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A Self-Interaction Leading to Fluctuations of Order $n^{5/6}$

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Abstract

In [1], we built and studied a Curie-Weiss model exhibiting self-organized criticality : it is a model with a self-interaction leading to fluctuations of order $n^{3/4}$ and a limiting law proportional to $\exp(-x^4/12)$. In this paper we modify our model in order to « kill the term x^4 » and to obtain a self-interaction leading to fluctuations of order $n^{5/6}$ and a limiting law $C \exp(-\lambda x^6) dx$, for suitable positive constants C and λ .

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1 Introduction

This paper is a sequel to the articles [1] and [4], in which we built and studied a Curie-Weiss model exhibiting self-organized criticality. It was the model given by the distribution

$$\frac{1}{Z_n} \exp\left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2}\right) \mathbb{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i),$$

where Z_n is a renormalisation factor. We proved rigorously that this model exhibits a simple phenomenon of self-organized criticality : if we build the model with a symmetric probability ρ on \mathbb{R} satisfying some integrability conditions, then the sum S_n of the random variables behaves as in the critical generalized Ising Curie-Weiss model (see [3]). More precisely, the fluctuations of S_n are of order $n^{3/4}$ and the limiting law is

$$\left(\frac{4}{3}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{s^4}{12}\right) ds.$$

The purpose of this article is to « kill the term x^4 ». We modify the distribution we studied in [1] and [4] in order to obtain a self-interaction leading to fluctuations of order $n^{5/6}$ and a limiting law

$$C \exp(-\lambda x^6) dx,$$

where C and λ are some positive constants.

To this end, we first focus on the reasons why the fluctuations of S_n in the model we studied in [1] are of order $n^{3/4}$. The interacting term of the model is

$$F(x_1 + \dots + x_n, x_1^2 + \dots + x_n^2),$$

where $F(x, y) = x^2/(2y)$ for $(x, y) \in \mathbb{R} \times]0, +\infty[$. Let I be the rate function for the large deviations of

$$\frac{1}{n} \sum_{k=1}^n (X_k, X_k^2), \quad n \geq 1,$$

where $(X_k)_{k \geq 1}$ is a sequence of independent random variables with common law ρ . By analysing the proofs in [1], we can see that the fluctuations of S_n are of order $n^{3/4}$ because, in the expansion of the function $I - F$ around its minimum, the first non-vanishing term with the variable x (corresponding to S_n/n) appears in the fourth order. More precisely, if σ^2 denotes the variance of ρ and μ_4 its fourth moment, this term is $\mu_4 x^4 / (12\sigma^8)$.

As a consequence, in order to « kill the term x^4 », we are going to modify the interacting function F of our model into some function H so that, in the expansion of the function $I - H$ around its minimum, the first term with the variable x only appears in the sixth order. We could consider

$$H(x, y) = F(x, y) + \frac{\mu_4 x^4}{12\sigma^8}, \quad (x, y) \in \mathbb{R} \times]0, +\infty[.$$

However we want to build a self-interaction, thus we estimate μ_4 by $(x_1^4 + \dots + x_n^4)/n$ (as we estimated σ^2 by $(x_1^2 + \dots + x_n^2)/n$ in order to build the model in [1]). That is why the interacting term we want to consider is

$$\begin{aligned} H(x_1 + \dots + x_n, x_1^2 + \dots + x_n^2, x_1^4 + \dots + x_n^4) \\ = \frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2} + \frac{1}{12} \frac{(x_1^4 + \dots + x_n^4)(x_1 + \dots + x_n)^4}{(x_1^2 + \dots + x_n^2)^4}. \end{aligned}$$

We observe with computer simulations that, with this interacting term, for several probability measures ρ , the fluctuations of the sum S_n are of order $n^{5/6}$ and the limiting law is proportional to $\exp(-\lambda x^6)$ for some $\lambda > 0$.

In sections 2 and 3 we initiate the proof of a fluctuations theorem for S_n with this interacting function H . We use the same techniques as in [1] : we compute the expansion of $I_\bullet - H$ where I_\bullet is the rate function for the large deviations of

$$\frac{1}{n} \sum_{k=1}^n (X_k, X_k^2, X_k^4), \quad n \geq 1.$$

Unfortunately we encountered several problems with the rest of the proof : the techniques we used in [1] have not been successful and we had to modify H . Our investigations to build an interacting function H leading to fluctuations of order $n^{5/6}$ and amenable to a mathematical analysis led us to consider the following model :

The model. Let ρ be a probability measure on \mathbb{R} which is not the Dirac mass at 0. Let H be the function given by

$$\forall (x, y, z) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \quad H(x, y, z) = \frac{x^2}{2y} + \frac{1}{12} \frac{zx^4y^5}{y^9 + x^{10} + zx^4y^4}.$$

For any $n \geq 1$, we denote

$$\begin{aligned} Z_{H,n} = \int_{\mathbb{R}^n} \exp\left(H(x_1 + \dots + x_n, x_1^2 + \dots + x_n^2, x_1^4 + \dots + x_n^4)\right) \\ \times \mathbf{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i). \end{aligned}$$

We consider $(X_k^n)_{1 \leq k \leq n}$ an infinite triangular array of real-valued random variables such that, for all $n \geq 1$, (X_n^1, \dots, X_n^n) has the law $\tilde{\mu}_{H,n,\rho}$, which is the distribution with density

$$\frac{1}{Z_{H,n}} \exp\left(H(x_1 + \dots + x_n, x_1^2 + \dots + x_n^2, x_1^4 + \dots + x_n^4)\right) \mathbf{1}_{\{x_1^2 + \dots + x_n^2 > 0\}}$$

with respect to $\rho^{\otimes n}$. We denote

$$S_n = X_n^1 + \dots + X_n^n, \quad T_n = (X_n^1)^2 + \dots + (X_n^n)^2 \quad \text{and} \quad U_n = (X_n^1)^4 + \dots + (X_n^n)^4.$$

This model is well-defined : $Z_{H,n}$ is finite for any $n \geq 1$. Indeed

$$\forall (x, y, z) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \quad H(x, y, z) \leq \frac{x^2}{2y} + \frac{zx^4}{12y^4} = \frac{x^2}{2y} + \left(\frac{x^2}{y}\right)^2 \frac{z}{12y^2}.$$

We have

$$\forall (x_1, \dots, x_n) \in \mathbb{R}^n \quad \left(\sum_{i=1}^n x_i^2 \right)^2 \geq \sum_{i=1}^n x_i^4$$

and, by convexity of the function $t \mapsto t^2$, we get

$$\forall n \geq 1 \quad 1 \leq Z_{H,n} \leq \exp \left(\frac{n}{2} + \frac{n^2}{12} \right) < +\infty.$$

We state next our main result :

Theorem 1. *Let ρ be a symmetric probability measure on \mathbb{R} whose support contains at least five points and such that*

$$\exists w_0 > 0 \quad \int_{\mathbb{R}} e^{w_0 z^4} d\rho(z) < +\infty.$$

We denote by σ^2 the variance of ρ , by μ_4 its fourth moment, by μ_6 its sixth moment and by μ_8 its eighth moment. We assume that

$$5\mu_4^2 > 2\sigma^2\mu_6.$$

Then, under $\tilde{\mu}_{H,n,\rho}$, $(S_n/n, T_n/n, U_n/n)$ converges in probability to $(0, \sigma^2, \mu_4)$.

Moreover, if ρ admits a bounded density with respect to the Lebesgue measure on \mathbb{R} , then, under $\mu_{H,n,\rho}$,

$$\left(\frac{\mu_4^2}{\sigma^2} - \frac{2\mu_6}{5} \right)^{1/6} \frac{S_n}{\sigma^2 n^{5/6}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\frac{81}{2} \right)^{1/6} \Gamma \left(\frac{1}{6} \right)^{-1} \exp \left(-\frac{s^6}{18} \right) ds.$$

In section 5.b), we will actually prove this theorem for more general interacting functions H and more general probability measures ρ .

After giving some preliminaries, simulations and notations in section 2, we study the smoothness of I_n and we compute its expansion around $(0, \sigma^2, \mu_4)$ in section 3. Next, in section 4, we explain the first problems we encounter and we investigate how to build an interacting term which is amenable to a mathematical analysis. Finally, in section 5, we give the proof of (an extended version of) theorem 1. We end this paper by a discussion about a model with fluctuations of order $n^{1-1/2k}$ for $k \geq 4$.

2 Preliminaries

We denote by F the function defined by

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \quad F(x, y) = \frac{x^2}{2y}.$$

We recall the following proposition, which is proved in section 5 of [1] :

Proposition 2. *Let ρ be a symmetric probability measure on \mathbb{R} with variance $\sigma^2 > 0$ such that the function*

$$\Lambda : (u, v) \in \mathbb{R}^2 \longmapsto \ln \int_{\mathbb{R}} e^{uz+vw} d\rho(z)$$

is finite in the neighbourhood of $(0, 0)$. We define I by

$$\forall (x, y) \in \mathbb{R}^2 \quad I(x, y) = \sup_{(u, v) \in \mathbb{R}^2} (ux + vy - \Lambda(u, v)).$$

Then the function $I - F$ has a unique minimum on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$ at $(0, \sigma^2)$, with $(I - F)(0, \sigma^2) = 0$. Moreover, if the support of ρ contains at least three points and if μ_4 denotes the fourth moment of ρ , then, when (x, y) goes to $(0, \sigma^2)$,

$$I(x, y) - F(x, y) \sim \frac{\mu_4 x^4}{12\sigma^8} + \frac{(y - \sigma^2)^4}{2(\mu_4 - \sigma^4)}.$$

This is the starting point for the construction of an interaction term. Indeed, as we explained in the introduction, in order to « kill the term x^4 », it is enough to add some function R to F so that the term $\mu_4 x^4 / (12\sigma^8)$ vanishes from the above expansion and so that

$$I(x, y) - (F + R)(x, y) \sim Ax^6 + \frac{(y - \sigma^2)^4}{2(\mu_4 - \sigma^4)},$$

for some $A > 0$. However we want to build a self-interaction, thus we have to estimate μ_4 by $(x_1^4 + \dots + x_n^4)/n$ (as we estimated σ^2 by $(x_1^2 + \dots + x_n^2)/n$ in order to build our model in [1]). Hence it seems natural to consider $H = F + R$, with

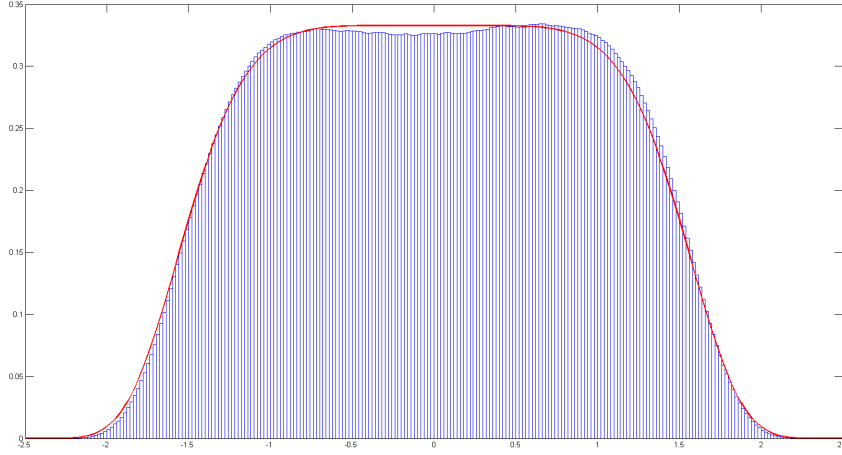
$$R : (x, y, z) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \longmapsto \frac{zx^4}{12y^4},$$

and this leads us to study the rate function I_\bullet of the large deviations for $\tilde{\nu}_{\bullet, n, \rho}$, the law of $(S_n/n, T_n/n, U_n/n)$ under $\rho^{\otimes n}$.

For $n \geq 1$ and $H = F + R$, let us consider $S_n = X_1^n + \dots + X_n^n$, where the law of (X_1^n, \dots, X_n^n) has the density

$$(x_1, \dots, x_n) \longmapsto \frac{1}{Z_{H, n}} \exp \left(H \left(x_1 + \dots + x_n, x_1^2 + \dots + x_n^2, x_1^4 + \dots + x_n^4 \right) \right)$$

with respect to $\rho^{\otimes n}$. We made computer simulations of this model which support us in the choice of $H = F + R$. We used Metropolis-within-Gibbs algorithms (cf. section 4 of [5]) and obtain :



IN BLUE, THE RENORMALIZED HISTOGRAM OF $6, 17 \times 10^{11}$ SIMULATIONS OF $S_n/n^{5/6}$, FOR $n = 10000$ AND ρ HAVING A DENSITY PROPORTIONAL TO $x \mapsto \exp(-x^4)$. IN RED, THE GRAPH OF THE DENSITY FUNCTION

$$x \mapsto \left(\frac{81}{2}\right)^{1/6} \Gamma\left(\frac{1}{6}\right)^{-1} \exp\left(-\frac{x^6}{18}\right).$$

We end this section by giving some notations. For a symmetric probability measure ρ on \mathbb{R} which is not the Dirac mass at 0, we denote by $\nu_{\bullet, \rho}$ the law of (Z, Z^2, Z^4) when Z is a random variable with distribution ρ . We define the Log-Laplace Λ_{\bullet} of $\nu_{\bullet, \rho}$ by

$$\forall (u, v, w) \in \mathbb{R}^3 \quad \Lambda_{\bullet}(u, v, w) = \ln \int_{\mathbb{R}} e^{uz+vwz^2+wz^4} d\rho(z).$$

If Λ_{\bullet} is finite in a neighbourhood of $(0, 0, 0)$ then the Cramér theorem (cf. [2]) states that $(\tilde{\nu}_{n, \rho})_{n \geq 1}$ satisfies the large deviations principle with speed n , governed by the Cramér transform I of $\nu_{\bullet, \rho}$ defined by

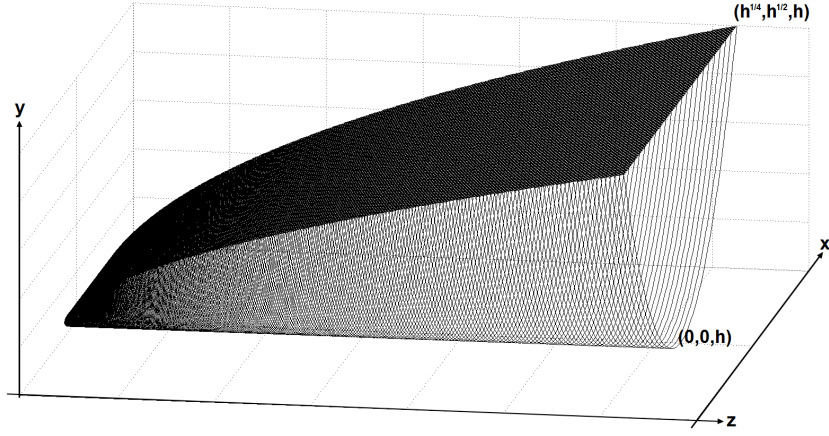
$$\forall (x, y, z) \in \mathbb{R}^3 \quad I_{\bullet}(x, y, z) = \sup_{(u, v, w) \in \mathbb{R}^3} (xu + yv + zw - \Lambda_{\bullet}(u, v, w)).$$

We denote by $D_{\Lambda_{\bullet}}$ and $D_{I_{\bullet}}$ the domains of \mathbb{R}^3 where the functions Λ_{\bullet} et I_{\bullet} are finite. We introduce next the subsets of \mathbb{R}^3

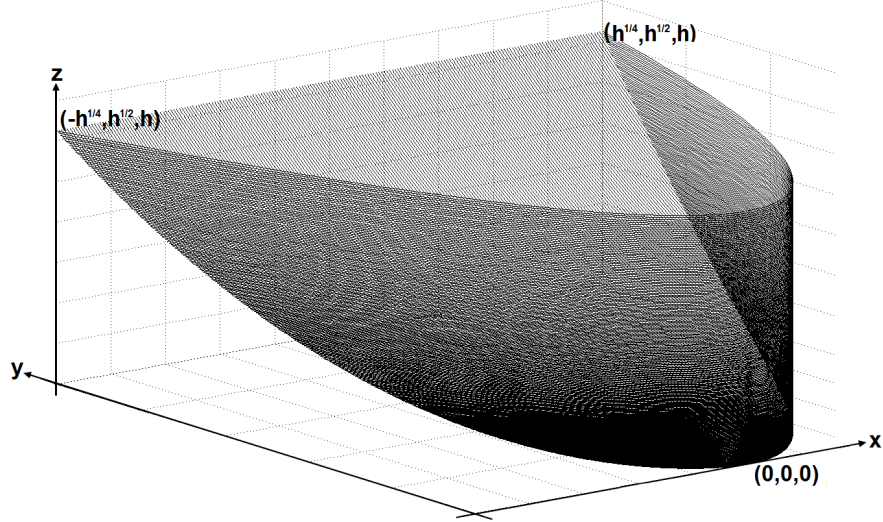
$$\Theta = \{(x, y, z) \in \mathbb{R}^3 : x^2 \leq y, y^2 \leq z\} \quad \text{and} \quad \Theta^* = \Theta \cap (\mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R}).$$

By convexity, we have that $\tilde{\nu}_{n, \rho}(\Theta) = 1$. We get that, under $\tilde{\mu}_{n, \rho}$, the distribution of $(S_n/n, T_n/n, U_n/n)$ is

$$\frac{1}{Z_{\bullet, n}} \exp(H(nx, ny, nz)) \mathbb{1}_{\Theta^*}(x, y, z) d\tilde{\nu}_{n, \rho}(x, y, z).$$



TWO VIEWS OF THE SET OF THE POINTS $(x, y, z) \in \partial\Theta$ SUCH THAT $z \leq h$.



We will proceed as we did in the article [1], i.e., we will study, for any $n \geq 1$, the function

$$G_n : (x, y, z) \mapsto I_\bullet(x, y, z) - \frac{1}{n}H(nx, ny, nz).$$

The Cramér transform I_\bullet has a unique minimum at $(0, \sigma^2, \mu_4)$ and the method we used in the section 5.b) of [1] allows us to compute the expansion of I_\bullet around its minimum.

In order to apply the Laplace's method, as in the section 7 of [1], we want to build H so that G_n also has a unique minimum at $(0, \sigma^2, \mu_4)$ for any $n \geq 1$, and so that its expansion around this minimum has the desired form :

$$Ax^6 + q(y - \sigma^2, z - \mu_4),$$

with $A > 0$ and q a positive definite quadratic form on \mathbb{R}^2 .

3 Expansion of I_\bullet around $(0, \sigma^2, \mu_4)$

Let ρ be a symmetric probability measure on \mathbb{R} with variance $\sigma^2 > 0$ and such that $(0, 0, 0) \in \overset{\circ}{D}_{\Lambda_\bullet}$. In this section, we first study the smoothness of I_\bullet , then we compute its expansion around its minimum $(0, \sigma^2, \mu_4)$. In the last subsection we give the expansion of $I_\bullet - F - R$ around $(0, \sigma^2, \mu_4)$.

a) Smoothness of I_\bullet

The function Λ_\bullet is finite in a neighbourhood of $(0, 0, 0)$ thus each moment of ρ is finite and the covariance matrix of $\nu_{\bullet, \rho}$ is

$$\begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \mu_4 - \sigma^4 & \mu_6 - \sigma^2 \mu_4 \\ 0 & \mu_6 - \sigma^2 \mu_4 & \mu_8 - \mu_4^2 \end{pmatrix}.$$

Lemma 3. *We assume that ρ is a symmetric probability measure on \mathbb{R} whose support contains at least five points. Then the support of $\nu_{\bullet, \rho}$ is not included in a hyperplane of \mathbb{R}^3 and thus*

$$(\mu_4 - \sigma^4)(\mu_8 - \mu_4^2) \neq (\mu_6 - \sigma^2 \mu_4)^2.$$

Proof. Since ρ is symmetric, its support contains the points $a, -a, b$ and $-b$ for some $a \neq b$. Therefore the support of $\nu_{\bullet, \rho}$ contains the points

$$(a, a^2, a^4), \quad (-a, a^2, a^4), \quad (b, b^2, b^4) \quad \text{and} \quad (-b, b^2, b^4).$$

We observe that these four points belong to the same plane \mathcal{P} whose equation is

$$-(a^2 + b^2)y + z + a^2b^2 = 0.$$

If c is a fifth point in the support of ρ then

$$-(a^2 + b^2)c^2 + c^4 + a^2b^2 = (c^2 - a^2)(c^2 - b^2) \neq 0.$$

Thus the point (c, c^2, c^4) , which is in the support of $\nu_{\bullet, \rho}$, is not included in \mathcal{P} . Hence the support of $\nu_{\bullet, \rho}$ is not included in a hyperplane of \mathbb{R}^3 . As a consequence the covariance matrix of $\nu_{\bullet, \rho}$ is invertible (see section III.5 of [14] for a proof), i.e., $(\mu_4 - \sigma^4)(\mu_8 - \mu_4^2) \neq (\mu_6 - \sigma^2 \mu_4)^2$. \square

We assume next that the support of ρ contains at least five points. The previous lemma and the proposition A.4¹ of [1] imply that $\nabla \Lambda_\bullet$ is a \mathcal{C}^∞ -diffeomorphism from $\overset{\circ}{D}_{\Lambda_\bullet}$ to A_{I_\bullet} , the admissible domain of I_\bullet . Moreover $A_{I_\bullet} \subset \Theta^*$ and

$$(0, \sigma^2, \mu_4) = \nabla \Lambda_\bullet(0, 0, 0) \in \nabla \Lambda_\bullet(\overset{\circ}{D}_{\Lambda_\bullet}) = A_{I_\bullet}.$$

The function I_\bullet is \mathcal{C}^∞ on A_{I_\bullet} and, if $(x, y, z) \mapsto (u(x, y, z), v(x, y, z), w(x, y, z))$ denotes the inverse function of $\nabla \Lambda_\bullet$, then, for any $(x, y, z) \in A_{I_\bullet}$,

$$\begin{aligned} I_\bullet(x, y, z) &= xu(x, y, z) + yv(x, y, z) + zw(x, y, z) \\ &\quad - \Lambda_\bullet(u(x, y, z), v(x, y, z), w(x, y, z)), \end{aligned}$$

1. Actually it is proposition 10 of the ARXIV version of [1].

$$\begin{aligned}\nabla I_{\bullet}(x, y, z) &= (\nabla \Lambda_{\bullet})^{-1}(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z)), \\ D_{(x, y, z)}^2 I_{\bullet} &= \left(D_{(u(x, y, z), v(x, y, z), w(x, y, z))}^2 \Lambda \right)^{-1}.\end{aligned}$$

In order to compute the derivatives of the previous terms, as in section 5.b) of [1], we introduce the functions f_j defined by

$$\forall j \in \mathbb{N} \quad \forall (u, v, w) \in \overset{\circ}{D}_{\Lambda_{\bullet}} \quad f_j(u, v, w) = \frac{\int_{\mathbb{R}} x^j e^{ux+vx^2+wx^4} d\rho(x)}{\int_{\mathbb{R}} e^{ux+vx^2+wx^4} d\rho(x)}.$$

The functions f_j , $j \in \mathbb{N}$, are C^∞ on $\overset{\circ}{D}_{\Lambda_{\bullet}}$ and they verify the following properties:
 $\star f_0$ is the identity function on \mathbb{R}^3 and

$$f_1 = \frac{\partial \Lambda}{\partial u}, \quad f_2 = \frac{\partial \Lambda}{\partial v} \quad \text{and} \quad f_4 = \frac{\partial \Lambda}{\partial w}.$$

\star For all $j \in \mathbb{N}$, $f_j(0, 0, 0) = \mu_j$ is the j -th moment of ρ . It is null if j is odd, since ρ is symmetric. Moreover, for any $j \in \mathbb{N}$,

$$\frac{\partial f_j}{\partial u} = f_{j+1} - f_j f_1, \quad \frac{\partial f_j}{\partial v} = f_{j+2} - f_j f_2 \quad \text{and} \quad \frac{\partial f_j}{\partial w} = f_{j+4} - f_j f_4.$$

For any $(u, v, w) \in \overset{\circ}{D}_{\Lambda_{\bullet}}$, we have

$$D_{(u, v, w)}^2 \Lambda_{\bullet} = \begin{pmatrix} f_2 - f_1^2 & f_3 - f_1 f_2 & f_5 - f_4 f_1 \\ f_3 - f_1 f_2 & f_4 - f_2^2 & f_6 - f_4 f_2 \\ f_5 - f_4 f_1 & f_6 - f_4 f_2 & f_8 - f_4^2 \end{pmatrix} (u, v, w).$$

We define

$$\begin{aligned}g &= (f_2 - f_1^2)(f_4 - f_2^2)(f_8 - f_4^2) + 2(f_3 - f_1 f_2)(f_6 - f_4 f_2)(f_5 - f_4 f_1) \\ &\quad - (f_4 - f_2^2)(f_5 - f_4 f_1)^2 - (f_2 - f_1^2)(f_6 - f_4 f_2)^2 - (f_8 - f_4^2)(f_3 - f_1 f_2)^2.\end{aligned}$$

This is a function which is positive on $\overset{\circ}{D}_{\Lambda_{\bullet}}$. Therefore

$$\forall (x, y, z) \in A_{\bullet} \quad D_{(x, y, z)}^2 I_{\bullet} = K(u(x, y, z), v(x, y, z), w(x, y, z)),$$

where K is a function from \mathbb{R}^3 to $\mathcal{S}_3(\mathbb{R})$, the set of the symmetric matrices of size 3, such that

$$\begin{aligned}K_{1,1} &= \frac{(f_4 - f_2^2)(f_8 - f_4^2) - (f_6 - f_4 f_2)^2}{g}, \\ K_{2,2} &= \frac{(f_2 - f_1^2)(f_8 - f_4^2) - (f_5 - f_4 f_1)^2}{g}, \\ K_{3,3} &= \frac{(f_2 - f_1^2)(f_4 - f_2^2) - (f_3 - f_1 f_2)^2}{g}, \\ K_{1,2} = K_{2,1} &= \frac{(f_5 - f_4 f_1)(f_6 - f_4 f_2) - (f_3 - f_1 f_2)(f_8 - f_4^2)}{g}, \\ K_{1,3} = K_{3,1} &= \frac{(f_3 - f_1 f_2)(f_6 - f_4 f_2) - (f_5 - f_4 f_1)(f_4 - f_2^2)}{g}, \\ K_{2,3} = K_{3,2} &= \frac{(f_3 - f_1 f_2)(f_5 - f_4 f_1) - (f_2 - f_1^2)(f_6 - f_4 f_2)}{g}.\end{aligned}$$

b) Computation of the terms of the expansion of I_\bullet

Notice that $g(0, 0, 0) = a\sigma^2$ with

$$a = (\mu_4 - \sigma^4)(\mu_8 - \mu_4^2) - (\mu_6 - \sigma^2\mu_4)^2 > 0.$$

We have

$$D_{(0, \sigma^2, \mu_4)}^2 I_\bullet = \begin{pmatrix} 1/\sigma^2 & 0 & 0 \\ 0 & (\mu_8 - \mu_4^2)/a & (\mu_4\sigma^2 - \mu_6)/a \\ 0 & (\mu_4\sigma^2 - \mu_6)/a & (\mu_4 - \sigma^4)/a \end{pmatrix}.$$

Let q be the positive definite quadratic form on \mathbb{R}^2 given by

$$\forall (y, z) \in \mathbb{R}^2 \quad q(y, z) = \frac{\mu_8 - \mu_4^2}{2a} y^2 + \frac{\mu_4\sigma^2 - \mu_6}{a} yz + \frac{\mu_4 - \sigma^4}{2a} z^2.$$

Taylor formula implies that, at the order 6, the expansion of I_\bullet in the neighbourhood of $(0, \sigma^2, \mu_4)$ is

$$\begin{aligned} I_\bullet(x, y, z) &= \frac{x^2}{2\sigma^2} + q(y - \sigma^2, z - \mu_4) \\ &+ \sum_{(\alpha, \beta, \gamma) \in \mathcal{T}} \frac{1}{\alpha! \beta! \gamma!} \frac{\partial^{\alpha+\beta+\gamma} I_\bullet}{\partial x^\alpha \partial y^\beta \partial z^\gamma}(0, \sigma^2, \mu_4) x^\alpha (y - \sigma^2)^\beta (z - \mu_4)^\gamma \\ &+ o(\|x, y - \sigma^2, z - \mu_4\|^6), \end{aligned}$$

with

$$\mathcal{T} = \{(\alpha, \beta, \gamma) \in \mathbb{N}^3 : \alpha + \beta + \gamma \in \{3, 4, 5, 6\}\}.$$

Thus we have to compute the terms

$$\frac{\partial^{\alpha+\beta+\gamma} I_\bullet}{\partial x^\alpha \partial y^\beta \partial z^\gamma}(0, \sigma^2, \mu_4)$$

for $(\alpha, \beta, \gamma) \in \mathcal{T}$. In order to optimize the computations, we will first determine the terms of the expansion of I_\bullet which are negligible compared to the term $Ax^6 + q(y - \sigma^2, z - \mu_4)$ with $A > 0$.

i) The non-negligible terms

Lemma 4. *Let $A > 0$ and q be a positive definite quadratic form on \mathbb{R}^2 . Then, in a neighborhood of $(0, 0, 0)$,*

$$\|x, y, z\|^6 = O(Ax^6 + q(y, z)).$$

Moreover, for any $(\alpha, \beta, \gamma) \in \mathbb{N}^3$, we have

$$\frac{\alpha}{3} + \beta + \gamma > 2 \implies \lim_{(x, y, z) \rightarrow (0, 0, 0)} \frac{x^\alpha y^\beta z^\gamma}{Ax^6 + q(y, z)} = 0.$$

Proof. For any $(x, y, z) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, there exists a unique (r, θ, φ) which belongs to $]0, +\infty[\times]0, 2\pi[\times]0, \pi[$ and satisfies

$$\begin{cases} x^3 = r \sin \varphi, \\ y = r \cos \theta \cos \varphi, \\ z = r \sin \theta \cos \varphi. \end{cases}$$

Thus

$$Ax^6 + q(y, z) = Ar^2 \sin^2 \varphi + r^2 \cos^2 \varphi q(\cos \theta, \sin \theta).$$

However the set $\{(\cos \theta, \sin \theta) : \theta \in [0, 2\pi[\}$ is compact in \mathbb{R}^2 and the continuous function q is positive on this set. As a consequence q has a minimum $m > 0$ and a maximum $M > m$. Hence

$$\min(A, m) r^2 \leq Ax^6 + q(y, z) \leq \max(A, m) r^2.$$

We get that

$$\frac{\|x, y, z\|^6}{Ax^6 + q(y, z)} = \frac{(x^2 + y^2 + z^2)^3}{Ax^6 + q(y, z)} \leq \frac{(r^{2/3} \sin^{2/3} \varphi + r^2 \cos^2 \varphi)^3}{\min(A, m) r^2} = \frac{(1 + r^{4/3})^3}{\min(A, m)}.$$

This is a bounded quantity when r tends to 0. Next

$$\left| \frac{x^\alpha y^\beta z^\gamma}{Ax^6 + q(y, z)} \right| \leq \frac{|r \sin \varphi|^{\alpha/3} |r \cos \theta \cos \varphi|^\beta |r \sin \theta \cos \varphi|^\gamma}{\min(A, m) r^2} = O(r^{\alpha/3 + \beta + \gamma - 2}).$$

Since the convergence of (x, y, z) to $(0, 0, 0)$ is equivalent to the convergence of r to 0, the lemma is proved. \square

This lemma states that the terms $x^\alpha y^\beta z^\gamma$, $(\alpha, \beta, \gamma) \in \mathcal{T}$, which are not negligible at $(0, \sigma^2, \mu_4)$ compared to $Ax^6 + q(y - \sigma^2, z - \mu_4)$, are such that

$$\frac{\alpha}{3} + \beta + \gamma \leq 2.$$

Thus, these terms are those for which (α, β, γ) is $(2, 1, 0)$, $(2, 0, 1)$, $(3, 0, 0)$, $(3, 1, 0)$, $(3, 0, 1)$, $(4, 0, 0)$, $(5, 0, 0)$ or $(6, 0, 0)$. Let us compute the coefficients of these terms in the expansion of I_\bullet . We denote

$$\begin{aligned} k_1 &= (f_4 - f_2^2)(f_8 - f_4^2) - (f_6 - f_4 f_2)^2, \\ k_2 &= (f_5 - f_4 f_1)(f_6 - f_4 f_2) - (f_3 - f_1 f_2)(f_8 - f_4^2), \\ k_3 &= (f_3 - f_1 f_2)(f_6 - f_4 f_2) - (f_5 - f_4 f_1)(f_4 - f_2^2). \end{aligned}$$

ii) The terms at the third order

Let us start with the terms at third order which might be non-negligible compared to $Ax^6 + q(y - \sigma^2, z - \mu_4)$:

$$\begin{aligned} \frac{\partial^3 I_\bullet}{\partial x^2 \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial^2 I_\bullet}{\partial x \partial y} \right) = \frac{\partial K_{1,2}(u, v, w)}{\partial x} \\ &= \frac{\partial u}{\partial x} \times \frac{\partial K_{1,2}}{\partial u}(u, v, w) + \frac{\partial v}{\partial x} \times \frac{\partial K_{1,2}}{\partial v}(u, v, w) + \frac{\partial w}{\partial x} \times \frac{\partial K_{1,2}}{\partial w}(u, v, w). \end{aligned}$$

We have

$$\begin{aligned}\frac{\partial v}{\partial x}(0, \sigma^2, \mu_4) &= \frac{\partial^2 I_\bullet}{\partial x \partial y}(0, \sigma^2, \mu_4) = 0 = \frac{\partial^2 I_\bullet}{\partial x \partial z}(0, \sigma^2, \mu_4) = \frac{\partial w}{\partial x}(0, \sigma^2, \mu_4), \\ \frac{\partial u}{\partial x}(0, \sigma^2, \mu_4) &= \frac{\partial^2 I_\bullet}{\partial x^2}(0, \sigma^2, \mu_4) = \frac{1}{\sigma^2},\end{aligned}$$

thus

$$\frac{\partial^3 I_\bullet}{\partial x^2 \partial y}(0, \sigma^2, \mu_4) = \frac{1}{\sigma^2} \left(\frac{1}{g(0, 0, 0)} \frac{\partial k_2}{\partial u}(0, 0, 0) - \frac{k_2(0, 0, 0)}{g^2(0, 0, 0)} \frac{\partial g}{\partial u}(0, 0, 0) \right).$$

We have $k_2(0, 0, 0) = k_3(0, 0, 0) = 0$ and $g(0, 0, 0) = \sigma^2 k_1(0, 0, 0)$ with

$$k_1(0, 0, 0) = \mu_8 \mu_4 - \mu_4^3 - \mu_8 \sigma^4 - \mu_6^2 + 2\mu_6 \mu_4 \sigma^2.$$

Using the properties of the functions f_i , $i \in \mathbb{N}$, for computing their partial derivatives, we get

$$\frac{\partial k_2}{\partial u}(0, 0, 0) = -k_1(0, 0, 0).$$

Hence

$$\frac{\partial^3 I_\bullet}{\partial x^2 \partial y}(0, \sigma^2, \mu_4) = \frac{-k_1(0, 0, 0)}{\sigma^4 k_1(0, 0, 0)} = -\frac{1}{\sigma^4}.$$

We compute next that

$$\frac{\partial k_1}{\partial u}(0, 0, 0) = \frac{\partial k_3}{\partial u}(0, 0, 0) = 0.$$

This implies that

$$\frac{\partial^3 I_\bullet}{\partial x^3}(0, \sigma^2, \mu_4) = \frac{\partial^3 I_\bullet}{\partial x^2 \partial z}(0, \sigma^2, \mu_4) = 0.$$

But we already knew that the third partial derivative of I_\bullet with respect to x is null at $(0, \sigma^2, \mu_4)$ since I_\bullet is even in its first variable.

We have shown that

$$\frac{1}{\alpha! \beta! \gamma!} \frac{\partial^{\alpha+\beta+\gamma} I_\bullet}{\partial x^\alpha \partial y^\beta \partial z^\gamma}(0, \sigma^2, \mu_4) = \begin{cases} -\frac{1}{2\sigma^4} & \text{if } (\alpha, \beta, \gamma) = (2, 1, 0), \\ 0 & \text{if } (\alpha, \beta, \gamma) = (2, 0, 1), \\ 0 & \text{if } (\alpha, \beta, \gamma) = (3, 0, 0). \end{cases}$$

iii) The terms at the fourth order

Let us focus now on the non-negligible terms at the fourth order :

$$\begin{aligned}\frac{\partial^4 I_\bullet}{\partial x^4} &= \frac{\partial^2 K_{1,1}(u, v, w)}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \frac{\partial K_{1,1}}{\partial u}(u, v, w) + \frac{\partial^2 v}{\partial x^2} \frac{\partial K_{1,1}}{\partial v}(u, v, w) \\ &+ \frac{\partial^2 w}{\partial x^2} \frac{\partial K_{1,1}}{\partial w}(u, v, w) + \left(\frac{\partial u}{\partial x} \right)^2 \frac{\partial^2 K_{1,1}}{\partial u^2}(u, v, w) + \left(\frac{\partial v}{\partial x} \right)^2 \frac{\partial^2 K_{1,1}}{\partial v^2}(u, v, w) \\ &+ \left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^2 K_{1,1}}{\partial w^2}(u, v, w) + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \frac{\partial^2 K_{1,1}}{\partial u \partial v}(u, v, w) \\ &+ 2 \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} \frac{\partial^2 K_{1,1}}{\partial u \partial w}(u, v, w) + 2 \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} \frac{\partial^2 K_{1,1}}{\partial v \partial w}(u, v, w).\end{aligned}$$

We have

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(0, \sigma^2, \mu_4) &= \frac{\partial^3 \mathbf{I}_\bullet}{\partial x^3}(0, \sigma^2, \mu_4) = 0 = \frac{\partial^3 \mathbf{I}_\bullet}{\partial x^2 \partial z}(0, \sigma^2, \mu_4) = \frac{\partial^2 w}{\partial x^2}(0, \sigma^2, \mu_4), \\ \frac{\partial^2 v}{\partial x^2}(0, \sigma^2, \mu_4) &= \frac{\partial^3 \mathbf{I}_\bullet}{\partial x^2 \partial y}(0, \sigma^2, \mu_4) = -\frac{1}{\sigma^4}.\end{aligned}$$

As a consequence

$$\frac{\partial^4 \mathbf{I}_\bullet}{\partial x^4}(0, \sigma^2, \mu_4) = \frac{1}{\sigma^4} \frac{\partial^2 K_{1,1}}{\partial u^2}(0, 0, 0) - \frac{1}{\sigma^4} \frac{\partial K_{1,1}}{\partial v}(0, 0, 0).$$

We have

$$\frac{\partial^2 K_{1,1}}{\partial u^2} = \frac{1}{g} \frac{\partial^2 k_1}{\partial u^2} - \frac{2}{g^2} \frac{\partial g}{\partial u} \frac{\partial k_1}{\partial u} - \frac{k_1}{g^2} \frac{\partial^2 g}{\partial u^2} + \frac{2k_1}{g^3} \left(\frac{\partial g}{\partial u} \right)^2,$$

and the properties of the functions f_i , $i \in \mathbb{N}$, for computing their partial derivatives give us

$$\frac{\partial g}{\partial u}(0, 0, 0) = 0,$$

so that

$$\frac{\partial^2 K_{1,1}}{\partial u^2}(0, 0, 0) = \frac{1}{\sigma^4 k_1(0, 0, 0)} \left(\sigma^2 \frac{\partial^2 k_1}{\partial u^2}(0, 0, 0) - \frac{\partial^2 g}{\partial u^2}(0, 0, 0) \right).$$

Moreover

$$\begin{aligned}\frac{\partial K_{1,1}}{\partial v}(0, 0, 0) &= \frac{1}{g(0, 0, 0)} \frac{\partial k_1}{\partial v}(0, 0, 0) - \frac{k_1(0, 0, 0)}{g^2(0, 0, 0)} \frac{\partial g}{\partial v}(0, 0, 0) \\ &= \frac{1}{\sigma^4 k_1(0, 0, 0)} \left(\sigma^2 \frac{\partial k_1}{\partial v}(0, 0, 0) - \frac{\partial g}{\partial v}(0, 0, 0) \right).\end{aligned}$$

We compute that

$$\begin{aligned}\frac{\partial k_1}{\partial v}(0, 0, 0) &= \frac{\partial^2 k_1}{\partial u^2}(0, 0, 0) = -\mu_8 \mu_6 + \mu_{10} \mu_4 - 2\mu_8 \mu_4 \sigma^2 - \mu_{10} \sigma^4 + 3\mu_8 \sigma^6 \\ &\quad - \mu_6 \mu_4^2 + 3\mu_4^3 \sigma^2 + 4\mu_6^2 \sigma^2 - 6\mu_6 \mu_4 \sigma^4.\end{aligned}$$

After factorising by $k_1(0, 0, 0)$, we get that this quantity is equal to

$$-3\sigma^2 k_1(0, 0, 0) + \mu_{10}(\mu_4 - \sigma^4) + \mu_8(\mu_4 \sigma^2 - \mu_6) + \mu_6(\mu_6 \sigma^2 - \mu_4^2).$$

We compute similarly that

$$\frac{\partial^2 g}{\partial u^2}(0, 0, 0) = -(\mu_4 + 4\sigma^4)k_1(0, 0, 0) + \sigma^2 \eta, \quad (1)$$

$$\frac{\partial g}{\partial v}(0, 0, 0) = (\mu_4 - 4\sigma^4)k_1(0, 0, 0) + \sigma^2 \eta. \quad (2)$$

where $\eta = \mu_{10}(\mu_4 - \sigma^4) + \mu_8(\mu_4 \sigma^2 - \mu_6) + \mu_6(\mu_6 \sigma^2 - \mu_4^2)$. Finally

$$\frac{\partial^4 \mathbf{I}_\bullet}{\partial x^4}(0, \sigma^2, \mu_4) = \frac{(-3\sigma^4 + \mu_4 + 4\sigma^4 + 3\sigma^4 + \mu_4 - 4\sigma^4)k_1(0, 0, 0)}{\sigma^8 k_1(0, 0, 0)} = \frac{2\mu_4}{\sigma^8}.$$

We have likewise

$$\begin{aligned}\frac{\partial^4 I_\bullet}{\partial x^3 \partial y}(0, \sigma^2, \mu_4) &= \frac{\partial^2 K_{1,2}(u, v, w)}{\partial x^2}(0, \sigma^2, \mu_4) \\ &= \frac{1}{\sigma^4} \frac{\partial^2 K_{1,2}}{\partial u^2}(0, 0, 0) - \frac{1}{\sigma^4} \frac{\partial K_{1,2}}{\partial v}(0, 0, 0) \\ &= \frac{1}{\sigma^8 k_1(0, 0, 0)} \left(\sigma^2 \frac{\partial^2 k_2}{\partial u^2} - \frac{k_2}{k_1} \frac{\partial^2 g}{\partial u^2} - \sigma^2 \frac{\partial k_2}{\partial v} + \frac{k_2}{k_1} \frac{\partial g}{\partial v} \right) (0, 0, 0)\end{aligned}$$

and

$$\frac{\partial^4 I_\bullet}{\partial x^3 \partial z}(0, \sigma^2, \mu_4) = \frac{1}{\sigma^8 k_1(0, 0, 0)} \left(\sigma^2 \frac{\partial^2 k_3}{\partial u^2} - \frac{k_3}{k_1} \frac{\partial^2 g}{\partial u^2} - \sigma^2 \frac{\partial k_3}{\partial v} + \frac{k_3}{k_1} \frac{\partial g}{\partial v} \right) (0, 0, 0).$$

But $k_2(0, 0, 0) = k_3(0, 0, 0) = 0$ and we compute that

$$\frac{\partial^2 k_2}{\partial u^2}(0, 0, 0) = \frac{\partial k_2}{\partial v}(0, 0, 0) = \frac{\partial^2 k_3}{\partial u^2}(0, 0, 0) = \frac{\partial k_3}{\partial v}(0, 0, 0) = 0.$$

Hence we have shown that

$$\frac{1}{\alpha! \beta! \gamma!} \frac{\partial^{\alpha+\beta+\gamma} I_\bullet}{\partial x^\alpha \partial y^\beta \partial z^\gamma}(0, \sigma^2, \mu_4) = \begin{cases} \frac{\mu_4 x^4}{12 \sigma^4} & \text{if } (\alpha, \beta, \gamma) = (4, 0, 0), \\ 0 & \text{if } (\alpha, \beta, \gamma) = (3, 1, 0), \\ 0 & \text{if } (\alpha, \beta, \gamma) = (3, 0, 1). \end{cases}$$

iv) The terms at the fifth and sixth orders

We still have to prove that

$$\frac{1}{120} \frac{\partial^5 I_\bullet}{\partial x^5}(0, \sigma^2, \mu_4) = 0 \quad \text{and} \quad \frac{1}{720} \frac{\partial^6 I_\bullet}{\partial x^4}(0, \sigma^2, \mu_4) = A > 0.$$

By symmetry of I_\bullet at its first variable, we obtain immediately that its fifth partial derivative with respect to x is null. Let us determine its sixth partial derivative with respect to x . We notice first that

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3}(0, \sigma^2, \mu_4) &= \frac{\partial^4 I_\bullet}{\partial x^4}(0, \sigma^2, \mu_4) = \frac{2\mu_4}{\sigma^8}, \\ \frac{\partial^3 v}{\partial x^3}(0, \sigma^2, \mu_4) &= \frac{\partial^4 I_\bullet}{\partial x^3 \partial y}(0, \sigma^2, \mu_4) = 0 = \frac{\partial^4 I_\bullet}{\partial x^3 \partial z}(0, \sigma^2, \mu_4) = \frac{\partial^3 w}{\partial x^3}(0, \sigma^2, \mu_4), \\ \frac{\partial^4 u}{\partial x^4}(0, \sigma^2, \mu_4) &= \frac{\partial^5 I_\bullet}{\partial x^5}(0, \sigma^2, \mu_4) = 0.\end{aligned}$$

Thus we know the partial derivatives with respect to x at $(0, \sigma^2, \mu_4)$ of the functions u , v , and w until the third order. We write then the sixth partial derivative I_\bullet with respect to x , taken at $(0, \sigma^2, \mu_4)$ and we only keep the terms

which do not vanish because of the symmetries:

$$\begin{aligned}
\frac{\partial^6 \mathbf{I}}{\partial x^6}(0, \sigma^2, \mu_4) &= \frac{\partial^4 K_{1,1}(u, v, w)}{\partial x^4}(0, \sigma^2, \mu_4) = \frac{\partial^4 v}{\partial x^4}(0, \sigma^2, \mu_4) \frac{\partial K_{1,1}}{\partial v}(0, 0, 0) \\
&+ \frac{\partial^4 w}{\partial x^4}(0, \sigma^2, \mu_4) \frac{\partial K_{1,1}}{\partial w}(0, 0, 0) + 3 \left(\frac{\partial^2 v}{\partial x^2}(0, \sigma^2, \mu_4) \right)^2 \frac{\partial^2 K_{1,1}}{\partial v^2}(0, 0, 0) \\
&+ \left(\frac{\partial u}{\partial x}(0, \sigma^2, \mu_4) \right)^4 \frac{\partial^4 K_{1,1}}{\partial u^4}(0, 0, 0) + 4 \frac{\partial^3 u}{\partial x^3}(0, \sigma^2, \mu_4) \frac{\partial u}{\partial x}(0, \sigma^2, \mu_4) \frac{\partial^2 K_{1,1}}{\partial u^2}(0, 0, 0) \\
&+ 6 \left(\frac{\partial u}{\partial x}(0, \sigma^2, \mu_4) \right)^2 \frac{\partial^2 v}{\partial x^2}(0, \sigma^2, \mu_4) \frac{\partial^3 K_{1,1}}{\partial u^2 \partial v}(0, 0, 0).
\end{aligned}$$

We computed above that

$$\frac{\partial K_{1,1}}{\partial v}(0, 0, 0) = \frac{\sigma^4 - \mu_4}{\sigma^4} \quad \text{and} \quad \frac{\partial^2 K_{1,1}}{\partial u^2}(0, 0, 0) = \frac{\mu_4 + \sigma^4}{\sigma^4}.$$

As a consequence

$$\begin{aligned}
\frac{\partial^6 \mathbf{I}}{\partial x^6}(0, \sigma^2, \mu_4) &= \frac{3}{\sigma^8} \frac{\partial^2 K_{1,1}}{\partial v^2}(0, 0, 0) + \frac{8\mu_4(\mu_4 + \sigma^4)}{\sigma^{14}} - \frac{6}{\sigma^8} \frac{\partial^3 K_{1,1}}{\partial u^2 \partial v}(0, 0, 0) \\
&+ \frac{1}{\sigma^8} \frac{\partial^4 K_{1,1}}{\partial u^4}(0, 0, 0) + \frac{\partial^4 v}{\partial x^4}(0, \sigma^2, \mu_4) \frac{\sigma^4 - \mu_4}{\sigma^4} + \frac{\partial^4 w}{\partial x^4}(0, \sigma^2, \mu_4) \frac{\partial K_{1,1}}{\partial w}(0, 0, 0). \quad (3)
\end{aligned}$$

We have

$$\begin{aligned}
\frac{\partial^4 v}{\partial x^4}(0, \sigma^2, \mu_4) &= \frac{\partial^3 K_{1,2}(u, v, w)}{\partial x^3}(0, \sigma^2, \mu_4) = \frac{\partial^3 u}{\partial x^3}(0, \sigma^2, \mu_4) \frac{\partial K_{1,2}}{\partial u}(0, 0, 0) \\
&+ \left(\frac{\partial u}{\partial x}(0, \sigma^2, \mu_4) \right)^3 \frac{\partial^3 K_{1,2}}{\partial u^3}(0, 0, 0) + 3 \frac{\partial u}{\partial x}(0, \sigma^2, \mu_4) \frac{\partial^2 v}{\partial x^2}(0, \sigma^2, \mu_4) \frac{\partial^2 K_{1,2}}{\partial u \partial v}(0, 0, 0) \\
&= \frac{1}{\sigma^8} \left(2\mu_4 \frac{\partial K_{1,2}}{\partial u}(0, 0, 0) + \sigma^2 \frac{\partial^3 K_{1,2}}{\partial u^3}(0, 0, 0) - 3\sigma^2 \frac{\partial^2 K_{1,2}}{\partial u \partial v}(0, 0, 0) \right)
\end{aligned}$$

and we have already computed that

$$\frac{\partial K_{1,2}}{\partial u}(0, 0, 0) = \frac{1}{g(0, 0, 0)} \frac{\partial k_2}{\partial u}(0, 0, 0) = -\frac{1}{\sigma^2}.$$

By differentiating and evaluating at $(0, 0, 0)$, we get

$$\begin{aligned}
\frac{\partial^2 K_{1,2}}{\partial u \partial v}(0, 0, 0) &= \frac{1}{g(0, 0, 0)} \frac{\partial^2 k_2}{\partial u \partial v}(0, 0, 0) - \frac{1}{g^2(0, 0, 0)} \frac{\partial k_2}{\partial u}(0, 0, 0) \frac{\partial g}{\partial v}(0, 0, 0) \\
&= \frac{1}{\sigma^4 k_1(0, 0, 0)} \left(\sigma^2 \frac{\partial^2 k_2}{\partial u \partial v}(0, 0, 0) + \frac{\partial g}{\partial v}(0, 0, 0) \right).
\end{aligned}$$

The properties of the functions f_i , $i \in \mathbb{N}$, and their partial derivatives give us

$$\frac{\partial^2 k_2}{\partial u \partial v}(0, 0, 0) = 3\sigma^2 k_1(0, 0, 0) - \mu_{10}(\mu_4 - \sigma^4) - \mu_8(\mu_4 \sigma^2 - \mu_6) - \mu_6(\mu_6 \sigma^2 - \mu_4^2)$$

and, by formula (2), we get

$$\frac{\partial^2 K_{1,2}}{\partial u \partial v}(0, 0, 0) = \frac{3\sigma^4 k_1(0, 0, 0) + (\mu_4 - 4\sigma^4) k_1(0, 0, 0)}{\sigma^4 k_1(0, 0, 0)} = \frac{\mu_4 - \sigma^4}{\sigma^4}.$$

Finally, by using the fact that the partial derivative of g with respect to u at $(0, 0, 0)$ vanishes, we obtain

$$\begin{aligned}\frac{\partial^3 K_{1,2}}{\partial u^3}(0, 0, 0) &= \frac{1}{g(0, 0, 0)} \frac{\partial^3 k_2}{\partial u^3}(0, 0, 0) - \frac{3}{g^2(0, 0, 0)} \frac{\partial k_2}{\partial u}(0, 0, 0) \frac{\partial^2 g}{\partial u^2}(0, 0, 0) \\ &= \frac{1}{\sigma^4 k_1(0, 0, 0)} \left(\sigma^2 \frac{\partial^3 k_2}{\partial u^3}(0, 0, 0) + 3 \frac{\partial^2 g}{\partial u^2}(0, 0, 0) \right).\end{aligned}$$

We compute that

$$\frac{\partial^3 k_2}{\partial u^3}(0, 0, 0) = 9\sigma^2 k_1(0, 0, 0) - 3\mu_{10}(\mu_4 - \sigma^4) - 3\mu_8(\mu_4 \sigma^2 - \mu_6) - 3\mu_6(\mu_6 \sigma^2 - \mu_4^2)$$

and, by formula (1), we obtain

$$\frac{\partial^2 K_{1,2}}{\partial u \partial v}(0, 0, 0) = \frac{9\sigma^4 k_1(0, 0, 0) - 3(\mu_4 + 4\sigma^4)k_1(0, 0, 0)}{\sigma^4 k_1(0, 0, 0)} = \frac{-3(\mu_4 + \sigma^4)}{\sigma^4}.$$

Finally

$$\frac{\partial^4 v}{\partial x^4}(0, \sigma^2, \mu_4) = \frac{1}{\sigma^8} \left(\frac{-2\mu_4}{\sigma^2} + \frac{-3(\mu_4 + \sigma^4)}{\sigma^2} - 3 \frac{\mu_4 - \sigma^4}{\sigma^2} \right) = \frac{-8\mu_4}{\sigma^{10}}.$$

Likewise we obtain

$$\begin{aligned}\frac{\partial^4 w}{\partial x^4}(0, \sigma^2, \mu_4) &= \frac{1}{\sigma^{10} k_1(0, 0, 0)} \left(2\mu_4 \frac{\partial k_3}{\partial u}(0, 0, 0) - 3\sigma^2 \frac{\partial^2 k_3}{\partial u \partial v}(0, 0, 0) \right. \\ &\quad + \frac{3}{k_1(0, 0, 0)} \frac{\partial k_3}{\partial u}(0, 0, 0) \frac{\partial g}{\partial v}(0, 0, 0) + \sigma^2 \frac{\partial^3 k_3}{\partial u^3}(0, 0, 0) \\ &\quad \left. - \frac{3}{k_1(0, 0, 0)} \frac{\partial k_3}{\partial u}(0, 0, 0) \frac{\partial^2 g}{\partial u^2}(0, 0, 0) \right).\end{aligned}$$

We compute that

$$\frac{\partial k_3}{\partial u}(0, 0, 0) = \frac{\partial^2 k_3}{\partial u \partial v}(0, 0, 0) = 0 \quad \text{and} \quad \frac{\partial^2 k_3}{\partial u \partial v}(0, 0, 0) = 2k_1(0, 0, 0),$$

so that

$$\frac{\partial^4 w}{\partial x^4}(0, \sigma^2, \mu_4) = \frac{2}{\sigma^8}.$$

Next we have

$$\frac{\partial K_{1,1}}{\partial w}(0, 0, 0) = \frac{1}{\sigma^4 k_1(0, 0, 0)} \left(\sigma^2 \frac{\partial k_1}{\partial w}(0, 0, 0) - \frac{\partial g}{\partial w}(0, 0, 0) \right)$$

and we compute that the partial derivative of k_1 with respect to w taken at $(0, 0, 0)$ is equal to

$$-3\mu_4 k_1(0, 0, 0) + 2\mu_{10}(\mu_4 \sigma^2 - \mu_6) + \mu_{12}(\mu_4 - \sigma^4) + \mu_8^2 + \mu_6^2 \mu_4 - 2\mu_8 \mu_4^2,$$

and that the partial derivative of g with respect to w at $(0, 0, 0)$ is equal to

$$\begin{aligned}(\mu_6 - 4\mu_4 \sigma^2)k_1(0, 0, 0) + 2\mu_{10}\sigma^2(\mu_4 \sigma^2 - \mu_6) + \mu_{12}\sigma^2(\mu_4 - \sigma^4) \\ + \sigma^2(\mu_8^2 + \mu_6^2 \mu_4 - 2\mu_8 \mu_4^2).\end{aligned}$$

Whence

$$\frac{\partial K_{1,1}}{\partial w}(0,0,0) = \frac{-3\sigma^2\mu_4k_1(0,0,0) - (\mu_6 - 4\mu_4\sigma^2)k_1(0,0,0)}{\sigma^4k_1(0,0,0)} = \frac{\mu_4\sigma^2 - \mu_6}{\sigma^4}.$$

We insert these previous results in the expression in the formula (3) of the fourth partial derivative of I_* with respect to x taken at $(0, \sigma^2, \mu_4)$:

$$\begin{aligned} \frac{\partial^6 I_*}{\partial x^6}(0, \sigma^2, \mu_4) &= \frac{3}{\sigma^8} \frac{\partial^2 K_{1,1}}{\partial v^2}(0,0,0) - \frac{6}{\sigma^8} \frac{\partial^3 K_{1,1}}{\partial u^2 \partial v}(0,0,0) + \frac{1}{\sigma^8} \frac{\partial^4 K_{1,1}}{\partial u^4}(0,0,0) \\ &\quad + \frac{16\mu_4^2 - 2\mu_6\sigma^2 + 2\mu_4\sigma^4}{\sigma^{14}}. \end{aligned}$$

We have

$$\frac{\partial^2 K_{1,1}}{\partial v^2} = \frac{1}{g} \frac{\partial^2 k_1}{\partial v^2} - \frac{2}{g^2} \frac{\partial g}{\partial v} \frac{\partial k_1}{\partial v} - \frac{k_1}{g^2} \frac{\partial^2 g}{\partial v^2} + \frac{2k_1}{g^3} \left(\frac{\partial g}{\partial v} \right)^2.$$

Thus

$$\begin{aligned} \frac{\partial^2 K_{1,1}}{\partial v^2}(0,0,0) &= \frac{1}{\sigma^4 k_1(0,0,0)} \left(\sigma^2 \frac{\partial^2 k_1}{\partial v^2}(0,0,0) - \frac{\partial^2 g}{\partial v^2}(0,0,0) \right) \\ &\quad - \frac{2}{\sigma^6 k_1^2(0,0,0)} \left(\sigma^2 \frac{\partial k_1}{\partial v}(0,0,0) - \frac{\partial g}{\partial v}(0,0,0) \right) \frac{\partial g}{\partial v}(0,0,0) \\ &= \frac{1}{\sigma^4 k_1(0,0,0)} \left(\sigma^2 \frac{\partial^2 k_1}{\partial v^2} - \frac{\partial^2 g}{\partial v^2} - 2\sigma^2 \frac{\partial K_{1,1}}{\partial v} \frac{\partial g}{\partial v} \right) (0,0,0). \end{aligned}$$

We already know the values at $(0,0,0)$ of the partial derivatives of g and k_1 with respect to v . Moreover, the properties of the function f_i , $i \in \mathbb{N}$, and their partial derivatives give us, after factorisation,

$$\begin{aligned} \sigma^2 \frac{\partial^2 k_1}{\partial v^2}(0,0,0) - \frac{\partial^2 g}{\partial v^2}(0,0,0) &= (7\sigma^2\mu_4 - \mu_6 - 8\sigma^6)k_1(0,0,0) - 2\mu_{10}(\mu_4 - \sigma^4)^2 \\ &\quad + 2\mu_6(\mu_4 - \sigma^4)(\mu_8 + \mu_4^2 - 2\sigma^2\mu_6) - 2\sigma^2(\mu_6\sigma^2 - \mu_4^2)^2. \end{aligned}$$

As a consequence

$$\begin{aligned} \sigma^4 k_1(0,0,0) \frac{\partial^2 K_{1,1}}{\partial v^2}(0,0,0) &= (7\sigma^2\mu_4 - \mu_6 - 8\sigma^6)k_1(0,0,0) - 2\mu_{10}(\mu_4 - \sigma^4)^2 \\ &\quad + 2\mu_6(\mu_4 - \sigma^4)(\mu_8 + \mu_4^2 - 2\sigma^2\mu_6) - 2\sigma^2(\mu_6\sigma^2 - \mu_4^2)^2 - 2(\sigma^4 - \mu_4) \left(\mu_{10}(\mu_4 - \sigma^4) \right. \\ &\quad \left. + \mu_8(\mu_4\sigma^2 - \mu_6) + \mu_6(\mu_6\sigma^2 - \mu_4^2) \right) - \frac{2(\sigma^4 - \mu_4)(\mu_4 - 4\sigma^4)k_1(0,0,0)}{\sigma^2}. \end{aligned}$$

Thus

$$\begin{aligned} \sigma^4 k_1(0,0,0) \frac{\partial^2 K_{1,1}}{\partial v^2}(0,0,0) &= 2\sigma^2 \left((\mu_4 - \sigma^4)(\mu_4\mu_8 - \mu_6^2) - (\mu_6\sigma^2 - \mu_4^2)^2 \right) \\ &\quad + \frac{2\mu_4^2 + \sigma^4\mu_4 + \sigma^2\mu_6 - 4\sigma^8}{\sigma^2} k_1(0,0,0). \end{aligned}$$

By developing we get $(\mu_4 - \sigma^4)(\mu_4\mu_8 - \mu_6^2) - (\mu_6\sigma^2 - \mu_4^2)^2 = \mu_4 k_1(0,0,0)$. Thus

$$\frac{\partial^2 K_{1,1}}{\partial v^2}(0,0,0) = \frac{2\mu_4^2 - \sigma^4\mu_4 - \sigma^2\mu_6}{\sigma^6}.$$

Next, since the partial derivatives of g and k_1 with respect to u are null at $(0, 0, 0)$, we get

$$\begin{aligned} \frac{\partial^3 K_{1,1}}{\partial u^2 \partial v}(0, 0, 0) &= \frac{1}{g(0, 0, 0)} \frac{\partial^3 k_1}{\partial u^2 \partial v}(0, 0, 0) - \frac{1}{g^2(0, 0, 0)} \frac{\partial^2 k_1}{\partial u^2}(0, 0, 0) \frac{\partial g}{\partial v}(0, 0, 0) \\ &\quad - \frac{1}{g^2(0, 0, 0)} \frac{\partial k_1}{\partial v}(0, 0, 0) \frac{\partial^2 g}{\partial u^2}(0, 0, 0) + \frac{2k_1(0, 0, 0)}{g^3(0, 0, 0)} \frac{\partial^2 g}{\partial u^2}(0, 0, 0) \frac{\partial g}{\partial v}(0, 0, 0) \\ &\quad - \frac{k_1(0, 0, 0)}{g^2(0, 0, 0)} \frac{\partial^3 g}{\partial u^2 \partial v}(0, 0, 0). \end{aligned}$$

As in the computation of $(\partial^2 K_{1,1}/\partial v^2)(0, 0, 0)$, we notice that this expression can be written as a function of the second partial derivative of $K_{1,1}$ with respect to u and of the partial derivative of $K_{1,1}$ with respect to v :

$$\begin{aligned} \frac{\partial^3 K_{1,1}}{\partial u^2 \partial v}(0, 0, 0) &= \frac{1}{\sigma^4 k_1(0, 0, 0)} \left(\sigma^2 \frac{\partial^3 k_1}{\partial u^2 \partial v}(0, 0, 0) - \frac{\partial^3 g}{\partial u^2 \partial v}(0, 0, 0) \right. \\ &\quad \left. - \sigma^2 \frac{\partial^2 K_{1,1}}{\partial u^2}(0, 0, 0) \frac{\partial g}{\partial v}(0, 0, 0) - \sigma^2 \frac{\partial K_{1,1}}{\partial v}(0, 0, 0) \frac{\partial^2 g}{\partial u^2}(0, 0, 0) \right). \end{aligned}$$

After factorising, this is equal to

$$\begin{aligned} \frac{1}{\sigma^4 k_1(0, 0, 0)} \left(\sigma^2 \frac{\partial^3 k_1}{\partial u^2 \partial v}(0, 0, 0) - \frac{\partial^3 g}{\partial u^2 \partial v}(0, 0, 0) - \frac{2(\mu_4^2 - 4\sigma^8)}{\sigma^2} k_1(0, 0, 0) \right. \\ \left. - 2\sigma^4 (\mu_{10}(\mu_4 - \sigma^4) + \mu_8(\mu_4 \sigma^2 - \mu_6) + \mu_6(\mu_6 \sigma^2 - \mu_4^2)) \right). \end{aligned}$$

and the properties of the functions f_i , $i \in \mathbb{N}$, and their partial derivatives, give us, after factorising by $k_1(0, 0, 0)$,

$$\begin{aligned} \sigma^2 \frac{\partial^3 k_1}{\partial u^2 \partial v}(0, 0, 0) - \frac{\partial^3 g}{\partial u^2 \partial v}(0, 0, 0) &= (\sigma^2 \mu_4 + \mu_6 - 8\sigma^6) k_1(0, 0, 0) \\ &\quad + 2\sigma^4 (\mu_{10}(\mu_4 - \sigma^4) + \mu_8(\mu_4 \sigma^2 - \mu_6) + \mu_6(\mu_6 \sigma^2 - \mu_4^2)). \end{aligned}$$

As a consequence

$$\begin{aligned} \frac{\partial^3 K_{1,1}}{\partial u^2 \partial v}(0, 0, 0) &= \frac{1}{\sigma^4 k_1(0, 0, 0)} \left(-\frac{2(\mu_4^2 - 4\sigma^8)}{\sigma^2} + \sigma^2 \mu_4 + \mu_6 - 8\sigma^6 \right) k_1(0, 0, 0) \\ &= \frac{-2\mu_4^2 + \sigma^4 \mu_4 + \sigma^2 \mu_6}{\sigma^6}. \end{aligned}$$

We finish this proof by computing the fourth partial derivative of $K_{1,1}$ with respect to u taken at $(0, 0, 0)$. Since the partial derivatives of g and k_1 with respect to u are null at $(0, 0, 0)$, we get

$$\begin{aligned} \frac{\partial^4 K_{1,1}}{\partial u^4}(0, 0, 0) &= \frac{1}{g(0, 0, 0)} \frac{\partial^4 k_1}{\partial u^4}(0, 0, 0) - \frac{k_1(0, 0, 0)}{g^2(0, 0, 0)} \frac{\partial^4 g}{\partial u^4}(0, 0, 0) \\ &\quad - \frac{6}{g^2(0, 0, 0)} \frac{\partial^2 k_1}{\partial u^2}(0, 0, 0) \frac{\partial^2 g}{\partial u^2}(0, 0, 0) + \frac{6k_1(0, 0, 0)}{g^3(0, 0, 0)} \left(\frac{\partial^2 g}{\partial u^2}(0, 0, 0) \right)^2. \end{aligned}$$

After factorisation, it is equal to

$$\frac{1}{\sigma^4 k_1(0,0,0)} \left(\sigma^2 \frac{\partial^4 k_1}{\partial u^4}(0,0,0) - \frac{\partial^4 g}{\partial u^4}(0,0,0) - 6\sigma^2 \frac{\partial^2 K_{1,1}}{\partial u^2}(0,0,0) \frac{\partial^2 g}{\partial u^2}(0,0,0) \right).$$

We compute that

$$\begin{aligned} \sigma^2 \frac{\partial^4 k_1}{\partial u^4}(0,0,0) - \frac{\partial^4 g}{\partial u^4}(0,0,0) &= -(23\sigma^2 \mu_4 + 5\mu_6 + 24\sigma^6) k_1(0,0,0) \\ &\quad + 6(\mu_4 + \sigma^4) (\mu_{10}(\mu_4 - \sigma^4) + \mu_8(\mu_4 \sigma^2 - \mu_6) + \mu_6(\mu_6 \sigma^2 - \mu_4^2)) \end{aligned}$$

and we have already computed that

$$\begin{aligned} \frac{\partial^2 g}{\partial u^2}(0,0,0) \frac{\partial^2 K_{1,1}}{\partial u^2}(0,0,0) &= \frac{\mu_4 + \sigma^4}{\sigma^4} \left(-(\mu_4 + 4\sigma^4) k_1(0,0,0) \right. \\ &\quad \left. + \sigma^2 \mu_{10}(\mu_4 - \sigma^4) + \sigma^2 \mu_8(\mu_4 \sigma^2 - \mu_6) + \sigma^2 \mu_6(\mu_6 \sigma^2 - \mu_4^2) \right). \end{aligned}$$

Thus we get

$$\begin{aligned} \sigma^4 k_1(0,0,0) \frac{\partial^4 K_{1,1}}{\partial u^4}(0,0,0) &= -(23\sigma^2 \mu_4 + 5\mu_6 + 24\sigma^6) k_1(0,0,0) \\ &\quad + \frac{6(\mu_4 + \sigma^4)(\mu_4 + 4\sigma^4)}{\sigma^2} k_1(0,0,0). \end{aligned}$$

Hence

$$\frac{\partial^4 K_{1,1}}{\partial u^4}(0,0,0) = \frac{7\sigma^4 \mu_4 - 5\mu_6 \sigma^2 + 6\mu_4^2}{\sigma^6}.$$

We have then

$$\begin{aligned} \frac{\partial^6 \mathbf{I}}{\partial x^6}(0, \sigma^2, \mu_4) &= \frac{3(2\mu_4^2 - \sigma^4 \mu_4 - \sigma^2 \mu_6)}{\sigma^{14}} - \frac{6(-2\mu_4^2 + \sigma^4 \mu_4 + \sigma^2 \mu_6)}{\sigma^{14}} \\ &\quad + \frac{7\sigma^4 \mu_4 - 5\mu_6 \sigma^2 + 6\mu_4^2}{\sigma^{14}} + \frac{16\mu_4^2 - 2\mu_6 \sigma^2 + 2\mu_4 \sigma^4}{\sigma^{14}}. \end{aligned}$$

We have shown that

$$\frac{\partial^6 \mathbf{I}}{\partial x^6}(0, \sigma^2, \mu_4) = \frac{40\mu_4^2 - 16\sigma^2 \mu_6}{\sigma^{14}}.$$

Thus

$$\frac{1}{\alpha! \beta! \gamma!} \frac{\partial^{\alpha+\beta+\gamma} \mathbf{I}}{\partial x^\alpha \partial y^\beta \partial z^\gamma}(0, \sigma^2, \mu_4) = \begin{cases} 0 & \text{if } (\alpha, \beta, \gamma) = (5, 0, 0), \\ \frac{5\mu_4^2 - 2\sigma^2 \mu_6}{90\sigma^{14}} & \text{if } (\alpha, \beta, \gamma) = (6, 0, 0). \end{cases}$$

v) Conclusion

The term A is then $(5\mu_4^2 - 2\sigma^2 \mu_6)/(90\sigma^{14})$. The computations of the previous section imply that

Proposition 5. *Let ρ be a symmetric probability measure on \mathbb{R} whose support contains at least five points. We suppose that $(0, 0, 0) \in \overset{\circ}{D}_\Lambda$ and that*

$$5\mu_4^2 > 2\sigma^2\mu_6.$$

Let q denote the definite positive quadratic form on \mathbb{R}^2 given by

$$\forall (y, z) \in \mathbb{R}^2 \quad q(y, z) = \frac{\mu_8 - \mu_4^2}{2a}y^2 + \frac{\mu_4\sigma^2 - \mu_6}{a}yz + \frac{\mu_4 - \sigma^4}{2a}z^2.$$

Then, in the neighbourhood of $(0, \sigma^2, \mu_4)$,

$$\mathbf{I}_\bullet(x, y, z) - \frac{x^2}{2\sigma^2} + \frac{x^2(y - \sigma^2)}{2\sigma^4} - \frac{\mu_4 x^4}{12\sigma^8} \sim \frac{(5\mu_4^2 - 2\sigma^2\mu_6)x^6}{90\sigma^{14}} + q(y - \sigma^2, z - \mu_4).$$

For many usual distributions, the term $5\mu_4^2 - 2\sigma^2\mu_6$ is positive. For example $5\mu_4^2 - 2\sigma^2\mu_6 = 12b^2c^8 > 0$, for $\rho = (1 - 2b)\delta_0 + b\delta_{-c} + b\delta_c$ with $c > 0$ and $b \in]0, 1/2[$. However we can find a probability measure on \mathbb{R} for which this term is non-positive. To this end, it is enough to take a measure whose sixth moment explodes compared to the fourth moment. Let us consider the measure with density

$$x \mapsto \frac{1}{1 + x^6} \mathbb{1}_{[-5, 5]}(x) \left(\int_{-5}^5 \frac{dy}{1 + y^6} \right)^{-1}$$

with respect to the Lebesgue measure on \mathbb{R} . Its moments can be computed in simple fractions $X^\alpha/(1 + X^6)$, $\alpha \in \{2, 4, 6\}$. We compute that $5\mu_4^2 - 2\sigma^2\mu_6$ is non-positive (an approaching value is -0.483).

4 Construction of an interaction term

In this section we investigate how to build an interaction term whose associated model is amenable to mathematical analysis. We first find criteria on ρ and H so that $\mathbf{I}_\bullet - H$ has a unique minimum at $(0, \sigma^2, \mu_4)$ in any compact subset of Θ^* whose interior contains $(0, \sigma^2, \mu_4)$, and so that the expansion of $\mathbf{I}_\bullet - F - R$ still holds for $\mathbf{I}_\bullet - H$. Next we extend the criteria on H in order to control what happens outside any compact of Θ^* , especially what happens around the line $x = y = 0$ of \mathbb{R}^3 . We use then a variant of Varadhan's lemma. We end this section by proving that the function H given in the introduction satisfies these criteria.

a) First investigations

Let us suppose that ρ is a symmetric probability measure on \mathbb{R} whose support contains at least five points. We assume that

$$(0, 0, 0) \in \overset{\circ}{D}_\Lambda \quad \text{and} \quad 5\mu_4^2 > 2\sigma^2\mu_6.$$

In section 2 we saw that it seems natural to consider the interacting function $H = F + R$ with

$$R : (x, y, z) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \mapsto \frac{zx^4}{12y^4}.$$

It satisfies

$$\forall n \geq 1 \quad \forall (x, y, z) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \quad \frac{1}{n} H(xn, yn, zn) = H(x, y, z).$$

In the neighbourhood of $(0, \sigma^2, \mu_4)$,

$$F(x, y) + R(x, y, z) = \frac{x^2}{2\sigma^2} \frac{1}{1+h} + \frac{\mu_4 x^4}{12\sigma^8} \frac{1}{(1+h)^4} + \frac{(z - \mu_4)x^4}{12\sigma^8} \frac{1}{(1+h)^4},$$

where $h = (y - \sigma^2)/\sigma^2$. In the neighbourhood of 0, we have

$$\begin{aligned} \frac{1}{1+h} &= 1 - h + h^2 - h^3 + h^4 + o(h^4), \\ \frac{1}{(1+h)^4} &= 1 - 4h + 10h^2 - 20h^3 + o(h^3). \end{aligned}$$

Thus, in the neighbourhood of $(0, \sigma^2, \mu_4)$,

$$\begin{aligned} F(x, y) + R(x, y, z) &= \frac{x^2}{2\sigma^2} - \frac{x^2(y - \sigma^2)}{2\sigma^4} + \frac{x^2(y - \sigma^2)^2}{2\sigma^6} + \frac{\mu_4 x^4}{12\sigma^8} \\ &\quad - \frac{x^2(y - \sigma^2)^3}{2\sigma^8} - \frac{\mu_4 x^4 (y - \sigma^2)}{3\sigma^{10}} + \frac{x^4(z - \mu_4)}{12\sigma^8} + \frac{x^2(y - \sigma^2)^4}{2\sigma^{10}} \\ &\quad + \frac{5\mu_4 x^4 (y - \sigma^2)^2}{6\sigma^{12}} - \frac{x^4 (y - \sigma^2)(z - \mu_4)}{3\sigma^{10}} + o(\|x, y - \sigma^2, z - \mu_4\|^6). \end{aligned}$$

Lemma 4 implies that, in the neighbourhood of $(0, \sigma^2, \mu_4)$,

$$\begin{aligned} F(x, y) + R(x, y, z) &= \frac{x^2}{2\sigma^2} - \frac{x^2(y - \sigma^2)}{2\sigma^4} + \frac{\mu_4 x^4}{12\sigma^8} \\ &\quad + o\left(\frac{(5\mu_4^2 - 2\sigma^2\mu_6)x^6}{90\sigma^{14}} + q(y - \sigma^2, z - \mu_4)\right) \end{aligned}$$

and then it follows from proposition 5 that, in the neighbourhood of $(0, \sigma^2, \mu_4)$,

$$I_\bullet(x, y, z) - F(x, y) - R(x, y, z) \sim \frac{(5\mu_4^2 - 2\sigma^2\mu_6)x^6}{90\sigma^{14}} + q(y - \sigma^2, z - \mu_4).$$

The computations of the previous section show that, in the neighbourhood of 0,

$$I_\bullet(x, \sigma^2, \mu_4) - F(x, \sigma^2) - R(x, \sigma^2, \mu_4) \sim \frac{5\mu_4^2 - 2\sigma^2\mu_6}{90\sigma^{14}} x^6.$$

As a consequence, if $5\mu_4^2 < 2\sigma^2\mu_6$ then $I_\bullet - F - R$ does not have a minimum at $(0, \sigma^2, \mu_4)$. Thus it is not possible to prove that $I_\bullet - F - R$ is non-negative for any symmetric probability measures on \mathbb{R} , as we did in [1] for $I - F$. The techniques we used there have not been successful and we have not been able to show that $I_\bullet - F - R$ has a unique minimum for an interesting class of probability measures on \mathbb{R} . We will go around this problem by modifying the interacting function H in order to « force » the function

$$G_n : (x, y, z) \mapsto I_\bullet(x, y, z) - \frac{1}{n} H(nx, ny, nz)$$

to have a unique minimum at $(0, \sigma^2, \mu_4)$ for any $n \geq 1$, and to have the same expansion we obtained above.

By analysing the essential ingredient of the proof of theorem 2 of [1], we consider the following hypothesis :

Hypothesis 6. *Let ρ be a symmetric probability measure on \mathbb{R} whose support contains at least five points. We assume that*

$$(0, 0, 0) \in \overset{\circ}{D}_\Lambda \quad \text{and} \quad 5\mu_4^2 > 2\sigma^2\mu_6.$$

Let H be a function from Θ^* to \mathbb{R} . We suppose that there exists $(R_n)_{n \geq 1}$ a sequence of upper semi-continuous functions from Θ^* to \mathbb{R} satisfying, for any $(x, y, z) \in \Theta^*$,

$$\begin{aligned} \forall n \geq 1 \quad 0 &\leq R_{n+1}(x, y, z) \leq R_n(x, y, z) \leq R(x, y, z), \\ \forall n \geq 1 \quad H(x, y, z) - F(x, y) &= nR_n\left(\frac{x}{n}, \frac{y}{n}, \frac{z}{n}\right), \\ R_n(x, y, z) &\xrightarrow[n \rightarrow +\infty]{} 0 \end{aligned}$$

and, for every $(x, y, z) \in \mathbb{R}^3$,

$$n(R - R_n)\left(\frac{x}{n^{1/6}}, \frac{y}{\sqrt{n}} + \sigma^2, \frac{z}{\sqrt{n}} + \mu_4\right) \xrightarrow[n \rightarrow +\infty]{} 0.$$

We have the following proposition :

Proposition 7. *Suppose that ρ and H satisfy the hypothesis 6. If q denotes the definite positive quadratic form of proposition 5, then, for any $(x, y, z) \in \mathbb{R}^3$,*

$$n(\mathbf{I} - F - R_n)\left(\frac{x}{n^{1/6}}, \frac{y}{\sqrt{n}} + \sigma^2, \frac{z}{\sqrt{n}} + \mu_4\right) \xrightarrow[n \rightarrow +\infty]{} q(y, z) + \frac{5\mu_4^2 - 2\sigma^2\mu_6}{90\sigma^{14}}x^6.$$

Let K be a compact subset of \mathbb{R}^3 included in Θ^* such that $(0, \sigma^2, \mu_4)$ belongs to the interior of K . There exists $n_0 \geq 1$ such that $\mathbf{I} - F - R_{n_0}$ has a unique minimum on K at $(0, \sigma^2, \mu_4)$.

We will use the following lemma, which is a variant of Dini's theorem :

Lemma 8. *Let $(g_n)_{n \geq 1}$ be a non-increasing sequence of functions defined on a compact set X and which converges pointwise to a function g defined on X . If the function $g_n - g$ is upper semi-continuous for any $n \geq 1$, then $(g_n)_{n \geq 1}$ converges uniformly over X towards g .*

Proof. For any $n \geq 1$, we put $h_n = g_n - g$. The sequence $(h_n)_{n \geq 1}$ is non-increasing and converges pointwise to the null function. For a fixed $\varepsilon > 0$ and for any $n \geq 1$, we denote

$$A_n(\varepsilon) = \{x \in X : h_n(x) < \varepsilon\}.$$

These sets are open since, for any $n \geq 1$, the function h_n is upper semi-continuous. The convergence of the sequence $(h_n)_{n \geq 1}$ implies that

$$X \subset \bigcup_{n \geq 1} A_n(\varepsilon).$$

We can extract a finite subcover : there exists $N \geq 1$ such that

$$X \subset \bigcup_{n \leq N} A_n(\varepsilon).$$

Since $(h_n)_{n \in \mathbb{N}}$ is non-decreasing, then $X \subset A_N(\varepsilon)$. Thus

$$\forall x \in X \quad \exists N > 0 \quad n \geq N \implies h_n(x) \leq h_N(x) < \varepsilon.$$

This proves the lemma. \square

Proof of proposition 7. In the neighbourhood of $(0, \sigma^2, \mu_4)$,

$$I_\bullet(x, y, z) - F(x, y) - R(x, y, z) \sim \frac{(5\mu_4^2 - 2\sigma^2\mu_6)x^6}{90\sigma^{14}} + q(y - \sigma^2, z - \mu_4).$$

For every $n \geq 1$, we denote

$$G_n = I_\bullet - F - R_n = (I_\bullet - F - R) + (R - R_n).$$

The expansion of $I_\bullet - F - R_n$ and the hypothesis 6 imply that, for $(x, y, z) \in \mathbb{R}^3$,

$$n(I_\bullet - F - R_n) \left(\frac{x}{n^{1/6}}, \frac{y}{\sqrt{n}} + \sigma^2, \frac{z}{\sqrt{n}} + \mu_4 \right) \xrightarrow{n \rightarrow +\infty} q(y, z) + \frac{5\mu_4^2 - 2\sigma^2\mu_6}{90\sigma^{14}} x^6.$$

Next the function $R - R_n$ is non-negative, thus $G_n \geq I_\bullet - F - R$ and there exists an open set U centered at $(0, \sigma^2, \mu_4)$ such that, for any $(x, y, z) \in U$,

$$G_n(x, y, z) \geq \frac{1}{2}q(y - \sigma^2, z - \mu_4) + \frac{5\mu_4^2 - 2\sigma^2\mu_6}{180\sigma^{14}} x^6.$$

The right term of this inequality is non-negative since $5\mu_4^2 > 2\sigma^2\mu_6$. Since q is a definite positive quadratic form, this term vanishes only at $(0, \sigma^2, \mu_4)$. Thus we proved that, for any $n \geq 1$, G_n has a unique minimum on U at $(0, \sigma^2, \mu_4)$ and it is equal to 0.

Without loss of generality, we can suppose that U is included in K . The set $K \cap U^c$ is a compact subset of \mathbb{R}^3 included in Θ^* . Let ν_ρ be the law of (Z, Z^2) when Z is a random variable with distribution ρ . We denote by Λ the Log-Laplace of ν_ρ and by I its Cramér transform. The measure ρ is symmetric and $(0, 0, 0) \in \overset{\circ}{D}_{\Lambda_\bullet}$. Moreover we have

$$\forall (u, v) \in \mathbb{R}^2 \quad \Lambda(u, v) = \Lambda_\bullet(u, v, 0).$$

As a consequence $(0, 0) \in \overset{\circ}{D}_\Lambda$ and proposition 2 implies that the function $I - F$ has a unique minimum at $(0, \sigma^2)$ on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$. Next, for any $(x, y, z, u, v) \in \mathbb{R}^5$,

$$I_\bullet(x, y, z) \geq xu + yv + z \times 0 - \Lambda_\bullet(u, v, 0) = xu + yv - \Lambda(u, v).$$

Taking the supremum over $(u, v) \in \mathbb{R}^2$, it comes that

$$\forall (x, y, z) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \quad I_\bullet(x, y, z) - \frac{x^2}{2y} \geq I(x, y) - \frac{x^2}{2y}.$$

Hence, for $(x, y, z) \in K \cap U^c$, there are two cases :

★ Either $(x, y) \neq (0, \sigma^2)$ and then $I_\bullet(x, y, z) - F(x, y) > 0$.

★ Or $(x, y) = (0, \sigma^2)$ and then $z \neq \mu_4$. The function I_\bullet has a unique minimum at $(0, \sigma^2, \mu_4)$ in which it is null (see chapter V of [6] for a proof of this result). Thus

$$I_\bullet(0, \sigma^2, z) - F(0, \sigma^2) = I_\bullet(0, \sigma^2, z) > 0.$$

In each case

$$\forall (x, y, z) \in K \cap U^c \quad I_\bullet(x, y, z) - \frac{x^2}{2y} > 0.$$

By hypothesis, the sequence of functions $(R_n + F - I_\bullet)_{n \geq 1}$ is non-increasing and converges pointwise to $F - I_\bullet$. Moreover, for any $n \geq 1$, $R_n + F - I_\bullet$ is upper semi-continuous. Hence the previous lemma implies that $(I_\bullet - F - R_n)_{n \geq 1}$ converges uniformly to $I_\bullet - F$ on $K \cap U^c$. As a consequence there exists $n_0 \geq 1$ such that $I_\bullet - F - R_{n_0}$ is positive on $K \cap U^c$. Hence $I_\bullet - F - R_{n_0}$ has a unique minimum on $K \cap U^c$ at $(0, \sigma^2, \mu_4)$. \square

b) Around Varadhan's lemma

We saw in section 2 that the law of $(S_n/n, T_n/n, U_n/n)$ under $\tilde{\mu}_{H,n,\rho}$ is

$$\frac{e^{nF(x,y)+nR_n(x,y,z)} \mathbb{1}_{\Theta^*}(x, y, z) d\tilde{\nu}_{\bullet,n,\rho}(x, y, z)}{\int_{\Theta^*} e^{nF(x,y)+nR_n(x,y,z)} d\tilde{\nu}_{\bullet,n,\rho}(x, y, z)}.$$

We search additional conditions on H and ρ so that, if A is a closed set which does not contain $(0, \sigma^2, \mu_4)$, then

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Theta^* \cap A} e^{nF(x,y)+nR_n(x,y,z)} d\tilde{\nu}_{\bullet,n,\rho}(x, y, z) < 0.$$

To this end, we need a variant of Varadhan's lemma. By proposition 7 we can conclude if A is a compact subset of Θ^* . We have to extend the criteria on H in order to control what happens around the line $x = y = 0$ of \mathbb{R}^3 . We proceed similarly as in [1].

Hypothesis 9. Assume that ρ and H satisfy the hypothesis 6. We suppose that ρ has a bounded support and that, for any $r > 0$, there exists $\delta > 0$ such that

$$\forall (x, y, z) \in \Theta \cap (\mathbb{R} \times]0, \delta] \times \mathbb{R}) \quad \forall n \geq 1 \quad R_n(x, y, z) \leq r.$$

Hypothesis 10. Assume that ρ and H satisfy the hypothesis 6. We suppose that there exists $c_0 > 0$ such that

$$\forall (x, y, z) \in \Theta^* \quad \forall n \geq 1 \quad R_n(x, y, z) \leq c_0 y.$$

Proposition 11. Let ρ and H fulfill the hypothesis 6. We have

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Theta^*} e^{nF(x,y)+nR_n(x,y,z)} d\tilde{\nu}_{\bullet,n,\rho}(x, y, z) \geq 0.$$

Suppose that ρ and H also satisfy either the hypothesis 9 or the hypothesis 10. Then, for any closed subset A of \mathbb{R}^3 which does not contain $(0, \sigma^2, \mu_A)$, we have

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Theta^* \cap A} e^{nF(x,y) + nR_n(x,y,z)} d\tilde{\nu}_{\bullet, \rho}(x, y, z) < 0.$$

Proof. The large deviations principle satisfied by $(\tilde{\nu}_{\bullet, \rho})_{n \geq 1}$ implies that

$$\begin{aligned} \frac{1}{n} \ln \int_{\Theta^*} e^{nF(x,y) + nR_n(x,y,z)} d\tilde{\nu}_{\bullet, \rho}(x, y, z) \\ \geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \tilde{\nu}_{\bullet, \rho}(\Theta^*) \geq -\inf \left\{ I_{\bullet}(x, y, z) : (x, y, z) \in \overset{\circ}{\Theta} \right\} = 0. \end{aligned}$$

Let us show the second inequality. Proposition 4 of [4] states that there exists $\gamma > 0$ such that, for $\delta \in]0, \sigma^2[$ small enough and n large enough,

$$\int_{\mathbb{R}^2} e^{nF(x,y)} \mathbf{1}_{x^2 \leq y} \mathbf{1}_{0 < y \leq \delta} d\tilde{\nu}_{n, \rho}(x, y) \leq e^{-n\gamma},$$

where $\tilde{\nu}_{n, \rho}$ denotes the law of $(S_n/n, T_n/n)$ under $\rho^{\otimes n}$.

The function H satisfies the hypothesis 9 or 10 thus we can choose δ small enough so that

$$\forall (x, y, z) \in \Theta \cap (\mathbb{R} \times]0, \delta] \times \mathbb{R}) \quad \forall n \geq 1 \quad R_n(x, y, z) \leq \frac{\gamma}{2}.$$

Hence

$$\begin{aligned} \int_{\Theta^*} e^{nF(x,y) + nR_n(x,y,z)} \mathbf{1}_{y \leq \delta} d\tilde{\nu}_{\bullet, \rho}(x, y, z) \\ \leq e^{n\gamma/2} \int_{\Theta^*} e^{nF(x,y)} \mathbf{1}_{y \leq \delta} d\tilde{\nu}_{\bullet, \rho}(x, y, z) \\ \leq e^{n\gamma/2} \int_{\mathbb{R}^3} e^{nF(x,y)} \mathbf{1}_{x^2 \leq y} \mathbf{1}_{0 < y \leq \delta} d\tilde{\nu}_{\bullet, \rho}(x, y, z) \\ = e^{n\gamma/2} \int_{\mathbb{R}^2} e^{nF(x,y)} \mathbf{1}_{x^2 \leq y} \mathbf{1}_{0 < y \leq \delta} d\tilde{\nu}_{n, \rho}(x, y) \\ \leq e^{n\gamma/2} e^{-n\gamma} = e^{-n\gamma/2}. \end{aligned}$$

Thus, for δ small enough,

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Theta^*} e^{nF(x,y) + nR_n(x,y,z)} \mathbf{1}_{y \leq \delta} d\tilde{\nu}_{\bullet, \rho}(x, y, z) \leq -\gamma/2.$$

We define $A_\delta = \{(x, y, z) \in \Theta \cap A : y \geq \delta\}$. We have

$$\Theta^* \cap A \subset \{(x, y, z) \in \Theta^* : y \leq \delta\} \cup A_\delta.$$

If ρ and H satisfy the hypothesis 10, we have, for any $(x, y, z) \in \Theta^*$ and $n \geq 1$,

$$I_{\bullet}(x, y, z) - F(x, y) - R_n(x, y, z) \geq I_{\bullet}(x, y, z) - \frac{1}{2} - c_0 y \geq I_{\bullet}(x, y, z) - \frac{1}{2} - c_0 \sqrt{z}.$$

Since $(0, 0, 0) \in \overset{\circ}{D}_{\Lambda_\bullet}$, there exists $w_0 > 0$ small enough so that $(0, 0, w_0) \in D_{\Lambda_\bullet}$. By the definition of the Cramér transform, we have

$$I_\bullet(x, y, z) \geq 0 \times x + 0 \times y + w_0 \times z - \ln \int_{\mathbb{R}} e^{0 \times t + 0 \times t^2 + w_0 \times t^4} d\rho(t).$$

As a consequence

$$I_\bullet(x, y, z) - F(x, y) - R_n(x, y, z) \geq w_0 z - c_0 \sqrt{z} - \frac{1}{2} - \ln \int_{\mathbb{R}} e^{w_0 t^4} d\rho(t).$$

The right term converges to $+\infty$ when z goes to $+\infty$ and it does not depend on x, y and n . As a consequence, there exists $z_0 > 0$ such that, for any $n \geq 1$,

$$\forall (x, y, z) \in \Theta^* \cap (\mathbb{R} \times \mathbb{R} \times [z_0, +\infty[) \quad I_\bullet(x, y, z) - F(x, y) - R_n(x, y, z) \geq 1.$$

We put $K = \{(x, y, z) \in \Theta : z \leq \max(z_0, 2\mu_4)\}$. The above inequality implies that

$$\inf_{n \geq 1} \inf_{A_\delta \cap K^c} (I_\bullet - F - R_n) \geq 1.$$

Moreover, we can reduce δ so that the set $\{(x, y, z) \in \Theta : y \geq \delta\} \cap K$ is a compact subset of \mathbb{R}^3 included in Θ^* and whose interior contains $(0, \sigma^2, \mu_4)$. Thus proposition 7 ensures the existence of $n_0 \geq 1$ such that $I_\bullet - F - R_{n_0}$ has a unique minimum in $\{(x, y, z) \in \Theta : y \geq \delta\} \cap K$ at $(0, \sigma^2, \mu_4)$. Since $I - F - R_{n_0}$ is a good rate function and $A_\delta \cap K$ does not contain $(0, \sigma^2, \mu_4)$, we have

$$\inf_{A_\delta \cap K} (I_\bullet - F - R_{n_0}) > 0.$$

As a consequence

$$\inf_{A_\delta} (I_\bullet - F - R_{n_0}) > 0.$$

If ρ and H satisfy the hypothesis 9 and if K' denotes the closed convex hull of the support of $\nu_{\bullet, \rho}$ (which is then compact), then we can also reduce δ in order to apply proposition 7 and find some $n_0 \geq 1$ such that

$$\inf_{A_\delta} (I_\bullet - F - R_{n_0}) = \inf_{A_\delta \cap K'} (I_\bullet - F - R_{n_0}) > 0.$$

In both cases, the usual Varadhan lemma (see [2]) implies that there exists $\gamma_1 > 0$ such that, for n large enough,

$$\int_{A_\delta} e^{nF(x, y) + nR_{n_0}(x, y, z)} d\tilde{\nu}_{\bullet, \rho}(x, y, z) \leq e^{-n\gamma_1}.$$

Finally, since $R_n \leq R_{n_0}$ for any $n \geq n_0$, we have

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{A_\delta} e^{nF(x, y) + nR_n(x, y, z)} d\tilde{\nu}_{\bullet, \rho}(x, y, z) \leq -\gamma_1.$$

Hence

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Theta^* \cap A} e^{nF(x, y) + nR_n(x, y, z)} d\tilde{\nu}_{\bullet, \rho}(x, y, z) \leq \max\left(-\frac{\gamma}{2}, -\gamma_1\right) < 0.$$

This ends the proof of the proposition. \square

c) One good candidate

Let ρ satisfies the hypothesis 6. One good candidate for H is

$$H : (x, y, z) \in \Theta^* \mapsto \frac{x^2}{2y} + \frac{1}{12} \frac{zx^4y^5}{y^9 + x^{10} + zx^4y^4}.$$

Indeed, the sequence $(R_n)_{n \geq 1}$ defined by

$$\forall (x, y, z) \in \Theta^* \quad \forall n \geq 1 \quad R_n(x, y, z) = \frac{1}{12} \frac{zx^4y^5}{y^9 + nx^{10} + zx^4y^4},$$

consists of upper semi-continuous functions and, for any $(x, y, z) \in \Theta^*$,

$$\forall n \geq 1 \quad 0 \leq R_{n+1}(x, y, z) \leq R_n(x, y, z) \leq R(x, y, z),$$

$$\forall n \geq 1 \quad H(x, y, z) - \frac{x^2}{2y} = nR_n\left(\frac{x}{n}, \frac{y}{n}, \frac{z}{n}\right),$$

$$R_n(x, y, z) \xrightarrow[n \rightarrow +\infty]{} 0.$$

We have next

$$(R - R_n)(x, y, z) = \frac{zx^4}{12y^4} - \frac{1}{12} \frac{zx^4y^5}{y^9 + nx^{10} + zx^4y^4} = \frac{zx^4(nx^{10} + zx^4y^4)}{12y^4(y^9 + nx^{10} + zx^4y^4)}.$$

Evaluating in $(x/n^{1/6}, y/\sqrt{n} + \sigma^2, z/\sqrt{n} + \mu_4)$, we get

$$n \left(\frac{x}{n^{1/6}}\right)^{10} + \left(\frac{z}{\sqrt{n}} + \mu_4\right) \left(\frac{x}{n^{1/6}}\right)^4 \left(\frac{y}{\sqrt{n}} + \sigma^2\right)^4 \sim \frac{x^{10} + \sigma^8 \mu_4}{n^{2/3}},$$

$$\left(\frac{y}{\sqrt{n}} + \sigma^2\right)^9 + n \left(\frac{x}{n^{1/6}}\right)^{10} + \left(\frac{z}{\sqrt{n}} + \mu_4\right) \left(\frac{x}{n^{1/6}}\right)^4 \left(\frac{y}{\sqrt{n}} + \sigma^2\right)^4 \sim \sigma^{18}$$

and

$$\frac{(z/\sqrt{n} + \mu_4)x^4n^{-2/3}}{12(y/\sqrt{n} + \sigma^2)^4} \sim \frac{\mu_4 x^4}{12\sigma^8 n^{2/3}}.$$

Thus

$$\begin{aligned} n(R - R_n)\left(\frac{x}{n^{1/6}}, \frac{y}{\sqrt{n}} + \sigma^2, \frac{z}{\sqrt{n}} + \mu_4\right) &\sim n \frac{\mu_4 x^4}{12\sigma^8 n^{2/3} \sigma^{18}} \frac{x^{10} + \sigma^8 \mu_4}{n^{2/3}} \\ &\sim \frac{\mu_4 x^4 (x^{10} + \sigma^8 \mu_4)}{12\sigma^{26} n^{1/3}} \xrightarrow[n \rightarrow +\infty]{} 0. \end{aligned}$$

Hence H satisfies the hypothesis 6. Finally, for any $(x, y, z) \in \Theta^*$, we have

$$zx^4y^5 \leq y^{10} + nyx^{10} + zx^4y^5 = y(y^9 + nx^{10} + zx^4y^4).$$

Thus H also satisfies the hypothesis 10.

5 Fluctuations theorem

In this section, we suppose that ρ has a density f with respect to the Lebesgue measure on \mathbb{R} . We will proceed as in section 7 of [1] to obtain our fluctuations result: we first compute an asymptotic expression of the density of $\nu_{\bullet, \rho}^{*n}$, for n large enough. Next we prove a generalisation of theorem 1 with the help of Laplace's method.

a) **Asymptotic expression of the density of $\nu_{\bullet,\rho}^{*n}$**

Proposition 12. *If ρ is a probability measure having a density f with respect to the Lebesgue measure on \mathbb{R} , then $\nu_{\bullet,\rho}^{*3}$ admits a density with respect to the Lebesgue measure on \mathbb{R}^3 . Suppose that, for some $p \in]1, 2]$,*

$$\int_{\mathbb{R}^3} \frac{f^p(x)f^p(y)f^p(z)}{|(x+y+z)(x-y)(y-z)(z-x)|^{p-1}} dx dy dz < +\infty. \quad (*)$$

Then, for n large enough, $\tilde{\nu}_{\bullet,n,\rho}$ has a density g_n with respect to the Lebesgue measure on \mathbb{R}^3 such that, for any compact subset K of A_{\bullet} , when n goes to $+\infty$, uniformly over $(x, y, z) \in K$,

$$g_n(x, y, z) \sim \left(\frac{n}{2\pi}\right)^{3/2} \left(\det D_{(x,y,z)}^3 I_{\bullet}\right)^{1/2} e^{-nI_{\bullet}(x,y,z)}.$$

Proof. Let Φ be a measurable positive function on \mathbb{R}^3 . We have

$$\begin{aligned} & \int_{\mathbb{R}^3} \Phi(x, y, z) d\nu_{\bullet,\rho}^{*3}(x, y, z) \\ &= \int_{\mathbb{R}^9} \Phi(u_1 + u_2 + u_3, v_1 + v_2 + v_3, w_1 + w_2 + w_3) \\ & \quad \times d\nu_{\bullet,\rho}(u_1, v_1, w_1) d\nu_{\bullet,\rho}(u_2, v_2, w_2) d\nu_{\bullet,\rho}(u_3, v_3, w_3) \\ &= \int_{\mathbb{R}^3} \Phi(x + y + z, x^2 + y^2 + z^2, x^4 + y^4 + z^4) d\rho(x) d\rho(y) d\rho(z) \\ &= \int_{\mathbb{R}^3} \Phi(x + y + z, x^2 + y^2 + z^2, x^4 + y^4 + z^4) f(x)f(y)f(z) dx dy dz. \end{aligned}$$

Let us make the change of variables given by

$$\varphi : (x, y, z) \in \mathbb{R}^3 \mapsto (x + y + z, x^2 + y^2 + z^2, x^4 + y^4 + z^4).$$

The function φ is \mathcal{C}^1 on \mathbb{R}^3 . We compute its Jacobian : for any $(x, y, z) \in \mathbb{R}^3$, we have

$$\begin{aligned} \text{Jac}_{(x,y,z)}\varphi &= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 4x^3 & 4y^3 & 4z^3 \end{vmatrix} = 8 \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix} \\ &= 8 \left((y-x)(z^3-x^3) - (z-x)(y^3-x^3) \right) \\ &= 8 \left((y-x)(z-x)(z^2+xz+x^2) - (z-x)(y-x)(y^2+xy+x^2) \right) \\ &= 8(y-x)(z-x) \left(z^2+xz+x^2 - y^2-xy-x^2 \right) \\ &= 8(y-x)(z-x)(z-y)(x+y+z). \end{aligned}$$

We introduce the set

$$\begin{aligned} \mathcal{H} &= \{ (x, y, z) \in \mathbb{R}^3 : x + y + z = 0 \} \cup \{ (x, y, z) \in \mathbb{R}^3 : x = y \} \\ & \quad \cup \{ (x, y, z) \in \mathbb{R}^3 : x = z \} \cup \{ (x, y, z) \in \mathbb{R}^3 : y = z \}. \end{aligned}$$

It is the union of four hyperplanes on which the Jacobian of φ vanishes. We define next

$$\begin{aligned} O_1 &= \{ (x, y, z) \in \mathbb{R}^3 : x < y < z \}, & O_2 &= \{ (x, y, z) \in \mathbb{R}^3 : x < z < y \}, \\ O_3 &= \{ (x, y, z) \in \mathbb{R}^3 : y < x < z \}, & O_4 &= \{ (x, y, z) \in \mathbb{R}^3 : y < z < x \}, \\ O_5 &= \{ (x, y, z) \in \mathbb{R}^3 : z < x < y \}, & O_6 &= \{ (x, y, z) \in \mathbb{R}^3 : z < y < x \}. \end{aligned}$$

The six open sets O_1, \dots, O_6 are a partition of $\mathbb{R}^3 \setminus \mathcal{H}$. On each of these open sets, the Jacobian of φ does not vanish. The set \mathcal{H} is negligible with respect to the Lebesgue measure on \mathbb{R}^3 , thus

$$\begin{aligned} \int_{\mathbb{R}^3} \Phi(x, y, z) d\nu_{\bullet, \rho}^{*3}(x, y, z) &= \sum_{i=1}^6 \int_{O_i} \Phi(\varphi(x, y, z)) f(x) f(y) f(z) dx dy dz \\ &= \sum_{i=1}^6 \int_{O_i} \Phi(\varphi(x, y, z)) g(x, y, z) |\text{Jac}_{(x, y, z)} \varphi| dx dy dz, \end{aligned}$$

where g is the function defined on $\mathbb{R}^3 \setminus \mathcal{H}$ by

$$\forall (x, y, z) \in \mathbb{R}^3 \setminus \mathcal{H} \quad g(x, y, z) = \frac{f(x)f(y)f(z)}{|8(x+y+z)(x-y)(y-z)(z-x)|}.$$

On each open set O_i , $i \in \{1, \dots, 6\}$, the function φ is \mathcal{C}^1 and its Jacobian does not vanish. In order to apply the global version of the inverse function theorem, we still have to prove that φ is one to one on each of these open sets. Let $(u, v, w) \in \mathbb{R}^3$ such that there exists $(x, y, z) \in \mathbb{R}^3 \setminus \mathcal{H}$ which verifies $(u, v, w) = \varphi(x, y, z)$. We have then

$$x + y = u - z, \quad x^2 + y^2 = v - z \quad \text{et} \quad x^4 + y^4 = w - z^4.$$

We search a polynomial equation satisfied by z . We have

$$\begin{aligned} (x + y)^2 (x^2 + y^2) &= x^4 + y^4 + 2x^2y^2 + 2x^3y + 2xy^3 \\ &= x^4 + y^4 + 2xy(xy + x^2 + y^2) \\ &= x^4 + y^4 + ((x + y)^2 - (x^2 + y^2)) \frac{(x + y)^2 + x^2 + y^2}{2}. \end{aligned}$$

Hence

$$(u - z)^2 (v - z^2) = w - z^4 + ((u - z)^2 - v + z^2) \frac{(u - z)^2 + v - z^2}{2}.$$

By developping, we get

$$4uz^3 - 4u^2z^2 + 2u(u^2 - v)z + u^2v - \frac{u^4}{2} + \frac{v^2}{2} - w = 0.$$

Since $(x, y, z) \notin \mathcal{H}$, we have $u = x + y + z \neq 0$ and thus $P_{(u, v, w)}(z) = 0$ with

$$P_{(u, v, w)}(X) = X^3 - uX^2 + \frac{u^2 - v}{2}X + \frac{uv}{4} - \frac{u^3}{8} + \frac{v^2}{8u} - \frac{w}{4u}.$$

Since x, y and z are exchangeable in the expression of φ , we also have that

$$P_{(u,v,w)}(x) = P_{(u,v,w)}(y) = 0.$$

Let $i \in \{1, \dots, 6\}$. We have shown that, if $(u, v, w) \in \mathbb{R}^3$ is such that there exists $(x, y, z) \in O_i$ satisfying $(u, v, w) = \varphi(x, y, z)$, then x, y and z are the zeros of the cubic polynomial $P_{(u,v,w)}$. As a consequence φ is one to one on O_i . Thus, by the global version of the inverse function theorem (see theorem 3.8.10 of [7]), for any $i \in \{1, \dots, 6\}$, the map φ is a \mathcal{C}^1 -diffeomorphism from O_i^+ to $\varphi(O_i^+)$. We denote by φ_i^{-1} its inverse function.

Since x, y and z are exchangeable in the expression of φ , we get that all the sets $\varphi(O_i)$, $i \in \{1, \dots, 6\}$, are equal to some set \mathcal{U} and, for any $(u, v, w) \in \mathcal{U}$, the coordinates of $\varphi_i^{-1}(u, v, w)$, $i \in \{1, \dots, 6\}$, are the same up to a non-trivial permutation. As a consequence

$$\forall i \in \{1, \dots, 6\} \quad g \circ \varphi_i^{-1} = g \circ \varphi_1^{-1}.$$

Let us make the change of variables given by φ on each open set O_i :

$$\int_{\mathbb{R}^3} \Phi(x, y, z) d\nu_{\rho}^{*3}(x, y, z) = \sum_{i=1}^6 \int_{\varphi(O_i)} \Phi(u, v, w) g \circ \varphi_i^{-1}(u, v, w) du dv dw.$$

The previous remarks about the symmetric structure of φ imply that

$$\int_{\mathbb{R}^3} \Phi(x, y, z) d\nu_{\rho}^{*3}(x, y, z) = \int_{\mathbb{R}^3} \Phi(u, v, w) 6 g \circ \varphi_1^{-1}(u, v, w) \mathbf{1}_{\mathcal{U}}(u, v, w) du dv dw.$$

Hence ν_{ρ}^{*3} admits the density $f_3 = 6 g \circ \varphi_1^{-1} \times \mathbf{1}_{\mathcal{U}}$ with respect to the Lebesgue measure on \mathbb{R}^3 .

Next, for any $p \in [1, +\infty[$, we have

$$\int_{\mathbb{R}^3} f_3^p(u, v, w) du dv dw = 6^p \int_{\mathcal{U}} (g \circ \varphi_1^{-1}(u, v, w))^p du dv dw.$$

Let us make the change of variables given by φ_1^{-1} :

$$\int_{\mathbb{R}^3} f_3^p(u, v, w) du dv dw = 6^p \int_{O_1} g^p(x, y, z) |\text{Jac}_{(x,y,z)}\varphi| dx dy dz.$$

By symmetry, we write this integral

$$\begin{aligned} \int_{\mathbb{R}^3} f_3^p(u, v, w) du dv dw &= 6^{p-1} \sum_{i=1}^6 \int_{O_i} g^p(x, y, z) |\text{Jac}_{(x,y,z)}\varphi| dx dy dz \\ &= 6^{p-1} \int_{\mathbb{R}^3} g^p(x, y, z) |\text{Jac}_{(x,y,z)}\varphi| dx dy dz. \end{aligned}$$

This is equal to

$$\left(\frac{3}{4}\right)^{p-1} \int_{\mathbb{R}^3} \frac{f^p(x)f^p(y)f^p(z)}{|(x+y+z)(x-y)(y-z)(z-x)|^{p-1}} dx dy dz.$$

As a consequence, if this integral is finite for some $p \in]1, 2]$, then $f_3 \in L^p$. Thus proposition A.6¹ of [1] implies that, for n large enough, $\tilde{\nu}_{\bullet, n, \rho}$ has a density g_n with respect to the Lebesgue measure on \mathbb{R}^3 such that, for any compact subset K of A_{\bullet} , when $n \rightarrow +\infty$, uniformly over $(x, y, z) \in K$,

$$g_n(x, y, z) \sim \left(\frac{n}{2\pi}\right)^{3/2} \left(\det D_{(x,y,z)}^3 I_{\bullet}\right)^{1/2} e^{-nI_{\bullet}(x,y,z)}.$$

This ends the proof of the proposition. \square

Let us prove that, if f is bounded, then there exists $p \in]1, 2]$ such that

$$\int_{\mathbb{R}^3} \frac{f^p(x)f^p(y)f^p(z)}{|(x+y+z)(x-y)(y-z)(z-x)|^p} dx dy dz < +\infty.$$

Young's inequality implies that, for any positive real numbers a, b, c and d ,

$$\frac{1}{abcd} \leq \frac{1}{2} \left(\frac{1}{(ab)^2} + \frac{1}{(cd)^2} \right) \leq \frac{1}{4} \left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} \right).$$

By this inequality and by the symmetry of the integral in x, y and z we have

$$\int_{\mathbb{R}^3} \frac{f^p(x)f^p(y)f^p(z)}{|(x+y+z)(x-y)(y-z)(z-x)|^{p-1}} dx dy dz \leq \frac{1}{4} (I_1 + 3I_2),$$

with

$$I_1 = \int_{\mathbb{R}^3} \frac{f^p(x)f^p(y)f^p(z)}{|x+y+z|^{4(p-1)}} dx dy dz,$$

$$I_2 = \int_{\mathbb{R}^3} \frac{f^p(x)f^p(y)f^p(z)}{|x-y|^{4(p-1)}} dx dy dz.$$

Making the change of variable $(x, y, z) \mapsto (x+y+z, y, z)$, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} \frac{f^p(u-v-w)f^p(v)f^p(w)}{|u|^{4(p-1)}} du dv dw \\ &= \int_{[-1,1] \times \mathbb{R}^2} \frac{f^p(u-v-w)f^p(v)f^p(w)}{|u|^{4(p-1)}} du dv dw \\ &\quad + \int_{[-1,1]^c \times \mathbb{R}^2} \frac{f^p(u-v-w)f^p(v)f^p(w)}{|u|^{4(p-1)}} du dv dw \\ &\leq \|f\|_{\infty}^p \int_{[-1,1] \times \mathbb{R}^2} \frac{f^p(v)f^p(w)}{|u|^{4(p-1)}} du dv dw \\ &\quad + \int_{[-1,1]^c \times \mathbb{R}^2} f^p(u-v-w)f^p(v)f^p(w) du dv dw. \end{aligned}$$

Fubini's theorem implies that

$$I_1 \leq \|f\|_{\infty}^p \left(\int_{-1}^1 \frac{du}{|u|^{4(p-1)}} \right) \left(\int_{\mathbb{R}} f^p(x) dx \right)^2 + \left(\int_{\mathbb{R}} f^p(x) dx \right)^3.$$

1. Actually it is proposition 16 of to the ARXIV version of [1].

We have

$$\int_{\mathbb{R}} f^p(x) dx = \int_{\mathbb{R}} f^{p-1}(x)f(x) dx \leq \|f\|_{\infty}^{p-1} \int_{\mathbb{R}} f(x) dx < +\infty.$$

Thus $I_1 < +\infty$ as soon as $p < 5/4$, since the function $u \mapsto |u|^{4(1-p)}$ is then integrable on $[-1, 1]$. We show similarly that $I_2 < +\infty$ as soon as $p < 5/4$. Hence

$$\forall p \in [1, 5/4[\quad \int_{\mathbb{R}^3} \frac{f^p(x)f^p(y)f^p(z)}{|(x+y+z)(x-y)(y-z)(z-x)|^p} dx dy dz < +\infty.$$

b) Proof of theorem 1

We prove in fact a more general fluctuation theorem than theorem 1.

Let ρ be a symmetric probability measure on \mathbb{R} and let H be a real-valued function defined on Θ^* such that, for any $n \geq 1$,

$$\begin{aligned} Z_{H,n} = \int_{\mathbb{R}^n} \exp\left(H\left(x_1 + \dots + x_n, x_1^2 + \dots + x_n^2, x_1^4 + \dots + x_n^4\right)\right) \\ \times \mathbf{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i) < +\infty. \end{aligned}$$

We consider $(X_n^k)_{1 \leq k \leq n}$ an infinite triangular array of real-valued random variables such that, for all $n \geq 1$, (X_n^1, \dots, X_n^n) has the law $\tilde{\mu}_{H,n,\rho}$, which is the distribution with density

$$\frac{1}{Z_{H,n}} \exp\left(H\left(x_1 + \dots + x_n, x_1^2 + \dots + x_n^2, x_1^4 + \dots + x_n^4\right)\right) \mathbf{1}_{\{x_1^2 + \dots + x_n^2 > 0\}}$$

with respect to $\rho^{\otimes n}$. We denote

$$S_n = X_n^1 + \dots + X_n^n, \quad T_n = (X_n^1)^2 + \dots + (X_n^n)^2 \quad \text{and} \quad U_n = (X_n^1)^4 + \dots + (X_n^n)^4.$$

We have the following general fluctuation theorem :

Theorem 13. *Let ρ be a symmetric probability measure on \mathbb{R} whose support contains at least five points and such that*

$$\exists w_0 > 0 \quad \int_{\mathbb{R}} e^{w_0 z^4} d\rho(z) < +\infty.$$

We denote by σ^2 the variance of ρ , by μ_4 its fourth moment, by μ_6 its sixth moment and by μ_8 its eighth moment. We assume that

$$5\mu_4^2 > 2\sigma^2\mu_6.$$

Suppose that H satisfies the hypothesis 6 and that ρ and H fulfill either the hypothesis 9 or the hypothesis 10. Then, under $\tilde{\mu}_{H,n,\rho}$, $(S_n/n, T_n/n, U_n/n)$ converges in probability towards $(0, \sigma^2, \mu_4)$.

Moreover, if ρ has a density f with respect to the Lebesgue measure on \mathbb{R} such that, for some $p \in]1, 2]$,

$$\int_{\mathbb{R}^3} \frac{f^p(x)f^p(y)f^p(z)}{|(x+y+z)(x-y)(y-z)(z-x)|^{p-1}} dx dy dz < +\infty, \quad (*)$$

then, under $\tilde{\mu}_{H,n,\rho}$

$$\left(\frac{\mu_4^2}{\sigma^2} - \frac{2\mu_6}{5}\right)^{1/6} \frac{S_n}{\sigma^2 n^{5/6}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\frac{81}{2}\right)^{1/6} \Gamma\left(\frac{1}{6}\right)^{-1} \exp\left(-\frac{s^6}{18}\right) ds.$$

We proved in the previous subsection that, if f is bounded, then it satisfies (*) for any $p \in]1, 5/4[$. We have also proved in section 4.c) that the function

$$H : (x, y, z) \in \Theta^* \mapsto \frac{x^2}{2y} + \frac{1}{12} \frac{zx^4 y^5}{y^9 + x^{10} + zx^4 y^4}$$

satisfies the hypothesis 6 and 10. Hence theorem 1 is a consequence of this theorem.

Considering our article [4], we could prove this fluctuation theorem for ρ having an absolutely continuous component (and not necessarily having a density which satisfies (*)) or, more generally, for ρ satisfying the Cramér condition

$$\forall \alpha > 0 \quad \sup_{\|(s,t,u)\| \geq \alpha} \left| \int_{\mathbb{R}} e^{isz + itz^2 + iuz^4} d\rho(z) \right| < 1. \quad (C)$$

However the proof would be much more technical.

Proof of proposition 13. We denote by $\theta_{\bullet,n,\rho}$ the law of $(S_n/n, T_n/n, U_n/n)$ under $\tilde{\mu}_{\bullet,n,\rho}$. Let U be an open neighbourhood of $(0, \sigma^2, \mu_4)$ in \mathbb{R}^3 . Suppose that ρ and H satisfy the hypothesis 6 and also either the hypothesis 9 or the hypothesis 10. Then proposition 11 implies that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \theta_{\bullet,n,\rho}(U^c) &= \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Theta^* \cap U^c} e^{nF(x,y) + nR_n(x,y,z)} d\tilde{\mu}_{\bullet,n,\rho}(x, y, z) \\ &\quad - \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Theta^*} e^{nF(x,y) + nR_n(x,y,z)} d\tilde{\mu}_{\bullet,n,\rho}(x, y, z) < 0. \end{aligned}$$

Hence there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for any $n > n_0$,

$$\theta_{\bullet,n,\rho}(U^c) \leq e^{-n\varepsilon} \xrightarrow[n \rightarrow \infty]{} 0.$$

Thus, for each open neighbourhood U of $(0, \sigma^2, \mu_4)$,

$$\lim_{n \rightarrow +\infty} \tilde{\mu}_{\bullet,n,\rho} \left(\left(\frac{S_n}{n}, \frac{T_n}{n}, \frac{U_n}{n} \right) \in U^c \right) = 0.$$

This means that, under $\tilde{\mu}_{\bullet,n,\rho}$, $(S_n/n, T_n/n, U_n/n)$ converges in probability to $(0, \sigma^2, \mu_4)$.

Next, in section 4.a), we proved that, in the neighbourhood of $(0, \sigma^2, \mu_4)$,

$$I_{\bullet}(x, y, z) - F(x, y) - R(x, y, z) \sim \frac{(5\mu_4^2 - 2\sigma^2\mu_6)x^6}{90\sigma^{14}} + q(y - \sigma^2, z - \mu_4).$$

Thus, there exists $\delta > 0$ such that, for any $(x, y, z) \in B_{\delta}$,

$$(I_{\bullet} - F - R)(x, y, z) \geq \frac{(5\mu_4^2 - 2\sigma^2\mu_6)x^6}{180\sigma^{14}} + \frac{1}{2}q(y - \sigma^2, z - \mu_4),$$

where B_δ denotes the open ball of radius δ centered at $(0, \sigma^2, \mu_4)$. We have $R_n \leq R$ for any $n \geq 1$, thus, for any $(x, y, z) \in B_\delta$ and $n \geq 1$,

$$(I_\bullet - F - R_n)(x, y, z) \geq \frac{(5\mu_4^2 - 2\sigma^2\mu_6)x^6}{180\sigma^{14}} + \frac{1}{2}q(y - \sigma^2, z - \mu_4). \quad (4)$$

We can reduce δ , in order to have $B_\delta \subset K$ where K is a compact subset of A_{I_\bullet} so that $B_\delta \subset A_{I_\bullet} \subset \Theta^*$.

Let $n \geq 1$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded continuous function. We have

$$\begin{aligned} \mathbb{E}_{\tilde{\mu}_{n,\rho}} \left(f \left(\frac{S_n}{n^{5/6}} \right) \right) &= \frac{1}{Z_{H,n}} \int_{\Theta^*} f(xn^{1/6}) e^{nF(x,y) + nR_n(x,y,z)} d\tilde{\nu}_{\bullet,n,\rho}(x, y, z) \\ &= \frac{A_n + B_n}{Z_{H,n}}, \end{aligned}$$

with

$$\begin{aligned} A_n &= \int_{B_\delta} f(xn^{1/6}) e^{nF(x,y) + nR_n(x,y,z)} d\tilde{\nu}_{\bullet,n,\rho}(x, y, z), \\ B_n &= \int_{\Theta^* \cap B_\delta^c} f(xn^{1/6}) e^{nF(x,y) + nR_n(x,y,z)} d\tilde{\nu}_{\bullet,n,\rho}(x, y, z). \end{aligned}$$

Suppose in addition that ρ has a density f with respect to the Lebesgue measure on \mathbb{R} such that, for some $p \in]1, 2]$, the function

$$(x, y, z) \mapsto \frac{f^p(x)f^p(y)f^p(z)}{|(x+y+z)(x-y)(y-z)(z-x)|^{p-1}}$$

is integrable. Then proposition 12 implies that, for n large enough, $\tilde{\nu}_{\bullet,n,\rho}$ has a density g_n with respect to the Lebesgue measure on \mathbb{R}^3 . Let us introduce the factor $e^{-nI_\bullet(x,y,z)}$ in the expression of A_n :

$$A_n = n^{3/2} \int_{B_\delta} f(xn^{1/6}) e^{-nG_n(x,y,z)} H_n(x, y, z) dx dy dz,$$

where $G_n = I_\bullet - F - R_n$ and $H_n : (x, y, z) \mapsto e^{nI_\bullet(x,y,z)} g_n(x, y, z) / n^{3/2}$. We define

$$B_{\delta,n} = \{ (x, y, z) \in \mathbb{R}^3 : x^2/n^{1/3} + y^2/n + z^2/n \leq \delta^2 \}.$$

Let us make the change of variables given by

$$(x, y, z) \mapsto (xn^{-1/6}, yn^{-1/2} + \sigma^2, zn^{-1/2} + \mu_4),$$

with Jacobian $n^{-7/6}$:

$$\begin{aligned} A_n &= n^{1/3} \int_{B_{\delta,n}} f(x) \exp \left(-nG_n \left(\frac{x}{n^{1/6}}, \frac{y}{\sqrt{n}} + \sigma^2, \frac{z}{\sqrt{n}} + \mu_4 \right) \right) \\ &\quad \times H_n \left(\frac{x}{n^{1/6}}, \frac{y}{\sqrt{n}} + \sigma^2, \frac{z}{\sqrt{n}} + \mu_4 \right) dx dy dz. \end{aligned}$$

We check now that we can apply the dominated convergence theorem to this integral. The uniform expansion of g_n given by proposition 12 means that for any $\alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$,

$$\forall (x, y, z) \in K \quad \left| H_n(x, y, z) (2\pi)^{3/2} \left(\det D_{(x,y,z)}^2 I_\bullet \right)^{-1/2} - 1 \right| \leq \alpha.$$

If $(x, y, z) \in B_{\delta, n}$, then $(x_n, y_n, z_n) = (xn^{-1/6}, yn^{-1/2} + \sigma^2, zn^{-1/2} + \mu_4) \in B_{\delta}$, which is included in K . Thus for all $n \geq n_0$ and $(x, y, z) \in B_{\delta, n}$,

$$\left| H_n(x_n, y_n, z_n) (2\pi)^{3/2} \left(\det D_{(x_n, y_n, z_n)}^2 I_{\bullet} \right)^{-1/2} - 1 \right| \leq \alpha.$$

Moreover $(x_n, y_n, z_n) \rightarrow (0, \sigma^2, \mu_4)$ thus, by continuity,

$$\left(D_{(x_n, y_n, z_n)}^2 I_{\bullet} \right)^{-1/2} \xrightarrow{n \rightarrow +\infty} \left(D_{(0, \sigma^2)}^2 I_{\bullet} \right)^{-1/2} = \left(D_{(0, 0)}^2 \Lambda_{\bullet} \right)^{1/2},$$

whose determinant is $\sqrt{\sigma^2 a}$, with $a = (\mu_4 - \sigma^4)(\mu_8 - \mu_4^2) - (\mu_6 - \sigma^2 \mu_4)^2$. It is positive according to lemma 3. Therefore

$$\mathbb{1}_{B_{\delta, n}}(x, y, z) H_n(x_n, y_n, z_n) \xrightarrow{n \rightarrow +\infty} \left((2\pi)^3 \sigma^2 a \right)^{-1/2}.$$

Next, proposition 7 implies that, for any $(x, y, z) \in \mathbb{R}^3$,

$$\exp(-nG_n(x_n, y_n, z_n)) \xrightarrow{n \rightarrow +\infty} \exp\left(-q(y, z) - \frac{5\mu_4^2 - 2\sigma^2 \mu_6}{90\sigma^{14}} x^6\right).$$

Let us check that the integrand is dominated by an integrable function, which is independent of n . The function

$$(x, y, z) \mapsto \left(D_{(x, y, z)}^2 I_{\bullet} \right)^{-1/2}$$

is bounded on B_{δ} by some $M_{\delta} > 0$. The uniform expansion of g_n implies that for all $(x, y, z) \in B_{\delta}$, $H_n(x, y, z) \leq C_{\delta}$ for some constant $C_{\delta} > 0$. Finally, the inequality (4) above yields that, for any $(x, y, z) \in \mathbb{R}^3$,

$$\begin{aligned} \mathbb{1}_{B_{\delta, n}}(x, y, z) f(x) \exp(-nG_n(x_n, y_n, z_n)) H_n(x_n, y_n, z_n) \\ \leq \|f\|_{\infty} C_{\delta} \exp\left(-\frac{1}{2}q(y, z) - \frac{5\mu_4^2 - 2\sigma^2 \mu_6}{180\sigma^{14}} x^6\right). \end{aligned}$$

The right term defines an integrable function on \mathbb{R}^3 , thus it follows from the dominated convergence theorem that

$$A_n \underset{+\infty}{\sim} n^{1/3} \int_{\mathbb{R}^3} \frac{f(x)}{\sqrt{2\pi\sigma^2}(2\pi\sqrt{a})} \exp\left(-q(y, z) - \frac{5\mu_4^2 - 2\sigma^2 \mu_6}{90\sigma^{14}} x^6\right) dx dy dz.$$

Fubini's theorem implies that, for some constant $k > 0$,

$$A_n \underset{+\infty}{\sim} kn^{1/3} \int_{\mathbb{R}} f(x) \exp\left(-\frac{5\mu_4^2 - 2\sigma^2 \mu_6}{90\sigma^{14}} x^6\right) dx.$$

Let us focus now on B_n . Proposition 11 implies that there exists $\varepsilon > 0$ such that, for n large enough, $B_n \leq \|f\|_{\infty} e^{-n\varepsilon}$ and thus $B_n = o(n^{1/3})$. Therefore

$$A_n + B_n \underset{+\infty}{\sim} kn^{1/3} \int_{\mathbb{R}} f(x) \exp\left(-\frac{5\mu_4^2 - 2\sigma^2 \mu_6}{90\sigma^{14}} x^6\right) dx.$$

Applying this to $f = 1$, we get

$$Z_n \underset{+\infty}{\sim} kn^{1/3} \int_{\mathbb{R}} \exp\left(-\frac{5\mu_4^2 - 2\sigma^2\mu_6}{90\sigma^{14}}x^6\right) dx = 3kn^{1/3} \left(\frac{5\mu_4^2 - 2\sigma^2\mu_6}{90\sigma^{14}}\right)^{-1/6} \Gamma\left(\frac{1}{6}\right).$$

Finally

$$\mathbb{E}_{\tilde{\mu}_{n,\rho}}\left(f\left(\frac{S_n}{n^{3/4}}\right)\right) \underset{+\infty}{\sim} \frac{\int_{\mathbb{R}} f(x) \exp\left(-\frac{5\mu_4^2 - 2\sigma^2\mu_6}{90\sigma^{14}}x^6\right) dx}{3 \left(\frac{5\mu_4^2 - 2\sigma^2\mu_6}{90\sigma^{14}}\right)^{-1/6} \Gamma\left(\frac{1}{6}\right)}.$$

The ultimate change of variables $y = (5\mu_4^2 - 2\sigma^2)^{1/6}x/(5\sigma^{14})^{1/6}$ gives us theorem 13. \square

6 Fluctuations of order $n^{1-1/2k}$?

Let $k \geq 4$. We denote by $I_{\bullet,k}$ the Cramér transform of $(Z, Z^2, Z^4, \dots, Z^{2k-2})$, where Z is a random variable with distribution ρ . We would like to find a large class of probability measures ρ on \mathbb{R} such that :

★ There exists an interacting function H_k from \mathbb{R}^k to \mathbb{R} such that, for any $n \geq 1$, the function

$$G_{n,k} : (y_1, y_2, \dots, y_{2k-2}) \mapsto I_{\bullet,k}(y_1, y_2, \dots, y_{2k-2}) - \frac{1}{n}H_k(ny_1, ny_2, \dots, ny_{2k-2})$$

admits a unique minimum at $(0, \sigma^2, \mu_4, \dots, \mu_{2k-2})$, where $\sigma^2, \mu_4, \dots, \mu_{2k-2}$ are the successive moments of ρ .

★ For any $n \geq 1$, we denote by $Z_{n,k}$ the integral

$$\int_{\mathbb{R}^n} \exp\left(H_k\left(x_1 + \dots + x_n, x_1^2 + \dots + x_n^2, \dots, x_1^{2k} + \dots + x_n^{2k}\right)\right) \prod_{i=1}^n d\rho(x_i)$$

and we suppose it is finite.

★ There exist $A_k > 0$ and a function q_k from \mathbb{R}^{k-1} to \mathbb{R} satisfying

$$\int_{\mathbb{R}} e^{-q_k(y_2, \dots, y_{2k-2})/2} dy_2 \dots dy_{2k-2} < +\infty$$

such that, in a neighbourhood of $(0, \sigma^2, \mu_4, \dots, \mu_{2k-2})$,

$$G_{n,k}(y_1, y_2, \dots, y_{2k-2}) \sim A_k y_1^{2k} + q_k(y_2 - \sigma^2, \dots, y_{2k-2} - \mu_{2k-2}).$$

In this case, we consider $(X_k^n)_{1 \leq k \leq n}$ an infinite triangular array of real-valued random variables such that, for all $n \geq 1$, (X_n^1, \dots, X_n^n) has the distribution

$$\frac{1}{Z_{n,k}} \exp\left(H_k\left(x_1 + \dots + x_n, x_1^2 + \dots + x_n^2, \dots, x_1^{2k} + \dots + x_n^{2k}\right)\right) \prod_{i=1}^n d\rho(x_i).$$

We denote $S_n = X_1^n + \dots + X_n^n$ for any $n \geq 1$. By using arguments as in the last sections, we could prove the following :

$$\frac{S_n}{n^{1-1/2k}} \underset{n \rightarrow \infty}{\mathcal{L}} \left(\int_{\mathbb{R}} \exp(-A_k y^{2k}) dy \right)^{-1} \exp(-A_k x^{2k}) dx.$$

Unfortunately the proof of such a result does not seem to be possible with the techniques we employed in this paper, for several reasons :

★ In order to obtain the expansion of $G_{n,k}$, in the case $k = 2$ or 3 , we made very long and tedious computations. Of course we could repeat these computations for $k = 4$, then $k = 5$, ... But it would be very complicated and this is not reasonable if we do not find a simple way to determine the variable A_k for any $k \geq 4$. Moreover we have not understood why, for $k = 2, 3$, the terms « we do not want » in the expansion of $G_{n,k}$ vanish.

★ For $k = 3$, there are probability measures such that A_k is negative. In the same way, there may exist $k_0 \geq 4$ such that $A_{k_0} < 0$ for any probability measure. In this case, G_{n,k_0} could not admit a minimum at $(0, \sigma^2, \mu_4, \dots, \mu_{2k-2})$ and we should find new criteria on H_{k_0} to solve this problem.

★ With the « natural » interacting function in the case $k = 3$, we have not managed to prove that $G_{n,3}$ has a unique minimum at $(0, \sigma^2, \mu_4)$ (while our simulations tend to conjecture this is true). We had to force the interacting function to have the desired behaviour by finding some suitable criteria. Moreover the candidate we propose for H is rather complicated. We also failed to make convincing computer simulations with our modified model (although it is amenable to mathematical analysis) : the convergence is too slow because

$$n(R - R_n) \left(\frac{x_1 + \dots + x_n}{n}, \frac{x_1^2 + \dots + x_n^2}{n}, \dots, \frac{x_1^{2k-2} + \dots + x_n^{2k-2}}{n} \right)$$

becomes negligible only for very large n .

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