partial^\alpha_{t,u}r\partial^\beta f_m|\leq C_{\alpha,\beta,l}(t^2+|u|^2)^{-(1+|\alpha|)/2}|\omega_m|^{-l},\quad |\partial^\alpha_{t,u}(r\sqrt{t^2+|u|^2})\partial^\beta f_m|\leq C_{\alpha,\beta,l}|\omega_m|^{-l}$$

by looking at the expression of d,r in (4.7). For $\lambda\notin-\mathbb{N}_0$ fixed, we take $N\gg 2|\Re(\lambda)|$, this proves that

$$t^{-\lambda}(t^2+|u|^2)^{-M}\int_{\mathbb{H}^{n-k+1}}d^\lambda\kappa_1 f_m(t',u')t'^{-n+k-1}(t'^2+|u'|^2)^{\frac{k}{2}}dt'du'$$

is C^N in $(t,u)\in\bar{\mathbb{H}}^{n-k+1}$ for $2M\ll N$ and all its derivatives of order α with $|\alpha|<N$ are bounded by $C_{l,N}|\omega_m|^{-l}$ for all l,N,m. Thus for M fixed, by taking $N\to\infty$ we see that this function is smooth in $\bar{\mathbb{H}}^{n-k+1}$ and its derivatives are rapidly decreasing in $|\omega_m|$.

We now have to deal with the integral kernel

$$\kappa_2:=\theta(t,u)\theta'(t',u')r^kd^\lambda\int_{\mathbb{R}^{n-k}}e^{-ir\omega_m.z}(1+|z|^2)^{-\lambda}F_{0,2}(\lambda,d(1+|z|^2)^{-1})dz$$

and we will show that

$$f'_m(t,u):=\int_{\mathbb{H}^{n-k+1}}\kappa_2f_m(t',u')t'^{-n+k-1}(t'^2+|u'|^2)^{\frac{k}{2}}dt'du'$$

satisfies

$$f'_m\in\dot{C}(\bar{\mathbb{H}}^{n-k+1}),\quad |\partial^\alpha_{t,u}f'_m|\leq C_{\alpha,l}|\omega_m|^{-l}. \tag{4.9}$$

First remark that, since $d<\frac{1}{2}$, we have $1-\chi(d(1+|z|^2)^{-1})=0$ if $|z|>C$ for some $C>0$ depending on χ. We use the change of variables $s=t/t', v=(u-u')/t'$ in this last integral. By elementary computations, it turns out that

$$d=(2\cosh(d_{\mathbb{H}^{n-k+1}}(t,u;t',u')))^{-1}=(2\cosh(d_{\mathbb{H}^{n-k+1}}(1,0_{\mathbb{R}^{n-k}};s,v)))^{-1}$$

but $F_{0,2}(\lambda,d(1+|z|^2)^{-1})$ is supported in $\{d>\epsilon\}$ for some $\epsilon>0$ depending on χ thus it is supported in $\{(s,v)\in K\}$ where K is a euclidean ball included in $\mathbb{H}^{n-k+1}$ (thus a compact of $\mathbb{H}^{n-k+1}$). Moreover in the variables (t,u,s,v),

$$\kappa_2=\theta(t,u)\theta'\Big(\frac{t}{s},u-\frac{t}{s}v\Big)r^kd^\lambda\int_{|z|<C}e^{-ir\omega_m\cdot z}(1+|z|^2)^{-\lambda}F_{0,2}(\lambda,d(1+|z|^2)^{-1})dz_{\mathbb{R}^k}$$

and all its derivatives with respect to (t,u) are in $L^1(K,s^{-1}dsdz)$, this fact is proved by Perry [23, Appendix] and is a direct consequence of the conormal singularity of $F_0(\lambda,\tau)$ at $\tau=\frac{1}{2}$. And from the expression of r, we see that the derivatives of r or order α are bounded by $C_\alpha t^{-1-|\alpha|}$ for $(t,u,s,v)\in\mathbb{H}^{n-k+1}\times K$. We deduce that

$$\int_K\kappa_2f_m\left(\frac{t}{s},u-\frac{t}{s}v\right)\left(\frac{t}{s}\right)^{n-k+1}\left(\left(\frac{t}{s}\right)^2+\left|u-\frac{t}{s}v\right|^2\right)^{\frac{k}{2}}s^{-1}dsdv$$

is in $\dot{C}^\infty(\mathbb{H}^{n-k+1})$ since $f_m(t,u)=O(t^\infty)$ and K is compact. In addition, its derivatives of order α are clearly bounded by $C_{\alpha,l}|\omega_m|^{-l}$ for all α,l. We have thus proved (4.9) and that

$$\sum_{m\neq0}R_m(\lambda)e^{i\omega_m\cdot(z-z')}f\in\rho^\lambda C^\infty_c(\bar{X}_k).$$

It remains now to study $\theta R_0(\lambda)\theta'f_0$ where $f_0:=\langle f,1\rangle_{T^k}$ is the zeroth Fourier term of f. But recall from [8] that $R_0(\lambda)$ acting on $\mathbb{H}^{n-k+1}$ is nothing more than the hyperbolic resolvent

$$R_0(\lambda;t,u;t'u')=\left(\frac{tt'}{(t^2+|u|^2)(t'^2+|u'|^2)}\right)^{\frac{k}{2}}R_{\mathbb{H}^{n-k+1}}\left(\lambda-\frac{k}{2};t,u;t',u'\right).$$

for $\lambda\notin(\frac{k}{2}-\mathbb{N}_0)$ if $n-k+1$ is odd and for $\lambda\in\mathbb{C}$ otherwise. Using the analysis of [17], we directly obtain that

$$\theta R_0(\lambda)\theta'f_0\in\rho^\lambda R_c^{-1}C^\infty(\bar{\mathbb{H}}^{n-k+1})\subset\rho^\lambda R_c^{-1}C^\infty_{acc}(\bar{X}_k)$$

where the inclusion means: consider the function on X_k as constant with respect to $z\in T^k$. As a conclusion (4.2) is proved and the proof of the lemma is achieved too, at least for $\lambda\notin-\mathbb{N}_0$. The points at $-\mathbb{N}_0$ can in fact be treated by taking $N>0$ large in (4.6) and essentially the same arguments than for $N=0$. □

Now we briefly review the construction of a parametrix for $R(\lambda)$ in [8, Prop 3.1 and 3.5] which can be continued to infinite order (at least formally, the problem of convergence will be discussed later). This is obtained by localizing in the neighbourhoods near infinity M_k and M_r. One can construct some operators $\mathcal{E}^k_\infty(\lambda)$ on M_k ($k=1,\dots,n-1$) and $\mathcal{E}^r_\infty(\lambda)$ on M_r such that

$$(\Delta_{M_k}-\lambda(n-\lambda))\mathcal{E}^k_\infty(\lambda)=\chi^k+\mathcal{K}^k_\infty(\lambda),$$
$$(\Delta_{M_r}-\lambda(n-\lambda))\mathcal{E}^r_\infty(\lambda)=\chi^r+\mathcal{K}^r_\infty(\lambda)$$

with $\mathcal{K}^k_\infty(\lambda)$, $\mathcal{K}^r_\infty(\lambda)$ having smooth Schwartz kernels $\mathcal{K}^k_\infty(\lambda;w,w')$ and $\mathcal{K}^r_\infty(\lambda;w,w')$) which vanish at all order when $\rho(w)\to 0$.

The first step of the parametrix construction of $\mathcal{E}^k_\infty(\lambda)$ is to take a smooth function χ^k_L with support in M_k which is equal to 1 in $\{x^2+|y|^2>4\}$ such that $\chi^k_L\chi^k=\chi^k$ and $1-\chi^k_L$ can be chosen as a product (see the construction in [8])

$$1-\chi^k_L(x,y,z)=\psi^k_L(y)\phi_L(x) \tag{4.10}$$

independent of the variable on T^k; then set

$$E^k_0(\lambda):=\chi^k_L R_{X_k}(\lambda)\chi^k,\quad K^k_0(\lambda)=[\Delta_{X_k},\chi^k_L]R_{X_k}(\lambda)\chi^k$$

and we obtain $(\Delta_{M_k}-\lambda(n-\lambda))E^k_0(\lambda)=\chi^k+K^k_0(\lambda)$ as a first parametrix in the neighbourhood M_k of $\partial\bar{X}$ in $\bar{X}$. The next steps of the construction in [8, Prop.3.1] involve only some operators with Schwartz kernels of the same type than $K^k_0(\lambda)$ but with additional decay at $\partial\bar{X}\times\bar{X}$ in $\bar{X}\times\bar{X}$. The part of the parametrix on M_r is done as in the work of Guillopé-Zworski [12] (and more generally [17]) by using at first step

$$E^r_0(\lambda):=\chi^r_L R_{\mathbb{H}^{n+1}}(\lambda)\chi^r,\quad K^r_0(\lambda)=[\Delta_{\mathbb{H}^{n+1}},\chi^r_L]R_{\mathbb{H}^{n+1}}(\lambda)\chi^r$$

with a function χ^r_L which is equal to 1 on the support of χ^r and which can be expressed as a product $\chi^r_L(x,y)=\phi^r_L(x)\varphi^r_L(y)$ in M_r. The other steps of the construction in M_r do not make more singular kernels than $K^r_0(\lambda)$ appear.

The previous lemma allows to deduce the following

Proposition 4.3. *Let $\theta,\theta'\in C^\infty(\bar{X})$ constant near c and such that supp$(\theta')\cap c=\emptyset$ and $\theta\theta'=0$. Then for λ not a resonance we have*

$$\theta R(\lambda)\theta'\in {R_c}^{-1}{\rho'}^{\lambda}\rho^\lambda C^\infty(\bar{X}\times\bar{X}),\quad \theta' R(\lambda)\theta\in {R'_c}^{-1}\rho^\lambda{\rho'}^{\lambda}C^\infty(\bar{X}\times\bar{X})$$

and $R(\lambda)$ has the mapping property

$$R(\lambda):\dot{C}^\infty(\bar{X})\to R_c^{-1}\rho^\lambda C^\infty_{acc}(\bar{X}). \tag{4.11}$$

Proof: if we carefully look at the expression of $\mathcal{K}_\infty(\lambda)$ following [8, Prop. 3.1 and 3.5] and we use previous lemma, it is not difficult to check that

$$(I_k)^*\mathcal{K}^k_\infty(\lambda)(I_k)_*\in\rho^\infty{\rho'}^{\lambda}{R'_c}^{-1}C^\infty(\bar{X}\times\bar{X}), \tag{4.12}$$

$$(I_k)^*\mathcal{K}^r_\infty(\lambda)(I_r)_*\in\rho^\infty{\rho'}^{\lambda}C^\infty(\bar{X}\times\bar{X}). \tag{4.13}$$

The second statement is essentially well-known (see [8, 12] for instance) and is a direct consequence of the explicit formula of $R_{\mathbb{H}^{n+1}}(\lambda)$. To prove the first one, we essentially use Lemma 4.2. It is not difficult to check (see again [8]) that $[\Delta_{X_k},\chi^k_L]$ is a first order operator with smooth coefficients supported in $\{1<x^2+|y|^2\leq 4, 0\leq x\}$ and vanishing at second order at $x=0$. Using the compactification coordinates (t,u) of (2.1), it is also a first order operator with smooth coefficients supported in $\{\epsilon<t^2+|y|^2\leq 1, 0\leq t\}$ for some $\epsilon>0$ and vanishing at second order at $t=0$, moreover its support does not intersect the support of χ^k. Therefore, using (4.3) in Lemma 4.2 we easily deduce that

$$(I_k)^*[\Delta_{X_k},\chi^k_L]R_{X_k}(\lambda)\chi^k(I_k)_*\in\rho^{\lambda+2}{\rho'}^{\lambda}{R'_c}^{-1}C^\infty(\bar{X}\times\bar{X}). \tag{4.14}$$

Now the iterative construction of [8, Prop. 3.1] corresponds to capture the Taylor expansion of this term at $\rho=0$ and the remaining error terms at each step are like (4.14) but with more decay in ρ; this finally implies (4.12). The terms appearing in the expression of $\mathcal{E}^k_\infty(\lambda)$ in [8, Prop. 3.1], are thus $\chi^k_L R_{X_k}\chi^k$ plus some operators whose Schwartz kernels are in $\rho^{\lambda+2}{\rho'}^{\lambda}{R'_c}^{-1}C^\infty(\bar{X}_k\times\bar{X}_k)$. Therefore $\mathcal{E}^k_\infty(\lambda)$ satisfies exactly the same properties than $R_{X_k}(\lambda)$ described in Lemma 4.2.

By standard pseudo-differential calculus on compact manifolds we can obtain the compact part of the parametrix $\mathcal{E}^i_\infty(\lambda)$ so that

$$(\Delta_X - \lambda(n-\lambda))\mathcal{E}^i_\infty(\lambda) = \chi + \mathcal{K}^i_\infty(\lambda)$$

with $\mathcal{K}^i_\infty(\lambda)$ having a smooth kernel with compact support in $X \times X$ and $\mathcal{E}^i_\infty(\lambda)$ being a pseudo-differential operator of order -2 supported in a compact set of $X \times X$.

Thus we obtain

$$(\Delta_X - \lambda(n-\lambda))\mathcal{E}_\infty(\lambda) = 1 + \mathcal{K}_\infty(\lambda)$$

with

$$\mathcal{E}_\infty(\lambda) := \mathcal{E}^i_\infty(\lambda) + \sum_{\alpha=1,\dots,n-1,r} (I_\alpha)^* \mathcal{E}^\alpha_\infty(\lambda)(I_\alpha)_*,$$

$$\mathcal{K}_\infty(\lambda) := \mathcal{K}^i_\infty + \sum_{\alpha=1,\dots,n-1,r} (I_\alpha)^* \mathcal{K}^\alpha_\infty(\lambda)(I_\alpha)_*.$$

Using Lemma 4.2, (4.12), (4.13) and the explicit formulae of the regular terms in $\mathcal{E}^r_\infty(\lambda)$ in [8, 12] it is straightforward to see that

$$\mathcal{K}_\infty(\lambda) \in \rho^\infty {\rho'}^\lambda {R'_c}^{-1} C^\infty(\bar{X} \times \bar{X}) \tag{4.15}$$

$$\theta \mathcal{E}_\infty(\lambda)\theta' \in {R_c}^{-1} \rho^\lambda {\rho'}^\lambda C^\infty(\bar{X} \times \bar{X}), \quad \theta' \mathcal{E}_\infty(\lambda)\theta \in \rho^\lambda {\rho'}^\lambda {R'_c}^{-1} C^\infty(\bar{X} \times \bar{X}). \tag{4.16}$$

Moreover using Lemma 4.2 for the mapping properties of the cusps terms and [7, Prop. 3.1] for the mapping properties of the regular terms, we have

$$\mathcal{E}_\infty(\lambda) : \dot{C}^\infty(\bar{X}) \to \rho^\lambda R_c^{-1} C^\infty_{\mathrm{acc}}(\bar{X}). \tag{4.17}$$

We can then write

$$R(\lambda) = \mathcal{E}_\infty(\lambda) - \mathcal{E}_\infty(\lambda)\mathcal{K}_\infty(\lambda) + \mathcal{E}_\infty(\lambda)\mathcal{K}_\infty(\lambda)(1 + \mathcal{K}_\infty(\lambda))^{-1}\mathcal{K}_\infty(\lambda) \tag{4.18}$$

and $(1 + \mathcal{K}_\infty(\lambda))^{-1} = 1 + F(\lambda)$ with

$$F(\lambda) = -\mathcal{K}_\infty(\lambda) - \mathcal{K}_\infty(\lambda)F(\lambda).$$

This proves that $F(\lambda)$ is Hilbert-Schmidt on $\rho^N L^2(X)$ for $\Re(\lambda) > \frac{n-1}{2}$ and N large, since $\mathcal{K}_\infty(\lambda)$ is. Using that ${\rho'}^n {R'_c}^{-1}$ is bounded, the composition $\mathcal{K}_\infty(\lambda)F(\lambda)\mathcal{K}_\infty(\lambda)$ has a Schwartz kernel in the same class than $\mathcal{K}_\infty(\lambda)$ (and $\mathcal{K}_\infty(\lambda)^2$ too). In view of its construction, we see that the range of $\mathcal{K}_\infty(\lambda)$ is composed of functions with support in $\bar{X} \setminus c$, thus we can find a smooth function $\theta' \in C^\infty(\bar{X})$ with $\mathrm{supp}(\theta') \cap c = \emptyset$ such that $\theta'\mathcal{K}_\infty(\lambda) = \mathcal{K}_\infty(\lambda)$. Thus if θ is a function in $C^\infty(\bar{X})$ such that $\theta = 1$ near c and $\theta\theta' = 0$ we have from (4.16), (4.15) that

$$\theta\mathcal{E}_\infty(\lambda)\mathcal{K}_\infty(\lambda) \in \rho^\lambda {\rho'}^\lambda R_c^{-1} {R'_c}^{-1} C^\infty(\bar{X} \times \bar{X}). \tag{4.19}$$

Now we can for example use Mazzeo's composition results in [15] to deal with the regular terms

$$(\mathcal{E}^i_\infty(\lambda) + (I_r)^*\mathcal{E}^r_\infty(\lambda)(I_r)_*)\mathcal{K}_\infty(\lambda) \in \rho^\lambda {\rho'}^\lambda C^\infty(\bar{X} \times \bar{X}).$$

Then $(1-\theta)(I_k)^*\mathcal{E}^k_\infty(\lambda)(I_k)_*\mathcal{K}_\infty(\lambda)$ can be studied exactly with the same method than for the proof of (4.2) in Lemma 4.2 and we see that

$$(1-\theta)(I_k)^*\mathcal{E}^k_\infty(\lambda)(I_k)_*\mathcal{K}_\infty(\lambda) \in \rho^\lambda {\rho'}^\lambda {R'_c}^{-1} C^\infty(\bar{X} \times \bar{X})$$

and we conclude, using (4.19), that

$$\mathcal{E}_\infty(\lambda)\mathcal{K}_\infty(\lambda) \in \rho^\lambda {\rho'}^\lambda R_c^{-1} {R'_c}^{-1} C^\infty(\bar{X} \times \bar{X})$$

and the same holds for $\mathcal{E}_\infty(\lambda)\mathcal{K}_\infty(\lambda)(1 + F(\lambda))\mathcal{K}_\infty(\lambda)$. We have completed the proof in view of (4.18) and the symmetry of the resolvent kernel.

Moreover we have also proved that

$$R(\lambda) - \mathcal{E}_\infty(\lambda) \in (\rho\rho')^\lambda (R_c R'_c)^{-1} C^\infty(\bar{X} \times \bar{X}). \tag{4.20}$$

The mapping property of $R(\lambda)$ is then easily deduced from (4.18) and (4.17) since $\mathcal{K}(\lambda)$ maps $\rho^N L^2(X)$ to $\dot{C}^\infty(\bar{X})$ if $N \gg |\Re(\lambda)|$ in view of the form (4.15) of its kernel. □

Remark: we did not study the convergence problem of the infinite order parametrix $\mathcal{E}_\infty(\lambda)$ but to avoid this problem, it suffices to take the parametrix $\mathcal{E}_N(\lambda)$ of [8] for large N and the same proof actually would show the same results for $R(\lambda)$ but with C^M regularity for some $M > N - C|\Re(\lambda)|$ (with $C > 0$) instead of C^∞ regularity. Since it is true for all N, we get the same results.

5. Poisson operator, Eisenstein function

5.1. **Poisson operator.** Using the product decomposition of the metric in Lemma 2.2, an indicial equation for the Laplacian and the mapping property of the resolvent, we can construct a Poisson operator following the method of Graham-Zworski [7].

Actually, we now work with the special boundary defining function ρ but every other choice of boundary defining function $\hat{\rho} \in C^\infty_{\text{acc}}(\bar{X})$ defined in Lemma 2.2 would induce an equivalent (but not the same) construction for the Poisson operator. We will simply add the necessary arguments when the generalization is not transparent.

With the metric under the form (2.13), the Laplacian is

$$\Delta_X = -(\rho\partial_\rho)^2 + n\rho\partial_\rho - \frac{1}{2}\text{Tr}(h^{-1}(\rho).\partial_\rho h(\rho))\rho^2\partial_\rho + \rho^2\Delta_{h(\rho)}. \tag{5.1}$$

In the neighbourhood M_k of the cusp c_k this gives

$$\Delta_X = -(\rho\partial_\rho)^2 + n\rho\partial_\rho - 2k(\rho^2 + |u|^2)^{-1}\rho^3\partial_\rho + \rho^2\Delta_{h(\rho)}$$

with $h(\rho) = du^2 + (\rho^2 + |u|^2)^2 dz^2$ a metric on $\{0 < |u| < 1\} \times T^k_z$, and by an elementary computation we obtain

$$R_c\Delta_X R_c^{-1} = -(\rho\partial_\rho)^2 + n\rho\partial_\rho + \rho^2(\Delta_u + (\rho^2 + |u|^2)^{-2}\Delta_z) \tag{5.2}$$

where Δ_u, Δ_z are the flat Laplacians on $\mathbb{R}^{n-k}_u, T^k_z$. Similarly with a function $\hat{\rho}$ of Lemma 2.2 we have

$$\Delta_X = -(\hat{\rho}\partial_{\hat{\rho}})^2 + n\hat{\rho}\partial_{\hat{\rho}} - \frac{1}{2}\text{Tr}(\hat{h}^{-1}(\hat{\rho}).\partial_{\hat{\rho}}\hat{h}(\hat{\rho}))\hat{\rho}^2\partial_{\hat{\rho}} + \hat{\rho}^2\Delta_{h(\hat{\rho})} + O(\hat{\rho}^\infty).$$

and in coordinates $(\hat{\rho}, v, \zeta)$ near c_k, we see from (2.11) that

$$R_c\Delta_X R_c^{-1} = -(\hat{\rho}\partial_{\hat{\rho}})^2 + n\hat{\rho}\partial_{\hat{\rho}} + P_1 + P_2 + \hat{\rho}^2 e^{-2\omega} r_c^{-4}\Delta_\zeta + O(\hat{\rho}^\infty)$$

for some differential operators

$$P_1 = P_1(\hat{\rho}, v, \hat{\rho}^2\partial_{\hat{\rho}}, \hat{\rho}\partial_v), \quad P_2 = P_2(\hat{\rho}, v, \zeta, \hat{\rho}\partial_v, \hat{\rho}\partial_\zeta) = O(r_c^\infty)$$

of order 2, with P_2 (resp. P_1) having smooth coefficents on $\bar{X}$ (resp. smooth outside c_k). By making the same change of coordinates (2.9) in (5.2), it would give some differential operators with smooth coefficients at c_k except the term with Δ_ζ thus P_1 has to be smooth at c_k.

We now use Graham-Zworski's construction [7] and we refer the reader to their paper for additional details. If $f \in C^\infty_{\text{acc}}(\partial\bar{X})$ we deduce from (5.1) and (5.2) the indicial equation in $\{\rho < \epsilon\}$

$$(\Delta_X - \lambda(n-\lambda))\rho^{n-\lambda+j}R_c^{-1}f - j(2\lambda - n - j)\rho^{n-\lambda+j}R_c^{-1}f \in \rho^{n-\lambda+j+1}R_c^{-1}C^\infty_{\text{acc}}(\bar{X}). \tag{5.3}$$

Here, the key fact is that the singular term $r_c^{-4}\Delta_z$ applied to $f \in C^\infty_{acc}(\partial\bar{X})$ gives a functions in $\dot{C}^\infty_c(\bar{X})$ by (2.7). Therefore for all $f \in R_c^{-1}C^\infty_{\text{acc}}(\partial\bar{X})$ one can construct by induction and Borel lemma (see again [7]) a function $\Phi(\lambda)f \in \rho^{n-\lambda}R_c^{-1}C^\infty_{\text{acc}}(\bar{X})$ for $\lambda \in \mathbb{C} \setminus \frac{1}{2}(n + \mathbb{N})$ such that

$$(\Delta_X - \lambda(n-\lambda))\Phi(\lambda)f \in \dot{C}^\infty(\bar{X}), \quad \rho^{\lambda-n}\Phi(\lambda)f|_{\rho=0} = f.$$

By construction, we have the formal Taylor expansion

$$\Phi(\lambda)f = \rho^{n-\lambda}\sum_{j=0}^{\infty}\rho^{2j}c_{j,\lambda}P_{j,\lambda}f, \quad \forall f\in C^\infty_{\mathrm{acc}}(\partial\bar{X}) \tag{5.4}$$

where $P_{j,\lambda}$ is a differential operator on B which is polynomial in λ and

$$c_{j,\lambda} := (-1)^j\frac{\Gamma(\lambda-\frac{n}{2}-j)}{2^{2j}j!\Gamma(\lambda-\frac{n}{2})}.$$

Now we can set for $\lambda\notin\frac{1}{2}(n+\mathbb{N})$ and λ not a resonance

$$\mathcal{P}(\lambda)f = \Phi(\lambda)f - R(\lambda)(\Delta_X-\lambda(n-\lambda))\Phi(\lambda)f \tag{5.5}$$

which satisfies

$$\begin{cases} (\Delta_X-\lambda(n-\lambda))\mathcal{P}(\lambda)f = 0\\ \mathcal{P}(\lambda)f = \rho^{n-\lambda}F(\lambda,f)+\rho^{\lambda}G(\lambda,f)\\ F(\lambda,f),G(\lambda,f)\in R_c^{-1}C^\infty_{\mathrm{acc}}(\bar{X})\\ F(\lambda,f)|_{\rho=0} = f\end{cases} \tag{5.6}$$

using Proposition 4.3. We have defined a family of operators

$$\mathcal{P}(\lambda): R_c^{-1}C^\infty_{\mathrm{acc}}(\partial\bar{X})\to\rho^{n-\lambda}R_c^{-1}C^\infty_{\mathrm{acc}}(\bar{X})+\rho^{\lambda}R_c^{-1}C^\infty_{\mathrm{acc}}(\bar{X})$$

and we will now prove the uniqueness of an operator satisfying (5.6) in $\{\Re(\lambda)\geq\frac{n}{2}\}$. The principle is the same than in [7]: if $\Re(\lambda)>\frac{n}{2}$, λ not a resonance and $\mathcal{P}_1(\lambda)f,\mathcal{P}_2(\lambda)f$ are two solutions of (5.6), then the previous indicial equation shows that $\mathcal{P}_1(\lambda)f-\mathcal{P}_2(\lambda)f\in\rho^{\lambda}R_c^{-1}C^\infty(\bar{X})$ but this function is in $L^2(X)$ using (2.6) so this must be 0; to treat the case $\Re(\lambda)=\frac{n}{2}$, we use a boundary pairing Lemma like Proposition 3.2 of [7]:

Lemma 5.1. *For $i=1,2$, let $u_i=\rho^{n-\lambda}F_i+\rho^{\lambda}G_i$ some functions satisfying*

$$(\Delta_X-\lambda(n-\lambda))u_i = r_i\in\dot{C}^\infty(\bar{X})$$

with $F_i,G_i\in R_c^{-1}C^\infty(\bar{X})$, then we have for $\Re(\lambda)=\frac{n}{2}$ and $\lambda\neq\frac{n}{2}$

$$\int_X(u_1\overline{r_2}-r_1\overline{u_2})\ dvol_g = (2\lambda-n)\int_B(F_1|_B\overline{F_2}|_B-G_1|_B\overline{G_2}|_B)\ dvol_{h_0}$$

Proof: we apply Green Lemma in $X_\epsilon=\{\rho\geq\epsilon\}$

$$\int_{X_\epsilon}(u_1\bar{r_2}-u_2\bar{r_1})\ \mathrm{dvol}_g = \epsilon^{-n+1}\int_{\rho=\epsilon}(u_1\partial_\rho\bar{u_2}-\bar{u_2}\partial_\rho u_1)\ \mathrm{dvol}_{h(\epsilon)} \tag{5.7}$$

and we will take the limit as $\epsilon\to0$. Using the asymptotics of u_1,u_2 we get

$$u_1\partial_\rho\bar{u_2}-\bar{u_2}\partial_\rho u_1 = (2\lambda-n)\rho^{n-1}(F_1\overline{F_2}-G_1\overline{G_2})+\rho^n(G_1\partial_\rho\overline{G_2}-G_2\partial_\rho\overline{G_1}+F_1\partial_\rho\overline{F_2}-\overline{F_2}\partial_\rho F_1).$$

Recall from (2.5) that $\mathrm{dvol}_{h(\epsilon)}=R_c(\epsilon)^2\mu_{\partial\bar{X}}$ with $R_c(\epsilon)=(|u|^2+\epsilon^2)^{\frac{1}{2}}$ in the neighbourhood B_k of the cusp submanifold c_k, so the only terms in the right hand side of (5.7) for which the limit are not apparent are

$$\epsilon\int_{\rho=\epsilon}(G_1\partial_\rho\overline{G_2}-G_2\partial_\rho\overline{G_1})\ \mathrm{dvol}_{h(\epsilon)},\quad \epsilon\int_{\rho=\epsilon}(F_1\partial_\rho\overline{F_2}-F_2\partial_\rho\overline{F_1})\ \mathrm{dvol}_{h(\epsilon)}.$$

The study of both terms when $\epsilon\to0$ is the same and can be clearly reduced to the limit of

$$\int_{T^k}\int_{|u|\leq1}G_1(\epsilon,u,z)\epsilon\partial_\epsilon\overline{G_2(\epsilon,u,z)}(|u|^2+\epsilon^2)^k du_{\mathbb{R}^{n-k}}dz_{T^k} \tag{5.8}$$

when $\epsilon\to0$, $G_i(\rho,u,z)$ being the function G_i in the coordinates of the neighbourhood B_k of c_k. Using that on $G_i\in R_c^{-1}C^\infty(\bar{X})$, it suffices to show that the limit of

$$\int_{|u|\leq1}\epsilon\partial_\epsilon[(|u|^2+\epsilon^2)^{-\frac{k}{2}}](|u|^2+\epsilon^2)^{\frac{k}{2}}du_{\mathbb{R}^{n-k}}$$

is 0 when $\epsilon \to 0$ to prove that the limit of (5.8) is 0. Now this last integral is equal to

$$C\int_0^1 \epsilon^2(r^2+\epsilon^2)^{-1}r^{n-k-1}dr \leq C\epsilon\int_0^\infty (1+r^2)^{-1}dr$$

for a constant C, this finally proves the lemma. □

Now using this lemma with $u_2 = R(n-\lambda)\varphi$ for $\varphi \in \dot{C}^\infty(\bar{X})$ and $u_1 = \mathcal{P}_1(\lambda)f - \mathcal{P}_2(\lambda)f$ this proves that $\langle u_1, \varphi\rangle = 0$ for all $\varphi \in \dot{C}^\infty(\bar{X})$, thus $u_1 = 0$. As a conclusion, we have

Proposition 5.2. *For $\Re(\lambda) \geq \frac{n}{2}$, $\lambda \notin \frac{1}{2}(n+\mathbb{N}_0)$, $\lambda(n-\lambda) \notin \sigma_{pp}(\Delta_X)$ there exists a unique linear operator*

$$\mathcal{P}(\lambda) : R_c^{-1}C^\infty_{acc}(\partial\bar{X}) \to \rho^{n-\lambda}R_c^{-1}C^\infty_{acc}(\bar{X}) + \rho^\lambda R_c^{-1}C^\infty_{acc}(\bar{X})$$

analytic in λ and solution of the Poisson problem (5.6). It is given by (5.5) and called Poisson operator.

By (5.5) it admits a meromorphic continuation with poles of finite multiplicity to $\mathbb{C}\setminus\frac{1}{2}(n+\mathbb{N}_0)$.

5.2. **Eisenstein functions.** In this part, we define Eisenstein functions as a weighted restriction of the Schwartz kernel of the resolvent at $B\times X$ and we prove that they are the Schwartz kernel of the transpose of the Poisson operator.

As a consequence of Proposition 4.3 and (4.20) we first obtain the

Corollary 5.3. *The Eisenstein function $E(\lambda) := (\rho^{-\lambda}R(\lambda))|_{B\times X}$ is well defined, meromorphic in $\lambda \in \mathbb{C}$ and satisfies*

$$E(\lambda) \in {R_c}^{-1}C^\infty(\partial\bar{X}\times X). \tag{5.9}$$

Moreover, if $E_{mod}(\lambda)$ is the 'model Eisenstein function' defined by

$$E_{mod}(\lambda) := (\rho^{-\lambda}\mathcal{E}_\infty(\lambda))|_{B\times X}$$

then

$$E(\lambda) - E_{mod}(\lambda) \in {\rho'}^\lambda(R_cR'_c)^{-1}C^\infty(\partial\bar{X}\times\bar{X}). \tag{5.10}$$

Let $E_{X_k}(\lambda)$ be the Eisenstein function for the model space X_k obtained from (4.4) and (4.6) (recall that $\rho = t = \frac{x}{x^2+|y|^2}$ with our choice in Lemma 2.2)

$$E_{X_k}(\lambda; y, z; x', y', z') = |y|^{2\lambda}{x'}^\lambda r^{-2\lambda+k}\sum_{m\in\mathbb{Z}^k} e^{i\omega_m\cdot(z-z')}F_{0,\lambda}(r|\omega_m|)$$

for $y \neq 0$, where by convention $r = (|y-y'|^2 + {x'}^2)^{\frac{1}{2}}$ denotes here the restriction of r to $x = 0$. In the compactification coordinates (t, u) of (2.1) this gives

$$E_{X_k}(\lambda; u, z; t', u', z') = {t'}^\lambda r^{-2\lambda+k}|u|^{-2\lambda}({t'}^2+|u'|^2)^{-\lambda}\sum_{m\in\mathbb{Z}^k} e^{i\omega_m\cdot(z-z')}F_{0,\lambda}(r|\omega_m|) \tag{5.11}$$

and r is expressed in these coordinates by

$$r^2 = \frac{{t'}^2+|u-u'|^2}{|u|^2({t'}^2+|u'|^2)}. \tag{5.12}$$

Similarly let $E_{\mathbb{H}^{n+1}}(\lambda)$ be the Eisenstein function on $\mathbb{H}^{n+1}$

$$E_{\mathbb{H}^{n+1}}(\lambda; y; x', y') = \frac{\pi^{-\frac{n}{2}}\Gamma(\lambda)}{(2\lambda-n)\Gamma(\lambda-\frac{n}{2})}\frac{{x'}^\lambda}{(|y-y'|^2+{x'}^2)^\lambda}. \tag{5.13}$$

Using the construction of the parametrix for the resolvent, we can deduce an expression for the model Eisenstein function

$$E_{mod}(\lambda) = \sum_{\alpha=1,\dots,n-1,r}(\iota_\alpha)^*E^\alpha_{mod}(\lambda)(I_\alpha)_* \tag{5.14}$$

with $\iota_\alpha := I_\alpha|_{\rho=0}$ and in M_k, M_r

$$E^k_{mod}(\lambda; y, z; w') := \psi^k_L(y) E_{X_k}(\lambda; y, z; w')\chi^k(w'),$$

$$E^r_{mod}(\lambda; y; w') := \psi^r_L(y)\gamma_r(y)^{-\lambda} E_{\mathbb{H}^{n+1}}(\lambda; y; w')\chi^r(w'). \tag{5.15}$$

with $\rho(x, y) = x\gamma_r(y) + O(x)$ in M_r for some positive smooth function γ_r in B_r and ψ^α_L defined in (4.10).

We show that the Eisenstein functions can be viewed as a Schwartz distributional kernel of an operator, that we also denote $E(\lambda)$, mapping $\dot{C}^\infty(\bar{X})$ to $C^{-\infty}(\bar{B})$, actually with weighted L^2 continuity results.

Lemma 5.4. *There exists $C > 1$ such that for $|\Re(\lambda) - \frac{n}{2}| \leq C^{-1}N$,*

$$E(\lambda) : \rho^N L^2(X) \to L^2(B)$$

is a meromorphic family of Hilbert-Schmidt operators with poles of finite multiplicity, included in the set of resonances. Moreover for $\Re(\lambda) < 0$ and λ not a resonance, $(b, w) \to \rho(w)^{-\lambda} E(\lambda; b; w)$ is a continuous function on $B \times (\bar{X} \setminus c)$.

Proof: the terms $E(\lambda) - E_{mod}(\lambda)$ and $(\iota_r)^* E^r_{mod}(\lambda)(I_r)_*$ in $E(\lambda)$ clearly satisfy those two properties, we thus only have to deal with $E^k_{mod}(\lambda)$ in X_k. From (5.11) and (5.12) we have

$$|t'^N E_{X_k}(\lambda; u, z; t', u', z')| \leq \frac{t'^{\Re(\lambda)+N}(|u - u'|^2 + t'^2)^{\frac{k}{2}-\Re(\lambda)}}{|u|^k|u'|^k} \sum_{m\in\mathbb{Z}^k} |F_{0,\lambda}(r|\omega_m|)|.$$

When $r|\omega_m| > 1$, the classical estimate $|K_s(z)| \leq Ce^{-C\Re(z)}$ for $\Re(z) > 1$ (with $C > 0$ depending on s) on Mac Donald's function shows that $|F_{0,\lambda}(r|\omega_m|)| \leq e^{-Cr|\omega_m|}$ thus

$$\sum_{|\omega_m|>1/r} |F_{0,\lambda}(r|\omega_m|)| \leq Cr^{-k} \leq Ct'^{-k}$$

where C depends on λ. Therefore we get for $N > 4|\Re(\lambda)|$

$$|t'^N E_{X_k}(\lambda)| \leq Ct'^{\frac{N}{2}}|u|^{-k}|u'|^{-k} + \frac{t'^{\Re(\lambda)+N}(|u - u'|^2 + t'^2)^{\frac{k}{2}-\Re(\lambda)}}{|u|^k|u'|^k} \sum_{|\omega_m|\leq 1/r} |F_{0,\lambda}(r|\omega_m|)|. \tag{5.16}$$

Now for $r|\omega_m| \leq 1$ we use the definition (6.4) of Mac Donald function $K_s(z)$ to decompose $F_{0,\lambda}(r|\omega_m|)$ under the form

$$F_{0,\lambda}(r|\omega_m|) = c(\lambda)(\varphi_{-\lambda+\frac{k}{2}}(r^2|\omega_m|^2) + r^{2\lambda-k}|\omega_m|^{2\lambda-k}\varphi_{\lambda-\frac{k}{2}}(r^2|\omega_m|^2))$$

with $\varphi_s(x)$ smooth on $x \in [0, \infty)$ and $c(\lambda)$ constant depending on λ. The term coming from $\varphi_{-\lambda+\frac{k}{2}}$ is treated exactly as before (the part with $r|\omega_m| > 1$) and for the term coming from $\varphi_{\lambda-\frac{k}{2}}$ we have

$$\sum_{|\omega_m|<1/r} (r|\omega_m|)^{2\Re(\lambda)-k}|\varphi_{\lambda-\frac{k}{2}}(r^2|\omega_m|^2)| \leq \begin{cases} C(r^{-k} + r^{2\Re(\lambda)-2k}) & \text{if } \Re(\lambda) - \frac{k}{2} \leq 0 \\ Cr^{-k} & \text{if } \Re(\lambda) - \frac{k}{2} > 0 \end{cases}$$

for some $C > 0$ depending on $|\lambda|$. In view of (5.16), we conclude that for $N > 4|\Re(\lambda)| + 2k$

$$|(\iota_k)^* t'^N E_{X_k}(\lambda)(I_k)_*| \leq C\rho'^{\frac{N}{2}} R_c^{-1} R_c'^{-1}$$

and this function is in $L^2(B\times X)$ if N is large enough using (2.6) (here R_c denotes the restriction of R_c to $B \times X$). The meromorphic property and the finiteness of the poles multiplicity comes from the discussion before the Lemma, using the formulae for the model Eisenstein functions and the fact that the poles of the resolvent have finite multiplicity.

The second statement of the Lemma is essentially treated in the same way. Using that for $\Re(\lambda) < 0$

$$r^{-2\lambda+k}F_{0,\lambda}(r|\omega_m|) = c(\lambda)(r^{-2\lambda+k}\varphi_{-\lambda+\frac{k}{2}}(r^2|\omega_m|^2) + |\omega_m|^{2\lambda-k}\varphi_{\lambda-\frac{k}{2}}(r^2|\omega_m|^2))$$

is continuous in $(u,t',u')\in\{u\neq 0, u'\neq 0, t'^2+|u'|^2<1, |u|<1\}$ (the power in $r^{-2\lambda+k}$ being negative) and that the sum $\sum_m r^{-2\lambda+k}F_{0,\lambda}(r|\omega_m|)$ is locally uniformly convergent in the same set by previous estimates, we deduce that $t'^{-\lambda}E_{X_k}(\lambda;u,z;t',u',z')$ is also continous there and this achieves the proof. □

The transpose ${}^tE(\lambda)$ is then well-defined from from $L^2(B)$ to $\rho^{-N}L^2(X)$ for some N depending on λ and its kernel is $E(\lambda;w,b)$. Let $\varphi\in\dot{C}^\infty(\bar{X})$ and $f\in\dot{C}^\infty_c(\partial\bar{X})\simeq\dot{C}^\infty(\bar{B})$, then for $\Re(\lambda)=\frac{n}{2}$ we use Lemma 5.1, identity $R(\lambda)={}^tR(\lambda)=R(n-\lambda)^*$ and Lemma 5.4 to deduce

$$\begin{aligned}\int_X \bar{\varphi}(\mathcal{P}(\lambda)f)\ \mathrm{dvol}_g &= (2\lambda-n)\int_B f(\overline{\rho^{\lambda-n}R(n-\lambda)\varphi})|_B\ \mathrm{dvol}_{h_0}\\ &= (2\lambda-n)\int_B f(\rho^{-\lambda}R(\lambda)\bar{\varphi})|_B\ \mathrm{dvol}_{h_0}\\ &= (2\lambda-n)\int_B f(E(\lambda)\bar{\varphi})\ \mathrm{dvol}_{h_0}\end{aligned}$$

which proves

Lemma 5.5. *The Schwartz kernel of* $\mathcal{P}(\lambda)$ *is* $(2\lambda-n)E(\lambda;w;b)\in C^\infty(X\times B)$.

This also implies that $\mathcal{P}(\lambda)$ admits a meromorphic continuation to $\mathbb{C}$ with poles of finite multiplicity, and in particular it is analytic in $\{\Re(\lambda)>\frac{n}{2}\}$ except a finite number of poles at points λ_0 such that $\lambda_0(n-\lambda_0)\in\sigma_{pp}(\Delta_X)$. By mimicking the proof of Graham-Zworski [7, Prop. 3.5] it is straightforward to see that, for $f\in R_c^{-1}C^\infty_{\mathrm{acc}}(\partial\bar{X})$, $\mathcal{P}(\frac{n}{2}+k)f$ has $\log(\rho)$ terms in the asymptotic expansion and it is the unique solution of the problem

$$\left\{\begin{array}{l}(\Delta_X-\frac{n^2}{4}+k^2)\mathcal{P}(\frac{n}{2}+k)f=0\\ \mathcal{P}(\frac{n}{2}+k)f=\rho^{\frac{n}{2}-k}F_k(f)+\rho^{\frac{n}{2}+k}\log(\rho)G_k(f)\\ F_k(f),G_k(f)\in R_c^{-1}C^\infty_{\mathrm{acc}}(\bar{X})\\ F_k(f)|_{\rho=0}=f\end{array}\right. \tag{5.17}$$

The Eisenstein functions are linked to the spectral projectors (via Stone's formula) of Δ_X in the following sense

Proposition 5.6. *If* $\Re(\lambda)=\frac{n}{2}$ *and* $\lambda\neq\frac{n}{2}$ *then*

$$R(\lambda;w;w')-R(n-\lambda;w;w')=(n-2\lambda)\int_B E(\lambda;b;w')E(n-\lambda;b;w)\ dvol_h(b) \tag{5.18}$$

where $h=(\rho^2g)|_B$. *Moreover there exists* $C>1$ *such that for* N *large, we have*

$$R(\lambda)-R(n-\lambda)=(2\lambda-n){}^tE(n-\lambda)E(\lambda)$$

in the strip $|\Re(\lambda)|\leq C^{-1}N$ *as operators from* $\rho^NL^2(X)$ *to* $\rho^{-N}L^2(X)$.

Proof: the proof of (5.18) contains nothing more than the proof of Theorem 1.3 of [3] or Proposition 2.1 of [11] in a simpler case. Note that the convergence of the integral in (5.18) is insured by (5.9) and (2.5). The second part of the Proposition is a consequence of the mapping properties of $R(\lambda),E(\lambda)$ proved before. □

Combined with Lemma 5.4, this relation implies that $E(\lambda)$ and $R(\lambda)$ have same poles, except possibly at the points λ such that $\lambda(n-\lambda)\in\sigma_{pp}(\Delta_X)$.

6. Scattering operator

Using notations of (5.6), we can define the scattering operator as the linear operator

$$S(\lambda):\left\{\begin{array}{ccc}R_c^{-1}C^\infty_{\mathrm{acc}}(\partial\bar{X}) & \to & R_c^{-1}C^\infty_{\mathrm{acc}}(\partial\bar{X})\\ f & \to & G(\lambda,f)|_B\end{array}\right. \tag{6.1}$$

for $\Re(\lambda) \geq \frac{n}{2}$, $\lambda \notin \frac{1}{2}(n+\mathbb{N})$ and λ not a resonance. With (5.5), one obtains a meromorphic continuation of $S(\lambda)$ to $\mathbb{C}$. Like $\mathcal{P}(\lambda)$, the scattering operator certainly depends on the choice of boundary defining function (here ρ), but any other choice $\hat{\rho} = e^{\omega}\rho \in C^{\infty}_{\rm acc}(\bar{X})$ of Lemma 2.2 induces an equivalent construction and two corresponding scattering operators $S(\lambda)$ and $\hat{S}(\lambda)$ are related by the covariant rule

$$\hat{S}(\lambda) = e^{-\lambda\omega_0}S(\lambda)e^{(n-\lambda)\omega_0}, \quad \omega_0 = \omega|_{\partial\bar{X}},$$

this is a trivial consequence of uniqueness of solution of Poisson problem. Therefore it suffices in this section to deal with the special boundary defining function ρ.

From Lemma 5.5, (5.5) and (6.1), we deduce that for $f \in \dot{C}^{\infty}_c(\partial\bar{X}) \simeq \dot{C}^{\infty}(\bar{B})$ and $\Re(\lambda) < 0$

$$S(\lambda)f = \lim_{\rho\to 0}[\rho^{-\lambda}((2\lambda-n)^tE(\lambda)f - \Phi(\lambda)f)] = (2\lambda-n)\lim_{\rho\to 0}[\rho^{-\lambda}({}^tE(\lambda)f)] \tag{6.2}$$

which is well defined in view of the continuity of $E(\lambda;b;w')$ proved in Lemma 5.4. As a consequence the distributional kernel of $S(\lambda)$ on B is

$$S(\lambda;b;b') = (2\lambda-n)\lim_{w'\to b'}(\rho(w')^{-\lambda}E(\lambda;b;w'))$$

which can be rewritten using the symmetry of the resolvent kernel as the restriction

$$S(\lambda) = (2\lambda-n)(\rho^{-\lambda}\rho'^{-\lambda}R(\lambda))|_{\rho=\rho'=0} \tag{6.3}$$

for $\Re(\lambda) < 0$ and λ not resonance. Moreover we deduce from (4.20) that

$$S(\lambda) - (\rho^{-\lambda}\rho'^{-\lambda}\mathcal{E}_{\infty}(\lambda))|_{\rho=\rho'=0} \in R_c^{-1}R_c'^{-1}C^{\infty}(\partial\bar{X}\times\partial\bar{X})$$

which is easily seen to be compact on $L^2(B)$ in view of (2.5), and this term extends meromorphically to $\mathbb{C}$ with poles of finite multiplicity.

We want to study the structure of the extendible distribution (6.3) on $\bar{B}\times\bar{B}$, which continues meromorphically to $\mathbb{C}$; it suffices actually to describe the singular part $(\rho^{-\lambda}\rho'^{-\lambda}\mathcal{E}_{\infty}(\lambda))|_{\rho=\rho'=0}$ of $S(\lambda)$. To analyze this singular part of $S(\lambda)$ in the neighbourhood of the cusp submanifolds, it turns out to be more convenient to work in the neighbourhood M_k with the coordinates (x,y,z) than in their compactified version (t,u,z). Indeed we will see that, up to conformal factors, the scattering operator for the model $X_k = \Gamma_k\backslash\mathbb{H}^{n+1}$ is $\Delta_{Y_k}^{\lambda-\frac{n}{2}}$ where again $Y_k = \mathbb{R}^{n-k}\times T^k$ with the flat metric. This is what Froese-Hislop-Perry used in [3] in dimension 3.

Using Fourier transform in the (y,z) variable on X_k we see that the Laplacian on X_k is transformed into the one dimensional operator

$$P_{\xi_m} = -x^2\partial_x^2 + (n-1)x\partial_x + x^2|\xi_m|^2$$

with $\xi_m = (\xi,\omega_m)$. We easily deduce that the resolvent can be expressed by

$$R_{X_k}(\lambda;w,w') = -(xx')^{\frac{n}{2}}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^{n-k}} e^{i\xi_m.(y-y',z-z')}G_{\xi_m}(\lambda;x,x')d\xi$$

$$G_{\xi_m}(\lambda;x,x') := K_{\lambda-\frac{n}{2}}(|\xi_m|x)I_{\lambda-\frac{n}{2}}(|\xi_m|x')H(x-x') + K_{\lambda-\frac{n}{2}}(|\xi_m|x')I_{\lambda-\frac{n}{2}}(|\xi_m|x)H(x'-x)$$

with H the Heaviside function, $(w;w') = (x,y,z;x',y',z')$ the coordinates on $X_k\times X_k$ and $I_\nu(z), K_\nu(z)$ the modified Bessel functions. Therefore using that $\rho = \frac{x}{x^2+|y|^2}$ and

$$I_\nu(z) = \frac{2^{-\nu}z^\nu}{\nu\Gamma(\nu)} + O(z^{\Re(\nu+2)}), \quad K_\nu(z) = -\frac{\nu}{2}\Gamma(\nu)\Gamma(-\nu)(I_\nu(z) - I_{-\nu}(z)) \tag{6.4}$$

as $z\to 0$, we obtain for $\Re(\lambda) < 0$ (using $\{\rho = 0\} = \{x = 0\}$ on B)

$$E_{X_k}(\lambda;y',z';w) = \frac{-|y'|^{2\lambda}2^{\frac{n}{2}-\lambda}}{\Gamma(\lambda-\frac{n}{2}+1)}x^{\frac{n}{2}}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^{n-k}} e^{i\xi_m.(y-y',z-z')}|\xi_m|^{\lambda-\frac{n}{2}}K_{\lambda-\frac{n}{2}}(|\xi_m|x)d\xi \tag{6.5}$$

and

$$\begin{aligned} S_{X_k}(\lambda;y,z;y',z') &:= (2\lambda-n)[\rho(x,y)^{-\lambda}E_{X_k}(\lambda;y',z';x,y,z)]|_{x=0} \\ &= 2^{n-2\lambda}\frac{\Gamma(\frac{n}{2}-\lambda)}{\Gamma(\lambda-\frac{n}{2})}|y|^{2\lambda}|y'|^{2\lambda}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^{n-k}}e^{i\xi_m.(y-y',z-z')}|\xi_m|^{2\lambda-n}d\xi \end{aligned}$$

where this last sum-integral is understood (by splitting the term with $\omega_m=0$ and the terms with $\omega_m\neq 0$) as the function on $\mathbb{R}^{n-k}_y\times T^k_z\times\mathbb{R}^{n-k}_{y'}\times T^k_{z'}$

$$\frac{2^{2\lambda-n}\pi^{-\frac{n-k}{2}}\Gamma(\lambda-\frac{k}{2})}{\Gamma(\frac{n}{2}-\lambda)}|y-y'|^{-2\lambda+k}+\sum_{m\neq 0}\int_{\mathbb{R}^{n-k}}e^{i\xi_m.(y-y',z-z')}|\xi_m|^{2\lambda-n}d\xi$$

which is continuous on $\{y\neq 0, y'\neq 0\}$. This last function continues meromorphically to $\lambda\in\mathbb{C}$ in the distribution sense thus

(6.6)

$$S^k_{mod}(\lambda;y,z;y',z') := [\rho(x',y')^{-\lambda}E^k_{mod}(\lambda;y,z;x',y',z')]|_{x=0} = \psi^k_L(y)S_{X_k}(\lambda;y,z;y',z')\psi^k(y')$$

continues meromorphically to $\mathbb{C}$ as a distribution. Note that the measure dvol_{h_0} on Y_k is

$$\mathrm{dvol}_{h_0}=|y|^{-2n}dydz.$$

To work on $Y_k=\mathbb{R}^{n-k}_y\times T^k_z$ with the natural measure $dydz$ corresponding to the flat metric $\widetilde{h}_0$, we have to multiply the kernel of $S_{X_k}(\lambda)$ by $|y|^{-n}|y'|^{-n}$, thus (6.6) can be rewritten, acting on $L^2(Y_k,dydz)$

$$S^k_{mod}(\lambda)=c(\lambda)\psi^k_L|y|^{2\lambda-n}\Delta_{Y_k}^{\lambda-\frac{n}{2}}|y|^{2\lambda-n}\psi^k \text{ with } c(\lambda):=2^{n-2\lambda}\frac{\Gamma(\frac{n}{2}-\lambda)}{\Gamma(\lambda-\frac{n}{2})}. \tag{6.7}$$

Note that it has poles at $\lambda=\frac{n}{2}+j$ (with $j\in\mathbb{N}$) with residue the differential operator on Y_k

$$\mathrm{Res}_{\frac{n}{2}+j}(S^k_{mod}(\lambda))=\frac{(-1)^{j+1}2^{-2j}}{j!(j-1)!}\psi^k_L|y|^{2j}\Delta^j_{Y_k}|y|^{2j}\psi^k \text{ on } L^2(Y_k,dydz).$$

For the singularity of the kernel of $S(\lambda)$ in the regular neighbourhood B_r on $L^2(B_r,\mathrm{dvol}_{h_0})$ (to see it acting on $L^2(B_r,\mathrm{dvol}_{\widetilde{h}_0})$ it suffices to multiply the kernel by $(r_cr'_c)^n$) we define the model scattering operator using (5.13)

$$S_{\mathbb{H}^{n+1}}(\lambda;y;y'):=(2\lambda-n)[x'^{-\lambda}E_{\mathbb{H}^{n+1}}(\lambda;y;x',y')]|_{x'=0}=\frac{\pi^{-\frac{n}{2}}\Gamma(\lambda)}{\Gamma(\lambda-\frac{n}{2})}|y-y'|^{-2\lambda}$$

and we get from (5.15)

$$S^r_{mod}(\lambda;y;y'):=[\rho(x',y')^{-\lambda}E^r_{mod}(\lambda;y;x',y')]|_{x'=0}=\frac{\psi^r_L(y)\psi^r(y')}{\gamma_r(y)^\lambda\gamma_r(y')^\lambda}S_{\mathbb{H}^{n+1}}(\lambda;y;y'), \tag{6.8}$$

which continues meromorphically to $\mathbb{C}$ with poles at $\frac{n}{2}+j$ (with j integers) and residue

$$\mathrm{Res}_{\frac{n}{2}+j}(S^r_{mod}(\lambda))=\frac{(-1)^{j+1}2^{-2j}}{j!(j-1)!}\psi^r_L\gamma_r^{-\frac{n}{2}-j}\Delta^j_{\mathbb{R}^n}\gamma_r^{-\frac{n}{2}-j}\psi^r.$$

With notations of (6.8), (6.6) we can now define the model scattering operator

$$S_{mod}(\lambda):=\sum_{\alpha=1,\dots,n-1,r}(\iota_\alpha)^*S^\alpha_{mod}(\lambda)(\iota_\alpha)_* \tag{6.9}$$

and we have

$$S(\lambda)-S^k_{mod}(\lambda)\in R_c^{-1}{R'_c}^{-1}C^\infty(\partial\bar{X}\times\partial\bar{X})$$

which is a compact operator on $L^2(B)$. From this study, it is straightforward to check that $S(\lambda)$ is a bounded operators on $L^2(B)$ in $\{\Re(\lambda)\leq\frac{n}{2}\}$ (and λ not resonance).

We summarize this discussion in the following

Lemma 6.1. *$S(\lambda)$ is meromorphic in $\mathbb{C}$ as an operator acting on $R_c^{-1}C^\infty_{acc}(\partial\bar{X})$, with Schwartz kernel the meromorphic continuation from $\{\Re(\lambda)<0\}$ to $\mathbb{C}$ of the distribution*

$$(2\lambda-n)(\rho^{-\lambda}\rho'^{-\lambda}R(\lambda))|_{B\times B}\in C^{-\infty}(\bar{X}\times\bar{X}).$$

Its poles in $\{\Re(\lambda)\leq\frac{n}{2}\}$ are included in the set of resonances and have finite multiplicity, whereas the poles in $\{\Re(\lambda)>\frac{n}{2}\}$ are first order poles with residue

$$Res_{\lambda_0}S(\lambda)=\begin{cases}-\frac{(-1)^{j+1}2^{-2j}}{j!(j-1)!}P_j+\Pi_{\lambda_0} & \text{if } \lambda_0=\frac{n}{2}+j, j\in\mathbb{N}\\ \Pi_{\lambda_0} & \text{if } \lambda_0\notin\frac{n}{2}+\mathbb{N}\end{cases}$$

where P_j is the differential operator on (B,h_0) with principal symbol $\sigma_0(P_j)=|\xi|^{2j}_{h_0}$, defined by

$$[Res_{\frac{n}{2}+j}\rho^{-\lambda}\Phi(\lambda)]|_{\rho=0}=\frac{(-1)^j2^{-2j}}{j!(j-1)!}P_j$$

and Π_{λ_0} is a finite-rank operator with Schwartz kernel $2j\left((\rho\rho')^{-\lambda_0}Res_{\lambda_0}R(\lambda)\right)|_{B\times B}$ satisfying rank $\Pi_{\lambda_0}=\dim\ker_{L^2}(\Delta_X-\lambda_0(n-\lambda_0))$.

Proof: the meromorphic property of $S(\lambda)$ and its Schwartz kernel have been discussed, the statement about the poles outside $\{\Re(\lambda)\leq\frac{n}{2}\}$ is also clear by (5.5) . For the case of a pole λ_0 with $\Re(\lambda_0)>\frac{n}{2}$, the proof is the same than [7, Prop 3.6]. The fact about the rank of Π_{λ_0} is quite straightforward by mimicking the proof of [10, Th. 1.1]: we only need the indicial equation (5.3) and that there is no solution of $(\Delta_X-\lambda_0(n-\lambda_0))u=0$ with $u\in\dot{C}^\infty(X)$, this last fact being already proved by Mazzeo [16]. □

Note that this Lemma also holds for any boundary defining function $\hat{\rho}\in C^\infty_{\rm acc}(\bar{X})$. The operators P_j will be discussed in next section.

We now give functional relations for Eisenstein functions and scattering operator:

Proposition 6.2. *If $\Re(\lambda)<0$, we have for $w\in X$, $b'\in B$,*

$$E(\lambda;b';w)=-\int_B S(\lambda;b';b)E(n-\lambda;b;w)\ dvol_{h_0}(b)$$

and there exists $C>1$ such that for N large the meromorphic identity

$$E(\lambda)=-S(\lambda)E(n-\lambda) \tag{6.10}$$

holds true in the strip $-C^{-1}N<\Re(\lambda)\leq\frac{n}{2}$ as operators from $\rho^NL^2(X)$ to $L^2(B)$.

Proof: if for $w\in X$ fixed and $\Re(\lambda)<0$ we multiply (5.18) by $\rho(w')^{-\lambda}$ and take the limit $w'\to b'\in B$, then we obtain the first result using the symmetry of the resolvent kernel (which also induces the symmetry of the kernel of $S(\lambda)$). The next part is just a meromorphic continuation using mapping properties of $E(\lambda)$ and $S(\lambda)$. □

We deduce easily from this Proposition and Proposition 5.6 the

Corollary 6.3. *If λ_0 is such that $\Re(\lambda_0)\leq\frac{n}{2}$, $\lambda_0(n-\lambda_0)\notin\sigma_{pp}(\Delta_X)$ and $S(\lambda)$ holomorphic at λ_0, then λ_0 is not a resonance.*

Here is another inmportant property of $S(\lambda)$:

Proposition 6.4. *For $\Re(\lambda)=\frac{n}{2}$, $S(\lambda)$ is invertible on $L^2(B)$ and we have*

$$S(\lambda)^{-1}=S(n-\lambda)=S(\lambda)^*$$

Proof: the unitarity of $S(\lambda)$ on the critical line comes directly from the density of $\dot{C}^\infty(\bar{B})\subset C^\infty_{\rm acc}(\partial\bar{X})$ in $L^2(B)$ and Lemma 5.1 whereas the equation $S(\lambda)^{-1}=S(n-\lambda)$ is a consequence of the definition of $S(\lambda)$ and again the density of $C^\infty_{\rm acc}(\partial\bar{X})$ in $L^2(B)$. □

We give a description of the scattering operator as a pseudo differential in the class defined in Section 3 and characterized by the type of singularity of its Schwartz kernel on the blown-up manifold $\bar{B}\times_\Phi \bar{B}$.

Theorem 6.5. *Let $\lambda\notin \frac{n}{2}+\mathbb{N}$ and λ not a resonance, then with definition (3.4), the scattering operator $S(\lambda)$ is a Φ-pseudo-differential operator on $\bar{B}$ of order*

$$S(\lambda)\in \Psi_\Phi^{2\lambda-n,E_\lambda}(\bar{B})+(R_cR_c')^{-1}C^\infty(\partial\bar{X}\times\partial\bar{X})$$

with respect to volume density $dvol_{h_0}$, where for $k=1,\dots,n-1$

$$E_\lambda(\mathcal{F}_k)=-2\lambda-k,\quad E_\lambda(\mathcal{I}_k)=-4\lambda,\quad E_\lambda(\mathcal{T}_k)=E_\lambda(\mathcal{B}_k)=-k.$$

Proof: for technical reasons, we begin by working with the density $\mathrm{dvol}_{\widetilde{h}_0}$ and it will suffice to multiply by the correct factors at the end. If $\eta\in C_0^\infty([0,\infty))$ is a function which is equal to 1 in a small neighbourhood of 0, we can decompose (6.7) as

$$S^k_{mod}(\lambda)=c(\lambda)\psi^k_L|y|^{2\lambda-n}\left(\eta(\Delta_y)\Delta_y^{\lambda-\frac{n}{2}}+(1-\eta(\Delta_{Y_k}))\Delta_{Y_k}^{\lambda-\frac{n}{2}}\right)\psi^k|y|^{2\lambda-n}$$

on $L^2(Y_k,dydz)$. The first term has a kernel

$$\psi^k_L(y)\psi^k(y')|y|^{2\lambda-n}|y'|^{2\lambda-n}\int_{\mathbb{R}^{n-k}}e^{i\xi.(y-y')}|\xi|^{2\lambda-n}\eta(|\xi|)d\xi$$

which is smooth for y,y' in $\mathbb{R}^{n-k}$ and since it is the Fourier transform of a distribution classically conormal to 0, it is straightforward to check that it can be expressed by

$$\psi^k_L(y)\psi^k(y')|y|^{2\lambda-n}|y'|^{2\lambda-n}F_\lambda(\sqrt{1+|y-y'|^2})\tag{6.11}$$

with $F_\lambda(x)$ smooth on $[0,\infty)$ and having an expansion

$$F_\lambda(x)\sim x^{-2\lambda+k}\sum_{j=0}^{\infty}a_j(\lambda)x^{-j}\tag{6.12}$$

when $x\to\infty$. To describe the singularity of this kernel on the manifold $\bar{B}$, we use near infinity the polar coordinates $v=|y|^{-1},\omega=y/|y|,v'=|y'|^{-1},\omega'=y'/|y'|$. Since $|y-y'|=|\frac{\omega}{v'}-\frac{\omega'}{v}|$ we deduce that the kernel (6.11)

$$\psi^k_L(\frac{\omega}{v})\psi^k(\frac{\omega'}{v'})v^{-2\lambda+n}v'^{-2\lambda+n}F_\lambda\left(\sqrt{1+\left|\frac{\omega}{v'}-\frac{\omega'}{v}\right|^2}\right).$$

First, it is clearly smooth in $B\times B$. By lifting $|\frac{\omega}{v'}-\frac{\omega'}{v}|,v,v'$ on $\bar{B}\times_\Phi\bar{B}$ we have that

$$\beta_\Phi{}^*\left(\sqrt{1+\left|\frac{\omega}{v'}-\frac{\omega'}{v}\right|^2}\right)\rho_{\mathcal{T}_k}\rho_{\mathcal{B}_k}\rho_{\mathcal{F}_k}\in C^\infty(\bar{B}\times_\Phi\bar{B})\tag{6.13}$$

does not vanish on $\mathcal{F}_k,\mathcal{B}_k,\mathcal{T}_k$ and

$$\beta_\Phi{}^*(vv')\rho_{\mathcal{T}_k}^{-1}\rho_{\mathcal{B}_k}^{-1}\rho_{\mathcal{F}_k}^{-2}\rho_{\mathcal{I}_k}^{-2}\in C^\infty(\bar{B}\times_\Phi\bar{B})\tag{6.14}$$

does not vanish on $\mathcal{T}_k,\mathcal{B}_k,\mathcal{F}_k,\mathcal{I}_k$. From this and (6.12) it is straightforward to check that

$$\psi^k_L|y|^{2\lambda-n}\eta(\Delta_y)\Delta_y^{\lambda-\frac{n}{2}}\psi^k|y|^{2\lambda-n}\in(\rho_{\mathcal{T}_k}\rho_{\mathcal{B}_k})^{n-k}\rho_{\mathcal{F}_k}^{2n-2\lambda-k}\rho_{\mathcal{I}_k}^{-4\lambda+2n}C^\infty(\bar{B}\times_\Phi\bar{B}).\tag{6.15}$$

To deal with the term $\psi^k_L|y|^{2\lambda-n}(1-\eta(\Delta_{Y_k}))\Delta_{Y_k}^{\lambda-\frac{n}{2}}\psi^k|y|^{2\lambda-n}$, we first analyze the operator

$$A(\lambda):=\psi^k_L|y|^{2\lambda-n}(1+\Delta_{Y_k})^{\lambda-\frac{n}{2}}\psi^k|y|^{2\lambda-n}.$$

For that we can begin to use a partition of unity $(\theta_i)_i$ associated to a covering by some euclidian ball on T^k and some functions $\theta_i'\in C_0^\infty(T^k)$ such that $\theta_i'=1$ on the support of θ_i, then it is standard to see that for $s\in\mathbb{C}\setminus[0,\infty)$

$$(\Delta_{Y^k}+1-s)^{-1}=\sum_i\theta_i'(\Delta_{\mathbb{R}^n}+1-s)^{-1}\theta_i+\kappa(s)\tag{6.16}$$

$$\kappa(s):=(\Delta_{Y^k}+1-s)^{-1}\sum_i[\Delta_z,\theta'_i](\Delta_{\mathbb{R}^n}+1-s)^{-1}\theta_i.$$

The kernel $\kappa(s;y,z;y',z')$ of $\kappa(s)$ can be written as the composition

$$\kappa(s;y,z;y'',z'')=(\Delta_{Y_k}+1-s)^p\int_{Y_k}\kappa_1(s;y-y',z-z')\kappa_2(s;y'-y'',z',z'')dy'dz' \tag{6.17}$$

with

$$\kappa_1(s;Y,Z):=\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}^{n-k}}e^{i(\xi.Y+\omega_m.Z)}(1+|\xi|^2+|\omega_m|^2)^{-1-p}d\xi$$

$$\kappa_2(s;y'-y'',z',z''):=\sum_i[\Delta_{z'},\theta'_i(z')](\Delta_{\mathbb{R}^n}+1-s)^{-1}(y',z';y'',z'')\theta_i(z'').$$

Since for some $\epsilon>0$ we have $[\Delta_{z'},\theta'_i(z')]\theta_i(z'')=0$ for $|z-z''|<\epsilon$, it suffices to use the explicit formula of the resolvent kernel of $\Delta_{\mathbb{R}^n}$ with Bessel functions to see that $\kappa_2(s)$ is smooth and satisfies the estimate

$$|\partial^\alpha_{Y,z',z''}\kappa_2(s;Y,z',z'')|\leq C_\alpha\exp(-C_\alpha\sqrt{\Re(s)(1+|Y|^2)})$$

for $\Re(s)\geq\frac{1}{2}$ and some constant $C_\alpha>0$. The kernel $\kappa_1(s)$ is continuous and uniformly bounded if p is large enough, moreover it satisfies for all $N>0$ the estimate

$$|\partial^\alpha_Y\kappa_2(s;Y,Z)|\leq C_{\alpha,N}(1+|Y|)^{-N}$$

for some constant $C_{\alpha,N}>0$. Therefore, using all these estimates and change of variables $y'=u+y$ in (6.17), it is straightforward to check that $\kappa(s;w;w')$ is smooth and satisfies the estimate for all $N>0$

$$|\partial^\alpha_{w,w'}\kappa(s;w;w')|\leq C_{\alpha,N}e^{-C'_\alpha\Re(s)}(1+|y-y'|)^{-N}. \tag{6.18}$$

for some constant $C_{\alpha,N},C'_\alpha>0$ and using the notation $w=(y,z),w'=(y',z')$.

Let Γ be the oriented contour in $\mathbb{C}$ defined by

$$\Gamma=\{\frac{1}{2}+re^{i\frac{\pi}{4}};\infty>r>0\}\cup\{\frac{1}{2}re^{-i\frac{\pi}{4}};0<r<\infty\}.$$

As a consequence of (6.16) and using Cauchy formula, the kernel of $A(\lambda)$ is (with the notation $w=(y,z),w'=(y',z')$)

$$A(\lambda;w;w')=A_1(\lambda;w,w')+A_2(\lambda;w;w'),$$

$$A_1(\lambda;w;w'):=\psi^k_L(y)|y|^{2\lambda-n}\psi^k(y')|y'|^{2\lambda-n}\sum_i\theta'_i(z)\theta_i(z')\int_{\mathbb{R}^n}e^{i\xi.(w-w')}(1+|\xi|^2)^{\lambda-\frac{n}{2}}d\xi,$$

$$A_2(\lambda;w;w'):=\psi^k_L(y)|y|^{2\lambda-n}\psi^k(y')|y'|^{2\lambda-n}\int_\Gamma s^{\lambda-\frac{n}{2}}\kappa(s;w;w')ds.$$

To analyze $A_1(\lambda)$, we use the polar coordinates $v=|y|^{-1},\omega=y/|y|,v'=|y'|^{-1},\omega'=y'/|y'|$ in the y,y' variables and we have $w-w'=(\frac{\omega}{v'}-\frac{\omega'}{v},z-z')$ which vanishes only (and at first order) on the lifted interior diagonal D_Φ of $\bar{B}\times_\Phi\bar{B}$. From the Fourier representation of $A_1(s;w;w')$, we deduce that $A_1(s;w;w')$ is a distribution which is polyhomogeneous conormal to D_Φ of order $2\lambda-n$, vanishes at all order on the boundaries $\mathcal{T}_k,\mathcal{B}_k,\mathcal{F}_k$ of $\bar{B}\times_\Phi\bar{B}$ and has a conormal singularity of order $-4\lambda+2n$ at $\mathcal{I}_k$ (this last one coming from the term $|y|^{2\lambda-n}|y'|^{2\lambda-n}$ as before):

$$\beta^*_\Phi A_1(\lambda)\in\rho_{\mathcal{I}_k}^{-4\lambda+2n}I^{2\lambda-n}(\bar{B}\times_\Phi\bar{B};D_\Phi).$$

The behaviour of $A_2(\lambda)$ comes directly from (6.18) using the polar coordinates and (6.13) and (6.14) as before: we see that

$$\beta^*_\Phi A_2(\lambda)\in\rho^\infty_{\mathcal{T}_k}\rho^\infty_{\mathcal{B}_k}\rho^\infty_{\mathcal{F}_k}\rho_{\mathcal{I}_k}^{-4\lambda+2n}C^\infty(\bar{B}\times_\Phi\bar{B})$$

thus

$$\beta^*_\Phi A(\lambda)\in\rho_{\mathcal{I}_k}^{-4\lambda+2n}I^{2\lambda-n}(\bar{B}\times_\Phi\bar{B};D_\Phi). \tag{6.19}$$

For $N > \Re(\lambda) - \frac{n}{2}$, we have

$$S^k_{mod}(\lambda) = c(\lambda)\psi^k_L |y|^{2\lambda-n}\Big(\eta(\Delta_y)\Delta_y^{\lambda-\frac{n}{2}} + (1+\Delta_{Y_K})^{\lambda-\frac{n}{2}} + (1+\Delta_{Y_k})^N\varphi(1+\Delta_{Y_k})\Big)\psi^k|y|^{2\lambda-\frac{n}{2}}$$

with

$$\varphi(x) = x^{-N}\Big((1-\eta(x-1))(x-1)^{\lambda-\frac{n}{2}} - (1-\eta(x))x^{\lambda-\frac{n}{2}}\Big)$$

which is a symbol in $(0,\infty)$ of order $\lambda - \frac{n}{2} - N - 1$ in the sense that it has a support in $[\epsilon,\infty)$ for some $\epsilon > 0$, it is smooth and satisfies

$$|\partial_x^l \varphi(x)| \leq C_l(1+x)^{\Re(\lambda)-\frac{n}{2}-1-N-l}.$$

Hence following the method of Helffer-Robert [13], we have

$$\varphi(1+\Delta_{Y_k}) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} M[\varphi](s)(1+\Delta_{Y_k})^{-s}ds$$

where $M[\varphi](s)$ is the Mellin transform of φ defined by

$$M[\varphi](s) := \int_0^\infty t^{s-1}\varphi(t)dt$$

and which is rapidly decreasing on $i\mathbb{R}$. From the previous study of $(1+\Delta_{Y_K})^{\lambda-\frac{n}{2}}$ and using Mellin's transform, we deduce that if $B(\lambda)$ is the operator

$$B(\lambda) := \psi^k_L|y|^{2\lambda-n}(1+\Delta_{Y_k})^N\varphi(1+\Delta_{Y_K})\psi^k|y|^{2\lambda-n}$$

then its kernel satisfies

$$B(\lambda;w;w') = B_1(\lambda;w;w') + B_2(\lambda;w;w')$$

$$B_1(\lambda;w,w') := \psi^k_L(y)|y|^{2\lambda-n}\psi^k(y')|y'|^{2\lambda-n}\sum_i \theta_i'(z)\theta_i(z')\int_{\mathbb{R}^n} e^{i\xi.(w-w')}(1+|\xi|^2)^N\varphi(1+|\xi|^2)d\xi$$

$$B_2(\lambda;w;w') := \psi^k_L(y)|y|^{2\lambda-n}\psi^k(y')|y'|^{2\lambda-n}\frac{(1+\Delta_w)^N}{2\pi i}\int_{-i\infty}^{i\infty} M[\varphi](s)\int_\Gamma \tau^{s-\frac{n}{2}}\kappa(\tau;w,w')d\tau ds.$$

In view of the estimate (6.18) on $\kappa(\tau;w;w')$ and its smoothness, we easily obtain that the kernel $B_2(\lambda;w;w')$, when lifted on $\bar{B}\times_\Phi \bar{B}$, has exactly the same properties than $A_2(\lambda;w,w')$. For the term $B_1(\lambda;w;w')$ we can proceed as for $A_1(\lambda;w,w')$ and it finally shows that

$$\beta_\Phi{}^*B(\lambda) \in \rho_{\mathcal{I}_k}^{-4\lambda+2n} I^{2\lambda-n-1}(\bar{B}\times_\Phi \bar{B}; D_\Phi).$$

Combined with (6.15), (6.19), this proves the Theorem after multiplying by the lift of $(r_c r_c')^{-n}$ to return with the correct density. □

Remark: As a consequence, we can obtain quite general mapping properties for $S(\lambda)$ (i.e. the actions of $S(\lambda)$ on extendible distributions on $\bar{B}$ conormal to $\partial\bar{B}$) using general theory for those operators, see for exemple Vaillant [26, Section 2.2].

7. Conformal theory on the boundary

As explained by Graham-Zworski [7], there is a strong connection between scattering theory on Einstein conformally compact manifolds (in particular convex co-compact hyperbolic quotients) and conformal theory of its boundary. We check here that similar results hold in this degenerate case.

First recall from Lemma 2.2 that for any $\hat{h}_0 := e^{2\omega_0}h_0 \in [h_0]_{\text{acc}}$, there exists a boundary defining function $\hat{\rho} = e^\omega\rho \in C^\infty_{\text{acc}}(\bar{X})$, unique up to $\dot{C}^\infty(\bar{X})$, such that $\omega|_{\partial\bar{X}} = \omega_0$ and which put the metric under the almost product form (2.12). This gives a way to identify special boundary defining functions of Lemma 2.2 with representatives of the subconformal class $[h_0]_{\text{acc}}$. Moreover we saw that the scattering operators $S(\lambda), \hat{S}(\lambda)$ obtained by solving Poisson problem respectively with ρ and $\hat{\rho}$ (i.e. for conformal representatives h_0 and $\hat{h}_0$) are related by

$$\hat{S}(\lambda)f = e^{-\lambda\omega_0}S(\lambda)e^{(n-\lambda)\omega_0}f. \tag{7.1}$$

In this sense, $S(\lambda)$ is a conformally covariant operator and by looking at the residues we have the rule

$$\hat{P}_j = e^{(-\frac{n}{2}-j)\omega_0}P_je^{(\frac{n}{2}-j)\omega_0}$$

which also makes this differential operator being conformally covariant.

Let us now give a few words about conformal GJMS Laplacians. In [6], Graham-Jenne-Manson-sparling defined, on any n-th dimensional Riemannian compact manifold (M, h_0), a family of "natural" conformally covariant differential operators $(P_j)_j$ with principal symbol $\Delta_{h_0}^j$. We call P_j the j-th GJMS Laplacian. They are defined for $j \in \mathbb{N}$ if n is odd and for $j \leq n/2$ integer if n is even and natural in the sense that they can be written in terms of covariant derivatives and curvature of h_0 and conformally covariant in the sense that the operator $\hat{P}_j$ obtained with the same expression than P_j but with a conformal metric $\hat{h}_0 = e^{2\omega_0}h_0$ is related to P_j by the identity

$$\hat{P}_j = e^{-(\frac{n}{2}+j)\omega_0}P_je^{(\frac{n}{2}-j)\omega_0}.$$

Moreover P_1 is Yamabe's Laplacian and P_2 is Paneitz operator. If h_0 is locally conformally flat and $n > 2$ is even, it is also proved in [6] that the P_j can be constructed without obstruction for any $j \in \mathbb{N}$, this is the case in particular of the conformal infinity of a convex co-compact hyperbolic quotients. Note that, since the expression of P_j is local with respect to the metric, these operators can also be defined on non-compact Riemannian manifolds. Graham and Zworski [7] show that on asymptotically Einstein manifolds (X, g) of dimension $n+1$ (with $\bar{X}$ the conformal closure), the residue $\mathrm{Res}_{\frac{n}{2}+j}S(\lambda)$ of the scattering operator obtained by solving Poisson problem with boundary defining function x is P_j on the conformal infinity $(\partial\bar{X}, x^2g|_{T\partial\bar{X}})$ for any j integer if n is odd (resp. for $j \leq \frac{n}{2}$ if n is even). Actually, we learnt from Robin Graham that this also holds for any j if $n > 2$ is even and if (X, g) has negative constant curvature outside a compact set, where in this case the conformal infinity is locally conformally flat. The reason, given in [5], which makes this special case working is that there is no obstruction to construct a hyperbolic conformally compact metric g on $(0, \epsilon]_x \times M$ with conformal infinity $(M \simeq \{x = 0\}, h_0)$ for any (M, h_0) locally conformally flat compact manifold, and actually g is necessary given by

$$g = x^{-2}(dx^2 + h_0 - x^2P + x^4(\frac{1}{4}Ph_0^{-1}P)) \tag{7.2}$$

where $P = (n-2)^{-1}(\mathrm{Ric} - (2n-2)^{-1}Kh_0)$ is the Schouten tensor of h_0, with K, Ric the scalar and Ricci curvatures of h_0. This is a consequence of the constant curvature equation.

Since in our case the metric on $X = \Gamma\backslash\mathbb{H}^{n+1}$ is also hyperbolic, the curvature equation (which is local!) implies again that the tensor $\hat{h}(\hat{\rho})$ in (2.12) has all its Taylor expansion with respect to $\hat{\rho}$ at $\hat{\rho} = 0$ determined by $\hat{h}_0 = \hat{h}(0)$ if $n > 2$: the expression of $\hat{h}(\hat{\rho})$ is explicit and, like (7.2),

$$\hat{h}(\hat{\rho}) = \hat{h}_0 - \hat{\rho}^2P + \hat{\rho}^4(\frac{1}{4}P\hat{h}_0^{-1}P)$$

with P is the Schouten tensor of $\hat{h}_0$.

If $n > 2$, we saw that the expression of $\mathrm{Res}_{\frac{n}{2}+j}S(\lambda)$ is obtained from the construction of $\Phi(\lambda)$ exactly like in the convex co-compact case (the construction is local in term of $\hat{h}(\hat{\rho})$ thus in term of $\hat{h}_0$). By equivalence of the construction of $\Phi(\lambda)$ in [7] and in our case, we obtain the

Proposition 7.1. *The operator P_j of Lemma 6.1 is the j-th conformal GJMS Laplacian defined in* [6] *on locally conformally flat compact manifolds in the sense that it has the same local expression in term of the metric h_0.*

As in the work of Graham-Zworski [7], there is a way to recover the Q-curvature from the scattering operator when n is even. Indeed the construction of the function $\Phi(\lambda)$ being entirely local, the arguments of Graham-Zworski show that, with $P_{j,\lambda}$ defined in (5.4), then $P_{j,\lambda}1$ can be defined as a smooth function on B and satisfies

$$P_{j,\lambda}1=(n-\lambda)Q_{j,\lambda}$$

with $Q_{j,\lambda}$ a smooth function on B depending polynomially on λ. Then one can define

$$Q:=Q_{\frac{n}{2},n} \tag{7.3}$$

and this function on B can be expressed in a natural way in function of the Riemannian tensor h_0 and its covariant derivatives, with the same local expression Branson's Q-curvature on compact manifold and for $n=2$ it is the scalar curvature of h_0. Moreover if $\hat{h}_0=e^{2\omega_0}h_0$ conformal to h_0, it is well-known (see for instance [7]) that its associated Q-curvature is

$$\hat{Q}=e^{-n\omega_0}(Q+P_{\frac{n}{2}}\omega_0). \tag{7.4}$$

We would like to show, like [7], that Q can be expressed by a constant time $S(n)1$ where $S(n)1$ has to be defined. It turns out from our previous analysis that if all cusps have even rank, then $R_c\in C^\infty_{\rm acc}(\bar{X})$ and thus $S(\lambda)1\in R_c^{-1}C^\infty_{\rm acc}(\bar{X})$. When a cusp has odd rank, we can use the Schwartz kernel of $\mathcal{P}(\lambda)$ and $S(\lambda)$ to define $\mathcal{P}(\lambda)1$ and $S(\lambda)1$.

Theorem 7.2. *For λ in a neighbourhood of n then $S(\lambda)1$ is an extendible distribution on $\bar{B}$ depending holomorphically on λ and satisfying*

$$S(\lambda)1\in\prod_{k\ odd}r_{c_k}^{n-2\lambda}C^\infty(\bar{B})+R_c^{-1}C^\infty_{acc}(\partial\bar{X}).$$

Moreover the Q-curvature defined in (7.3) satisfies

$$Q=\frac{(-1)^{\frac{n}{2}}2^{-n}}{\frac{n}{2}!(\frac{n}{2}-1)!}S(n)1. \tag{7.5}$$

Proof: the fact that $S(\lambda)1$ can be meromorphically defined is an easy consequence of the expression of $S(\lambda)$ near the cusp submanifolds in (6.7) since only the zeroth-Fourier term plays a role in $S^k_{mod}(\lambda)1$. The function 1 on $L^2(B)=L^2(B,\mathrm{dvol}_{h_0})$ becomes r_c^n in $L^2(B,\mathrm{dvol}_{\widetilde{h}_0})$ then $S^k_{mod}(\lambda)1$ is the function

$$c(\lambda)\psi^k_L(y)|y|^{2\lambda-n}\Delta_y^{\lambda-\frac{n}{2}}(|y|^{2\lambda-2n}\psi^k(y))$$

which, by using the variable $u=-y/|y|^2$, can be seen to be

$$c(\lambda)|u|^{-k}\psi^k_L(-u/|u|^2)\Delta_u^{\lambda-\frac{n}{2}}(|u|^k\psi^k(-u/|u|^2)). \tag{7.6}$$

This is clearly an element of $R_c^{-1}C^\infty(\partial\bar{X})$ if k is even since $|u|^k\psi(-u/|u|^2)$ is a smooth compactly supported function in $\mathbb{R}^{n-k}$. Now if k is odd, this last function has a classical conormal singularity at $u=0$ of order k thus it is straigthforward to see via Fourier transform that (7.6) has an expansion of the form

$$|u|^{n-2\lambda}\sum_{i=0}^\infty f_i(u/|u|)|u|^i$$

for some f_i smooth. Moreover $S(\lambda)1$ is holomorphic near $\lambda=n$ since the residue $P_{\frac{n}{2}}$ of $S(\lambda)$ at n is a differential operator with no constant term (see [7]). We have proved first part of the Proposition since the other terms in the expression of $S(\lambda)1$ are clearly functions in $R_c^{-1}C^\infty(\bar{X})$.

To prove (7.5), we will use the same kind of arguments than [7, Th. 2]. We will show that for $\lambda\neq n$ near n and $\phi\in\dot{C}^\infty_c(\partial\bar{X})$

$$\int_{\partial\bar{X}}(\mathcal{P}(\lambda)1)(\rho,m)\phi(m)\ \mathrm{dvol}_{h_0}(m)= \tag{7.7}$$

$$\rho^{n-\lambda}\sum_{j=0}^{\frac{n}{2}}\rho^{2j}c_{j,\lambda}\int_{\partial\bar{X}}(P_{j,\lambda}1)\phi \mathrm{dvol}_{h_0}+\rho^{\lambda}\int_{\partial\bar{X}}(S(\lambda)1)\phi\ \mathrm{dvol}_{h_0}+O(\rho^{n+\frac{1}{2}})$$

with $O(\rho^{n+\frac{1}{2}})$ holomorphic at $\lambda=n$ and $c_{0,n}P_{0,n}=1$, and we will show that $\mathcal{P}(n)1=1$. This implies (7.5) in the extendible distribution sense on $\bar{B}$ by taking the limit $\lambda\to n$ in (7.7), $c_{j,\lambda}P_{j,\lambda}$ being holomorphic at $\lambda=n$.

Let $\epsilon_0>0$ and for all $\epsilon\in(0,\epsilon_0]$ we define $f_\epsilon\in\dot{C}_c^\infty(\partial\bar{X})$ which is equal to 1 in $\{r_c\geq 2\epsilon\}$ and 0 in some small neighbourhood $\{r_c<\epsilon\}$ of c. We can also suppose that f_ϵ does not depend on $z\in T^k$ by taking $\epsilon>0$ small and we define $f_0:=1$ which is the pointwise limit of f_ϵ as $\epsilon\to 0$. Then we know from (5.4), (5.5) and the definition of $S(\lambda)$ that for λ near n

$$\mathcal{P}(\lambda)f_\epsilon=\rho^{n-\lambda}\sum_{j=0}^{\frac{n}{2}}\rho^{2j}c_{j,\lambda}P_{j,\lambda}f_\epsilon+\rho^{\lambda}S(\lambda)f_\epsilon+O(\rho^{n+\frac{1}{2}}) \tag{7.8}$$

for all $\epsilon\in(0,\epsilon_0]$ (but not $\epsilon=0$) and the $O(\rho^{n+\frac{1}{2}})$ is holomorphic in $\lambda=n$ since $(c_{j,\lambda})_{j<n/2}$, $\mathcal{P}(\lambda)$ are holomorphic and $\mathrm{Res}_nS(\lambda)=\mathrm{Res}_n(c_{n/2,\lambda}P_{n/2,\lambda})$. One way to compute $\mathcal{P}(\lambda)f$ for $f=f_\epsilon$ with $\epsilon\in[0,\epsilon_0]$ is to use

$$\mathcal{P}(\lambda)f_\epsilon(w)=(2\lambda-n)\int_B E(\lambda;b;w)f_\epsilon(b)\ \mathrm{dvol}_{h_0}(b)$$

with the local representations (5.10). The terms involving $E^r_{mod}(\lambda)$ are standard and we have an expansion for $\lambda\neq n$ but near n

$$\int_B E^r_{mod}(\lambda;b;w)f_\epsilon(b)\ \mathrm{dvol}_{h_0}(b)\in\rho^{n-\lambda}\sum_{j=0}^{\frac{n}{2}}\rho^j\dot{C}_c^\infty(\partial\bar{X})+\rho^{\lambda}\dot{C}_c^\infty(\partial\bar{X})+O(\rho^{n+\frac{1}{2}}) \tag{7.9}$$

continuous in $\epsilon\in[0,\epsilon_0]$ and the $O(\rho^{n+\frac{1}{2}})$ holomorphic in $\lambda=n$. Now to deal with the term

$$\int_B E^k_{mod}(\lambda;b;w)f_\epsilon(b)\ \mathrm{dvol}_{h_0}(b)$$

we use the fact that f_ϵ are independent of $z\in T^k$ thus only the zeroth-Fourier coefficient in $E_{X_k}(\lambda)$ play a role and, using the formula of $E_{X_k}(\lambda)$ in (5.11) or (6.5), we are lead to study the function

$$H_\epsilon:(t,u)\to|u|^kt^\lambda\int_{\mathbb{R}^{n-k}}(t^2+|u-u'|^2)^{-\lambda+\frac{k}{2}}\psi_L^k(-u'/|u'|)f_\epsilon(u')|u'|^kdu'$$

in $\{t^2+|u|^2<1\}$. Let $F_\epsilon(u')=\psi^k_L(-u'/|u'|)f_\epsilon(u')|u'|^k$ and $\phi\in\dot{C}_c^\infty(\partial\bar{X})$ with support in $\{|u|^2<1\}$. Then for $t>0,\epsilon>0$ we use Fourier transform in $u\in\mathbb{R}^{n-k}$ to get

$$\int_{\mathbb{R}^{n-k}}\int_{T^k}H_\epsilon(t,u)\phi(u,z)dudz=d(\lambda)\int_{\mathbb{R}^{n-k}}\int_{T^k}|\xi|^{\lambda-\frac{n}{2}}t^{\frac{n}{2}}K_{\lambda-\frac{n}{2}}(t|\xi|)\mathcal{F}(F_\epsilon)(\xi)\mathcal{F}(\phi|u|^k)(\xi,z)d\xi dz \tag{7.10}$$

where $K_s(z)$ is the modified Bessel function, $d(\lambda)$ is analytic and $\mathcal{F}$ means Fourier transform in variable u. If $\epsilon>0$ or k even, then F_ϵ is a smooth compactly supported function in $\mathbb{R}^{n-k}$ whereas if $\epsilon=0$, F_0 has a classical conormal singularity of order k at $u=0$ thus its Fourier transform in u is a polyhomogeneous symbol of order $-n$ in $\xi\in\mathbb{R}^{n-k}$:

$$\mathcal{F}(F_0)(\xi)\sim|\xi|^{-n}\sum_{j=0}^{\infty}|\xi|^{-j}\theta_j(\xi/|\xi|),\quad \theta_j\in C^\infty(S^{n-k-1}) \tag{7.11}$$

and we deduce that (7.10) extends continuously to $\epsilon\in[0,\epsilon_0]$. In any case, we can use (6.4) and the definition

$$I_\lambda(z)=(\frac{z}{2})^\lambda\sum_{j=0}^{\infty}\frac{(\frac{z^2}{4})^j}{j!\Gamma(\lambda+j+1)}$$

to prove that (7.10) has an asymptotic expansion of the form

$$(7.12)\quad t^{n-\lambda}\sum_{j=0}^{\frac{n}{2}}a_j(\lambda)t^{2j}\int\phi|u|^{-k}(\Delta_u^jF_\epsilon)\ \mathrm{dvol}_{h_0}+t^\lambda b(\lambda)\int\phi|u|^{-k}(\Delta_u^{\lambda-\frac{n}{2}}F_\epsilon)\ \mathrm{dvol}_{h_0}+O(t^{n+\frac12})$$

for $\lambda\neq n$ near n and where $(a_j(\lambda))_{j<\frac{n}{2}}$ and $O(t^{n+\frac12})$ are holomorphic at n and $a_{n/2}(\lambda),b(\lambda)$ have a first order pole at n. This proves with (7.9) that for $\lambda\neq n$ but near n, $\epsilon\in[0,\epsilon_0]$ and $\phi\in\dot{C}_c^\infty(\partial\bar{X})$, the function

$$\rho\to\int_{\partial\bar{X}}(\mathcal{P}(\lambda)f_\epsilon)(\rho,m)\phi(m)\ \mathrm{dvol}_{h_0}(m)$$

has an expansion of the form

$$\rho^{n-\lambda}\sum_{j=0}^{\frac{n}{2}}\rho^{2j}\int_{\partial\bar{X}}f^\epsilon_{j,\lambda}\phi\ \mathrm{dvol}_{h_0}+\rho^\lambda\int_{\partial\bar{X}}g^\epsilon_\lambda\phi\ \mathrm{dvol}_{h_0}+O(\rho^{n+\frac12})$$

where $f^\epsilon_{j,\lambda},g^\epsilon_\lambda$ are some smooth functions in B and extendible distributions on $\bar{B}$ and the $O(\rho^{n+\frac12})$ is holomorphic at $\lambda=n$. Moreover it is important to note that all terms are continuous in $\epsilon\in[0,\epsilon_0]$ in view of the rapid decreasing of ϕ in the cusp. For $\epsilon>0$, the expansion in (7.8) shows that, in the distribution sense,

$$f^\epsilon_{j,\lambda}=c_{j,\lambda}P_{j,\lambda}f_\epsilon\quad g^\epsilon_\lambda=S(\lambda)f_\epsilon.$$

Thus for $\lambda\neq n$ fixed near n, we can take the limit as $\epsilon\to0$ in the distribution sense and using the continuity of $\mathcal{P}(\lambda),P_{j,\lambda}$ and the fact that $S(\lambda)1$ is the limit in the extendible distribution topology of $S(\lambda)f_\epsilon$, we get (7.7) since we also have $c_{0,n}P_{0,n}=1$ by construction of $\Phi(\lambda)$.

It remains to check that $\mathcal{P}(n)1=1$. It clearly suffices to show that

$$(7.13)\qquad (\mathcal{P}(n)1)-1\in L^2(X)$$

since $\mathcal{P}(n)1$ and 1 are both solutions of $\Delta_Xu=0$ and 0 is not an L^2 eigenvalue of Δ_X. Let $J=[\frac{n}{2}]$ be the integer part of $\frac{n}{2}$. Then it is easy to check that there exist $f_j^r\in\dot{C}_c^\infty(\partial\bar{X})$ such that

$$\int_BE^r_{mod}(n;b;w)\ \mathrm{dvol}_{h_0}(b)-\sum_{j=0}^Jf_j^r\rho^j\in L^2(X).$$

We now consider (7.10) with $\lambda=n$ and we split the integral in two parts I_1,I_2 corresponding to $\{t|\xi|<1\}$ and $\{t|\xi|>1\}$. Setting $\xi=r\omega$ with $r=|\xi|$, changing the variable $R=tr$ and using (7.11) and $|K_{n/2}(z)|\leq e^{-Cz}$ for $z>1$ in I_2 yields

$$\begin{aligned}|I_2|&\leq t^k\int_1^\infty\int_{S^{n-k-1}}\int_{T^k}R^{\frac{n}{2}}e^{-CR}|\mathcal{F}(\phi|u|^k)(\frac{R}{t}\omega,z)|R^{n-k-1}dRd\omega dz\\&\leq Ct^{\frac{n+k}{2}}||\phi||_{L^2(B)}.\end{aligned}$$

To deal with I_1 we use for $z\in(0,1)$

$$z^{\frac{n}{2}}K_{n/2}(z)=\sum_{j=0}^Ja_jz^j+M(z)\text{ with }M(z)=O(z^{J+1})$$

and the change of variables $R=rt$ to deduce the expansion

$$I_1=\sum_{j=0}^Jt^j\int\phi|u|^{-k}f_j^k\ \mathrm{dvol}_{h_0}+R_1-R_2$$

for some $f_j^k\in C^\infty(B)$ and

$$R_1:=t^{-n+k}\int_0^1\int_{S^{n-k-1}}\int_{T^k}M(R)\mathcal{F}(F_0)(\frac{R}{t}\omega)\mathcal{F}(\phi|u|^k)(\frac{R}{t}\omega,z)R^{n-k-1}dRd\omega dz,$$

$$R_2 := t^{-n+k}\sum_{j=0}^{J} a_j t^j \int_1^\infty \int_{S^{n-k-1}} \int_{T^k} R^j \mathcal{F}(F_0)(\frac{R}{t}\omega)\mathcal{F}(\phi|u|^k)(\frac{R}{t}\omega, z)R^{n-k-1}dRd\omega dz$$

Since $M(R) \in L^2((0,1), R^{n-k-1}dR)$ and $R^{j-n} \in L^2((1,\infty), R^{n-k-1}dR)$ for $j \leq J$ we can use (7.11) to obtain

$$|R_1| + |R_2| \leq Ct^{\frac{n+k}{2}}||\phi||_{L^2(B)}.$$

We deduce from these estimates that there exist some $f_j \in C^\infty(B)$ such that

$$\mathcal{P}(n)1 - \sum_{j=0}^{J} f_j \rho^j \in L^2(X).$$

To get (7.13) and conclude the proof of the Proposition, is suffices to remark that $f_0 = 1$ by taking the limit $\lambda \to n$ in (7.7) and to check that $f_j = 0$ for $j = 2, \dots, J$. But this last identity is an easy consequence of the indicial equation $\Delta_X(\rho^j f_j) - j(n-j)\rho^j f_j \in \rho^{j+1}C^\infty(B)$. □

If we change the conformal representative $\hat{h}_0 = e^{2\omega_0}h_0 \in [h_0]_{\rm acc}$, let $\hat{\rho} = e^\omega \rho$ be the associated boundary defining function obtained by Lemma 2.2 (unique modulo $\dot{C}^\infty(\bar{X})$) with $\omega|_{\partial\bar{X}} = \omega_0$, then the related scattering operator $\hat{S}(\lambda)$ satisfies

$$\hat{S}(n)1 = e^{-n\omega_0}\Big(S(n)1 + \frac{\frac{n}{2}!(\frac{n}{2}-1)!}{(-1)^{\frac{n}{2}}2^{-n}}P_{\frac{n}{2}}\omega_0\Big)$$

in view of (7.1). Thus with (7.4) we deduce that identity (7.5) still holds with a different choice of conformal representative for the associated scattering operator.

On X one can define a renormalized volume when n is even, like for asymptotically Einstein manifolds [4, 7]. Let $\hat{h}_0$ be a conformal representative in $[h_0]_{\rm acc}$ and let $\hat{\rho} \in C^\infty_{\rm acc}(\bar{X})$ be the boundary defining function of Lemma 2.2 uniquely defined modulo $\dot{C}^\infty(\bar{X})$, which puts the metric under the form

$$g = \frac{d\hat{\rho}^2 + \hat{h}(\hat{\rho})}{\hat{\rho}^2} + O(\hat{\rho}^\infty), \quad \hat{h}(0) = \hat{h}_0,$$

$\hat{h}(\rho)$ is a smooth family of metrics on B, with an even Taylor expansion at $\hat{\rho} = 0$. Let us consider

$$I(\epsilon) := \int_{\hat{\rho}>\epsilon} \mathrm{dvol}_g$$

for $\epsilon > 0$ small.

Lemma 7.3. *As $\epsilon \to 0$, we have the expansion*

$$I(\epsilon) \sim \sum_{j=1}^{\frac{n}{2}} \alpha_j \epsilon^{-n+2j-2} + L\log(\epsilon^{-1}) + V + o(1)$$

for some $a_j, L, V \in \mathbb{R}$ where L (resp. V) does not depend on the choice of representative $\hat{h}_0 \in [h_0]_{acc}$ if n is even (resp. n is odd).

Proof: the existence of the expansion for $\hat{\rho} = \rho$ (i.e. $\hat{h}_0 = h_0$) is quite direct and the general case is relatively similar using expression (2.11). Define the density

$$dv(\hat{\rho}) := \left(\frac{\det(\hat{h}(\hat{\rho}))}{\det(\hat{h}_0)}\right)^{\frac{1}{2}} d\hat{\rho}\ \mathrm{dvol}_{\hat{h}_0}$$

on $[0,\epsilon] \times \partial\bar{X}$, then $dv(\rho)$ has a Taylor expansion at $\hat{\rho} = 0$ (and out of c) of the form

$$dv(\hat{\rho}) = d\hat{\rho}\ \mathrm{dvol}_{\hat{h}_0}(1 + \sum_{i=1}^{\frac{n}{2}} \hat{\rho}^{2i} v_{2i} + O(\hat{\rho}^{n+2}))$$

where v_{2i} are smooth functions on $B = \partial\bar{X} \setminus c$. We return to the expression of $\hat{h}(\hat{\rho})$ near c_k, detailed in (2.11) and (2.9):

$$\hat{h}(s) = h_1(s,v,dv) + h_2(s,v,z,dv,d\zeta) + e^{2\omega}(s^2e^{-2\omega} + |u|^2)^2 d\zeta^2 + O(r_c^\infty)$$

where $u, \omega \in C^\infty_{\rm acc}(\bar{X})$ even in s, $u|_{s=0} = v$ and $h_2 = O(r_c^\infty) = O((s^2+|v|^2)^\infty)$. We see that near c_k there exists $\alpha(s,v,\zeta)$ smooth, even in s, such that $\alpha(0,v,\zeta) = 1$ and

$$\frac{dv(s)}{\mathrm{dvol}_{\hat{h}_0}} = \alpha(s,v,\zeta)\frac{(s^2e^{-2\omega}+|u|^2)^k}{|v|^{2k}}ds = 1 + |v|^{-2k}\sum_i f_{2i}s^{2i}$$

for some $f_{2i} \in C^\infty(\partial\bar{X})$, where we used that k is integer. Thus we have the expansion

$$dv(\hat{\rho}) = (1 + \sum_{i=1}^{\frac{n}{2}} \hat{\rho}^{2i}v_{2i})d\hat{\rho}\ \mathrm{dvol}_{\hat{h}_0} + O(R_c^{-2}\hat{\rho}^{n+2})d\hat{\rho}\ \mathrm{dvol}_{\hat{h}_0}, \tag{7.14}$$

with $v_{2i} \in R_c^{-2}C^\infty(\partial\bar{X})$. Integrating $dv(\hat{\rho})$ on $\{\epsilon_0 > \hat{\rho} > \epsilon\}$ gives the searched expansion for $I(\epsilon)$ since each v_{2i} is in $L^1(\partial\bar{X}, dvol_{\hat{h}_0})$. We also clearly have

$$L = \int_{\partial\bar{X}} v_n\ \mathrm{dvol}_{\hat{h}_0}.$$

To prove independence of L if n is even (and V if n is odd) with respect to the choice of boundary defining function (or conformal representative) considered in Lemma (2.2), it suffices to mimick the same proof than in [4, Th. 3.1], the essential argument being that for any choice of boundary defining function $\hat{\rho}$ of Lemma 2.2, $\hat{\rho}$ is an odd function of ρ in the sense that $\partial_\rho^{2j}\hat{\rho} = 0$. □

We conclude by a result similar to Graham-Zworski's Theorem relating integral of Q with renormalized volume L

Theorem 7.4. *The Q curvature on B is in $L^1(B, dvol_{h_0})$ and we have*

$$\frac{(-1)^{\frac{n}{2}}2^{1-n}}{\frac{n}{2}!(\frac{n}{2}-1)!}\int_B Q\ dvol_{h_0} = L$$

Proof: the proof is essentially similar to the proof of [7, Th. 2] but the expansions have to be done "again some function $\phi \in \dot{C}^\infty_c(\partial\bar{X})$" in the spirit of the proof of Proposition 7.2. Let $\epsilon_0 > 0$, $\chi \in C_0^\infty([0,\epsilon_0))$ which is equal to 1 on $[0,\epsilon_0/2]$ and let $\phi \in \dot{C}^\infty_c(\partial\bar{X})$. We now define the function $\psi(\rho,b) = \chi(\rho)\phi(b)$ in the collar neighbourhood of the boundary $(0,\epsilon_0)_\rho \times \partial\bar{X}$, we set $u_\lambda := \mathcal{P}(\lambda)1$ and we will use the notation "pf" for "finite part as $\epsilon \to 0$". Using Green Formula as in [7, Prop. 3.3] we check that for $\lambda \neq n$ but near n

$$\mathrm{pf}\int_{\rho>\epsilon}(|du_\lambda|^2 - \lambda(n-\lambda)u_\lambda^2)\psi\ \mathrm{dvol}_g = -n\int_B (S(\lambda)1)\phi\ \mathrm{dvol}_{h_0} - \frac{1}{2}\mathrm{pf}\int_{\rho>\epsilon} u_\lambda^2\Delta_X(\psi)\ \mathrm{dvol}_g. \tag{7.15}$$

But following line by line the proof of [7, Th. 2] and using the expansion (7.14) of the volume form as $\rho \to 0$, it is straightforward to see that

$$\mathrm{pf}\int_{\rho>\epsilon}(|du_\lambda|^2 - \lambda(n-\lambda)u_\lambda^2)\psi\ \mathrm{dvol}_g = \int_B H_\lambda\phi\ \mathrm{dvol}_{h_0}$$

for some function $H_\lambda \in C^\infty(B)$ depending continuously of λ in a complex ball containing n, and such that $H_n = -\frac{n}{2}v_n$ with v_n defined in (7.14). Since $v_n \in L^1(B, \mathrm{dvol}_{h_0})$, this gives

$$\lim_{\phi\to 1}\lim_{\lambda\to n}\mathrm{pf}\int_{\rho>\epsilon}(|du_\lambda|^2 - \lambda(n-\lambda)u_\lambda^2)\psi\ \mathrm{dvol}_g = -\frac{n}{2}L.$$

But from 7.5, we have

$$\lim_{\phi\to 1}\lim_{\lambda\to n}\int_B (S(\lambda)1)\phi\ \mathrm{dvol}_{h_0} = \frac{(-1)^{\frac{n}{2}}2^{-n}}{\frac{n}{2}!(\frac{n}{2}-1)!}\int_B Q\ \mathrm{dvol}_{h_0}$$

thus it remains to deal with the last term in (7.15). First observe that $\Delta_X(\chi\phi) = \phi\Delta_X(\chi) + \chi\Delta_X(\phi)$ and that $\Delta_X(\chi)$ has compact support in X thus

$$\mathrm{pf}\int_{\rho>\epsilon} u_\lambda^2\phi\Delta_X(\chi)\ \mathrm{dvol}_g = \int_X u_\lambda^2\phi\Delta_X(\chi)\ \mathrm{dvol}_g.$$

Using that $u_\lambda \to 1$ when $\lambda \to n$ and Green formula, we deduce that

$$\lim_{\phi\to 1}\lim_{\lambda\to n}\int_X u_\lambda^2\phi\Delta_X(\chi)\ \mathrm{dvol}_g = \lim_{\phi\to 1}\int_X \phi\Delta_X(\chi)\ \mathrm{dvol}_g = 0.$$

Now from (5.1) we have $\Delta_X(\phi) = \rho^2\Delta_{h(\rho)}(\phi)$ and let us take for some small $\delta > 0$, $\phi = \phi_\delta$ depending only on r_c and which is equal to 1 in $\{r_c > 2\delta\}$ and 0 in $\{r_c < \delta\}$. Recall that r_c is a function which is equal to $|u|$ in the neighbourhood $B_k = \{(u,z) \in \mathbb{R}^{n-k}\times T^k; |u| < 1\}$ of c_k in $\partial\bar{X}$ and we will write, by abuse of notation, r_c for the function $|u|$ on $(0,\epsilon_0)_\rho \times B_k$. We will show that

$$\lim_{\delta\to 0}\lim_{\lambda\to n}\mathrm{pf}\int_\epsilon^{\epsilon_0}\int_B u_\lambda^2\chi\rho^{-n+1}\Delta_{h(\rho)}(\phi_\delta)\ \mathrm{dvol}_{h(\rho)}d\rho = 0 \tag{7.16}$$

and the Theorem will be proved. Since $\chi = 1$ near $\rho = 0$, we can suppose, using again Green formula, that $\chi = 1$ in (7.16). In view of the assumptions on ϕ_δ it suffices the work in neighbourhoods $(0,\epsilon_0)_\rho \times \{|u| < 2\delta\} \times T_z^k \subset M_k$ of c_k where the metric $h(\rho)$ on B has the form

$$h(\rho) = dr_c^2 + r_c^2 d\theta^2 + (r_c^2+\rho^2)^2 dz^2, \quad \text{with } r_c = |u|, \theta := u/|u|. \tag{7.17}$$

Again from the proof of Theorem 2 in [7], we have for $\lambda \neq n$

$$u_\lambda^2 = \rho^{2n-2\lambda}\Big(1 + \sum_{p=1}^{\frac{n}{2}-1}(n-\lambda)u_{p,\lambda}\rho^{2p} + u_{\frac{n}{2},\lambda}\rho^n\Big) + \rho^n S(\lambda)1 + O((n-\lambda)\rho^{n+\frac12}) \tag{7.18}$$

with $u_{p,\lambda}$ holomorphic in λ and smooth in B, $u_{\frac{n}{2},n} = -S(n)1$ and the big O depends on δ. Moreover using (7.12) with $\epsilon = 0$ ($F_0 = |u|^k = r_c^k$ near c_k), we see that in $\{r_c < 2\delta\}$

$$u_{p,\lambda} = d_{p,\lambda} r_c^{-2p} \tag{7.19}$$

for some constants $d_{p,\lambda}$ holomorphic in λ. Since ϕ_δ has compact support, observe by Green formula that

$$\int_B \Delta_{h(\rho)}(\phi_\delta)\ \mathrm{dvol}_{h(\rho)} = 0. \tag{7.20}$$

Thus it remains to compute the finite part of $\int_\epsilon^{\epsilon_0}(u_\lambda^2 - \rho^{2n-2\lambda})\rho^{-n+1}\Delta_{h(\rho)}(\phi_\delta)\ \mathrm{dvol}_{h(\rho)}d\rho$. For that we use (7.17) and (7.14) to see that near c_k

$$\Delta_{h(\rho)}(\phi_\delta)\frac{\mathrm{dvol}_{h(\rho)}}{\mathrm{dvol}_{h_0}} = \sum_{j,l,m\in\mathbb{N}_0} c_{j,l,m,k}\rho^{2(j+l+m)} r_c^{-k-n+1-2j-2m}\partial_{r_c}(r_c^{n+k-2l-1}\partial_{r_c}(\phi_\delta)) \tag{7.21}$$

for some constants $c_{j,l,m,k}$. Then we multiply the expansion (7.18) of $u_\lambda^2 - \rho^{2n-2\lambda}$ by (7.21) and obtain

$$(u_\lambda^2 - \rho^{2n-2\lambda})\Delta_{h(\rho)}(\phi_\delta)\ \mathrm{dvol}_{h(\rho)} = O((n-\lambda)\rho^n)\ \mathrm{dvol}_{h_0} +$$
$$(n-\lambda)\sum_{\substack{j,l,m,p\\ p\leq \frac{n}{2}-1}}\rho^{2(m+l+p+j+n-\lambda)}u_{p,\lambda}c_{j,m,l,k}r_c^{-k-n+1-2j-2m}\partial_{r_c}(r_c^{n+k-2l-1}\partial_{r_c}(\phi_\delta))\ \mathrm{dvol}_{h_0}.$$

Multiplying this by ρ^{-n+1}, integrating on $\rho \in (\epsilon,\epsilon_0)$ and computing the finite part as $\epsilon \to 0$, it is straightforward to check that we obtain near

$$\mathrm{pf}\int_\epsilon^{\epsilon_0}\int_B (u_\lambda^2 - \rho^{2n-2\lambda})\Delta_{h(\rho)}(\phi_\delta)\ \mathrm{dvol}_{h(\rho)} = O(n-\lambda) +$$
$$\sum_{\substack{m+l+p+j=\frac{n}{2}-1\\ k=1,\dots,n-1}} c_{j,m,l,k}d_{p,\lambda}\int_B r_c^{-k-n+1-2j-2m-2p}\partial_{r_c}(r_c^{n+k-2l-1}\partial_{r_c}(\phi_\delta))\ \mathrm{dvol}_{h_0}.$$

where we have also used (7.19). We thus take $\lambda = n$ and we have to prove that

$$\lim_{\delta\to 0}\int_B r_c^{-k-n+1-2j-2m-2p}\partial_{r_c}(r_c^{n+k-2l-1}\partial_{r_c}(\phi_\delta))\ \mathrm{dvol}_{h_0} = 0$$

when $m+l+p+j = \frac{n}{2}-1$. But writing $\mathrm{dvol}_{h_0} = r_c^{n+k-1}dr_c d\theta$ near c_k and choosing $\phi_\delta(r_c) = \phi(r_c/\delta)$ for some ϕ such that $\mathrm{supp}(\partial_{r_c}\phi)\subset[1,2]$, this is reduced to the limit of

$$\int_\delta^{2\delta}(r_c^{k+1}\delta^{-2} + r_c^k\delta^{-1})dr_c$$

when $\delta\to 0$ and this is 0. The proof is achieved. □

Note that the result still holds by changing the conformal representative $\hat{h}_0 = e^{2\omega_0}h_0$ since by (7.4), the self-adjointness of $P_{n/2}$ and $P_{n/2}1 = 0$, we have

$$\int_B \hat{Q}\ \mathrm{dvol}_{\hat{h}_0} = \int_B (Q + P_{\frac{n}{2}}\omega_0)\ \mathrm{dvol}_{h_0} = \int_B Q\ \mathrm{dvol}_{h_0}.$$

As a corollary of this theorem we prove that the renormalized volume of X is the Euler characteristic of $\bar{X}$.

Proof of Corollary 1.4: in this case where $n+1=3$, the Q curvature is Gauss curvature on the boundary B. Thus it suffices to use Gauss-Bonnet theorem on the manifold with boundary $\{r_c\geq\delta\}$ in B, with the fact that B is a finite volume manifold with ends isometric to

$$\left((0,\epsilon)_{r_c}\times S^1_\theta; dr_c^2 + r_c^4 d\theta^2\right)$$

and we easily obtain the result when $\delta\to 0$ since the integral of the geodesic curvature on $\{r_c=\delta\}$ tends to 0. This gives that $\int_B Q\ \mathrm{dvol}_{h_0} = 2\pi\chi(\bar{B})$ and the result $L = \pi\chi(\bar{B})$ is deduced from Theorem 7.4. It is easy to check that $\partial\bar{X}$ is obtained from $\bar{B}$ by gluing two by two the circles of the boundary of $\bar{B}$, thus $\chi(\bar{B}) = \chi(\partial\bar{X})$ and since $n+1$ is odd, we have $2\chi(\bar{X}) = \chi(\partial\bar{X})$. □

With Figure 1, there is an intuitive interpretation of this result since by taking D_3 and D_4 not tangent but ϵ-close, we are in the convex co-compact case and we know from Graham-Zworski [7] that the integral of the curvature Q_ϵ is the renormalized volume, thus by Epstein formula this is $-\pi\chi(B_\epsilon)$ where B_ϵ is the boundary (compact) of the convex co-compact manifold. The curvature Q_ϵ and the measure μ_ϵ on the boundary depend continuously on $\epsilon\in[0,\epsilon_0)$, one could use Lebesgue theorem after checking a uniform bound of the integral of $Q_\epsilon\mu_\epsilon$. At last we see that B_ϵ is a 1-genus torus, like the compactification $\bar{B}$ of the limit $B = \lim_{\epsilon\to 0} B_\epsilon$.

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