Twisting Property (T) and the Baum-Connes morphism by a non-unitary representation

Maria-Paula Gomez-Aparicio, Université de Paris Sud-XI, Orsay

Topics in Noncommutative Geometry
Buenos Aires, Argentina
August 2010

Plan

1. Property (T)
   - Kazhdan's property (T)
   - C^*-algebraic caracterisation of (T)
   - The case of non-unitary representations : Twisted property (T)
2. Twisted Baum-Connes morphism
   - The Baum-Connes conjecture
   - Twisted Baum-Connes morphism
3. $R_F(G)$ acts on $K^{\mathrm{top}}(G)$

Let G be a locally compact (σ-compact) group and dg be the Haar mesure on G.

- (π, H) is a unitary representation of G if $\pi : G \to \mathcal{U}(H)$ is a strongly continuous group morphism i.e $\forall \xi \in H$, $g \mapsto \pi(g)\xi$ is continuous.
- (π, H) is irreducible if the only close subspaces of H which are G-invariant are $\{0\}$ and H.

Example

- The **trivial representation** $\left\{ \begin{array}{cccc} 1_G : & G & \to & \mathcal{U}(\mathbb{C}) \simeq \mathbb{S}^1 \\ & g & \mapsto & 1 \end{array} \right.$
- The **left regular representation**

$$\left\{ \begin{array}{l} \lambda_G : G \to \mathcal{U}(L^2(G)) \\ \lambda_G(g)(f)(t) = f(g^{-1}t), \end{array} \right. \quad \text{for } f \in L^2(G) \text{ and } g, t \in G.$$

Let (π, H_π) and (σ, H_σ) two unitary representations of G.

- π and σ are (unitarily) equivalent, $\pi \simeq \sigma$, if

$$\exists T : H_\pi \to H_\sigma \quad \text{isomorphism such that} \quad \forall g \in G, \quad \pi(g)T = T\sigma(g).$$

- π is contained in σ, $\pi \subset \sigma$, if

$$\exists\, H' \subset H_\sigma, \quad \text{closed and } G\text{-invariant such that} \quad \pi \simeq \sigma_{|H'}.$$

- π is weakly contained in σ, $\pi \prec \sigma$, if
$\forall \xi \in H_\pi$, $\|\xi\| = 1$, $\forall K \subset G$ compact, $\forall \varepsilon > 0$, $\exists \eta_1, \dots, \eta_n \in H_\sigma$, such that

$$|\langle \pi(g)\xi, \xi\rangle - \sum_{i=1}^{n} \langle \sigma(g)\eta_i, \eta_i\rangle| < \varepsilon, \quad \forall g \in K.$$

Example

- $1_G \subset \pi$ if and only if, $\exists \xi \in H_\pi$ non trivial, G-invariant.
- We can show $1_G \prec \pi$ if and only if, π almost has non trivial invariant vectors, ie. $\forall \varepsilon > 0$, $K \subset G$ compact $\exists \xi \in H_\pi$, $\|\xi\| = 1$ such that

$$\|\pi(g)\xi - \xi\| < \varepsilon \quad \forall g \in K.$$

Consider the unitary dual of G,

$$\widehat{G} = \{\text{equivalent classes of unitary irreducible representations of } G\}$$

endowed with the Fell topology :

$$\text{If} \quad S \subset \widehat{G} \quad \text{and} \quad \pi \in \widehat{G}. \quad \text{Then,} \quad \pi \in \overline{S} \quad \text{if} \quad \pi \prec S.$$

Example

- If G is abelian, $\widehat{G} = \{\text{equivalent classes of } \chi : G \to \mathbb{S}^1\}$ and the Fell topology is the topology of uniform convergence on compacts subset of G.

Definition (Kazhdan'67)

G has property (T) if 1_G is isolated in $\widehat{G}$.

Wang '75,

- G has property (T), if and only if, every unitary finite dimensional representation of G is isolated in $\widehat{G}$.
- G has (T) if and only if, for all unitary representation π of G,

$$1_G \prec \pi \quad \Rightarrow \quad 1_G \subset \pi.$$

Example

- If G is compact then G has property (T). In this case, the Fell topology on $\widehat{G}$ is the discrete topology.
- If G is a noncompact amenable group then G doesn't has (T).
- $\mathbb{R}$ and $\mathbb{Z}$ don't have (T).

Theorem (Kazhdan'67, Delaroche-Kirilov'68, Kostant'75,)

- *If G is a real connected simple Lie group with finite center and* $\mathrm{rank}_{\mathbb{R}} G \geq 2$*, then G has property (T).*

  In rank 1*,*

- $Sp(n,1)$*,* $n \geq 2$ *and* $F_{4(-20)}$ *have (T),*
- $SO(n,1)$ *and* $SU(n,1)$*, for* $n \geq 2$*, don't have (T).*

Theorem (Kazhdan'67, Wang '75)

- *If G has property (T), then $G/\overline{[G,G]}$ is compact.*
- *If Γ is a discrete group with property (T), then Γ is finitely generated.*
- *If Γ is a lattice in G, then*

$$G \text{ has property (T)} \Leftrightarrow \Gamma \text{ has property (T)}.$$

Example

1. $SL_n(\mathbb{R})$, $n \geq 3$ has property (T).
2. $SL_n(\mathbb{Z})$, $n \geq 3$ has property (T).
3. The free group with n generators $\mathbb{F}_n$ doesn't have property (T).
4. $SL_2(\mathbb{Z})$ doesn't have property (T) (because $\mathbb{F}_2$ is of finite index in $SL_2(\mathbb{Z})$), neither do $SL_2(\mathbb{R})$.

C^*-algebraic caracterisation of (T)

Let $C_c(G)$ be the space of continuous compactly supported functions on G. Then every unitary (π, H) of G can be extended to a representation of $C_c(G)$

$$\begin{aligned} \pi : C_c(G) &\longrightarrow \mathcal{L}(H) \\ f &\mapsto \pi(f) = \int_G f(g)\pi(g)dg. \end{aligned}$$

Definition

The maximal C^*-algebra associated to G, $C^*(G)$, is the completion of $C_c(G)$ for the norm :

$$\|f\|_{C^*(G)} = \sup_{(\pi, H_\pi) unitary} \|\pi(f)\|_{\mathcal{L}(H_\pi)}$$

Every unitary representation of G extends to a rep. of $C^*(G)$.

Examples

If G is an abelian group, then $C^*(G) = C_0(\widehat{G})$. In particular,

$$C^*(\mathbb{Z}) = C(\mathbb{S}^1).$$

Theorem (Akemann-Walter '81)

The following properties are equivalent :

1. *G has property (T)*
2. *$\exists p_G \in C^*(G)$ such that*

$$p_G^2 = p_G \quad \textit{and} \quad \begin{cases} \pi(p_G) = 0, & \forall \pi \in \widehat{G} \backslash \{1_G\} \\ 1_G(p_G) = 1 \end{cases}$$

Remark : For all (σ, H_σ) unitary representation of G, $\sigma(p_G)$ is the orthogonal projection on $H_\sigma^G = \{\xi \in H_\sigma \quad | \quad \sigma(g)\xi = \xi \quad \forall g \in G\}$.

In general,

$$\big((\pi, H_\pi) \in \widehat{G} \quad \text{isolated} \ \big) \longmapsto \quad p_\pi \in C^*(G) \quad \text{s.t.} \quad p_\pi^2 = p_\pi, \text{ and } \begin{cases} \pi(p_\pi) = \mathrm{Id}_{H_\pi} \\ \sigma(p_\pi) = 0, \text{if } \sigma \not\simeq \pi \end{cases}$$

Hence, if G has property (T) and ρ is an unitary finite dimensional irreducible representation of G :

$$\rho \quad \longmapsto \quad p_\rho \in C^*(G) \quad \text{and} \quad C^*(G) = I \oplus \mathrm{End}(V),$$

where I is a closed bilateral ideal.

Question : What if ρ is non-unitary ?

For example $SL_n(\mathbb{R})$, $n \geq 3$ has lots of non-unitary finite dimensional representations

Example : Standard representation $\rho : SL_n(\mathbb{R}) \to GL_n(\mathbb{C})$.

Twisted group algebras

Let $\rho : G \to \mathrm{Aut}(V)$ be a finite dimensional representation of G, and V endowed with an hermitian structure.

Definition (G. '07)

The maximal twisted group algebra $\mathcal{A}^{\rho}(G)$ is the completion of $C_c(G)$ for the norm given by :

$$\|f\|_{\mathcal{A}^{\rho}(G)} = \sup_{(\pi, H_\pi) unitaire} \|\rho \otimes \pi(f)\|_{\mathcal{L}(V \otimes H_\pi)}$$

Remarks :

- $\mathcal{A}^{\rho}(G)$ is a Banach algebra.
- If ρ is unitary then $\mathcal{A}^{\rho}(G) = C^*(G)$.
- If G is compact, then $\mathcal{A}^{\rho}(G) = C^*(G)$.

Examples

Let $G = \mathbb{Z}$ and let $\lambda_1, \lambda_2 \in \mathbb{C}^*$ such that $|\lambda_1| < |\lambda_2|$. Let,

$$\begin{aligned} \rho_i : \mathbb{Z} &\to \mathbb{C}^* \\ 1 &\mapsto \lambda_i, \end{aligned}$$

for $i = 1, 2$ be two characters of $\mathbb{Z}$. Denote,

$$\mathbb{S}^{\rho_i} := \{z \in \mathbb{C} \quad | \quad |z| = |\lambda_i|\},$$

then,

1. $\mathcal{A}^{\rho_i}(\mathbb{Z}) = C\left(\mathbb{S}^{\rho_i}\right)$,
2. $\mathcal{A}^{\rho_1 \oplus \rho_2}(\mathbb{Z}) = \mathcal{H}ol\left(\{z \in \mathbb{C} \quad | \quad |\lambda_1| < |z| < |\lambda_2|\}\right)$.

Twisted property (T)

Suppose that ρ is irreducible.

Definition (G. '07)

G has property (T) twisted by ρ (denoted by (T$\otimes\rho$)) if $\exists p_\rho \in \mathcal{A}^\rho(G)$ such that

$$p_\rho^2 = p_\rho \qquad \text{and} \qquad \begin{cases} (\rho\otimes\pi)(p_\rho) = 0, & \forall \pi \in \widehat{G}\setminus\{1_G\} \\ \rho(p_\rho) = Id_V \end{cases}$$

- If G has (T$\otimes\rho$) then for all unitary (σ, H), $(\rho\otimes\sigma)(p_\rho)$ is the orthogonal projection on $V\otimes H^G$.
- $$G \quad \text{has} \quad (T\otimes\rho) \Longleftrightarrow \mathcal{A}^\rho(G) = \ker(\rho)\oplus \mathrm{End}(V),$$

  then, ρ is isolated among representations of the form $\rho\otimes\pi$, where π runs over the irreducible unitary representations of G.

Results :

Theorem (G. '07)

If G has ($T\otimes\rho$) then G has (T).

For many Lie groups we have some kind of “converse” :

Theorem (G. '07)

If G is a simple connected real Lie group such that $\mathrm{rank}_{\mathbb{R}} G \geq 2$ or if G is locally isomorphic to $Sp(n,1)$, for $n \geq 2$, or $F_{4(-20)}$ then G has property ($T\otimes\rho$), for every irreducible finite dimensional representation ρ.

Idea : Use the fact that simple Lie groups satisfying (T) verify a stronger property : uniform decay of matrix coefficients of unitary representations not containing 1_G (Result of Cowling).

Theorem (G '07)

If Γ is a cocompact lattice in a group G having property $(T{\otimes}\rho)$ then Γ has property $(T{\otimes}\rho|_\Gamma)$.

Some other works on the strengthening of property (T)

- Lafforgue '07,
- Bader-Furman-Gelander-Monod '07,
- Fisher-Hichtman.

The Baum-Connes morphism

Definition

The reduce C^*-algebra of G, $C_r^*(G)$, is the completion of $C_c(G)$ for the norm given by :

$$\|f\|_{C_r^*(G)} = \|\lambda_G(f)\|_{\mathcal{L}(L^2(G))}.$$

The Baum-Connes conjecture predicts the K-theory of $C_r^*(G)$.

$K(C_r^*(G))$ ⟶

- Operator algebras
- Non-commutative geometry

$K(C_r^*(G))$ ⟶ $C_r^*(G)$captures the topology of $\widehat{G}_r = \{\text{tempered unitary rep. of } G\}$

$\longrightarrow$ • Harmonic Analysis

More precisely, for a locally compact group G, Baum, Connes and Higson have constructed an assembly map

$$\mu_r : K^{\mathrm{top}}(G) \to K(C_r^*(G)),$$

where

- $K^{\mathrm{top}}(G)$ is a "topological object" associated to G : the equivariant K-homology with compact support of the universal classifying space $\underline{E}G$ for proper G-actions ;
- $K(C_r^*(G))$ is an "analytic object" : the K-theory of the reduced C^*-algebra associated to G.

- Injectivity of $\mu_r \Rightarrow$ • Novikov conjecture
- Surjectivity of $\mu_r \Rightarrow \begin{cases} \bullet\text{Kadison-Kaplansky conjecture} \\ \bullet\text{Discrete series classification} \end{cases}$

Baum-Connes conjecture :

μ_r is an isomorphism for all locally compact group G.

Can we do the same with $C^*(G)$?

The left regular representation $\lambda_G : G \to \mathcal{U}(L^2(G))$ induces a C^*-morphism $\lambda_G : C^*(G) \to C^*_r(G)$ that is equal to the identity on $C_c(G)$.
Then there is a morphism

$$\tilde{\mu} : K^{\mathrm{top}}(G) \to K(C^*(G))$$

such that

$$\begin{array}{ccc} K^{\mathrm{top}}(G) & \xrightarrow{\mu_r} & K(C^*_r(G)) \\ & \searrow^{\tilde{\mu}} & \uparrow{\scriptstyle \lambda_{G,*}} \\ & & K(C^*(G)) \end{array}$$

is commutative.

Suppose G is not compact and has property (T). Let $p_G \in C^*(G)$ such that $\forall(\sigma, H)$ unitary, $\sigma(p_G)$ is the projection on H^G.
Then

$$\lambda_G(p_G) = 0 \quad \text{because} \quad L^2(G)^G = 0,$$

and λ_G is not injective.

- In this case, if μ_r is bijective then $\tilde{\mu}$ is not surjective.
- For a long time, all the proofs known of the Baum-Connes conjecture implied that $\tilde{\mu}$ (and so λ_G) is an isomorphism, so they are not valid for property (T) groups.
- Baum-Connes is also about distinguishing $K(C^*(G))$ and $K(C_r^*(G))$.

Example

- amenable groups (Kasparov),
- $SL_2(\mathbb{R})$, $SO(n,1)$, $SU(n,1)$, and more generally all a-T-menable groups (Julg-Kasparov, Higson-Kasparov).

Today, many property (T) groups are known to verify the Baum-Connes conjecture :

Example

- real semi-simple Lie groups (Wassermann, Lafforgue),
- cocompact lattices in $Sp(n,1)$, $n \geq 2$, in $F_{4,(-20)}$, in $SL_3(\mathbb{R})$, $SL_3(\mathbb{C})$, in $SL_3(\mathbb{H})$, in $E_{6(-26)}$ (Chatterji, Jolissaint, Lafforgue).

Remark : The conjecture is not known for $SL_3(\mathbb{Z})$.

Twisted Baum-Connes map

Let ρ be a (non-unitary) finite dimensional representation of G.

Definition

The twisted reduced group algebra $\mathcal{A}_r^{\rho}(G)$ is the completion of $C_c(G)$ by the norm given by

$$\|f\|_{\mathcal{A}_r^{\rho}(G)} = \|(\rho \otimes \lambda_G)(f)\|_{\mathcal{L}(V\otimes L^2(G))},$$

where $\lambda_G : G \to L^2(G)$ is the left regular representation of G.

Construction (G '08)

There is a *twisted Baum-Connes* morphism

$$\mu_r^{\rho} : K^{\mathrm{top}}(G) \to K(\mathcal{A}_r^{\rho}(G)),$$

which coincides with μ_r when ρ is unitary.

- Using Lafforgue's KK^{ban}.

Important ingredients :

1. Define twisted crossed products $A \rtimes_r^{\rho} G$, for A a G-C^*-algebra, such that $\mathbb{C} \rtimes_r^{\rho} G = \mathcal{A}_r^{\rho}(G)$.
2. For A and B two G-C^*-algebras, we have a descent morphism

$$j_{r,\rho} : KK_G(A,B) \to KK^{\mathrm{ban}}(A \rtimes_r^{\rho} G, B \rtimes_r^{\rho} G)$$

3. For X proper G-compact space, let $[p] \in K_0(C_0(X) \rtimes G)$ be the Mishchenko element (ie. the classe of $p \in C_c(G, C_0(X))$, $p(g)(x) := \sqrt{c(x)c(g^{-1}x)}$, where $c \in C_c(X, \mathbb{R}^+)$, such that $\int_G c(g^{-1}x)dg = 1, \forall x \in X$).

$$\begin{array}{ccc} KK_G(C_0(X), \mathbb{C}) & \xrightarrow{j_{r,\rho}} & KK^{\mathrm{ban}}(C_0(X) \rtimes_r^{\rho} G, \mathcal{A}_r^{\rho}(G)) \\ & \searrow^{\mu_r^{\rho}} & \downarrow \Sigma(.)([p]) \\ & & K(\mathcal{A}_r^{\rho}(G)) \end{array}$$

Passing to the inductive limit we get :

$$\mu_r^{\rho} : K^{\mathrm{top}}(G) \to K(\mathcal{A}_r^{\rho}(G)).$$

Theorem (G '08)

If G is one of the following groups :

- *an amenable group, or more generally, an a-T-menable group (e.g. :* $SL_2(\mathbb{R})$, $SO(n,1)$, $SU(n,1)$*),*
- *a real semi-simple Lie group (e.g.* $SL_n(\mathbb{R})$ *for* $n \geq 2$*),*
- *a cocompact lattice in* $Sp(n,1)$, $n \geq 2$*, or in* $F_{4(-20)}$*, or in* $SL_3(\mathbb{R})$*, in* $SL_3(\mathbb{C})$*, in* $SL_3(\mathbb{H})$*, in* $E_{6(-26)}$*,*

then μ_r^ρ *is an isomorphism.*

Proposition (G'08)

If Γ is a discret group and ρ is such that $\sum_{\gamma\in\Gamma}\frac{1}{\|\rho(\gamma)\|}$ converges, then

$$\mathcal{A}_r^{\rho}(\Gamma)\subset\ell^1(\Gamma)\subset C_r^*(\Gamma).$$

Remark

- Bost's conjecture : $\mu_{\ell^1}:K^{\mathrm{top}}(\Gamma)\to K(\ell^1(\Gamma))$ is an isomorphism, is true for $\Gamma=SL_3(\mathbb{Z})$.

Question : Do all twisted group algebras $\mathcal{A}_r^{\rho}(G)$ have the same K-theory for all ρ ?

Theorem (Bost '90)

If $G=\mathbb{Z}$ and $\rho_i:1\mapsto\lambda_i$ for $i=1,2$ and $|\lambda_1|<1<|\lambda_2|$ then, the restriction map

$$i:\mathcal{A}_r^{\rho_1\oplus\rho_2}(G)\to C_r^*(G),$$

induces an isomorphism in K-theory.

$R_F(G)$ acts on $K^{\mathrm{top}}(G)$

For G a locally compact group, we denote by $R_F(G)$ the set of finite dimensional representations of G.

- If G is a compact group, then $K^{\mathrm{top}}(G) = R(G)$, and we have a map

$$R_F(G) \to \mathrm{End}(R(G))$$
$$\rho \mapsto \left(\begin{array}{rcl} \Upsilon_\rho : R(G) & \longrightarrow & R(G) \\ \sigma & \longmapsto & \rho \otimes \sigma. \end{array} \right)$$

- In the general case,

$$K^{\mathrm{top}}(G) = \lim_{\rightarrow} KK_G(C_0(X), \mathbb{C}),$$

where X is a proper G-space such that X/G is compact.

Let $(\rho, V) \in R_F(G)$ and X a proper G-compact space.
The trivial vector bundle

$$\begin{array}{c} \mathcal{V} := X \times V \\ \big\downarrow \\ X \end{array}$$

can be endowed with a G-invariant hermitian metric.

Then

$$[\mathcal{V}] := [C_0(X, \mathcal{V})] \in KK_G\big(C_0(X), C_0(X)\big)$$

and taking the Kasparov's product by $[\mathcal{V}]$:

$$\begin{array}{rcl} KK_G\big(C_0(X), C_0(X)\big) \times KK_G\big(C_0(X), \mathbb{C}\big) & \to & KK_G\big(C_0(X), \mathbb{C}\big) \\ \big([\mathcal{V}], \alpha\big) & \mapsto & [\mathcal{V}] \otimes_{C_0(X)} \alpha, \end{array}$$

we get a morphism

$$KK_G\big(C_0(X), \mathbb{C}\big) \xrightarrow{[\mathcal{V}]\otimes .} KK_G\big(C_0(X), \mathbb{C}\big)$$

Hence passing to the inductive limit we get,

$$\Upsilon_\rho : K^{\mathrm{top}}(G) \to K^{\mathrm{top}}(G),$$

which coincides with the tensor product of representations if G is compact. On the other hand,

$$\begin{aligned} C_c(G) &\to C_c(G) \otimes \mathrm{End}(V) \\ f &\mapsto \big(g \mapsto f(g) \otimes \rho(g)\big), \end{aligned}$$

can be extended to a continuous map

$$\tau_\rho : \mathcal{A}_r^\rho(G) \to C_r^*(G) \otimes \mathrm{End}(V),$$

and, by Morita equivalence :

$$\tau_{\rho,*} : K(\mathcal{A}_r^\rho(G)) \to K(C_r^*(G)).$$

Theorem (G'08)

The following diagramm

$$\begin{array}{ccc} K^{\mathrm{top}}(G) & \xrightarrow{\mu_{\rho,r}} & K(\mathcal{A}_r^{\rho}(G)) \\ \Upsilon_\rho \downarrow & & \downarrow \tau_{\rho,*} \\ K^{\mathrm{top}}(G) & \xrightarrow{\mu_r} & K(C_r^*(G)), \end{array}$$

is commutative.