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A Curie-Weiss Model of Self-Organized Criticality : The Gaussian Case

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Abstract

We try to design a simple model exhibiting self-organized criticality, which is amenable to a rigorous mathematical analysis. To this end, we modify the generalized Ising Curie-Weiss model by implementing an automatic control of the inverse temperature. With the help of exact computations, we show that, in the case of a centered Gaussian measure with positive variance σ^2, the sum S_n of the random variables has fluctuations of order $n^{3/4}$ and that $S_n/n^{3/4}$ converges to the distribution $C\exp(-x^4/(4\sigma^4))\,dx$ where C is a suitable positive constant.

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1 Introduction

In their famous article [4], Per Bak, Chao Tang and Kurt Wiesenfeld showed that certain complex systems are naturally attracted by critical points, without any external intervention. The amplification of small internal fluctuations can lead to a critical state and cause a chain reaction leading to a radical change of the system behavior. These systems exhibit the phenomenon of self-organized criticality (SOC). Although there is no universal SOC theory, it can be well understood with the archetype of SOC : the sandpile model, first introduced in [4]. We consider a pile of sand and the constant drop of new sand grains, which randomly slide down the slope of sand. We observe local avalanches with different and unpredictable sizes which are not proportional to the input. Such phenomenon can be observed in nature (e.g., forest fires, earthquakes, species evolution). In general SOC can be observed empirically or simulated on a computer in various models. However the mathematical analysis of these models turns out to be extremely difficult, even for the sandpile model whose definition is yet simple. Self-organized criticality have been reviewed in recent works [1,2,7,11,15]. Other challenging models are the models for forest fires [12], which are built with the help of percolation process. Some simple models of evolutions also lead to critical behaviours [6].

Our goal here is to design a model exhibiting SOC, which is as simple as possible, and which is amenable to a rigorous mathematical analysis. The simplest models exhibiting self-organized criticality can be obtained by forcing standard critical transitions into a self-organized state (see [14] section 15.4.2). The idea is to start with the Ising Curie-Weiss model (see [8]), which presents a phase transition, and to create a feedback from the configuration to the control parameters in order to converge towards a critical point.

The generalized Ising Curie-Weiss model (see [9]) associated to a probability measure ρ on $\mathbb{R}$ (with some « sub-Gaussian » conditions) and the inverse temperature $\beta > 0$ is defined through an infinite triangular array of real-valued random variables $(X_n^k)_{1\leq k\leq n}$ such that, for all $n \geq 1$, $(X_n^1,\dots,X_n^n)$ has the distribution

$$\frac{1}{Z_n(\beta)}\exp\left(\frac{\beta}{2}\frac{(x_1+\cdots+x_n)^2}{n}\right)\prod_{i=1}^{n}d\rho(x_i),$$

where $Z_n(\beta)$ is a normalization. For any $n\geq 1$, we set $S_n = X_n^1+\cdots+X_n^n$. Let σ^2 be the variance of ρ. Ellis and Newman [9] have proved the following result. If $\beta < 1/\sigma^2$, then the fluctuations of S_n are of order $\sqrt{n}$ and $S_n/\sqrt{n}$ converges towards a specific Gaussian distribution. If $\beta = 1/\sigma^2$, then the fluctuations of S_n are of order $n^{1-1/2k}$, where k is an integer depending on the distribution ρ. The point $1/\sigma^2$ is the critical value of the generalized Ising Curie-Weiss model.

In order to obtain a model which presents SOC we transform the previous probability distribution by « replacing β by $n\,(x_1^2+\cdots+x_n^2)^{-1}$ ». Hence the model we consider is given by the distribution

$$\frac{1}{Z_n}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}\right)\prod_{i=1}^{n}d\rho(x_i).$$

We refer to [5] for a more detailed construction. This model can be defined for any distribution ρ, in particular for any Gaussian measure (contrary to the generalized Ising Curie-Weiss model). In this paper, we consider the case where ρ is the centered Gaussian measure with variance σ^2 and we show that $S_n/n^{3/4}$ converges to the distribution

$$\left(\frac{\sigma}{\sqrt{2}}\,\Gamma\left(\frac{1}{4}\right)\right)^{-1}\exp\left(-\frac{x^4}{4\sigma^4}\right)\,dx.$$

This fluctuation result shows that our model is a self-organized model exhibiting critical behaviour. Indeed it has the same behaviour than the critical generalized Ising Curie-Weiss model and, by construction, it does not depend on any external parameter. In this sense, we can conclude that this is a Curie-Weiss model of self-organized criticality.

The proof of this fluctuation result relies on the study of the sums S_n and $T_n=(X_n^1)^2+\cdots+(X_n^n)^2$. With the help of the Fourier analysis, we compute that the law of (S_n,T_n) has the density

$$(x,y)\longmapsto\frac{1}{\sigma^nC_n}\exp\left(\frac{x^2}{2y}-\frac{y}{2\sigma^2}\right)\left(y-\frac{x^2}{n}\right)^{(n-3)/2}\mathbb{1}_{x^2<ny}$$

with respect to the Lebesgue measure on $\mathbb{R}^2$, where $C_n>0$ is an adequate normalization constant. It is then straightforward to compute the expansion of this density function around its minimum $(0,\sigma^2)$ and we conclude with the help of Laplace's method. However the computations we make here are not possible for more general probability measures. In [5] we consider a class of distributions having an even density with respect to the Lebesgue measure and satisfying some integrability conditions and we prove a similar convergence result with more technical methods using the Cramér theory. These methods are more robust, but in this more general situation, we do not have the nice formulas available in the Gaussian case. The Gaussian case we handle here is the nicest instance of our model and it will serve to examine other questions on this model.

In section 2 we define properly our model for Gaussian measures and we state our main result. The proof is split in the two remaining sections.

2 Main result

The model. We denote by ρ_σ the Gaussian distribution with mean 0 and variance $\sigma^2>0$. We consider $(X_n^k)_{1\le k\le n}$ an infinite triangular array of real-valued random variables such that, for all $n\ge 1$, $(X_n^1,\dots,X_n^n)$ has the distribution $\widetilde{\mu}_{n,\sigma}$, where

$$\begin{aligned}d\widetilde{\mu}_{n,\sigma}(x_1,\dots,x_n)&=\frac{1}{Z_n}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}\right)\prod_{i=1}^n d\rho_\sigma(x_i)\\&=\frac{1}{(2\pi\sigma^2)^{n/2}Z_n}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}-\frac{x_1^2+\cdots+x_n^2}{2\sigma^2}\right)\prod_{i=1}^n dx_i\end{aligned}$$

and

$$Z_n=\int_{\mathbb{R}^n}(2\pi)^{-n/2}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}-\frac{1}{2}(x_1^2+\cdots+x_n^2)\right)\prod_{i=1}^n dx_i.$$

We define $S_n = X_n^1 + \cdots + X_n^n$ and $T_n = (X_n^1)^2 + \cdots + (X_n^n)^2$.

We notice that the event $\{x_1^2 + \cdots + x_n^2 = 0\}$ is negligible for the measure $\rho_\sigma^{\otimes n}$, so that the denominator in the exponential is almost surely positive. Moreover, $t \longmapsto t^2$ is a convex function, thus for any $n \geq 1$, $1 \leq Z_n \leq e^{n/2} < +\infty$.

Theorem 1. *Under $\widetilde{\mu}_{n,\sigma}$, $(S_n/n, T_n/n)$ converges in probability towards $(0, \sigma^2)$. Moreover*

$$\frac{S_n}{n^{3/4}} \xrightarrow[n\to+\infty]{\mathcal{L}} \left(\frac{\sigma}{\sqrt{2}} \, \Gamma\left(\frac{1}{4}\right) \right)^{-1} \exp\left(-\frac{y^4}{4\sigma^4} \right) \, dx.$$

To prove this theorem, we first compute, in section 3, the exact density of the law of (S_n, T_n) under $\widetilde{\mu}_{n,\sigma}$, for n large enough. Next, in section 4, we end the proof by using Laplace's method.

3 Computation of the law of (S_n, T_n)

In this section we compute the law of (S_n, T_n) under $\widetilde{\mu}_{n,\sigma}$.

Lemma 2. *We denote by ν_σ the law of (Z, Z^2) where Z is a Gaussian random variable with mean 0 and variance $\sigma^2 > 0$. Under $\widetilde{\mu}_{n,\sigma}$, the law of (S_n, T_n) is*

$$\frac{1}{Z_n} \exp\left(\frac{x^2}{2y} \right) \, d\nu_\sigma^{*n}(x, y).$$

Proof. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bounded measurable function. We have

$$\mathbb{E}_{\tilde{\mu}_{n,\sigma}}(f(S_n, T_n)) = \frac{1}{Z_n} \int_{\mathbb{R}^n} f(x_1 + \cdots + x_n, x_1^2 + \cdots + x_n^2) \exp\left(\frac{1}{2} \frac{(x_1 + \cdots + x_n)^2}{x_1^2 + \cdots + x_n^2} \right) \prod_{i=1}^n d\rho_\sigma(x_i).$$

The function $h : (x, y) \in \mathbb{R} \times \mathbb{R}\backslash\{0\} \longmapsto f(x, y) \exp(x^2/(2y))$ is measurable. Therefore

$$\begin{aligned} \mathbb{E}_{\tilde{\mu}_{n,\rho}}(f(S_n, T_n)) &= \frac{1}{Z_n} \int_{\mathbb{R}^n} h(x_1 + \cdots + x_n, x_1^2 + \cdots + x_n^2) \prod_{i=1}^n d\rho_\sigma(x_i) \\ &= \frac{1}{Z_n} \int_{\mathbb{R}^{2n}} h(z_1 + \cdots + z_n) \prod_{i=1}^n d\nu_\sigma(z_i) = \frac{1}{Z_n} \int_{\mathbb{R}^2} h(z) \, d\nu_\sigma^{*n}(z). \end{aligned}$$

Hence the announced law of (S_n, T_n), under $\widetilde{\mu}_{n,\sigma}$. □

We denote by Γ the gamma function defined by

$$\forall z > 0 \qquad \Gamma(z) = \int_0^{+\infty} x^{z-1} e^{-x} dx.$$

We compute next the density of ν_σ^{*n}:

Proposition 3. *For $n \geq 5$, under $\widetilde{\mu}_{n,\sigma}$, the law of (S_n, T_n) is*

$$\frac{1}{\sigma^n C_n} \exp\left(\frac{x^2}{2y} - \frac{y}{2\sigma^2}\right)\left(y - \frac{x^2}{n}\right)^{(n-3)/2} \mathbb{1}_{x^2<ny}\, dx\, dy,$$

where $C_n = Z_n \sqrt{2^n \pi n}\, \Gamma((n-1)/2)$.

For simplicity, we assume that $\sigma^2 = 1$. We just write ν^{*n} for ν_σ^{*n}. We denote by Φ_n its characteristic function. To get the previous proposition, we use the method of residue to compute ν^{*n} and a Fourier inversion formula to get the density of ν^{*n}. For $(u, v) \in \mathbb{R}^2$, we have

$$\Phi_n(u, v) = (\Phi_1(u, v))^n = \left(\mathbb{E}(e^{iuZ+ivZ^2})\right)^n = \left(\int_{\mathbb{R}} e^{iux+ivx^2} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}\right)^n.$$

We need some preliminary results:

The Gamma distribution with shape $k > 0$ and scale $\theta > 0$, denoted by $\Gamma(k, \theta)$, is the probability distribution with density function

$$x \longmapsto \frac{x^{k-1} e^{-x/\theta}}{\Gamma(k)\, \theta^k} \mathbb{1}_{x>0}$$

with respect to the Lebesgue measure on $\mathbb{R}$.

The complex logarithm function (or the principle value of complex logarithm), denoted by Log, is defined on $\Omega = \mathbb{C} \backslash]-\infty, 0]$ by

$$\forall z = x + iy \in \Omega \qquad \mathrm{Log}(z) = \frac{1}{2}\ln(x^2 + y^2) + 2i \arctan\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right).$$

If $\alpha \in \mathbb{C}$ and $z \in \Omega$, then the α-exponentiation of z is defined by

$$z^\alpha = \exp(\alpha \mathrm{Log}(z)).$$

By chapter XV of [10], for $k, \theta > 0$, the characteristic function of $\Gamma(k, \theta)$ is

$$u \in \mathbb{R} \longmapsto (1 - \theta i u)^{-k}.$$

We can now prove the following key lemma:

Lemma 4. *Let $t \in \mathbb{R}$ and $\zeta \in \mathbb{C}$ such that $\mathfrak{Re}(\zeta) > 0$. Then*

$$\int_{\mathbb{R}} e^{itx - \zeta x^2/2}\, dx = \sqrt{\frac{2\pi}{\mathfrak{Re}(\zeta)}} \exp\left(-\frac{t^2}{2\zeta}\right)\left(1 + i\,\frac{\mathfrak{Im}(\zeta)}{\mathfrak{Re}(\zeta)}\right)^{-1/2}.$$

Proof. Let $t \in \mathbb{R}$ and $\zeta = a + ib \in \mathbb{C}$ such that $\mathfrak{Re}(\zeta) > 0$. We define

$$K(t, \zeta) = \int_{\mathbb{R}} e^{itx - \zeta x^2/2}\, dx.$$

We factorize:

$$ixt - \frac{1}{2}\zeta x^2 = -\frac{1}{2}\zeta\left(x - \frac{it}{\zeta}\right)^2 - \frac{t^2}{2\zeta} = -\frac{1}{2}\zeta\left(x - \frac{tb}{|\zeta|} - i\frac{ta}{|\zeta|}\right)^2 - \frac{t^2}{2\zeta}.$$

Thus

$$e^{t^2/2\zeta}K(t,\zeta) = \int_{\mathbb{R}} e^{-\zeta(x-tb/|\zeta|-ita/|\zeta|)^2/2}\,dx.$$

The change of variables $y = x - tb/|\zeta|$ gives us

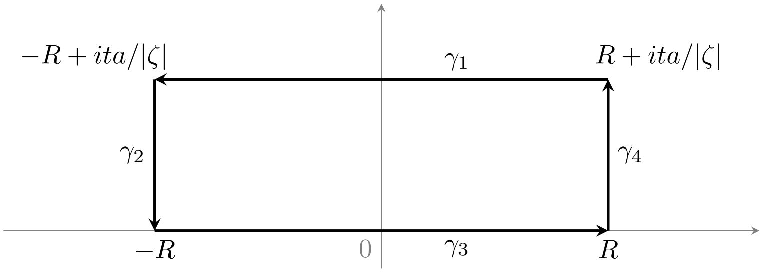
$$e^{t^2/2\zeta}K(t,\zeta) = \int_{\mathbb{R}} e^{-\zeta(y-ita/|\zeta|)^2/2}\,dy = -\lim_{R\to+\infty}\int_{\gamma_1} e^{-\zeta z^2/2}\,dz,$$

where the last integral is the contour integral of the entire function $z \longmapsto e^{-\zeta z^2/2}$, along the segment γ_1 in the complex plane with end points $R + ita/|\zeta|$ and $-R + ita/|\zeta|$.

Let γ be the rectangle in the complex plane joining successively the points $R + ita/|\zeta|$, $-R + ita/|\zeta|$, $-R$ and R. We apply the residue theorem:

$$\int_{\gamma} e^{-\zeta z^2/2}\,dz = 0$$

since $z \longmapsto \exp(-\zeta z^2/2)$ has no pole (see [13]). We denote γ_1, γ_2, γ_3 and γ_4 the successive edges of the rectangle γ.



$$\int_{\gamma_3} e^{-\zeta z^2/2}\,dz = \int_{-R}^{R} e^{-\zeta x^2/2}\,dx \underset{R\to+\infty}{\longrightarrow} \int_{\mathbb{R}} e^{-\zeta x^2/2}\,dx = 2\int_0^{+\infty} e^{-\zeta x^2/2}\,dx.$$

We make the change of variables $y = x^2$ on $]0,+\infty[$:

$$\begin{aligned} 2\int_0^{+\infty} e^{-\zeta x^2/2}\,dx = \int_0^{+\infty} e^{-\zeta y/2}\frac{dy}{\sqrt{y}} &= \int_0^{+\infty} e^{-iby/2}e^{-ay/2}\frac{dy}{\sqrt{y}} \\ &= \sqrt{\frac{2}{a}}\,\Gamma\left(\frac{1}{2}\right)\left(1+i\frac{b}{a}\right)^{-1/2} \end{aligned}$$

since we recognize, up to a normalization factor, the characteristic function of the Gamma distribution with shape $1/2$ and scale $2/a$. Moreover we have

$$\begin{aligned} \left|\int_{\gamma_4} e^{-\zeta z^2/2}\,dz\right| &= \left|\int_0^1 \exp\left(-\frac{\zeta}{2}\left(R+\frac{iat}{|\zeta|}x\right)^2\right)\frac{iat}{|\zeta|}\,dx\right| \\ &\le \frac{a|t|}{|\zeta|}\int_0^1 \exp\left(-\frac{aR^2}{2}+\frac{Ratbx}{|\zeta|}+\frac{a}{2}\left(\frac{atx}{|\zeta|}\right)^2\right)dx \\ &\le \frac{a|t|}{|\zeta|}\exp\left(-\frac{aR^2}{2}+\frac{Ra|tb|}{|\zeta|}+\frac{a}{2}\left(\frac{at}{|\zeta|}\right)^2\right) \underset{R\to+\infty}{\longrightarrow} 0. \end{aligned}$$

Likewise

$$\int_{\gamma_2} e^{-\zeta z^2/2}\,dz \underset{R\to+\infty}{\longrightarrow} 0.$$

Letting R go to $+\infty$, we conclude that

$$\sqrt{\frac{2}{a}}\,\Gamma\left(\frac{1}{2}\right)\left(1+i\frac{b}{a}\right)^{-1/2} + 0 - e^{t^2/2\zeta}K(t,\zeta) + 0 = 0.$$

Since $\Gamma(1/2)=\sqrt{\pi}$, we obtain the identity stated in the lemma. □

For $(u,v)\in\mathbb{R}^2$, setting $\zeta = 1-2iv \in \{\, z\in\mathbb{C} : \mathfrak{Re}(z)>0\,\}$, we have

$$\Phi_n(u,v) = \frac{1}{(2\pi)^{n/2}}\left(\int_{\mathbb{R}} e^{iux-\zeta x^2/2}\,dx\right)^n.$$

Applying lemma 4 with u and ζ, we obtain the following proposition:

Proposition 5. *The characteristic function Φ_n of the distribution ν^{*n} is*

$$(u,v)\in\mathbb{R}^2 \longmapsto \exp\left(-\frac{n}{2}\left(\frac{u^2}{1-2iv}+\mathrm{Log}(1-2iv)\right)\right).$$

Once we know the characteristic function Φ_n of the law ν^{*n}, a Fourier inversion formula gives us its density. We first have to check that Φ_n is integrable with respect to the Lebesgue measure on $\mathbb{R}^2$.

Let $(u,v)\in\mathbb{R}^2$. Since $(1-2iv)^{-1} = (1+2iv)/(1+4v^2)$, we have

$$\mathfrak{Re}\left(\frac{u^2}{1-2iv}+\mathrm{Log}(1-2iv)\right) = \frac{u^2}{1+4v^2} + \ln(\sqrt{1+4v^2}).$$

Using Fubini's theorem, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^2} |\Phi_n(u,v)|\,du\,dv &= \int_{\mathbb{R}^2} \exp\left(-\frac{nu^2}{2(1+4v^2)}\right)(1+4v^2)^{-n/4}\,du\,dv \\
&= \int_{\mathbb{R}} (1+4v^2)^{-n/4}\left(\int_{\mathbb{R}} \exp\left(-\frac{nu^2}{2(1+4v^2)}\right)du\right)dv \\
&= \int_{\mathbb{R}} (1+4v^2)^{-n/4}\sqrt{\frac{2\pi(1+4v^2)}{n}}\,dv \\
&= \sqrt{\frac{2\pi}{n}}\int_{\mathbb{R}} (1+4v^2)^{-(n-2)/4}\,dv.
\end{aligned}$$

The function $v\longmapsto(1+4v^2)^{-(n-2)/4}$ is continuous on $\mathbb{R}$ and integrable in the neighbourhood of $+\infty$ and $-\infty$ if and only if $n>4$.

Proposition 6. *If $n\geq 5$ then ν^{*n} has the density*

$$(x,y)\in\mathbb{R}^2 \longmapsto \left(\sqrt{2^n\pi n}\,\Gamma\left(\frac{n-1}{2}\right)\right)^{-1}\exp\left(-\frac{y}{2}\right)\left(y-\frac{x^2}{n}\right)^{(n-3)/2}\mathbb{1}_{x^2<ny}$$

with respect to the Lebesgue measure on $\mathbb{R}^2$.

Proof. We have seen that, if $n \geq 5$, then Φ_n is integrable on $\mathbb{R}^2$. The Fourier inversion formula (see [13]) implies that ν_σ^{*n} has the density

$$f_n : (x,y) \longmapsto \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ixu-iyv}\, \Phi_n(u,v)\, du\, dv$$

with respect to the Lebesgue measure on $\mathbb{R}^2$. Let $(x,y) \in \mathbb{R}^2$. By Fubini's theorem,

$$\begin{aligned} f_n(x,y) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \frac{e^{-iyv}}{(1-2iv)^{n/2}} \left(\int_{\mathbb{R}} \exp\left(-ixu - \frac{nu^2}{2(1-2iv)} \right) du \right) dv \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \frac{e^{-iyv}}{(1-2iv)^{n/2}} K\left(-x, \frac{n}{1-2iv} \right) dv, \end{aligned}$$

where K is defined by

$$\forall a > 0 \qquad \forall (t,b) \in \mathbb{R}^2 \qquad K(t, a+ib) = \int_{\mathbb{R}} e^{itz-(a+ib)z^2/2}\, dz.$$

Lemma 4 implies that for any $v \in \mathbb{R}$,

$$\begin{aligned} K\left(-x, \frac{n}{1-2iv}\right) &= \sqrt{\frac{2\pi(1+4v^2)}{n}} \exp\left(-\frac{x^2(1-2iv)}{2n} \right) (1+2iv)^{-1/2} \\ &= \sqrt{\frac{2\pi}{n}} \exp\left(-\frac{x^2(1-2iv)}{2n} \right) \left(\frac{1+4v^2}{1+2iv} \right)^{1/2} \\ &= \sqrt{\frac{2\pi}{n}} \exp\left(-\frac{x^2(1-2iv)}{2n} \right) (1-2iv)^{1/2}. \end{aligned}$$

Thus

$$\begin{aligned} f_n(x,y) &= \frac{1}{(2\pi)^2} \sqrt{\frac{2\pi}{n}} \int_{\mathbb{R}} \exp\left(-iyv - \frac{x^2(1-2iv)}{2n} \right) (1-2iv)^{-(n-1)/2}\, dv \\ &= \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{x^2}{2n} \right) \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(-iv \left(y - \frac{x^2}{n} \right) \right) (1-2iv)^{-(n-1)/2}\, dv. \end{aligned}$$

Therefore $\sqrt{2\pi n}\exp(x^2/2n) f_n(x,y)$ is the inverse Fourier transform of the distribution $\Gamma((n-1)/2, 2)$ taken at the point $y - x^2/n$. Hence

$$\begin{aligned} \sqrt{2\pi n} \exp\left(\frac{x^2}{2n} \right) f_n(x,y) = \left(\Gamma\left(\frac{n-1}{2} \right) 2^{(n-1)/2} \right)^{-1} &\left(y - \frac{x^2}{n} \right)^{(n-3)/2} \\ &\times \exp\left(-\frac{y}{2} + \frac{x^2}{2n} \right) \mathbb{1}_{y > x^2/n}. \end{aligned}$$

Simplifying this expression, we get the proposition. □

This previous result and proposition 2 imply that, for $n \geq 5$, under $\widetilde{\mu}_{n,\sigma}$, the law of (S_n, T_n) on $\mathbb{R}^2$ is

$$C_n^{-1} \exp\left(\frac{x^2}{2y} - \frac{y}{2} \right) \left(y - \frac{x^2}{n} \right)^{(n-3)/2} \mathbb{1}_{x^2 < ny}\, dx\, dy.$$

We observe next that $(\sigma X_n^1, \ldots, \sigma X_n^n)$ has the distribution $\widetilde{\mu}_{n,\sigma}$ if and only if $(X_n^1, \ldots, X_n^n)$ has the distribution $\widetilde{\mu}_{n,1}$. Hence a straightforward change of variables gives us proposition 3.

4 Proof of theorem 1

Let $\alpha, \beta \in]0,1]$, $n \geq 5$ and f a bounded measurable function. The change of variables $(x,y) \longmapsto (n^\alpha x, n^\beta y)$ yields

$$\mathbb{E}_{\tilde{\mu}_{n,1}}\left(f\left(\frac{S_n}{n^\alpha}, \frac{T_n}{n^\beta}\right)\right) = \frac{n^{\alpha+\beta}}{C_n}\int_{\mathbb{R}^2} f(x,y)\exp\left(\frac{n^{2\alpha-\beta}x^2}{2y} - \frac{n^\beta y}{2}\right) \times \left(n^\beta y - n^{2\alpha-1}x^2\right)^{(n-3)/2} \mathbb{1}_{n^{2\alpha}x^2 < n^{\beta+1}y}\, dx\, dy.$$

Factorizing by $n^{(n-3)/2}$, we notice that all the terms in the integral are functions of $x^2/n^{2-2\alpha}$ and $y/n^{1-\beta}$. We obtain the following proposition.

Proposition 7. *Let $\alpha, \beta \in]0,1]$. If $\sigma^2 = 1$ and $n \geq 5$ then, under $\widetilde{\mu}_{n,\sigma}$, the distribution of $(S_n/n^\alpha, T_n/n^\beta)$ is*

$$\frac{n^{\alpha+\beta}n^{(n-3)/2}}{C_n}\exp\left(-n\psi\left(\frac{x^2}{n^{2-2\alpha}}, \frac{y}{n^{1-\beta}}\right)\right)\varphi\left(\frac{x^2}{n^{2-2\alpha}}, \frac{y}{n^{1-\beta}}\right) dx\, dy,$$

where ψ and φ are the functions defined on $D^+ = \{\, (x,y) \in \mathbb{R}^2 : y > x \geq 0 \,\}$ by

$$\psi : (x,y) \longmapsto \frac{1}{2}\left(-\frac{x}{y} + y - \ln(y-x)\right),$$

$$\varphi : (x,y) \longmapsto (y-x)^{-3/2}\, \mathbb{1}_{D^+}(x,y).$$

We give next some properties of the map ψ. Especially they show why we choose $\alpha = 3/4$ and $\beta = 1$ in the previous proposition in order to prove theorem 1.

Lemma 8. *The map ψ has a unique minimum at $(0,1)$ and, in the neighbourhood of $(0,1)$,*

$$\psi(x,y) - \frac{1}{2} = \frac{1}{4}(x^2 + (y-1)^2) + o(\|x, y-1\|^2).$$

Moreover, we have

$$\forall \delta > 0 \qquad \inf\{\, \psi(x,y) : |x| \geq \delta \quad or \quad |y-1| \geq \delta \,\} > 1/2.$$

Proof. The map ψ is $\mathcal{C}^2$ on D^+ and, for fixed $y > 0$,

$$\frac{\partial \psi}{\partial x}(x,y) = \frac{1}{2}\left(-\frac{1}{y} + \frac{1}{y-x}\right) \geq 0.$$

Equality holds if and only if $x = 0$. Thus $x \longmapsto \psi(x,y)$ is increasing on $]0,y[$ and $\psi(0,y) = (y - \ln(y))/2$. Hence for any $(x,y) \in D^+\backslash\{(0,1)\}$,

$$\psi(x,y) > \frac{1}{2}(y - \ln(y)) > \frac{1}{2} = \psi(0,1).$$

Therefore ψ has a unique minimum at $(0,1)$. In the neighbourhood of $(0,0)$,

$$\begin{aligned}\psi(x, 1+h) &= \frac{1}{2}(-x(1-h+o(h^2)) + 1 + h - (h - x - \frac{1}{2}(h-x)^2 + o((h-x)^2)) \\ &= \frac{1}{2} + \frac{h^2}{4} + \frac{x^2}{4} + o(\|x,h\|^2).\end{aligned}$$

Hence the announced expansion of ψ in the neighbourhood of $(0,1)$. Moreover, if $|y-1|\geq\delta$ and $x\in[0,y[$, then

$$\psi(x,y)\geq\frac{1}{2}(1+\delta-\ln(1+\delta))>\frac{1}{2}.$$

If $x\geq\delta$ and $y>x$, then

$$2\psi(x,y)\geq-\frac{\delta}{y}+y-\ln(y-\delta)>\inf_{y>\delta}\left(-\frac{\delta}{y}+y-\ln(y-\delta)\right)>1$$

since $\delta\neq0$. Therefore $\inf\{\,\psi(x,y):|x|\geq\delta\quad\text{or}\quad|y-1|\geq\delta\,\}>1/2$. $\square$

By this lemma, for fixed (x,y), when n goes to $+\infty$,

$$\psi\left(\frac{x^2}{n^{2-2\alpha}},\frac{y}{n^{1-\beta}}\right)-\frac{1}{2}\sim\frac{x^4}{4}n^{3-4\alpha}+\frac{n}{4}\left(\frac{y}{n^{1-\beta}}-1\right)^2.$$

That is why we take $\alpha=3/4$ and $\beta=1$.

Let us prove theorem 1. Let $n\geq1$ and let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be a continuous bounded function. By proposition 7, we have

$$\mathbb{E}_{\tilde{\mu}_{n,1}}\left(f\left(\frac{S_n}{n^{3/4}},\frac{T_n}{n}\right)\right)=\frac{n^{7/4}n^{(n-3)/2}}{C_n}\int_{\mathbb{R}^2}f(x,y)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)$$
$$\times\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy.$$

It follows from the expansion of ψ in lemma 8 that there exists $\delta>0$ such that for $(x,y)\in D^+$, if $|x|<\delta$ and $|y-1|<\delta$, then,

$$\psi(x,y)-\frac{1}{2}\geq\frac{1}{8}(x^2+(y-1)^2).$$

We denote

$$A_n=\int_{x^2<\delta\sqrt{n}}\int_{|y-1|<\delta}f(x,y)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy.$$

The change of variables $(x,y)\longmapsto(x,y/\sqrt{n}+1)$ gives

$$\sqrt{n}e^{n/2}A_n=\int_{x^2<\delta\sqrt{n}}\int_{|y|<\delta\sqrt{n}}f\left(x,\frac{y}{\sqrt{n}}+1\right)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},\frac{y}{\sqrt{n}}+1\right)\right)$$
$$\exp\left(\frac{n}{2}\right)\varphi\left(\frac{x^2}{\sqrt{n}},\frac{y}{\sqrt{n}}+1\right)\mathbb{1}_{y+\sqrt{n}>x^2}\,dx\,dy.$$

Lemma 8 implies that

$$n\psi\left(\frac{x^2}{\sqrt{n}},\frac{y}{\sqrt{n}}+1\right)-\frac{n}{2}\underset{n\to+\infty}{\longrightarrow}\frac{x^4}{4}+\frac{y^2}{4}.$$

Moreover the continuity of f and φ on D^+ gives us

$$f\left(x,\frac{y}{\sqrt{n}}+1\right)\varphi\left(\frac{x^2}{\sqrt{n}},\frac{y}{\sqrt{n}}+1\right)\mathbb{1}_{y+\sqrt{n}>x^2}\mathbb{1}_{x^2<\delta\sqrt{n}}\mathbb{1}_{|y|<\delta\sqrt{n}}\underset{n\to+\infty}{\longrightarrow}f(x,1).$$

Finally the function inside the integral defining $\sqrt{n}e^{n/2}A_n$ is dominated by

$$(x,y) \longmapsto \|f\|_\infty \exp\left(-\frac{1}{8}(x^4+y^2)\right),$$

which is independent of n and integrable with respect to the Lebesgue measure on $\mathbb{R}^2$. By Lebesgue's dominated convergence theorem, we have

$$\sqrt{n}e^{n/2}A_n \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} f(x,1)e^{-x^4/4}e^{-y^2/4}\,dx\,dy = \sqrt{4\pi}\int_{\mathbb{R}} f(x,1)e^{-x^4/4}\,dx.$$

We define

$$\mathrm{B}_\delta = \{\,(x,y)\in D^+ : |x|<\delta,\ |y-1|<\delta\,\}$$

and

$$B_n = \int_{(x^2/\sqrt{n},y)\in \mathrm{B}_\delta^c} f(x,y)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy.$$

Let $\varepsilon = \inf\{\,\psi(x,y) : (x,y)\in \mathrm{B}_\delta^c\}$,

$$|B_n| \le e^{-(n-2)\varepsilon}\|f\|_\infty \int_{\mathbb{R}^2}\exp\left(-2\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy.$$

The change of variables $(x,y)\longmapsto(xn^{1/4},y)$ yields

$$\sqrt{n}e^{n/2}|B_n| \le e^{2\varepsilon}\|f\|_\infty e^{-n(\varepsilon-1/2)}n^{3/4}\int_{\mathbb{R}^2} e^{-2\psi(x^2,y)}\varphi(x^2,y)\mathbb{1}_{x^2<y}\,dx\,dy.$$

Lemma 8 guarantees that $\varepsilon > 1/2$ and, using the change of variables given by the function $(x,y)\longmapsto(x,y+x^2)$, we get

$$\int_{\mathbb{R}^2} e^{-2\psi(x^2,y)}\varphi(x^2,y)\mathbb{1}_{x^2<y}\,dx\,dy \le e\left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)\left(\int_0^{+\infty}\frac{e^{-y}}{\sqrt{y}}\,dy\right) < +\infty.$$

Therefore $\sqrt{n}e^{n/2}B_n$ goes to 0 as n goes to $+\infty$. Finally

$$\int_{\mathbb{R}^2} f(x,y)\exp\left(-n\psi\left(\frac{x^2}{\sqrt{n}},y\right)\right)\varphi\left(\frac{x^2}{\sqrt{n}},y\right)\mathbb{1}_{\sqrt{n}y>x^2}\,dx\,dy$$
$$= A_n + B_n \underset{+\infty}{=} \frac{e^{-n/2}}{\sqrt{n}}\left(\sqrt{4\pi}\int_{\mathbb{R}} f(x,1)e^{-x^4/4}\,dx + o(1) + o(1)\right).$$

If $f=1$, we have

$$\frac{C_n}{n^{7/4}n^{(n-3)/2}} \underset{+\infty}{\sim} \sqrt{\frac{4\pi}{n}}e^{-n/2}\int_{\mathbb{R}} e^{-x^4/4}\,dx.$$

Hence

$$\mathbb{E}_{\tilde{\mu}_{n,1}}\left(f\left(\frac{S_n}{n^{3/4}},\frac{T_n}{n}\right)\right) \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}} f(x,1)\frac{e^{-x^4/4}\,dx}{\int_{\mathbb{R}} e^{-u^4/4}\,du}$$
$$= \int_{\mathbb{R}^2} f(x,y)\left(\frac{e^{-x^4/4}\,dx}{\int_{\mathbb{R}} e^{-u^4/4}\,du}\otimes\delta_1(y)\right).$$

Since $(\sigma X_n^1, \dots, \sigma X_n^n)$ has the distribution $\widetilde{\mu}_{n,\sigma}$ if and only if $(X_n^1, \dots, X_n^n)$ has the distribution $\widetilde{\mu}_{n,1}$, we obtain that, under $\widetilde{\mu}_{n,\sigma}$,

$$\frac{S_n}{n^{3/4}} \xrightarrow[n\to+\infty]{\mathcal{L}} \frac{e^{-x^4/4\sigma^4}\,dx}{\int_{\mathbb{R}} e^{-y^4/4\sigma^4}\,dy} \qquad \text{and} \qquad \frac{T_n}{n} \xrightarrow[n\to+\infty]{\mathcal{L}} \sigma^2.$$

We also get that S_n/n converges in law (thus in probability) to 0. Finally the ultimate change of variable $y = \sqrt{2}\sigma x^{1/4}$ implies that

$$\int_{\mathbb{R}} e^{-y^4/4\sigma^4}\,dy = 2\int_0^{+\infty} e^{-y^4/4\sigma^4}\,dy = \frac{\sigma}{\sqrt{2}}\int_0^{+\infty} x^{1/4-1}e^{-x}\,dx = \frac{\sigma}{\sqrt{2}}\,\Gamma\left(\frac{1}{4}\right).$$

This ends the proof of theorem 1.

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