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Spectral and Scattering Theory for Space-Cutoff $P(\varphi)_2$ Models with Variable Metric

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Abstract. We consider space-cutoff $P(\varphi)_2$ models with a variable metric of the form

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}} g(x) : P\big(x, \varphi(x)\big) : \mathrm{d}x\,,$$

on the bosonic Fock space $L^2(\mathbb{R})$, where the kinetic energy $\omega = h^{\frac{1}{2}}$ is the square root of a real second order differential operator

$$h = Da(x)D + c(x)\,,$$

where the coefficients $a(x), c(x)$ tend respectively to 1 and m_∞^2 at ∞ for some $m_\infty > 0$.

The interaction term $\int_{\mathbb{R}} g(x) : P(x, \varphi(x)) : \mathrm{d}x$ is defined using a bounded below polynomial in λ with variable coefficients $P(x, \lambda)$ and a positive function g decaying fast enough at infinity.

We extend in this paper the results of [2] where h had constant coefficients and $P(x, \lambda)$ was independent of x.

We describe the essential spectrum of H, prove a Mourre estimate outside a set of thresholds and prove the existence of asymptotic fields. Our main result is the *asymptotic completeness* of the scattering theory, which means that the CCR representation given by the asymptotic fields is of Fock type, with the asymptotic vacua equal to bound states of H. As a consequence H is unitarily equivalent to a collection of second quantized Hamiltonians.

An important role in the proofs is played by the *higher order estimates*, which allow to control powers of the number operator by powers of the resolvent. To obtain these estimates some conditions on the eigenfunctions and generalized eigenfunctions of h are necessary. We also discuss similar models in higher space dimensions where the interaction has an ultraviolet cutoff.

1. Introduction

1.1. Space-cutoff $P(\varphi)_2$ models with variable metric

The $P(\varphi)_2$ *model* describes a self-interacting field of scalar bosons in 2 space-time dimensions with the interaction given by a bounded below polynomial $P(\varphi)$ of degree at least 4. Its construction in the seventies by Glimm and Jaffe (see e.g. [6]) was one of the early successes of constructive field theory. The first step of the construction relied on the consideration of a *spatially cutoff* $P(\varphi)_2$ interaction, where the cutoff is defined with a positive coupling function $g(x)$ of compact support. The formal expression

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}} g(x) : P\big(\varphi(x)\big) : \mathrm{d}x \,,$$

where $\omega = (D^2 + m^2)^{\frac{1}{2}}$ for $m > 0$ and $:\ :$ denotes the *Wick ordering*, can be given a rigorous meaning as a bounded below selfadjoint Hamiltonian on the Fock space $\Gamma(L^2(\mathbb{R}))$.

The spectral and scattering theory of H was studied in [2] by adapting methods originally developed for N-particle Schrödinger operators.

Concerning spectral theory, an HVZ theorem describing the essential spectrum of H and a Mourre positive commutator estimate were proved in [2]. As consequences of the Mourre estimate, one obtains as usual the local finiteness of point spectrum outside of the threshold set and, under additional assumptions, the limiting absorption principle.

The scattering theory of H was treated in [2] by the standard approach consisting in constructing first the *asymptotic fields*, which roughly speaking are the limits

$$\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} \phi(\mathrm{e}^{-\mathrm{i}t\omega} h) \mathrm{e}^{-\mathrm{i}tH} =: \phi^{\pm}(h) \,, \quad h \in L^2(\mathbb{R}) \,,$$

where $\phi(h)$ for $h \in L^2(\mathbb{R})$ are the Segal field operators. Since the model is massive, it is rather easy to see that the two CCR representations

$$h \mapsto \phi^{\pm}(h)$$

are unitarily equivalent to a direct sum of Fock representations. The central problem of scattering theory becomes then the description of the space of *vacua* for these asymptotic representations. The main result of [2] is the *asymptotic completeness*, which says that the asymptotic vacua coincide with the bound states of H. It implies that under time evolution any initial state eventually decays into a superposition of bound states of H and a finite number of asymptotically free bosons.

Although the Hamiltonians H do not describe any real physical system, they played an important role in the development of constructive field theory. Moreover they have the important property that the interaction is *local*. As far as we know, the $P(\varphi)_2$ models and the (non-relativistic) *Nelson model* are the only models with local interactions which can be constructed on Fock space by relatively easy arguments.

Our goal in this paper is to extend the results of [2] to the case where both the one-particle kinetic energy ω and the polynomial P have *variable coefficients*. More precisely we consider Hamiltonians

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}} g(x) :P\big(x, \varphi(x)\big): \mathrm{d}x,$$

on the bosonic Fock space $L^2(\mathbb{R})$, where the kinetic energy $\omega = h^{\frac{1}{2}}$ is the square root of a real second order differential operator

$$h = Da(x)D + c(x)\,,$$

$P(x, \lambda)$ is a variable coefficients polynomial

$$P(x, \lambda) = \sum_{p=0}^{2n} a_p(x)\lambda^p\,, \quad a_{2n}(x) \equiv a_{2n} > 0\,,$$

and $g \geq 0$ is a function decaying fast enough at infinity. We assume that $a(x)$, $c(x) > 0$ and $a(x) \to 1$ and $c(x) \to m_\infty^2$ when $x \to \infty$. The constant m_∞ has the meaning of the *mass at infinity*. Most of the time we will assume that $m_\infty > 0$.

As is well known, the Hamiltonian H appears when one tries to quantize the following non linear Klein–Gordon equation with variable coefficients:

$$\partial_t^2\varphi(t, x) + \big(Da(x)D + c(x)\big)\varphi(t, x) + g(x)\frac{\partial P}{\partial \lambda}\big(x, \varphi(t, x)\big) = 0\,.$$

Note that in [3], Dimock has considered perturbations of the full (translation-invariant) φ_2^4 model by lower order perturbations $\rho(t, x) :\varphi(t, x):$ where $\rho(t, x)$ has compact support in space-time.

We now describe in more details the content of the paper.

1.2. Content of the paper

The first difference between the $P(\varphi)_2$ models with a variable metric considered in this paper and the constant coefficients ones considered in [2] is that the polynomial $P(\lambda)$ is replaced by a variable coefficients polynomial $P(x, \lambda)$ in the interaction. The second is that the constant coefficients one-particle energy $(D^2 + m^2)^{\frac{1}{2}}$ is replaced by a variable coefficients energy $(Da(x)D + c(x))^{\frac{1}{2}}$.

Replacing $P(\lambda)$ by $P(x, \lambda)$ is rather easy. Actually, conditions on the function g and coefficients a_p needed to make sense of the Hamiltonian can be found in [13].

On the contrary replacing $(D^2 + m^2)^{\frac{1}{2}}$ by $(Da(x)D + c(x))^{\frac{1}{2}}$ leads to new difficulties. The construction of the Hamiltonian H is still rather easy, using hypercontractivity arguments.

However an essential tool to study the spectral and scattering theory of H is the so called *higher order estimates*, originally proved by Rosen [11], an example being the bound

$$N^{2p} \leq C_p(H + b)^{2p}\,, \quad p \in \mathbb{N}\,.$$

These bounds are very important to control various error terms and are a substitute for the lack of knowledge of the domain of H.

An substantial part of this paper is devoted to the proof of the higher order estimates in the variable metric case.

Let us now describe in more details the content of the paper.

In Section 2 we recall various well-known results, like standard Fock space notations, the notion of Wick polynomials and results on contractive and hypercontractive semigroups. We also recall some classical results on pseudodifferential calculus.

The space-cutoff $P(\varphi)_2$ model with a variable metric is described in Section 3 and its existence and basic properties are proved in Theorems 3.1 and 3.2.

In the massive case $m_\infty > 0$ we show using standard arguments on perturbations of hypercontractive semigroups that H is essentially self-adjoint and bounded below. The necessary properties of the interaction

$$V = \int_{\mathbb{R}} g(x) :P\big(x, \varphi(x)\big): \mathrm{d}x$$

as a multiplication operator are proved in Subsection 6.1 using pseudodifferential calculus and the analogous results known in the constant coefficients case.

The massless case $m_\infty = 0$ leads to serious difficulties, even to obtain the existence of the model. In fact the free semigroup $\mathrm{e}^{-t\mathrm{d}\Gamma(\omega)}$ is no more hypercontractive if $m_\infty = 0$ but only L^p-contractive. Using a result from Klein and Landau [9] we can show that H is essentially self-adjoint if for example g is compactly supported. Again the necessary properties of the interaction are proved in Subsection 6.2. The property that H is bounded below remains an open question and massless models will not be further considered in this paper.

Section 4 is devoted to the spectral and scattering theory of $P(\varphi)_2$ Hamiltonians with variable metric. It turns out that many arguments of [2] do not rely on the detailed properties of $P(\varphi)_2$ models but can be extended to an abstract framework.

In [4] we consider abstract bosonic QFT Hamiltonians of the form

$$H = \mathrm{d}\Gamma(\omega) + \mathrm{Wick}(w)\,,$$

acting on a bosonic Fock space $\Gamma(\mathfrak{h})$, where the one-particle energy ω is a self-adjoint operator on the one particle Hilbert space $\mathfrak{h}$ and the interaction term $\mathrm{Wick}(w)$ is a *Wick polynomial* associated to some kernel w. The spectral and scattering theory of such Hamiltonians is studied in [4] under a rather general set of conditions.

The first type of conditions requires that H is essentially self-adjoint and bounded below and satisfies *higher order estimates*, allowing to bound $\mathrm{d}\Gamma(\omega)$ and powers of the number operator N by sufficiently high powers of H.

The second type of conditions concern the one-particle energy ω. Essentially one requires that ω is *massive* i.e. $\omega \geq m > 0$ and has a nice spectral and scattering theory.

The last type of conditions concern the kernel w of the interaction $\mathrm{Wick}(w)$ and requires some decay properties of w at infinity.

The core of the present paper consists in proving that our $P(\varphi)_2$ Hamiltonians satisfy the hypotheses of [4], so that the results here follow from the abstract theorems in [4].

The essential spectrum of H is described in Theorem 4.3. As a consequence one obtains that H has a ground state. The Mourre estimate is shown in Theorem 4.4. We do not prove the limiting absorption principle, but note that for example the absence of singular continuous spectrum will follow from unitarity of the wave operators and asymptotic completeness.

The scattering theory and asymptotic completeness of wave operators, formulated as explained in Subsection 1.1 using asymptotic fields, is proved in Theorem 4.5.

Note that even in the constant coefficients case, we improve the results of [2]. No smoothness of the coupling function g is required and we can remove an unpleasant technical assumption on the coupling function g (condition (Bm) in [2, Subsection 6.2]) which excluded for example compactly supported g.

Analogous results for higher dimensional models where the interaction has also an ultraviolet cutoff are described in Section 5.

The properties of the interaction $\int_{\mathbb{R}} g(x) :P(x,\varphi(x)): \mathrm{d}x$ needed in Section 3 are proved in Section 6. In this section the interaction is considered as a Wick polynomial.

In Section 7 we prove some lower bounds on perturbations of $P(\varphi)_2$ Hamiltonians which will be needed in Section 8.

Section 8 is devoted to the proof of the higher order estimates. It turns out that the method of Rosen [11] uses in an essential way the fact that D^2+m^2 has the family $\{\mathrm{e}^{\mathrm{i}kx}\}_{k\in\mathbb{R}}$ as a basis of generalized eigenfunctions and that the functions $\mathrm{e}^{\mathrm{i}kx}$ are *uniformly bounded* both in x and k. In our case we have to use instead of $\{\mathrm{e}^{\mathrm{i}kx}\}_{k\in\mathbb{R}}$ a family of eigenfunctions and generalized eigenfunctions for $Da(x)D+c(x)$. It is necessary to impose some bounds on these functions to substitute for the uniform boundedness property in the constant coefficients case. These bounds are stated in Section 8 as conditions $(BM1)$, $(BM2)$ and deal respectively with the eigenfunctions and generalized eigenfunctions of $Da(x)D+c(x)$. Corresponding assumptions on the coupling function g and the polynomial $P(x,\lambda)$ are described in condition $(BM3)$.

Fortunately as we show in Appendices A and B, these conditions hold for a large class of second order differential operators.

Appendices A and B are devoted to conditions $(BM1)$, *(BM2)*. In Appendix A we discuss condition $(BM1)$ and show that we can always reduce ourselves to the case where h is a Schrödinger operator $D^2+V(x)+m_\infty^2$, where $V(x)\to 0$ at $\pm\infty$. We also prove that it is possible to find generalized eigenfunctions such that the associated unitary operator diagonalizing h on the continuous spectral subspace is *real*. This property is important in connection with Section 8.

Appendix B is devoted to condition $(BM2)$. It turns out that $(BM2)$ is actually a condition on the behavior of generalized eigenfunctions $\psi(x,k)$ for k near 0. If $h=D^2+V(x)$ and $V(x)\in O(\langle x\rangle^{-\mu})$ for some $\mu>0$, it is well known that the two cases $\mu > 2$ and $\mu \le 2$ lead to different behaviors of generalized eigenfunctions near $k = 0$.

We discuss condition $(BM2)$ if $\mu > 2$ using standard arguments based on *Jost solutions* which we recall for the reader's convenience. The case $0 < \mu \le 2$ is discussed using *quasiclassical solutions* by adapting results of Yafaev [17].

Finally Appendix C contains some technical estimates.

2. Preparations

In this section we collect various well-known results which will be used in the sequel.

2.1. Functional calculus

If $\chi \in C_0^\infty(\mathbb{R})$, we denote by $\tilde{\chi} \in C_0^\infty(\mathbb{C})$ an almost analytic extension of χ, satisfying

$$\tilde{\chi}_{|\mathbb{R}} = \chi\,,$$
$$|\partial_{\overline{z}}\tilde{\chi}(z)| \le C_n |Imz|^n\,, \quad n \in \mathbb{N}\,.$$

We use the following functional calculus formula for $\chi \in C_0^\infty(\mathbb{R})$ and A self-adjoint:

$$\chi(A) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(z-A)^{-1}\mathrm{d}z \wedge \mathrm{d}\,\overline{z}\,. \tag{2.1}$$

2.2. Fock spaces

In this subsection we recall various definitions on bosonic Fock spaces.

Bosonic Fock spaces

If $\mathfrak{h}$ is a Hilbert space then

$$\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n \mathfrak{h}\,,$$

is the *bosonic Fock space* over $\mathfrak{h}$. $\Omega \in \Gamma(\mathfrak{h})$ will denote the *vacuum vector*.

In all this paper the one-particle space $\mathfrak{h}$ will be equal to $L^2(\mathbb{R}, \mathrm{d}x)$. We denote by $\mathcal{F} : L^2(\mathbb{R}, \mathrm{d}x) \to L^2(\mathbb{R}, \mathrm{d}k)$ the unitary Fourier transform

$$\mathcal{F}u(k) = (2\pi)^{-\frac{1}{2}} \int \mathrm{e}^{-\mathrm{i}xk} u(x)\mathrm{d}x\,.$$

The *number operator* N is defined as

$$N\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}} = n\mathbb{1}\,.$$

We define the space of *finite particle vectors*:

$$\Gamma_{\mathrm{fin}}(\mathfrak{h}) = \mathcal{H}_{\mathrm{comp}}(N) := \big\{u \in \Gamma(\mathfrak{h}) \mid \text{for some } \; n \in \mathbb{N}, \;\; \mathbb{1}_{[0,n]}(N)u = u\big\}\,.$$

The *creation-annihilation* operators on $\Gamma(\mathfrak{h})$ are denoted by $a^*(h)$ and $a(h)$. The *field operators* are

$$\phi(h) := \frac{1}{\sqrt{2}}\big(a^*(h) + a(h)\big)\,,$$

which are essentially self-adjoint on $\Gamma_{\rm fin}(\mathfrak{h})$, and the *Weyl operators* are

$$W(h) := \mathrm{e}^{\mathrm{i}\phi(h)} .$$

dΓ operators

If $r : \mathfrak{h}_1 \to \mathfrak{h}_2$ is an operator one sets:

$$\mathrm{d}\Gamma(r) : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2) ,$$
$$\mathrm{d}\Gamma(r)\big|_{\bigotimes_{\rm s}^n \mathfrak{h}} := \sum_{j=1}^{n} \mathbb{1}^{\otimes(j-1)} \otimes r \otimes \mathbb{1}^{\otimes(n-j)} ,$$

with domain $\Gamma_{\rm fin}(\mathcal{D}(r))$. If r is closable, so is $\mathrm{d}\Gamma(r)$.

Γ operators

If $q : \mathfrak{h}_1 \mapsto \mathfrak{h}_2$ is bounded one sets:

$$\Gamma(q) : \Gamma(\mathfrak{h}_1) \mapsto \Gamma(\mathfrak{h}_2)$$
$$\Gamma(q)\big|_{\bigotimes_{\rm s}^n \mathfrak{h}_1} = q \otimes \cdots \otimes q .$$

$\Gamma(q)$ is bounded iff $\|q\| \le 1$ and then $\|\Gamma(q)\| = 1$.

2.3. Wick polynomials

We now recall the definition of Wick polynomials. We set

$$B_{\rm fin}\big(\Gamma(\mathfrak{h})\big) := \Big\{ B \in B\big(\Gamma(\mathfrak{h})\big) \mid \text{for some } n \in \mathbb{N} \ \ \mathbb{1}_{[0,n]}(N) B \mathbb{1}_{[0,n]}(N) = B \Big\} .$$

Let $w \in B(\otimes_{\rm s}^p \mathfrak{h}, \otimes_{\rm s}^q \mathfrak{h})$. The *Wick monomial* associated to the symbol w is:

$$\mathrm{Wick}(w) : \Gamma_{\rm fin}(\mathfrak{h}) \to \Gamma_{\rm fin}(\mathfrak{h})$$

defined as

$$\mathrm{Wick}(w)\big|_{\bigotimes_{\rm s}^n \mathfrak{h}} := \frac{\sqrt{n!(n+q-p)!}}{(n-p)!} w \otimes_{\rm s} \mathbb{1}^{\otimes(n-p)} . \tag{2.2}$$

This definition extends to $w \in B_{\rm fin}(\Gamma(\mathfrak{h}))$ by linearity. The operator $\mathrm{Wick}(w)$ is called a *Wick polynomial* and the operator w is called the *symbol* of $\mathrm{Wick}(w)$.

For example if $h_1, \dots, h_p, g_1, \dots, g_q \in \mathfrak{h}$ then:

$$\mathrm{Wick}(|\, g_1 \otimes_{\rm s} \cdots \otimes_{\rm s} g_q)(h_p \otimes_{\rm s} \cdots \otimes_{\rm s} h_1|) = a^*(q_1) \cdots a^*(g_q) a(h_p) \cdots a(h_1) .$$

If $\mathfrak{h} = L^2(\mathbb{R}, \mathrm{d}k)$ then any $w \in B(\otimes_{\rm s}^p \mathfrak{h}, \otimes_{\rm s}^q \mathfrak{h})$ is a bounded operator from $\mathcal{S}(\mathbb{R}^p)$ to $\mathcal{S}'(\mathbb{R}^q)$, where $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ denote the Schwartz spaces of functions and temperate distributions. It has hence a distribution kernel

$$w(k_1, \dots, k_q, k'_p, \dots, k'_1) \in S'(\mathbb{R}^{p+q}) ,$$

which is separately symmetric in the variables k and k'. It is then customary to denote the Wick monomial $\mathrm{Wick}(w)$ by:

$$\int w(k_1, \dots, k_q, k'_p, \dots, k'_1) a^*(k_1) \cdots a^*(k_q) a(k'_p) \cdots a(k'_1) \mathrm{d}k_1 \cdots \mathrm{d}k_q \mathrm{d}k'_p \cdots \mathrm{d}k'_1 .$$

If $\mathfrak{h} = L^2(\mathbb{R}, \mathrm{d}x)$, we will use the same notation, tacitly identifying $L^2(\mathbb{R}, \mathrm{d}x)$ and $L^2(\mathbb{R}, \mathrm{d}k)$ by Fourier transform.

2.4. Q-space representation of Fock space

Let $\mathfrak{h}$ be a Hilbert space and $c : \mathfrak{h} \to \mathfrak{h}$ a *conjugation* on $\mathfrak{h}$, *i.e.,* an anti-unitary involution. If $\mathfrak{h} = L^2(\mathbb{R}, \mathrm{d}x)$, we will take the standard conjugation $c : u \to \overline{u}$.

We denote by $\mathfrak{h}_c \subset \mathfrak{h}$ the real subspace of real vectors for c and $\mathfrak{M}_c \subset B(\Gamma(\mathfrak{h}))$ be the abelian Von Neumann algebra generated by the Weyl operators $W(h)$ for $h \in \mathfrak{h}_c$. The following result follows from the fact that Ω is a cyclic vector for $\mathfrak{M}_c$ (see e.g. [14]).

Theorem 2.1. *There exists a compact Hausdorff space Q, a probability measure μ on Q and a unitary map U such that*

$$\begin{aligned} U : \Gamma(\mathfrak{h}) &\to L^2(Q, \mathrm{d}\mu)\,, \\ U\Omega &= 1\,, \\ U\mathfrak{M}_c U^* &= L^\infty(Q, \mathrm{d}\mu)\,, \end{aligned}$$

where $1 \in L^2(Q, \mathrm{d}\mu)$ is the constant function equal to 1 on Q. Moreover:

$$U\Gamma(c)u = \overline{Uu},\ u \in \Gamma(\mathfrak{h})\,.$$

The space $L^2(Q, \mathrm{d}\mu)$ is called the Q-*space representation* of the Fock space $\Gamma(\mathfrak{h})$ associated to the conjugation c.

2.5. Contractive and hypercontractive semigroups

We collect now some standard results on contractive and hypercontractive semigroups.

We fix a probability space (Q, μ).

Definition 2.2. Let $H_0 \geq 0$ be a self-adjoint operator on $\mathcal{H} = L^2(Q, \mathrm{d}\mu)$.

The semigroup e^{-tH_0} is L^p-*contractive* if e^{-tH_0} extends as a contraction in $L^p(Q, \mathrm{d}\mu)$ for all $1 \leq p \leq \infty$ and $t \geq 0$.

The semigroup e^{-tH_0} is *hypercontractive* if

i) e^{-tH_0} is a contraction on $L^1(Q, \mathrm{d}\mu)$ for all $t > 0$,
ii) $\exists\, T, C$ such that

$$\|\mathrm{e}^{-TH_0}\psi\|_{L^4(Q,\mathrm{d}\mu)} \leq C\|\psi\|_{L^2(Q,\mathrm{d}\mu)}\,.$$

If e^{-tH_0} is *positivity preserving* (i.e. $f \geq 0$ a.e. implies $\mathrm{e}^{-tH_0}f \geq 0$ a.e.) and $\mathrm{e}^{-tH_0}1 \leq 1$ then e^{-tH_0} is L^p-contractive (see e.g. [8, Proposition 1.2]).

2.6. Perturbations of hypercontractive semigroups

The abstract result used to construct the $P(\varphi)_2$ Hamiltonian is the following theorem, due to Segal [12].

Theorem 2.3. *Let* e^{-tH_0} *be a hypercontractive semigroup. Let* V *be a real measurable function on* Q *such that* $V \in L^p(Q, \mathrm{d}\mu)$ *for some* $p > 2$ *and* $\mathrm{e}^{-tV} \in L^1(Q, \mathrm{d}\mu)$ *for all* $t > 0$. *Let* $V_n = \mathbb{1}_{\{|V| \leq n\}} V$ *and* $H_n = H_0 + V_n$. *Then the semigroups* e^{-tH_n} *converge strongly on* $\mathcal{H}$ *when* $n \to \infty$ *to a strongly continuous semigroup on* $\mathcal{H}$ *denoted by* e^{-tH}. *Its infinitesimal generator* H *has the following properties:*

i) H *is the closure of* $H_0 + V$ *defined on* $\mathcal{D}(H_0) \cap \mathcal{D}(V)$,

ii) H *is bounded below:*

$$H \geq -c - \ln \|\mathrm{e}^{-\delta V}\|_{L^1(Q, \mathrm{d}\mu)},$$

where c *and* δ *depend only on the constants* C *and* T *in Definition* 2.2.

We will also need the following result [14, Theorem 2.21].

Proposition 2.4. *Let* e^{-tH_0} *be a hypercontractive semigroup. Let* V, V_n *be real measurable functions on* Q *such that* $V_n \to V$ *in* $L^p(Q, \mathrm{d}\mu)$ *for some* $p > 2$, $\mathrm{e}^{-tV}, \mathrm{e}^{-tV_n} \in L^1(Q, \mathrm{d}\mu)$ *for each* $t > 0$ *and* $\|\mathrm{e}^{-tV_n}\|_{L^1}$ *is uniformly bounded in* n *for each* $t > 0$. *Then for* b *large enough*

$$(H_0 + V_n + b)^{-1} \to (H_0 + V + b)^{-1} \quad \text{in norm}\,.$$

The following lemma (see [13, Lemma V.5] for a proof) will be used later to show that a given function V on Q verifies $\mathrm{e}^{-tV} \in L^1(Q, \mathrm{d}\mu)$.

Lemma 2.5. *Let for* $\kappa \geq 1$, V_κ, V *be functions on* Q *such that for some* $n \in \mathbb{N}$:

$$\begin{aligned} \|V - V_\kappa\|_{L^p(Q, \mathrm{d}\mu)} &\leq C_1 (p-1)^n \kappa^{-\epsilon}\,, \quad \forall\, p \geq 2\,, \\ V_\kappa &\geq -C_2 - C_3 (\ln \kappa)^n\,. \end{aligned} \tag{2.3}$$

Then there exists constants κ_0, C_4 *and* $\alpha > 0$ *such that*

$$\mu\big\{q \in Q | V(q) \leq -C_4 (\ln \kappa)^n\big\} \leq \mathrm{e}^{-\kappa^\alpha}\,, \quad \forall \kappa \geq \kappa_0\,.$$

Consequently $\mathrm{e}^{-tV} \in L^1(Q, \mathrm{d}\mu)$, $\forall t > 0$ *with a norm depending only on* t *and the constants* C_i *in* (2.3).

The following theorem of Nelson (see [13, Theorem 1.17]) establishes a connection between contractions on $\mathfrak{h}$ and hypercontractive semigroups on the Q-space representation $L^2(Q, \mathrm{d}\mu)$ associated to a conjugation c.

Theorem 2.6. *Let* $r \in B(\mathfrak{h})$ *be a self-adjoint contraction commuting with* c. *Then*

i) $U\Gamma(r)U^*$ *is a positivity preserving contraction on* $L^p(Q, \mathrm{d}\mu)$, $1 \leq p \leq \infty$.

ii) *if* $\|r\| \leq (p-1)^{\frac{1}{2}}(q-1)^{-\frac{1}{2}}$ *for* $1 < p \leq q < \infty$ *then* $U\Gamma(r)U^*$ *is a contraction from* $L^p(Q, \mathrm{d}\mu)$ *to* $L^q(Q, d\mu)$.

Combining Theorem 2.6 with Theorem 2.3, we obtain the following result.

Theorem 2.7. *Let $\mathfrak{h}$ be a Hilbert space with a conjugation c. Let a be a self-adjoint operator on $\mathfrak{h}$ with*

$$[a, c] = 0\,, \quad a \geq m > 0\,. \tag{2.4}$$

Let $L^2(Q, \mathrm{d}\mu)$ be the Q-space representation of $\Gamma(\mathfrak{h})$ and let V be a real function on Q with $V \in L^p(Q, \mathrm{d}\mu)$ for some $p > 2$ and $\mathrm{e}^{-tV} \in L^1(Q, \mathrm{d}\mu)$ for all $t > 0$. Then:

i) *the operator sum $H = d\Gamma(a) + V$ is essentially self-adjoint on $\mathcal{D}(d\Gamma(a)) \cap \mathcal{D}(V)$.*
ii) *$H \geq -C$, where C depends only on m and $\|\mathrm{e}^{-V}\|_{L^p(Q,\mathrm{d}\mu)}$, for some p depending only on m.*

Note that by applying Theorem 2.6 to $a = (q-1)^{-\frac{1}{2}}\mathbb{1}_{\mathfrak{h}}$ for $q > 2$, we obtain the following lemma about the L^p properties of finite vectors in $\Gamma(\mathfrak{h})$ (see [13, Theorem 1.22]).

Lemma 2.8. *Let $\psi \in \otimes_{\mathrm{s}}^n \mathfrak{h}$ and $q \geq 2$. Then*

$$\|U\psi\|_{L^q(Q,\mathrm{d}\mu)} \leq (q-1)^{n/2}\|\psi\|\,.$$

2.7. Perturbations of L^p-contractive semigroups

The following theorem is shown in [9, Section II.2].

Theorem 2.9. *Let e^{-tH_0} be an L^p-contractive semigroup and V a real measurable function on Q such that $V \in L^{p_0}(Q, \mathrm{d}\mu)$ for some $p_0 > 2$ and $e^{-\delta V} \in L^1(Q, \mathrm{d}\mu)$ for some $\delta > 0$. Then $H_0 + V$ is essentially self-adjoint on $\mathcal{A}(H_0) \cap L^q(Q, \mathrm{d}\mu)$ for any $(\frac{1}{2} - \frac{1}{p_0})^{-1} \leq q < \infty$ where $\mathcal{A}(H_0)$ is the space of analytic vectors for H_0.*

2.8. Pseudodifferential calculus on $L^2(\mathbb{R}^d)$

We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz class of functions on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R}^d)$ the Schwartz class of tempered distributions on $\mathbb{R}^d$. We denote by $H^s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ the Sobolev spaces on $\mathbb{R}^d$.

We set as usual $D = \mathrm{i}^{-1}\partial_x$ and $\langle s\rangle = (s^2+1)^{\frac{1}{2}}$.

For $p, m \in \mathbb{R}$ and $0 \leq \epsilon < \frac{1}{2}$,we denote by $S_\epsilon^{p,m}$ the class of symbols $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_k^\beta a(x,k)| \leq C_{\alpha,\beta}\langle k\rangle^{p-|\beta|}\langle x\rangle^{m-(1-\epsilon)|\alpha|+\epsilon|\beta|}\,, \quad \alpha, \beta \in \mathbb{N}^d\,.$$

The symbol class $S_0^{p,m}$ will be simply denoted by $S^{p,m}$. The symbol classes above are equipped with the topology given by the seminorms equal to the best constants in the estimates above.

For $a \in S_\epsilon^{p,m}$, we denote by $\mathrm{Op}^{1,0}(a)$ (resp. $\mathrm{Op}^{0,1}(a)$) the *Kohn–Nirenberg* (resp. *anti Kohn–Nirenberg*) quantization of a defined by:

$$\mathrm{Op}^{1,0}(a)(x,D)u(x) := (2\pi)^{-d}\int\int \mathrm{e}^{\mathrm{i}(x-y)k}a(x,k)u(y)\mathrm{d}y\mathrm{d}k\,,$$

$$\mathrm{Op}^{0,1}(a)(x,D)u(x) := (2\pi)^{-d}\int\int \mathrm{e}^{\mathrm{i}(x-y)k}a(y,k)u(y)\mathrm{d}y\mathrm{d}k\,,$$

which are well defined as continuous maps from $S(\mathbb{R}^d)$ to $S'(\mathbb{R}^d)$. We denote by $\mathrm{Op}^{\mathrm{w}}(a)$ the *Weyl* quantization of a defined by:

$$\mathrm{Op}^{\mathrm{w}}(a)(x,D)u(x) := (2\pi)^{-1}\int\int \mathrm{e}^{\mathrm{i}(x-y)k} a\left(\frac{x+y}{2},k\right)u(y)\mathrm{d}y\mathrm{d}k\,.$$

We recall that as operators from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$:

$$\mathrm{Op}^{0,1}(m)^* = \mathrm{Op}^{1,0}(\overline{m})\,,\quad \mathrm{Op}^{\mathrm{w}}(m)^* = \mathrm{Op}^{\mathrm{w}}(\overline{m})\,.$$

We will also need the following facts (see [7, Theorem 18.5.4]):

$$\big[\mathrm{Op}^{\mathrm{w}}(b_1), \mathrm{i}\mathrm{Op}^{\mathrm{w}}(b_2)\big] = \mathrm{Op}^{\mathrm{w}}(\{b_1,b_2\}) + \mathrm{Op}^{\mathrm{w}}(S_\epsilon^{p_1+p_2-3,m_1+m_2-3(1-2\epsilon)})\,, \tag{2.5}$$

$$\begin{aligned}\mathrm{Op}^{\mathrm{w}}(b_1)\mathrm{Op}^{\mathrm{w}}(b_2) &+ \mathrm{Op}^{\mathrm{w}}(b_2)\mathrm{Op}^{\mathrm{w}}(b_1)\\ &= 2\mathrm{Op}^{\mathrm{w}}(b_1b_2) + \mathrm{Op}^{\mathrm{w}}(S_\epsilon^{p_1+p_2-2,m_1+m_2-2(1-2\epsilon)})\,,\end{aligned} \tag{2.6}$$

if $b_i \in S_\epsilon^{p_i,m_i}$ and $\{\ ,\ \}$ denotes the Poisson bracket.

The following two propositions will be proved in Appendix C.

Proposition 2.10. *Let $b \in S^{2,0}$ a real symbol such that for some $C_1, C_2 > 0$*

$$b(x,k) \geq C_1\langle k\rangle^2 - C_2\,.$$

Then:

i) $\mathrm{Op}^{\mathrm{w}}(b)(x,D)$ *is self-adjoint and bounded below on $H^2(\mathbb{R}^d)$.*

ii) *Let C such that $\mathrm{Op}^{\mathrm{w}}(b)(x,D)+C > 0$ and $s \in \mathbb{R}$. Then there exist $m_i \in S^{2s,0}$ for $i = 1,2,3$ such that*

$$\big(\mathrm{Op}^{\mathrm{w}}(b)(x,D)+C\big)^s = \mathrm{Op}^{\mathrm{w}}(m_1)(x,D) = \mathrm{Op}^{1,0}(m_2)(x,D) = \mathrm{Op}^{0,1}(m_3)(x,D)\,.$$

Proposition 2.11. *Let a_{ij}, c are real such that:*

$$\begin{aligned} [a_{ij}](x) \geq c_0\mathbb{1}\,,\quad c(x)\geq c_0 \quad &\textit{for some} \quad c_0 > 0\,,\\ [a_{ij}] - \mathbb{1}\,,\quad c(x) - m_\infty^2 \in S^{0,-\mu} \quad &\textit{for some} \quad m_\infty, \mu > 0\,.\end{aligned} \tag{2.7}$$

Set:

$$b(x,k) := \sum_{1\leq i,j\leq d} k_i a_{ij}(x)k_j + c(x)\,,$$

and

$$h := \sum_{1\leq i,j\leq d} D_i a_{ij}(x) D_j + c(x) = \mathrm{Op}^{\mathrm{w}}(b)\,.$$

Then:

i)

$$\omega := h^{\frac12} = \mathrm{Op}^{\mathrm{w}}(b^{\frac12}) + \mathrm{Op}^{\mathrm{w}}(S^{0,-1-\mu})\,.$$

ii) *there exists $0 < \epsilon < \frac12$ such that:*

$$\Big[\omega, \mathrm{i}\big[\omega,\mathrm{i}\langle x\rangle\big]\Big] = \mathrm{Op}^{\mathrm{w}}(\gamma)^2 + \mathrm{Op}^{\mathrm{w}}(r_{-1-\epsilon})\,,\quad \textit{for}\quad \gamma\in S_\epsilon^{0,-\frac12}\,,\quad r_{-1-\epsilon}\in S_\epsilon^{0,-1-\epsilon}\,.$$

3. The space-cutoff $P(\varphi)_2$ model with variable metric

In this section we define the space-cutoff $P(\varphi)_2$ Hamiltonians with variable metric and we prove some of their basic properties.

3.1. The $P(\varphi)_2$ model with variable metric

For $\mu \in \mathbb{R}$ we denote by S^μ the class of symbols $a \in C^\infty(\mathbb{R})$ such that

$$|\partial_x^\alpha a(x)| \leq C_\alpha \langle x\rangle^{\mu-\alpha}\,, \quad \alpha \in \mathbb{N}\,.$$

Let a, c two real symbols such that for some $\mu > 0$:

$$a - 1 \in S^{-\mu}\,, \quad a(x) > 0\,, \quad c - m_\infty^2 \in S^{-\mu}\,, \quad c(x) > 0\,, \tag{3.1}$$

where the constant $m_\infty \geq 0$ has the meaning of the *mass at infinity*. For most of the paper we will assume that the model is *massive* i.e. $m_\infty > 0$. The existence of the Hamiltonian in the massless case $m_\infty = 0$ will be proved in Theorem 3.2.

We consider the second order differential operator

$$h = Da(x)D + c(x)\,,$$

which is self-adjoint on $H^2(\mathbb{R})$. Clearly $h \geq m$ for some $m > 0$ if $m_\infty > 0$ and for $m = 0$ if $m_\infty = 0$. Note that h is a *real* operator i.e. $[h, c] = 0$, if c is the standard conjugation.

The *one-particle space* is

$$\mathfrak{h} = L^2(\mathbb{R}, \mathrm{d}x)\,,$$

and the *one-particle energy* is

$$\omega := \big(Da(x)D + c(x)\big)^{\frac{1}{2}}\,, \quad \text{acting on} \quad \mathfrak{h}\,.$$

The kinetic energy of the field is

$$H_0 := \mathrm{d}\Gamma(\omega)\,, \quad \text{acting on} \quad \Gamma(\mathfrak{h})\,.$$

To define the interaction we fix a real polynomial with x-dependent coefficients:

$$P(x, \lambda) = \sum_{p=0}^{2n} a_p(x)\lambda^p\,, \quad a_{2n}(x) \equiv a_{2n} > 0\,, \tag{3.2}$$

and a measurable function g with:

$$g(x) \geq 0\,, \quad \forall\, x \in \mathbb{R}\,,$$

and set for $1 \leq \kappa < \infty$ an UV-cutoff parameter:

$$V_\kappa := \int g(x) : P\big(x, \varphi_\kappa(x)\big) : \mathrm{d}x\,,$$

where : : denotes the *Wick ordering* and $\varphi_\kappa(x)$ are the *UV-cutoff fields*.

In the massive case, they are defined as:

$$\varphi_\kappa(x) := \phi(f_{\kappa,x})\,, \tag{3.3}$$

for

$$f_{\kappa,x} = \sqrt{2}\omega^{-\frac{1}{2}}\chi\left(\frac{\omega_\infty}{\kappa}\right)\delta_x\,, \quad x \in \mathbb{R}\,. \tag{3.4}$$

Here $\chi \in C_0^\infty(\mathbb{R})$ is a cutoff function equal to 1 near 0, $\omega_\infty = (D^2 + m_\infty^2)^{\frac{1}{2}}$, and δ_x is the δ distribution centered at x.

In the massless case we take:

$$f_{\kappa,x} = \sqrt{2}\omega^{-\frac{1}{2}}\chi\left(\frac{\omega}{\kappa}\right)\delta_x\,, \quad x \in \mathbb{R}\,.$$

Note that one can also use the above definition in the massive case (see Lemma 6.4).

Note also that since ω is a real operator, $f_{\kappa,x}$ are real vectors, which implies that V_κ is affiliated to $\mathfrak{M}_c$. Therefore in the Q-space representation associated to c, V_κ becomes a measurable function on (Q,μ).

We will see later that under appropriate conditions on the functions ga_p (see Theorems 3.1 and 3.2) the functions V_κ converge in $L^2(Q,\mathrm{d}\mu)$ when $\kappa \to \infty$ to a function V which will be denoted by

$$V := \int_{\mathbb{R}} g(x) :P\big(x,\varphi(x)\big): \mathrm{d}x\,.$$

3.2. Existence and basic properties

We consider first the massive case $m_\infty > 0$.

Theorem 3.1. *Let $\omega = (Da(x)D + c(x))^{\frac{1}{2}}$ where $a, c > 0$ and $a-1$, $c - m_\infty^2 \in S^{-\mu}$ for some $\mu > 0$. Assume that*

$$m_\infty > 0\,.$$

Let:

$$P(x,\lambda) = \sum_{p=0}^{2n} a_p(x)\lambda^p\,, \quad a_{2n}(x) \equiv a_{2n} > 0\,.$$

Assume:

$$\begin{aligned} ga_p \in L^2(\mathbb{R})\,, \quad &\text{for} \quad 0 \le p \le 2n\,, \quad g \in L^1(\mathbb{R})\,, \quad g \ge 0\,, \\ g(a_p)^{2n/(2n-p)} \in L^1(\mathbb{R}) \quad &\text{for} \quad 0 \le p \le 2n-1\,. \end{aligned} \tag{3.5}$$

Then

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}} g(x) :P\big(x,\varphi(x)\big): \mathrm{d}x = H_0 + V$$

is essentially self-adjoint and bounded below on $\mathcal{D}(H_0)\cap\mathcal{D}(V)$.

Proof. We apply Theorem 2.7 to $a = \omega$. We need to show that $V \in L^p(Q)$ for some $p > 2$ and $\mathrm{e}^{-tV} \in L^1(Q)$ for all $t > 0$. The first fact follows from Lemma 6.2 and Lemma 2.8. To prove that $\mathrm{e}^{-tV} \in L^1(Q)$ we use Lemma 2.5: we know from Lemma 6.2 i) that $\|V - V_\kappa\|_{L^2(Q)} \in O(\kappa^{-\epsilon})$ for some $\epsilon > 0$. Since $V\Omega$ and $V_\kappa\Omega$ are finite particle vectors, we deduce from Lemma 2.8 that for all $p \ge 2$ one has

$$\|V - V_\kappa\|_{L^p(Q)} \le C(p-1)^n\kappa^{-\epsilon}\,.$$

Hence the first estimate of (2.3) is satisfied. The second follows from Lemma 7.1. □

We now consider the massless case $m_\infty = 0$. For simplicity we assume that $a(x) \equiv 1$.

Theorem 3.2. *Let $\omega = (D^2 + c(x))^{\frac{1}{2}}$ where $c > 0$ and $c \in S^{-\mu}$ for some $\mu > 0$. Let:*

$$P(x, \lambda) = \sum_{p=0}^{2n} a_p(x)\lambda^p\,, \quad a_{2n}(x) \equiv a_{2n} > 0\,.$$

Assume:

$$g \quad \text{is compactly supported}\,, \tag{3.6}$$

$$\begin{aligned} g a_p \in L^2(\mathbb{R})\,, \quad \text{for} \quad 0 \le p \le 2n\,, \quad g \ge 0\,, \\ g(a_p)^{2n/(2n-p)} \in L^1(\mathbb{R}) \quad \text{for} \quad 0 \le p \le 2n-1\,. \end{aligned} \tag{3.7}$$

Then

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}} g(x) : P\big(x, \varphi(x)\big) : \mathrm{d}x = H_0 + V$$

is essentially self-adjoint on $\mathcal{A}(H_0) \cap L^q(Q, \mathrm{d}\mu)$ for q large enough, where $\mathcal{A}(H_0)$ is the space of analytic vectors for H_0.

Remark 3.3. It is not necessary to assume that g is compactly supported. In fact if we replace the cutoff function χ in the proof of Lemma 6.5 by the function $\langle x\rangle^{-\mu/2}$ we see that Lemma 6.5 still holds if:

$$c(x) \ge C\langle x\rangle^{-\mu}\,, \quad \text{for some} \quad C > 0\,. \tag{3.8}$$

Similarly Lemma 6.6 ii) still holds if we replace the conditions

$$g a_p \in L^2(\mathbb{R})\,, \quad g \quad \text{compactly supported}\,,$$

by

$$g a_p \langle x\rangle^{p\mu/2} \in L^2(\mathbb{R})\,.$$

The estimate iii) in Lemma 6.6 is replaced by:

$$\langle x\rangle^{-\mu/2}\omega^{-\frac{1}{2}} F\left(\frac{h}{k^2}\right)\delta_x \in O\big((\ln\kappa)^{\frac{1}{2}}\big)\,, \quad \text{uniformly in} \quad x \in \mathbb{R}\,.$$

Following the proof of Lemma 7.1, we see that Theorem 3.2 still holds if we assume (3.8), $g \in L^1(\mathbb{R})$ and if conditions (3.7) hold with a_p replaced by $a_p\langle x\rangle^{p\mu/2}$.

Remark 3.4. We believe that H is still bounded below in the massless case. For example using arguments similar to those in Lemma 6.5, one can check that the second order term in formal perturbation theory of the ground state energy $E(\lambda)$ of $H_0 + \lambda V$ is finite.

Proof. Since $\omega \ge 0$ is a real operator, we see from Theorem 2.6 that e^{-tH_0} is an L^p-contractive semigroup. Applying Theorem 2.9, it suffices to show that $V \in L^p(Q)$ for some $p > 2$ and $\mathrm{e}^{-\delta V} \in L^1(Q)$ for some $\delta > 0$. The first fact follows from

Lemma 6.6 and Lemma 2.8. To prove that $\mathrm{e}^{-tV} \in L^1(Q)$ we use again Lemma 2.5: the fact that for all $p \geq 2$

$$\|V - V_\kappa\|_{L^p(Q)} \leq C(p-1)^n \kappa^{-\epsilon} ,$$

follows as before from Lemma 6.6. The second condition in (2.3) follows from Lemma 6.6 iii), arguing as in the proof of Lemma 7.1. □

4. Spectral and scattering theory of $P(\varphi)_2$ Hamiltonians

In this section, we state the main results of this paper. We consider a $P(\varphi)_2$ Hamiltonian as in Theorem 3.1. We need first to state some conditions on the eigenfunctions and generalized eigenfunctions of $h = \omega^2$. These conditions will be needed to obtain *higher order estimates* in Section 8, an important ingredient in the proof of Theorems 4.3, 4.4 and 4.5.

We will say that the families $\{\psi_l(x)\}_{l\in I}$ and $\{\psi(x,k)\}_{k\in\mathbb{R}}$ form a basis of (generalized) eigenfunctions of h if:

$$\begin{gathered}
\psi_l(\,\cdot\,) \in L^2(\mathbb{R}) , \quad \psi(\,\cdot\,,k) \in \mathcal{S}'(\mathbb{R}) , \\
h\psi_l = \epsilon_l \psi_l , \quad \epsilon_l \leq m_\infty^2 , \quad l \in I \\
\psi(\,\cdot\,,k) = (k^2 + m_\infty^2)\psi(\,\cdot\,,k) , \quad k \in \mathbb{R} , \\
\sum_{l\in I} |\psi_l) \left(\psi_l \middle| + \frac{1}{2\pi}\int_{\mathbb{R}} \middle| \psi(\,\cdot\,,k)\right) \big(\psi(\,\cdot\,,k)\big| \mathrm{d}k = \mathbb{1} .
\end{gathered}$$

Here I is equal either to $\mathbb{N}$ or to a finite subset of $\mathbb{N}$. The existence of such bases follows easily from the spectral theory and scattering theory of the second order differential operator h, using hypotheses (3.1).

Let $M : \mathbb{R} \to [1 + \infty[$ a locally bounded Borel function. We introduce the following assumption on such a basis:

$$\begin{gathered}
(BM1) \quad \sum_{l\in I} \|M^{-1}(\,\cdot\,)\psi_l(\,\cdot\,)\|_\infty^2 < \infty , \\
(BM2) \quad \|M^{-1}(\,\cdot\,)\psi(\,\cdot\,,k)\|_\infty \leq C , \quad \forall\, k \in \mathbb{R} .
\end{gathered}$$

For a given weight function M, we introduce the following hypotheses on the coefficients of $P(x,\lambda)$:

$$(BM3) \quad g a_p M^s \in L^2(\mathbb{R}) , \quad g(a_p M^s)^{\frac{2n}{2n-p+s}} \in L^1(\mathbb{R}) , \quad \forall\, 0 \leq s \leq p \leq 2n-1 .$$

Remark 4.1. Hypotheses (BMi) for $1 \leq i \leq 3$ have still a meaning if M takes values in $[1, +\infty]$, if we use the convention that $(+\infty)^{-1} = 0$. Of course in order for $(BM3)$ to hold M must take finite values on $\operatorname{supp} g$.

Remark 4.2. The results below still hold if we replace $(BM2)$ by

$$(BM2') \quad |M^{-1}(\,\cdot\,)\psi(\,\cdot\,,k)\|_\infty \leq C \sup(1, |k|^{-\alpha}) , \quad k \in \mathbb{R} .$$

for some $0 \leq \alpha < \frac{1}{2}$.

The results of the paper are summarized in the following three theorems.

Theorem 4.3 (HVZ Theorem). *Let H be as in Theorem 3.1 and assume that there exists a basis of eigenfunctions $\{\psi_l(x)\}_{l\in I}$ and generalized eigenfunctions $\{\psi(x,k)\}_{k\in\mathbb{R}}$ of h such that conditions (BM1), (BM2), (BM3) hold. Then the essential spectrum of H equals $[\inf\sigma(H)+m_\infty,+\infty[$. Consequently H has a ground state.*

Theorem 4.4 (Mourre estimate). *Let H be as in Theorem 3.1 and assume in addition to the hypotheses of Theorem 4.3 that*

$$\langle x\rangle^s g a_p \in L^2(\mathbb{R}), \quad 0\le p\le 2n, \quad s>1.$$

let $a=\frac{1}{2}(\langle D\rangle^{-1}D\cdot x+\mathrm{h.c.})$ and $A=\mathrm{d}\Gamma(a)$. Let

$$\tau=\sigma_{\mathrm{pp}}(H)+m_\infty\mathbb{N}^*$$

be the set of thresholds *of H. Then:*

i) *the quadratic form $[H,\mathrm{i}A]$ defined on $\mathcal{D}(H)\cap\mathcal{D}(A)$ uniquely extend to a bounded quadratic form $[H,\mathrm{i}A]_0$ on $\mathcal{D}(H^m)$ for some m large enough.*

ii) *if $\lambda\in\mathbb{R}\backslash\tau$ there exists $\epsilon>0$, $c_0>0$ and a compact operator K such that*

$$1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H,\mathrm{i}A]_0 1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)\ge c_0 1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)+K.$$

iii) *for all $\lambda_1\le\lambda_2$ such that $[\lambda_1,\lambda_2]\cap\tau=\emptyset$ one has:*

$$\dim 1\!\!1_{[\lambda_1,\lambda_2]}(H)<\infty.$$

Consequently $\sigma_{\mathrm{pp}}(H)$ can accumulate only at τ, which is a closed countable set.

iv) *if $\lambda\in\mathbb{R}\backslash(\tau\cup\sigma_{\mathrm{pp}}(H))$ there exists $\epsilon>0$ and $c_0>0$ such that*

$$1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H,\mathrm{i}A]_0 1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)\ge c_0 1\!\!1_{[\lambda-\epsilon,\lambda+\epsilon]}(H).$$

Theorem 4.5 (Scattering theory). *Let H be as in Theorem 3.1 and assume that the hypotheses of Theorem 4.4 hold. Let us denote by $\mathfrak{h}_{\mathrm{c}}(\omega)$ the continuous spectral subspace of $\mathfrak{h}$ for ω. Then:*

1. *The* asymptotic Weyl operators*:*

$$W^\pm(h):=\text{s-}\lim_{t\pm\infty}\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH} \quad \textit{exist for all} \quad h\in\mathfrak{h}_{\mathrm{c}}(\omega),$$

and define a regular CCR representation over $\mathfrak{h}_{\mathrm{c}}(\omega)$.

2. *There exist unitary operators $\Omega^\pm$, called the* wave operators*:*

$$\Omega^\pm:\mathcal{H}_{\mathrm{pp}}(H)\otimes\Gamma\big(\mathfrak{h}_{\mathrm{c}}(\omega)\big)\to\Gamma(\mathfrak{h})$$

such that

$$W^\pm(h)=\Omega^\pm 1\!\!1\otimes W(h)\Omega^{\pm *}, \quad h\in\mathfrak{h}_{\mathrm{c}}(\omega),$$
$$H=\Omega^\pm\big(H_{|\mathcal{H}_{\mathrm{pp}}(H)}\otimes 1\!\!1+1\!\!1\otimes\mathrm{d}\Gamma(\omega)\big)\Omega^{\pm *}.$$

Remark 4.6. Appendices A and B are devoted to conditions $(BM1)$, $(BM2)$. For example condition $(BM1)$ is always satisfied for $M(x) = \langle x\rangle^{\alpha}$ if $\alpha > \frac{1}{2}$ and is satisfied for $M(x) = 1$ if h has a finite number of eigenvalues (see Proposition A.1).

Concerning condition $(BM2)$, we show in Lemma A.3 that it suffices to consider the case where $a(x) \equiv 1$. For example if $c(x) - m_\infty^2 \in O(\langle x\rangle^{-\mu})$ for $\mu > 2$ and h has no zero energy resonances, then $(BM2)$ is satisfied for $M(x) = 1$ (see Proposition B.3).

If $c(x) - m_\infty^2 \in O(\langle x\rangle^{-\mu})$ for $0 < \mu < 2$, is *negative near infinity* and has no zero energy resonances, then $(BM2)$ is satisfied for $M(x) = \langle x\rangle^{\mu/4}$ (see Proposition B.10).

If $c(x) - m_\infty^2$ is *positive near infinity*, holomorphic in a conic neighborhood of $\mathbb{R}$ and has no zero energy resonances, then $(BM2)$ is satisfied for $M(x) = 1$ in $\{|x| \le R\}$ and $M(x) = +\infty$ in $\{|x| > R\}$ (see Proposition B.14).

Remark 4.7. A typical situation in which all the assumptions are satisfied is when $a(x) - 1$, $c(x) - m_\infty^2$ and g, a_p are all in the Schwartz class $\mathcal{S}(\mathbb{R})$.

Proofs of Theorems 4.3*,* 4.4 *and* 4.5*.* It suffices to check that H belongs to the class of abstract QFT Hamiltonians considered in [4]. We check that H satisfies all the conditions in [4, Theorem 4.1], introduced in [4, Section 3].

Since $\omega \ge m > 0$, condition $(H1)$ in [4, Subsection 3.1] is satisfied. The interaction term V is clearly a Wick polynomial. By Theorem 3.1, H is essentially selfadjoint and bounded below on $\mathcal{D}(H_0) \cap \mathcal{D}(V)$, i.e. condition $(H2)$ in [4, Subsection 3.1] holds. Next by Theorem 8.1 the higher order estimates hold for H, i.e. condition $(H3)$ in [4, Subsection 3.1] is satisfied.

The second set of conditions concern the one-particle energy ω. Conditions $(G1)$ in [4, Subsection 3.2] are satisfied for $\mathcal{S} = \mathcal{S}(\mathbb{R})$ and $\langle x\rangle = (x^2+1)^{\frac{1}{2}}$. This follows immediately from the fact that $\omega \in \mathrm{Op}(S^{1,0})$ shown in Proposition 2.10 and pseudodifferential calculus. Condition $(G2)$ in [4, Subsection 3.2] has been checked in Proposition 2.11.

Let us now consider the conjugate operator a. To define a without ambiguity, we set $\mathrm{e}^{-\mathrm{i}ta} := \mathcal{F}^{-1} u_t \mathcal{F}$, where u_t is the unitary group on $L^2(\mathbb{R}, \mathrm{d}k)$ generated by the vector field $-\frac{k}{\langle k\rangle} \cdot \partial_k$. We see that u_t preserves the spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{F}\mathcal{D}(\omega) = \mathcal{D}(\langle k\rangle)$. This implies first that a is essentially self-adjoint on $\mathcal{S}(\mathbb{R})$, by Nelson's invariant subspace theorem. Moreover $\mathrm{e}^{\mathrm{i}ta}$ preserves $D(\omega)$ and $[\omega, a]$ is bounded on $L^2(\mathbb{R})$. By [1, Proposition 5.1.2], $\omega \in C^1(a)$ and condition $(M1\ i)$ in [4, Subsection 3.2] holds.

We see also that $a \in \mathrm{Op}(S^{0,1})$, so conditions $(G3)$ and $(G4)$ in [4, Subsection 3.2] hold. For $\omega_\infty = (D^2 + m_\infty^2)^{\frac{1}{2}}$, we deduce as above from pseudodifferential calculus that

$$[\omega, \mathrm{i}a]_0 = \omega_\infty^{-1}\langle D\rangle^{-1} D^2 + \mathrm{Op}(S^{0,-\mu})\,.$$

Since $\chi(\omega) - \chi(\omega_\infty)$ is compact, we obtain that

$$\chi(\omega)[\omega, \mathrm{i}a]_0\chi(\omega) = \chi^2(\omega_\infty)\omega_\infty^{-1}\langle D\rangle^{-1} D^2 + K\,, \quad \text{where} \quad K \quad \text{is compact}\,.$$

This implies that $\rho_\omega^a \geq 0$ and $\tau_a(\omega) = \{m_\infty\}$, hence (*M1 ii*) in [4, Subsection 3.2] holds.

Property (*C*) in [4, Subsection 3.2] follows from the fact that $\omega - \omega_\infty \in \mathrm{Op}(S^{1,-\mu})$ and pseudodifferential calculus. Finally property *(S)* in [4, Subsection 3.2] can be proved as explained in [4, Subsection 3.2].

The last set of conditions concern the decay properties of the Wick kernel of V. We see that condition *(D)* in [4, Subsection 3.2] is satisfied, using Lemma 6.2 and the fact that $\langle x\rangle^s g a_p \in L^2(\mathbb{R})$ for all $0 \leq p \leq 2n$.

Applying then [4, Theorem 4.1] we obtain Theorems 4.3, 4.4 and 4.5. □

5. Higher dimensional models

In this section we briefly discuss similar models in higher space dimension, when the interaction term has an ultraviolet cutoff.

We work now on $L^2(\mathbb{R}^d, \mathrm{d}x)$ for $d \geq 2$ and consider

$$h = \sum_{1\leq i,j\leq d} D_i a_{ij}(x) D_j + c(x)\,, \quad \omega = h^{\frac{1}{2}}\,.$$

where a_{ij}, c satisfy (2.7). The free Hamiltonian is as above

$$H_0 = \mathrm{d}\Gamma(\omega)\,,$$

acting on the Fock space $\Gamma(L^2(\mathbb{R}^d))$.

Since $d \geq 2$ it is necessary to add an ultraviolet cutoff to make sense out of the formal expression

$$\int_{\mathbb{R}^d} g(x) P\big(x, \varphi(x)\big)\mathrm{d}x\,.$$

We set

$$\varphi_\kappa(x) := \phi\left(\omega^{-\frac{1}{2}}\chi\left(\frac{\omega}{\kappa}\right)\delta_x\right),$$

where $\chi \in C_0^\infty([-1,1])$ is a cutoff function equal to 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and $\kappa \gg 1$ is an ultraviolet cutoff parameter. Since $\omega^{-\frac{1}{2}}\chi(\frac{\omega}{\kappa})\delta_x \in L^2(\mathbb{R}^d)$, $\varphi_\kappa(x)$ is a well defined selfadjoint operator on $\Gamma(L^2(\mathbb{R}^d))$.

If $P(x,\lambda)$ is as in (3.2) and $g \in L^1(\mathbb{R}^d)$, then

$$V := \int_{\mathbb{R}^d} g(x) P\big(x, \varphi_\kappa(x)\big)\mathrm{d}x\,,$$

is a well defined selfadjoint operator on $\Gamma(L^2(\mathbb{R}^d))$.

Lemma 5.1. *Assume that* $g \geq 0$, $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *and* $g a_p \in L^2(\mathbb{R}^d)$, $g a_p^{2n(2n-p)} \in L^1(\mathbb{R}^d)$ *for* $0 \leq p \leq 2n-1$. *Then*

$$V \in \bigcap_{1\leq p<\infty} L^p(Q, \mathrm{d}\mu)\,, \quad V \quad \textit{is bounded below}.$$

Proof. It is easy to see that $\Omega \in \mathcal{D}(V)$ hence $V \in L^2(Q, \mathrm{d}\mu)$. Using that $V\Omega$ is a finite particle vector we obtain by Lemma 2.8 that $V \in \bigcap_{1 \leq p < \infty} L^p(Q, \mathrm{d}\mu)$.

To prove that V is bounded below, we use the inequality:

$$a^p b^{n-p} \leq \epsilon b^n + C_\epsilon a^n \,, \quad a, b \geq 0 \,,$$

and obtain as an inequality between functions on Q:

$$|a_p(x)\varphi_\kappa(x)^p| \leq \epsilon \varphi_\kappa(x)^{2n} + C_\epsilon |a_p(x)|^{2n/(2n-p)} \,.$$

Integrating this bound for ϵ small enough we obtain that V is bounded below. □

Applying then Theorem 2.7, we obtain that:

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}^d} g(x) P\big(x, \varphi_\kappa(x)\big) \mathrm{d}x$$

is essentially selfadjoint and bounded below.

We have then the following theorem. As before we consider a generalized basis $\{\psi_l(x)\}_{l\in I}$ and $\{\psi(x,k)\}_{k\in\mathbb{R}^d}$ of eigenfunctions of h.

Theorem 5.2. *Assume that:*

$$\begin{aligned} ga_p \in L^2(\mathbb{R}^d) \,, &\quad 0 \leq p \leq 2n \,, \quad g \in L^1(\mathbb{R}^d) \,, \quad g \geq 0 \,, \\ g(a_p)^{2n/(2n-p)} \in L^1(\mathbb{R}^d) \,, &\quad 0 \leq p \leq 2n-1 \,, \\ \langle x\rangle^s g a_p \in L^2(\mathbb{R}^d) &\quad \forall\, 0 \leq p \leq 2n \,, \quad \text{for some} \quad s > 1 \,. \end{aligned}$$

Assume moreover that for a measurable function $M : \mathbb{R}^d \to \mathbb{R}^+$ with $M(x) \geq 1$ there exists a generalized basis of eigenfunctions of h such that:

$$\begin{cases} \sum_{l\in I} \|M^{-1}(\,\cdot\,)\psi_l(\,\cdot\,)\|_\infty^2 < \infty \,, \\ \|M^{-1}(\,\cdot\,)\psi(\,\cdot\,,k)\|_\infty \leq C \,, \quad k \in \mathbb{R} \,, \end{cases}$$

$$ga_p M^s \in L^2(\mathbb{R}^d) \,, \quad g(a_p M^s)^{2n/(2n-p+s)} \in L^1(\mathbb{R}^d) \,, \quad \forall\, 0 \leq s \leq p \leq 2n-1 \,.$$

Then the analogs of Theorems 4.3, 4.4 *and* 4.5 *hold for the Hamiltonian:*

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}^d} g(x) P\big(x, \varphi_\kappa(x)\big) \mathrm{d}x \,.$$

Remark 5.3. As in the one-dimensional case, the hypotheses concerning generalized eigenfunctions can be checked in some cases. An example is if $d = 3$, $[a_{ij}](x) = \mathbb{1}$ $c(x) - m_\infty^2 \in O(\langle x\rangle^{-3-\epsilon})$) and $h - m_\infty^2$ has no zero resonance or eigenvalue, where we can take $M(x) \equiv 1$. (See e.g. [15, Proposition 2.5 *iv*)]).

We will sketch the proof of Theorem 5.2, which again consists in showing that the conditions of [4, Theorem 4.1] are satisfied.

The condition on the one-particle operators can be checked exactly as in the one-dimensional case, as can the decay of the interaction kernel. To prove the higher order estimates, , we can argue as in Section 8 working now with the family $\{\psi(x)_l\}_{l\in I} \cup \{\psi(x,k)\}_{k\in\mathbb{R}^d}$. The various integrals in k occurring in the proof of the higher order estimates are convergent because the domain of integration is included in $\{|k|^2 \leq \kappa - m_\infty^2\}$ due to the energy cutoff $\chi(\kappa^{-1}\omega)$ in the definition of $\varphi_\kappa(x)$.

6. Properties of the interaction kernel

In this section prove some properties of the interaction $V = \int_{\mathbb{R}} g(x) :P(x, \varphi(x)): \mathrm{d}x$, considering V as a Wick polynomial.

6.1. Massive case

In this subsection we consider the massive case $m_\infty > 0$.

Lemma 6.1. *Let $g \in S(\mathbb{R})$. Then for $\kappa < \infty$:*

$$\int g(x) :\varphi_\kappa(x)^p: \mathrm{d}x = \sum_{r=0}^{p} \binom{p}{r}$$

$$\int w_{p,\kappa}(k_1, \ldots, k_r, k_{r+1}, \ldots, k_p) a^*(k_1) \ldots a^*(k_r) a(-k_{r+1}) \ldots a(-k_p) \mathrm{d}k_1 \ldots \mathrm{d}k_p \,,$$

where:

$$w_{p,\kappa}(k_1, \ldots, k_p) = (2\pi)^{-p/2} \int g(x) \prod_{j=1}^{p} \mathrm{e}^{-\mathrm{i}k_j x} m_\kappa(x, k_j) \mathrm{d}x \tag{6.1}$$

and $m_\kappa(x, k)$ is the anti Kohn–Nirenberg *symbol of $\omega^{-\frac{1}{2}} \chi(\frac{\omega_\infty}{\kappa})$.*

Proof. If $m_\kappa(x, k)$ is the anti Kohn–Nirenberg symbol of $\omega^{-\frac{1}{2}} \chi(\frac{\omega_\infty}{\kappa})$ we have:

$$\mathcal{F}\left(\omega^{-\frac{1}{2}} \chi\left(\frac{\omega_\infty}{\kappa}\right) \delta_x\right)(k) = (2\pi)^{-\frac{1}{2}} \mathrm{e}^{-\mathrm{i}xk} m_\kappa(x, k) \,.$$

Note that it follows from Proposition 2.10 that $m_\kappa \in S^{-r,0}$ for each $r \in \mathbb{N}$. We observe moreover that $\omega^{-\frac{1}{2}} \chi(\frac{\omega_\infty}{\kappa})$ is a real operator which implies that $m_\kappa(x, k) = \overline{m}_\kappa(x, -k)$ and hence

$$\varphi_\kappa(x) = (2\pi)^{-\frac{1}{2}} \int \mathrm{e}^{-\mathrm{i}kx} m_\kappa(x, k) \big(a^*(k) + a(-k)\big) \mathrm{d}k \,,$$

from which the lemma follows. □

We extend the above notation to $\kappa = \infty$ by denoting by $m_\infty(x, k)$ the anti Kohn–Nirenberg symbol of $\omega^{-\frac{1}{2}}$ and by $w_{p,\infty}$ the function in (6.1) with m_κ replaced by m_∞. Note that by Proposition 2.10 $m_\infty \in S^{-\frac{1}{2},0}$ so $w_{p,\infty}$ is a well defined function on $\mathbb{R}^d$ if $g \in S(\mathbb{R})$.

To study the properties of $w_{\kappa,p}$ it is convenient to introduce the following maps:

$$\begin{aligned} T_\kappa : \mathcal{S}(\mathbb{R}) &\to \mathcal{S}'(\mathbb{R}^p) \,, \quad 1 \leq \kappa \leq \infty \,, \\ g &\mapsto w_{\kappa,p} \,. \end{aligned}$$

Lemma 6.2. i) *T_κ is bounded from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^p)$ for each $1 \leq \kappa \leq \infty$ and there exists $\epsilon > 0$ such that*

$$\|T_\kappa - T_\infty\|_{B(L^2(\mathbb{R}), L^2(\mathbb{R}^p))} \in O(\kappa^{-\epsilon}) \,.$$

ii) *the map* $\langle D_{x_i}\rangle^s T_\infty \langle x\rangle^{-s}$ *is bounded from* $L^2(\mathbb{R})$ *to* $L^2(\mathbb{R}^p)$ *for each* $s \geq 0$ *and* $1 \leq i \leq p$.

iii) *one has*

$$\|f_{\kappa,x}\| \in 0\big((\ln \kappa)^{\frac{1}{2}}\big)\,, \quad \textit{uniformly for} \quad x \in \mathbb{R}\,.$$

Proof. The operator T_κ has the distribution kernel

$$(2\pi)^{-p/2} \prod_{j=1}^{p} \mathrm{e}^{-\mathrm{i}k_j x} m_\kappa(x, k_j)\,,$$

hence for $f \in S(\mathbb{R}^p)$ we have:

$$T_\kappa^* f(x) = (2\pi)^{-p/2} \int \prod_{j=1}^{p} \mathrm{e}^{\mathrm{i}k_j x} \overline{m}_\kappa(x, k_j) f(k_1, \ldots, k_p) \mathrm{d}k_1 \cdots \mathrm{d}k_p\,.$$

If $R : C^\infty(\mathbb{R}^p) \to C^\infty(\mathbb{R})$ is the operator of restriction to the diagonal

$$Rf(x) = f(x, \ldots, x)\,,$$

we see that

$$T_\kappa^* f = RM_\kappa \mathcal{F}_p^{-1} f\,,$$

where

$$M_\kappa = \prod_{j=1}^{p} \mathrm{Op}^{1,0}(\overline{m}_\kappa)(x_j, D_{x_j})\,,$$

and we have denoted by $\mathcal{F}_p$ the unitary Fourier transform on $L^2(\mathbb{R}^p)$. Since $\mathcal{F}$ is the unitary Fourier transform on $L^2(\mathbb{R})$, we have with obvious identification $\Gamma(\mathcal{F}) = \mathcal{F}_p$. Since $\mathrm{Op}^{1,0}(\overline{m}) = \mathrm{Op}^{0,1}(m)^*$, we see that

$$M_\kappa = \Gamma\left(\chi\left(\frac{\omega_\infty}{\kappa}\right)\omega^{-\frac{1}{2}}\right)_{|\otimes_{\mathrm{s}}^p L^2(\mathbb{R})}\,,$$

where we have used the Fock space notation. This yields

$$T_\kappa^* = R\Gamma\left(\chi\left(\frac{\omega_\infty}{\kappa}\right)\omega_\infty^{-\frac{1}{2}}\mathcal{F}^{-1}\right)\Gamma(\mathcal{F}\omega_\infty^{\frac{1}{2}}\omega^{-\frac{1}{2}}\mathcal{F}^{-1}) =: T_\kappa^{0*}\Gamma(\mathcal{F}\omega_\infty^{\frac{1}{2}}\omega^{-\frac{1}{2}}\mathcal{F}^{-1})\,,$$

where T_κ^0 is the analog of T_κ with ω replaced by ω_∞. This yields:

$$T_\kappa = \Gamma(\mathcal{F}\omega^{-\frac{1}{2}}\omega_\infty^{\frac{1}{2}}\mathcal{F}^{-1})T_\kappa^0\,. \tag{6.2}$$

By pseudodifferential calculus, we know that $\omega^{-\frac{1}{2}}\omega_\infty^{\frac{1}{2}} \in \mathrm{Op}(S^{0,0})$ and hence is bounded on $\mathcal{D}(\langle x\rangle^s)$ for all s. Therefore it suffices to prove i) and ii) for T_κ^0, T_∞^0. i) for T_κ^0 is shown in [2, Lemma 6.1]. To check ii) for T_κ^0 for integer s we use that

$$T_\infty^0(g)(k_1, \ldots, k_p) = \hat{g}(k_1 \cdots + k_p) \prod_{i=1}^{p} \omega_\infty^{-\frac{1}{2}}(k_i)\,.$$

Then $\partial_{k_1}^s T_\infty^0(k_1, \ldots, k_p)$ is a sum of terms

$$\partial_{k_1}^{s_1}\hat{g}(k_1 \cdots + k_p)\partial_{k_1}^{s_2}\omega_\infty^{-\frac{1}{2}}(k_1) \prod_{i=2}^{p} \omega_\infty^{-\frac{1}{2}}(k_i)$$

for $s_1+s_2=s$. We note that $\partial_k^s\omega_\infty^{-\frac{1}{2}}\in O(\langle k\rangle^{-\frac{1}{2}-s})$ for all $s\in\mathbb{N}$. This implies that if $\partial_k^s\hat{g}\in L^2(\mathbb{R})$ then $\partial_{k_1}^s T_\infty^0(g)\in L^2(\mathbb{R}^p)$. This proves ii) for integer s. We extend it to all $s\geq 0$ by interpolation.

Finally a direct computation shows that $\|\omega_\infty^{-\frac{1}{2}}\chi(\frac{\omega_\infty}{\kappa})\delta_x\| = O((\ln\kappa)^{\frac{1}{2}})$, which implies iii) since $\omega^{-\frac{1}{2}}\omega_\infty^{\frac{1}{2}}$ is bounded on $L^2(\mathbb{R})$. □

The following proposition follows easily from Lemmas 6.1 and 6.2.

Proposition 6.3. i) *Assume that $ga_p\in L^2(\mathbb{R})$ for $0\leq p\leq 2n$. Then*

$$\lim_{\kappa\to\infty} V_\kappa =: V \quad \textit{exists in} \quad \bigcap_{1\leq p<\infty} L^p(Q,\mathrm{d}\mu)\,.$$

ii) *V is a Wick polynomial with a Hilbert–Schmidt symbol.*

Proof. From Lemma 6.2 i) it follows that $V_\kappa\Omega\to V\Omega$ in $L^2(Q,\mathrm{d}\mu)$. The convergence in all L^p spaces for $p<\infty$ follows from the fact that V, V_κ are finite particle vectors, using Lemma 2.8. Part ii) follows also from Lemmas 6.1 and 6.2. □

It will be useful later to define the interaction term using an alternative definition of the UV-cutoff fields, namely:

$$\varphi_\kappa^{\mathrm{mod}}(x) := \phi(f_{\kappa,x}^{\mathrm{mod}})\,, \quad \text{for} \quad f_{\kappa,x}^{\mathrm{mod}} = \sqrt{2}\omega^{-\frac{1}{2}}\chi\left(\frac{\omega}{\kappa}\right)\delta_x\,, \tag{6.3}$$

leading to the UV-cutoff interaction

$$V_\kappa^{\mathrm{mod}} = \int_{\mathbb{R}} g(x) : P\big(x,\varphi_\kappa^{\mathrm{mod}}(x)\big) : \mathrm{d}x\,.$$

Clearly V_κ^{mod} is also affiliated to $\mathcal{M}_{\mathrm{c}}$. We will use later the following lemma.

Lemma 6.4. i) *V_κ^{mod} converges to V in $L^2(Q,\mathrm{d}\mu)$ when $\kappa\to\infty$.*

ii)

$$\|f_{\kappa,x}^{\mathrm{mod}}\| = O\big((\ln\kappa)^{\frac{1}{2}}\big)\,,$$

uniformly for $x\in\mathbb{R}$.

Proof. Let us denote by T_κ^{mod} the analog of T_κ for the alternative definition of UV-cutoff fields. We claim that

$$\text{s-}\lim_{\kappa\to\infty} T_\kappa^{\mathrm{mod}} = T_\infty\,. \tag{6.4}$$

which implies i). In fact arguing as in the proof of Lemma 6.2 we have

$$T_k^{\mathrm{mod}} = \Gamma\left(\mathcal{F}\chi\left(\frac{\omega}{\kappa}\right)\omega^{-\frac{1}{2}}\omega_\infty^{\frac{1}{2}}\mathcal{F}^{-1}\right)T_\infty^0\,,$$

which implies (6.4) since $\chi(\frac{\omega}{\kappa})\omega^{-\frac{1}{2}}\omega_\infty^{\frac{1}{2}}$ is uniformly bounded and converges strongly to $\omega^{-\frac{1}{2}}\omega_\infty^{\frac{1}{2}}$ when $\kappa \to \infty$. To prove ii) we write with obvious notation:

$$\begin{aligned}\omega^{-\frac{1}{2}}\chi\left(\frac{\omega}{\kappa}\right)\delta_x &= \omega^{-\frac{1}{2}}\chi\left(\frac{\omega}{\kappa}\right)F(\omega_\infty \leq C\kappa)\delta_x + \omega^{-\frac{1}{2}}\chi\left(\frac{\omega}{\kappa}\right)F(\omega_\infty \geq C\kappa)\delta_x \\ &= \chi\left(\frac{\omega}{\kappa}\right)\omega^{-\frac{1}{2}}\omega_\infty^{\frac{1}{2}}\omega_\infty^{-\frac{1}{2}}F(\omega_\infty \leq C\kappa)\delta_x \\ &\quad + \omega^{-\frac{1}{2}}\chi\left(\frac{\omega}{\kappa}\right)F(\omega_\infty \geq C\kappa)\omega_\infty\omega_\infty^{-1}\delta_x\,.\end{aligned}$$

The first term in the last line is $O((\ln\kappa)^{\frac{1}{2}})$ uniformly in x, the second is $O(1)$ if C is large enough, using Lemma C.1 and the fact that $\omega_\infty^{-1}\delta_x$ is in $L^2(\mathbb{R})$ uniformly in x. □

6.2. Massless case

We consider now the massless case $m_\infty = 0$. For simplicity we will assume that $a(x) \equiv 1$, i.e.

$$\omega = \left(D^2 + c(x)\right)^{\frac{1}{2}}, \quad c(x) > 0\,, \quad c \in S^{-\mu}\,.$$

We set as above

$$h = D^2 + c(x)\,, \quad \omega_1 = (h+1)^{\frac{1}{2}}\,.$$

Lemma 6.5. *Let $\chi \in C_0^\infty(\mathbb{R})$. Then:*

$$i) \quad \omega_1^{\frac{1}{2}}\chi(x)\omega^{-\frac{1}{2}}\,, \quad \omega_1\chi(x)\omega^{-1} \quad \textit{are bounded}.$$

If $F \in C_0^\infty(\mathbb{R})$ then

$$ii) \quad \omega_1^\delta\left[\chi(x), F\left(\frac{h}{\kappa^2}\right)\right]\omega^{-\frac{1}{2}} \in O(\kappa^{\delta-3/2}) \quad \forall\, 0 \leq \delta < 3/2\,.$$

Proof. Set $\chi = \chi(x)$. Then $\chi D^2\chi = D\chi^2 D - \chi''\chi$ and hence $\chi D^2\chi \leq CD^2 + C\chi_1$, for $\chi_1 \in C_0^\infty(\mathbb{R})$. This implies that

$$\chi(h+1)\chi \leq C(D^2 + \chi_1) \leq Ch\,,$$

since $c(x) > 0$. Therefore $\omega_1\chi\omega^{-1}$ is bounded, which proves the second statement of i). Since $\omega_1\chi^2\omega_1 \leq C(h+1)$, we also have

$$\chi\omega_1\chi^2\omega_1\chi \leq C\omega^2\,,$$

which by Heinz theorem implies that $\chi\omega_1\chi \leq C\omega$ and proves the first statement of i).

To prove ii) we write using (2.1):

$$\begin{aligned}&\omega_1^\delta\left[\chi, F\left(\frac{h}{\kappa^2}\right)\right]\omega^{-\frac{1}{2}} \\ &\quad = \frac{\mathrm{i}}{2\pi\kappa^2}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{F}(z)\left(z - \frac{h}{\kappa^2}\right)^{-1}\omega_1^\delta[\chi, h]\omega^{-\frac{1}{2}}\left(z - \frac{h}{\kappa^2}\right)^{-1}\mathrm{d}z \wedge \mathrm{d}\,\overline{z}\,.\end{aligned}$$

Since $[\chi, h] = 2D\chi' - \chi''$ we see using i) that $\omega_1^\delta[\chi, h]\omega^{-\frac{1}{2}} = \omega_1^{\delta+\frac{1}{2}}B$, where B is bounded. Using the bound $\langle h\rangle^\alpha(z - \frac{h}{\kappa^2})^{-1} \in O(\kappa^{-2\alpha})|\mathrm{Im}z|^{-1}$ for $z \in \operatorname{supp}\tilde{F}$, we obtain ii). □

To define the interaction in the massless case, we set:

$$\varphi_\kappa(x) := \sqrt{2}\phi\left(\omega^{-\frac{1}{2}}F\left(\frac{h}{\kappa^2}\right)\delta_x\right) \quad x \in \mathbb{R},$$

where $F \in C_0^\infty(\mathbb{R})$ equals 1 near 0, $\kappa \gg 1$ is again an UV cutoff parameter, and:

$$V_\kappa := \int_{\mathbb{R}} g(x) : P\big(x, \varphi_\kappa(x)\big) : \mathrm{d}x\,.$$

Lemma 6.6. *Assume that g is compactly supported and $ga_p \in L^2(\mathbb{R})$ for $0 \le p \le 2n$. Then:*

i) $\omega^{-\frac{1}{2}}F(\frac{h}{\kappa^2})\delta_x \in L^2(\mathbb{R})$ *for $x \in \operatorname{supp} g$ so the UV cutoff fields $\varphi_\kappa(x)$ are well defined.*

ii) V_κ *converges in $\bigcap_{1\le p<\infty} L^p(Q, \mathrm{d}\mu)$ to a real function V and there exists $\epsilon > 0$ such that:*

$$\|V - V_\kappa\|_{L^p(Q,\mathrm{d}\mu)} \le C(p-1)^n\kappa^{-\epsilon}, \quad \forall\, p \ge 2\,.$$

iii) *one has*

$$\left\|\omega^{-\frac{1}{2}}F\left(\frac{h}{\kappa^2}\right)\delta_x\right\| \in O\big((\ln\kappa)^{\frac{1}{2}}\big), \quad \textit{uniformly for} \quad x \in \operatorname{supp} g\,.$$

The function V in Lemma 6.6 will be denoted by:

$$V =: \int_{\mathbb{R}} g(x) : P\big(x, \varphi(x)\big) : \mathrm{d}x\,.$$

Proof. To simplify notation we set $F_\kappa = F(\frac{h}{\kappa^2})$. We take $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 on $\operatorname{supp} g$. Then for $x \in \operatorname{supp} g$, we have

$$\omega^{-\frac{1}{2}}F_\kappa\delta_x = \omega^{-\frac{1}{2}}F_\kappa\chi\delta_x = \omega^{\frac{1}{2}}F_\kappa\omega^{-1}\chi\omega_1\omega_1^{-1}\delta_x \in L^2(\mathbb{R})\,,$$

since $\omega_1^{-1}\delta_x \in L^2$ and $\omega^{-1}\chi\omega_1$ is bounded by Lemma 6.5 i).

To prove ii) we may assume that $P(x, \lambda) = \lambda^p$. We express the kernel $w_{p,\kappa}(k_1, \ldots, k_p)$ as in Lemma 6.1 and set $w_{p,\kappa} =: T_\kappa g$. Since $g = \chi^p g$, we have $w_{p,\kappa} = T_\kappa\chi^p g$, and hence $w_{p,\kappa} = \tilde{T}_\kappa g$, where:

$$\tilde{T}_\kappa^* = R\Gamma(\chi\omega^{-\frac{1}{2}}F_\kappa\mathcal{F}^{-1}) = R\Gamma(\omega_1^{-\frac{1}{2}})\Gamma\big(a(\kappa)\mathcal{F}^{-1}\big)\,,$$

for $a(\kappa) = \omega_1^{\frac{1}{2}}\chi F_\kappa\omega^{-\frac{1}{2}}$. We set also

$$\tilde{T}_\infty = R\Gamma(\chi\omega^{-\frac{1}{2}}\mathcal{F}^{-1})\,,$$

and we claim that

$$\|\tilde{T}_\kappa^* - \tilde{T}_\infty^*\| \in O(\kappa^{-\epsilon}) \quad \text{for some} \quad \epsilon > 0\,, \tag{6.5}$$

which clearly implies ii).

If we set

$$a_0(\kappa) = F_\kappa \omega_1^{\frac{1}{2}} \chi \omega^{-\frac{1}{2}} ,$$

then using Lemma 6.5 ii) , we obtain:

$$a(\kappa) = a_0(\kappa) + a_1(\kappa) , \quad \text{and}$$

$$a_0(\kappa) \in O(1) , \quad a_1(\kappa) \in O(\kappa^{-\delta}) , \quad \text{for some} \quad \delta > 0 . \tag{6.6}$$

Clearly on $\otimes^p \mathfrak{h}$, one has:

$$\Gamma(a_0 + a_1) = \sum_{I \subset \{1,\dots,p\}} a_{I(1)} \otimes \cdots \otimes a_{I(p)} =: \Gamma(a_0) + S(\kappa) , \tag{6.7}$$

for $I(j) = \mathbb{1}_I(j)$. By (6.6) the terms in (6.7) for $I \neq \emptyset$ are $O(\kappa^{-\delta})$ hence $S(\kappa)$ is $O(\kappa^{-\delta})$. Since by Lemma 6.2 $R\Gamma(\omega_1^{-\frac{1}{2}})$ is bounded, it follows that $R\Gamma(\omega_1^{-\frac{1}{2}})S(\kappa)$ is $O(\kappa^{-\delta})$. Therefore we only have to estimate

$$R\Gamma(\chi\omega^{-\frac{1}{2}}) - R\Gamma(\omega_1^{-\frac{1}{2}})\Gamma\big(a_0(\kappa)\big) = \big(R\Gamma(\omega_1^{-\frac{1}{2}}) - R\Gamma(\omega_1^{-\frac{1}{2}} F_\kappa)\big)\Gamma(\omega_1^{\frac{1}{2}} \chi \omega^{-\frac{1}{2}}) .$$

By Lemma 6.5 i) , $\Gamma(\omega_1^{\frac{1}{2}} \chi \omega^{-\frac{1}{2}})$ is bounded, and by Lemma 6.2

$$R\Gamma(\omega_1^{-\frac{1}{2}}) - R\Gamma(\omega_1^{-\frac{1}{2}} F_\kappa) \in O(\kappa^{-\epsilon}) .$$

This completes the proof of ii).

It remains to prove iii). We write for $x \in \operatorname{supp} g$:

$$\begin{aligned} \omega^{-\frac{1}{2}} F_\kappa \delta_x &= \omega^{-\frac{1}{2}} F_k \chi \delta_x \\ &= \omega^{-\frac{1}{2}} \chi F_\kappa \delta_x + \omega^{-\frac{1}{2}} [F_\kappa, \chi] \omega_1 \omega_1^{-1} \delta_x \\ &= \omega^{-\frac{1}{2}} \chi \omega_1^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} F_\kappa \delta_x + \omega^{-\frac{1}{2}} [F_\kappa, \chi] \omega_1 \omega_1^{-1} \delta_x . \end{aligned} \tag{6.8}$$

By Lemma 6.4 ii) , $\omega_1^{-\frac{1}{2}} F_\kappa \delta_x \in O((\ln \kappa)^{\frac{1}{2}})$, uniformly for $x \in \operatorname{supp} g$. Moreover by Lemma 6.5, $\omega^{-\frac{1}{2}} \chi \omega_1^{\frac{1}{2}}$ is bounded, hence the first term in the r.h.s. of (6.8) is $O(\ln \kappa)^{\frac{1}{2}}$. Next $\omega_1^{-1} \delta_x$ is in $L^2(\mathbb{R})$ uniformly in x, so the second term is $O(\kappa^{-\delta})$ for some $\delta > 0$ by Lemma 6.5 ii). This completes the proof of iii). □

7. Lower bounds

In this section we prove some lower bounds on the UV cutoff interaction V_κ. As explained in Section 3, V_κ is now considered as a function on Q. In all this section we assume that $m_\infty > 0$.

As consequence we prove Proposition 7.2, which will be needed in Section 8.

We recall from (3.2) that:

$$P(x, \lambda) = \sum_{p=0}^{2n} a_p(x) \lambda^p ,$$

for $a_{2n}(x) \equiv a_{2n} > 0$.

Lemma 7.1. *Let $f_{\kappa,x}$ and $f^{\rm mod}_{\kappa,x}$ be defined in* (3.4), (6.3). *Assume that*

$$g \geq 0\,, \quad g \in L^1(\mathbb{R})\,, \quad g a_p^{\frac{2n}{2n-p}} \in L^1(\mathbb{R})\,, \quad 0 \leq p \leq 2n-1\,.$$

Then there exists $C>0$ such that if

$$D_2 := C\left(1 + \sup_{0\leq p\leq 2n-1} \int g(x)|a_p(x)|^{\frac{2n}{2n-p}}\mathrm{d}x\right)\,, \quad D_3 = C\left(1+\int g(x)\mathrm{d}x\right)\,,$$

one has

$$\int g(x) :P\big(x,\phi(f_{\kappa,x})\big):\mathrm{d}x \geq -D_2 - D_3(\ln\kappa)^n\,, \quad \forall \kappa \geq 2\,,$$

and the analogous result for $f_{\kappa,x}$ replaced by $f^{\rm mod}_{\kappa,x}$.

Proof. We prove the lemma for $f_{\kappa,x}$, the proof for $f^{\rm mod}_{\kappa,x}$ being the same, using Lemma 6.4 ii) instead of Lemma 6.2 iii). Note first from by Lemma 6.2 iii) $\|f_{\kappa,x}\| \in O((\ln\kappa)^{\frac{1}{2}})$ uniformly in x. We will use the inequality

$$a^p b^{n-p} \leq \epsilon b^n + C_\epsilon a^n \quad \forall \epsilon > 0\,, \quad a,b \geq 0\,, \tag{7.1}$$

valid for $n,p \in \mathbb{N}$ with $p \leq n$. In fact (7.1) follows from

$$\lambda^p \leq \epsilon\lambda^n + C_\epsilon\,, \quad \forall\epsilon>0\,, \quad \lambda \geq 0\,,$$

by setting $\lambda = ba^{-1}$.

We recall the well-known *Wick identities*:

$$:\phi(f)^n:= \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m!)}\phi(f)^{n-2m}\left(-\frac{1}{2}\|f\|^2\right)^m\,. \tag{7.2}$$

We apply (7.2) to $f = f_{\kappa,x}$. Picking first ϵ small enough in (7.1) we get:

$$:\phi(f_{\kappa,x})^{2n}:\geq \frac{1}{2}\big(\phi(f_{\kappa,x})^{2n} - C(\ln\kappa)^n\big)\,.$$

Using again (7.1) for $\epsilon = 1$, we get also:

$$|:\phi(f_{\kappa,x})^p:| \leq C_2\big(|\phi(f_{\kappa,x})|^p + (\ln\kappa)^{p/2}\big) \quad 0 \leq p < 2n\,,$$

which yields:

$$\begin{aligned} :P\big(x,\phi(f_{\kappa,x})\big): \geq \frac{1}{2}&\left(\phi(f_{\kappa,x})^{2n} - C\sum_{p=0}^{2n-1} a_p(x)|\phi(f_{\kappa,x})|^p\right) \\ &- C\left((\ln\kappa)^n + \sum_{p=0}^{2n-1} a_p(x)(\ln\kappa)^{p/2}\right)\,. \end{aligned}$$

Using again (7.1), we get:

$$\begin{aligned} a_p(x)|\phi(f_{\kappa,x})|^p &= a_p(x)^{\frac{2n-p}{2n-p}}|\phi(f_{\kappa,x})|^p \leq \epsilon\phi(f_{\kappa,x})^{2n} + C_\epsilon a_p(x)^{\frac{2n}{2n-p}}\,, \\ a_p(x)(\ln\kappa)^{p/2} &= a_p(x)^{\frac{2n-p}{2n-p}}(\ln\kappa)^{p/2} \leq C\big((\ln\kappa)^n + a_p(x)^{\frac{2n}{2n-p}}\big)\,, \end{aligned}$$

which yields for ϵ small enough:

$$:P\big(x,\phi(f_{\kappa,x})\big):\geq -C\sum_{p=0}^{2n-1} a_p(x)^{\frac{2n}{2n-p}} - C(\ln\kappa)^n\,.$$

Integrating this estimate we obtain the lemma. □

As a consequence of Lemma 7.1, we have the following proposition, which allows to control a lower order polynomial by the $P(\varphi)_2$ Hamiltonian H.

Proposition 7.2. *Let $P(x,\lambda)$ be as in* (3.2). *Let*

$$H = \mathrm{d}\Gamma(\omega) + \int_{\mathbb{R}} g(x)\, P\big(x,\varphi(x)\big): \mathrm{d}x$$

and

$$Q(x,\lambda) = \sum_{r=0}^{2n-1} b_r(x)\lambda^r$$

where $gb_r \in L^2(\mathbb{R})$, $gb_r^{\frac{2n}{2n-r}} \in L^1(\mathbb{R})$. Let $D>0$ such that

$$\sup_{0\leq p\leq 2n}\|ga_p\|_2 + \sup_{0\leq r\leq 2n-1}\|gb_r\|_2 + \|g\|_1 + \sup_{0\leq p\leq 2n-1}\|ga_p^{\frac{2n}{2n-p}}\|_1 + \sup_{0\leq r\leq 2n-1}\|gb_r^{\frac{2n}{2n-r}}\|_1 \leq D\,.$$

Then

$$\pm\int_{\mathbb{R}} g(x)\, :Q\big(x,\varphi(x)\big): \mathrm{d}x \leq H + C(D)\,.$$

Proof. Set $R(x,\lambda) = P(x,\lambda)\pm Q(x,\lambda)$ and

$$W = \int_{\mathbb{R}} g(x)\, :R\big(x,\varphi(x)\big): \mathrm{d}x\,,\quad W_\kappa = \int_{\mathbb{R}} g(x)\, :R\big(x,\varphi_\kappa(x)\big): \mathrm{d}x\,.$$

It follows from Lemma 6.2, Lemma 2.8, and Lemma 7.1 that W, W_κ satisfy the conditions in Lemma 2.5 with constants C_i depending only on D. It follows then from Theorem 2.3 that

$$H \pm \int_{\mathbb{R}} g(x)\, :Q\big(x,\varphi(x)\big): \mathrm{d}x = H_0 + W \geq -C(D)\,,$$

for some constant $C(D)$ depending only on D. □

8. Higher order estimates

This section is devoted to the proof of *higher order estimates* for variable coefficients $P(\varphi)_2$ Hamiltonians. Higher order estimates are important for the spectral and scattering theory of H, because they substitute for the lack of knowledge of the domain of H.

The higher order estimates were originally proved by Rosen [11] in the constant coefficients case $\omega = (D^2+m^2)^{\frac{1}{2}}$ for $g\in C_0^\infty(\mathbb{R})$ and $P(x,\lambda)$ independent on x. The proof was later extended in [2] to the natural class $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The extension of these results to x-dependent polynomials is straightforward.

Analysing closely the proof of Rosen, one notes that a crucial role is played by the fact that the generalized eigenfunctions of the one-particle energy $(D^2 + m^2)^{\frac{1}{2}}$, namely the exponentials $\mathrm{e}^{\mathrm{i}k \cdot x}$ are *uniformly bounded* both in x and k.

To extend Rosen's proof to the variable coefficients case, it is convenient to diagonalize the one-particle energy ω in terms of eigenfunctions and generalized eigenfunctions of $\omega^2 = Da(x)D + c(x)$. However some bounds on eigenfunctions and generalized eigenfunctions are needed to replace the uniform boundedness of the exponentials in the constant coefficients case. These bounds are given by conditions (*BM*1), (*BM*2).

In this section, we will prove the following theorem.

Theorem 8.1. *Let H be a variable coefficients $P(\varphi)_2$ Hamiltonian as in Theorem* 3.1. *Assume that hypotheses* (*BM*1), (*BM*2), (*BM*3) *hold. Then there exists $b > 0$ such that for all $\alpha \in \mathbb{N}$, the following* higher order estimates *hold:*

$$\begin{aligned} \|N^\alpha (H+b)^{-\alpha}\| &< \infty \,, \\ \|H_0 N^\alpha (H+b)^{-n-\alpha}\| &< \infty \,, \\ \|N^\alpha (H+b)^{-1}(N+1)^{1-\alpha}\| &< \infty \,. \end{aligned} \tag{8.1}$$

The rest of the section is devoted to the proof of Theorem 8.1.

8.1. Diagonalization of ω

Let h, ω as in Theorem 3.1. By Subsection A.3, h is unitarily equivalent (modulo a constant term) to a Schrödinger operator $D^2 + V(x)$ for $V \in S^{-\mu}$.

Applying then standard results on the spectral theory of one dimensional Schrödinger operators, we know that there exists $\{\psi_l\}_{l\in I}$ and $\{\psi(\,\cdot\,, k)\}_{k\in\mathbb{R}}$ such that

$$\begin{gathered} \psi_l(\,\cdot\,) \in L^2(\mathbb{R}) \,, \quad \psi(\,\cdot\,, k) \in \mathcal{S}'(\mathbb{R}) \,, \\ h\psi_l = (\lambda_l + m_\infty^2)\psi_l \,, \quad \lambda_l < 0 \,, \quad \psi_l \in L^2(\mathbb{R}) \,, \\ h\psi(\,\cdot\,, k) = (k^2 + m_\infty^2)\psi(\,\cdot\,, k) \,, \quad k \in \mathbb{R}^* \,, \\ \sum_{l\in I} |\psi_l)(\psi_l| + \frac{1}{2\pi}\int_{\mathbb{R}} \big|\psi(\,\cdot\,, k)\big)\big(\psi(\,\cdot\,, k)\big|\mathrm{d}k = \mathbb{1} \,. \end{gathered}$$

Moreover using the results of Subsection A.2 and the fact that h is a real operator we can assume that

$$\overline{\psi}_l = \psi_l \,, \quad \overline{\psi}(x, k) = \psi(x, -k) \,. \tag{8.2}$$

The index set I equals either $\mathbb{N}$ or a finite subset of $\mathbb{N}$ depending on the number of negative eigenvalues of $D^2 + V$.

Let

$$\tilde{\mathfrak{h}} := l^2(I) \oplus L^2(\mathbb{R}, \mathrm{d}k) \,,$$

and

$$
\begin{aligned}
&W : L^2(\mathbb{R}, \mathrm{d}x) \to \tilde{\mathfrak{h}} , \\
&Wu := ((\psi_l|u))_{l\in I} \oplus \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \overline{\psi}(y,k)u(y)\mathrm{d}y .
\end{aligned}
\tag{8.3}
$$

Clearly W is unitary and

$$
W\omega W^* =: (\tilde{\omega}_{\mathrm{d}} \oplus \tilde{\omega}_{\mathrm{c}}) ,
$$

for

$$
\tilde{\omega}_{\mathrm{d}} = \oplus_{l\in I}(\lambda_l + m_\infty^2)^{\frac{1}{2}} , \quad \tilde{\omega}_{\mathrm{c}} = (k^2 + m_\infty^2)^{\frac{1}{2}} .
$$

If we set $\tilde{c} = WcW^*$, then it follows from (8.2) that

$$
\tilde{c}\big((u_l)_{l\in I} \oplus u(k)\big) = (\overline{u}_l)_{l\in I} \oplus \overline{u}(-k) ,
$$

i.e. $\tilde{c}$ is the direct sum of the canonical conjugation on $l^2(I)$ and the standard conjugation on $L^2(\mathbb{R}, \mathrm{d}k)$ used for the constant coefficients $P(\varphi)_2$ model.

8.2. Reduction of H

We will consider in the rest of this section the transformed Hamiltonian:

$$
\tilde{H} := \Gamma(W)H\Gamma(W)^* .
$$

In this subsection we determine the explicit form of $\tilde{H}$.

Let $(\tilde{Q}, \tilde{\mu})$ be the Q-space associated to the couple $(\tilde{h}, \tilde{c})$. We can extend $\Gamma(W) : \Gamma(\mathfrak{h}) \to \Gamma(\tilde{\mathfrak{h}})$ to a unitary map $T : L^2(Q, \mathrm{d}\mu) \to L^2(\tilde{Q}, \mathrm{d}\tilde{\mu})$.

Lemma 8.2. *T is an isometry from $L^p(Q,\ \mu)$ to $L^p(\tilde{Q}, \mathrm{d}\tilde{\mu})$ for all $1 \le p \le \infty$ and $T1 = 1$.*

Proof. If F is a real measurable function on Q, and $m(F)$ the operator of multiplication by F on $\Gamma(\mathfrak{h})$, then $m(TF) = \Gamma(W)m(F)\Gamma(W)^*$, which shows that T is positivity preserving. Since $T1 = T^*1 = 1$, T is doubly Markovian, hence a contraction on all L^p spaces (see [13]). We use the same argument for T^{-1}. □

Coming back to $\tilde{H}$ we have:

$$
\tilde{H} = \overline{\tilde{H}_0 + \tilde{V}} ,
$$

for

$$
\tilde{H}_0 := \Gamma(W)H_0\Gamma(W)^* = \mathrm{d}\Gamma(\tilde{\omega}_{\mathrm{d}} \oplus \tilde{\omega}_{\mathrm{c}}) , \quad \tilde{V} := \Gamma(W)V\Gamma(W)^* .
$$

We know from Lemma 6.4 that V is the limit in $\bigcap_{1\le p<\infty} L^p(Q, \mathrm{d}\mu)$ of V_κ^{mod}, where V_κ is a sum of terms of the form

$$
\int_{\mathbb{R}} g a_p(x) \prod_1^r a^*(f_{\kappa,x}^{\mathrm{mod}}) \prod_{r+1}^p a(f_{\kappa,x}^{\mathrm{mod}})\mathrm{d}x ,
$$

where $f_{\kappa,x}^{\mathrm{mod}} = \omega^{-\frac{1}{2}}\chi(\frac{\omega}{\kappa})\delta_x$. This implies using Lemma 8.2 that

$$
\tilde{V} = \lim_{\kappa\to\infty} \tilde{V}_\kappa , \quad \text{in} \quad \bigcap_{1\le p<\infty} L^p(Q, \mathrm{d}\mu)
$$

where $\tilde{V}_\kappa$ is a sum of terms of the form

$$\int_{\mathbb{R}} g a_p(x) \prod_1^r a^*(W f^{\mathrm{mod}}_{\kappa,x}) \prod_{r+1}^p a(W f^{\mathrm{mod}}_{\kappa,x}) \mathrm{d}x \,.$$

Another useful expression of $\tilde{V}$ is

$$\tilde{V} = \int_{\mathbb{R}} g(x) :P\big(x, \tilde{\varphi}(x)\big): \mathrm{d}x \,, \tag{8.4}$$

for

$$\tilde{\varphi}(x) = \varphi(W\delta_x) \,.$$

Therefore we see that $\tilde{H}$ is very similar to a $P(\varphi)_2$ Hamiltonian with constant coefficients, the only differences being that in addition to the usual one-particle energy $(k^2 + m_\infty^2)^{\frac{1}{2}}$ we have the diagonal operator $\tilde{\omega}_{\mathrm{d}}$, and in the interaction the delta function δ_x is replaced by $W\delta_x$.

From now on we will work with $\tilde{H}$ and to simplify notation we will omit the tildes on the objects $\tilde{Q}$, $\tilde{\mu}$, $\tilde{H}$, $\tilde{H}_0$, $\tilde{V}$, $\tilde{\mathfrak{h}}$, $\tilde{\omega}_{\mathrm{d}}$, $\tilde{\omega}_{\mathrm{c}}$. The one-particle energy $\omega_{\mathrm{d}} \oplus \omega_{\mathrm{c}}$ will be denoted simply by ω.

8.3. Cutoff Hamiltonians

We first recall some facts from [2].

Let $\mathfrak{h}$ be a Hilbert space equipped with a conjugation c. Let $\pi_1 : \mathfrak{h} \to \mathfrak{h}_1$ be an orthogonal projection on a closed subspace $\mathfrak{h}_1$ of $\mathfrak{h}$ with $[\pi_1, c] = 0$. Let $\mathfrak{h}_1^\perp$ be the orthogonal complement of $\mathfrak{h}_1$. In all formulas below we will consider π_1 as an element of $B(\mathfrak{h}, \mathfrak{h}_1)$. With this convention the orthogonal projection on $\mathfrak{h}_1$, considered as an element of $B(\mathfrak{h}, \mathfrak{h})$, is equal to $\pi_1^*\pi_1$.

Let $U : \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_1^\perp) \to \Gamma(\mathfrak{h})$ the canonical unitary map. We denote by $L^2(Q_1, \mathrm{d}\mu_1)$, $L^2(Q_1^\perp, \mathrm{d}\mu_1^\perp)$ the Q-space representations of $\Gamma(h_1)$, $\Gamma(\mathfrak{h}_1^\perp)$. Recall that by [2, Proposition 5.3], we may take as Q-space representation of $\Gamma(\mathfrak{h})$ the space $L^2(Q, \mathrm{d}\mu)$ for $Q = Q_1 \times Q_1^\perp$, $\mu = \mu_1 \otimes \mu_1^\perp$. Accordingly we denote by $(q_1, q_1^\perp)$ the elements of $Q = Q_1 \times Q_1^\perp$.

If $W \in B(\Gamma(\mathfrak{h}))$ we set:

$$B\big(\Gamma(\mathfrak{h})\big) \ni \Pi_1 W := U\big(\Gamma(\pi_1) W \Gamma(\pi_1^*) \otimes \mathbb{1}\big) U^* \,.$$

The following lemma is shown in [2, Subsection 7.1].

Lemma 8.3. i) *If $w \in B_{\mathrm{fin}}(\Gamma(\mathfrak{h}))$ then*

$$\Pi_1 \mathrm{Wick}(w) = \mathrm{Wick}\big(\Gamma(\pi_1^*\pi_1) w \Gamma(\pi_1^*\pi_1)\big) \,. \tag{8.5}$$

ii) *If V is a multiplication operator by a function in $L^2(Q, \mathrm{d}\mu)$ then $\Pi_1 V$ is the operator of multiplication by the function*

$$\Pi_1 V(q_1) = \int_{Q_1^\perp} V(q_1, q_1^\perp) \mathrm{d}\mu_1^\perp \,. \tag{8.6}$$

In particular if $W = \Pi_1^q a^*(h_i)\Pi_1^p a(g_i)$, then

$$\Pi_1 W = \Pi_1^q a^*(\pi_1^*\pi_1 h_i)\Pi_1^p a(\pi_1^*\pi_1 g_i)\,. \tag{8.7}$$

Let now $\{\pi_n\}_{n\in\mathbb{N}}$ be a sequence of orthogonal projections on $\mathfrak{h}$ such that

$$\pi_n \leq \pi_{n+1},\ [\pi_n, c] = 0\,, \quad \text{s-}\lim_{n\to+\infty} \pi_n = 1\!\!1\,, \tag{8.8}$$

and let Π_n the associated maps defined by (8.5). Using the representation (8.6) it is shown in [14, Proposition 4.9] that

$$\begin{array}{l} \text{i)}\ \ \Pi_n V \to V \ \text{ in } \ L^p(Q,\mathrm{d}\mu)\,, \ \text{ when } \ n\to\infty\,, \ \text{ if } \ V\in L^p(Q,\mathrm{d}\mu)\,, \ \ 1\leq p<\infty \\ \text{ii)}\ \ \|\mathrm{e}^{-t\Pi_n V}\|_{L^1(Q,\mathrm{d}\mu)} \leq \|\mathrm{e}^{-tV}\|_{L^1(Q,d\mu)}\,. \end{array} \tag{8.9}$$

8.4. Notation

Index sets

An element $u \in \mathfrak{h}$ is of the form $(u_l) \oplus u(k) \in l^2(I) \oplus L^2(\mathbb{R}, \mathrm{d}k)$. We put together the variables $l \in I$ and $k \in \mathbb{R}$ into a single variable $K \in I \sqcup \mathbb{R}$. We denote by $\mathrm{d}K$ the measure on $I \sqcup \mathbb{R}$ equal to the sum of the counting measure on I and the Lebesgue measure on $\mathbb{R}$. Then $\mathfrak{h} = L^2(I \sqcup \mathbb{R}, \mathrm{d}K)$ and

$$(u|v)_{\mathfrak{h}} = \sum_{l\in I} \overline{u}_l v_l + \int_{\mathbb{R}} \overline{u}(k)v(k)\mathrm{d}k = \int_{I\sqcup\mathbb{R}} \overline{u}(K)v(K)\mathrm{d}K\,.$$

For $K \in I \sqcup \mathbb{R}$ we set:

$$\omega(K) := \begin{cases} (k^2 + m_\infty^2)^{\frac{1}{2}} & \text{if} \quad K = k \in \mathbb{R}\,, \\ (\lambda_l + m_\infty^2)^{\frac{1}{2}} & \text{if} \quad K = l \in I\,, \end{cases}$$

so that the operator ω is the operator of multiplication by $\omega(K)$ on $L^2(I\sqcup\mathbb{R}, \mathrm{d}K)$. We set also:

$$|K| := \begin{cases} |k| & \text{if} \quad K = k \in \mathbb{R}\,, \\ l & \text{if} \quad K = l \in I\,. \end{cases}$$

$$a^\sharp(K) := \begin{cases} a^\sharp(k) & \text{if} \quad K = k \in \mathbb{R}\,, \\ a^\sharp(e_l) & \text{if} \quad K = l \in I\,, \end{cases}$$

where $\{e_l\}_{l\in I}$ is the canonical basis of $l^2(I)$.

Lattices

For $\nu \geq 1$, we consider the lattice $\nu^{-1}\mathbb{Z}$ and let

$$\mathbb{R} \ni k \mapsto [k]_\nu \in \nu^{-1}\mathbb{Z}$$

be the integer part of k defined by $-(2\nu)^{-1} < k - [k]_\nu \leq (2\nu)^{-1}$. We extend the function $[\,\cdot\,]_\nu$ to $I \sqcup \mathbb{R}$ by setting

$$[K]_\nu := \begin{cases} [k]_\nu & \text{if} \quad K = k \in \mathbb{R}\,, \\ l & \text{if} \quad K = l \in I\,. \end{cases}$$

As above we put together the variables $l \in I$ and $\gamma \in \nu^{-1}\mathbb{Z}$ into a single variable $\delta \in I \sqcup \nu^{-1}\mathbb{Z}$. For $\kappa \in [1, +\infty[$ an UV cutoff parameter, we denote by $\Gamma_{\kappa,\nu}$ the finite lattice $\nu^{-1}\mathbb{Z} \cap \{|\gamma| \leq \kappa\}$.

As in [2, Section 7.1] we choose increasing sequences κ_n, ν_n tending to $+\infty$ in such a way that

$$\Gamma_{\kappa_n,\nu_n} \subset \Gamma_{\kappa_{n+1},\nu_{n+1}} .$$

We denote by Γ_n the finite lattice Γ_{κ_n,ν_n}. The finite subset of $I \sqcup \nu^{-1}\mathbb{Z}$:

$$T_n := \{l \in I | \;\; l \leq \kappa_n\} \sqcup \Gamma_n$$

can be rewritten as

$$T_n = \{\delta \in I \sqcup \nu^{-1}\mathbb{Z} | \;\; |\delta| \leq \kappa_n\} .$$

Finite dimensional subspaces

For $\gamma \in \nu^{-1}\mathbb{Z}$ we denote by $e_\gamma \in L^2(\mathbb{R}, \mathrm{d}k)$ the vector $e_\gamma(k) = \nu^{\frac{1}{2}} \mathbb{1}_{]-(2\nu)^{-1},(2\nu)^{-1}]} (k - \gamma)$.

Following our previous convention we set for $\delta \in I \sqcup \nu^{-1}\mathbb{Z}$:

$$e_\delta := \begin{cases} 0 \oplus e_\gamma & \text{if} \quad \delta = \gamma \in \nu^{-1}\mathbb{Z} , \\ e_l \oplus 0 & \text{if} \quad \delta = l \in I . \end{cases}$$

Clearly (e_δ) is an orthonormal family in $\mathfrak{h}$.

For $n \in \mathbb{N}$ we denote by $\mathfrak{h}_n$ the finite dimensional subspace of $\mathfrak{h}$ spanned the e_δ for $\delta \in T_n$, and denote by $\pi_n : \mathfrak{h} \to \mathfrak{h}_n$ the orthogonal projection on the finite dimensional subspace $\mathfrak{h}_n$. Note that $\mathfrak{h}_n$ is invariant under the conjugation c.

Finally we set

$$a^\sharp(\delta) := \begin{cases} a^\sharp(e_\gamma) & \text{if} \quad \delta = \gamma \in \nu^{-1}\mathbb{Z} , \\ a^\sharp(e_l) & \text{if} \quad \delta = l \in I . \end{cases}$$

8.5. Proof of the higher order estimates

For $0 \leq \tau \leq 1$ and $n \in \mathbb{N}$ we set:

$$N_n^\tau = \int \omega\big([K]_{\nu_n}\big)^\tau a^*(K) a(K) \mathrm{d}K .$$

Note that with the notation in Subsection 8.1:

$$N_n^\tau = \mathrm{d}\Gamma\bigg(\Big(\omega_{\mathrm{d}} \oplus \omega_{\mathrm{c}}\big([k]_{\nu_n}\big)\Big)^\tau\bigg) = \mathrm{d}\Gamma\Big(\omega\big([K]_{\nu_n}\big)^\tau\Big) .$$

We set also

$$H_{0,n} = N_n^1 , \quad H_n = H_{0,n} + V_n ,$$

where

$$V_n = \Pi_n V .$$

Lemma 8.4. *There exists $C > 0$ such that*

i) $(N_n^\tau + C)^{-1} \to (N^\tau + C)^{-1}$,
ii) $(H_n + C)^{-1} \to (H + c)^{-1}$,

in norm when $n \to \infty$.

Proof. To prove i) we note that $\omega_{\mathrm{c}}([k]_{\nu_n})^\tau$ converges in norm to ω_{c}^τ for $0 \le \tau \le 1$, which implies that $N^{-\frac{1}{2}}(N_n^\tau - N^\tau)N^{-\frac{1}{2}}$ tends to 0 in norm. Since $\omega([K]_{\nu_n}) \ge c > 0$ uniformly in n, we know that $N^{\frac{1}{2}}(N_n^\tau + 1)^{-1}$ is bounded uniformly in n. This implies that $(N_n^\tau + 1)^{-1}$ converges in norm to $(N^\tau + 1)^{-1}$ when $n \to \infty$.

To prove ii) we follow the proof of [14, Proposition 4.8]: we have seen above that $N^{-\frac{1}{2}}(H_{0,n} - H_0)N^{-\frac{1}{2}}$ tends to 0 in norm. Moreover $\omega_{\mathrm{d}} \oplus \omega([k]_{\nu_n}) \ge C > 0$ uniformly w.r.t. n. This implies that $\mathrm{e}^{-tH_{0,n}}$ is hypercontractive with hypercontractivity bounds uniform in n. This implies that if $W \in L^p(Q, \mathrm{d}\mu)$ and $\mathrm{e}^{-TW} \in L^1(Q, \mathrm{d}\mu)$ there exists C such that $N \le C(H_{0,n} + W + C)$, uniformly in n. Writing

$$(H_{0,n}+W+C)^{-1}-(H_0+W+C)^{-1} = (H_{0,n}+W+C)^{-1}(H_0-H_{0,n})(H_0+W+C)^{-1}$$

and using the above bound, we obtain that $(H_{0,n} + W + C)^{-1}$ converges in norm to $(H + W + C)^{-1}$. Moreover it follows from Theorem 2.3 ii) that the constant C above depend only on $\|e^{-tW}\|_{L^1}$ for some $t > 0$.

Since by (8.9) e^{-tV_m} is uniformly bounded in $L^1(Q)$, we see that $(H_{0,n} + V_m + C)^{-1}$ converges in norm to $(H_0 + V_m + C)^{-1}$ when $n \to \infty$, uniformly w.r.t. m. Again by (8.9) $V_m \to V$ in L^p for some $p > 2$ and e^{-tV_m} is uniformly bounded in L^1, so by Proposition 2.4 we obtain that $(H_0 + V_m + C)^{-1}$ converges to $(H_0 + V + C)^{-1}$ when $m \to \infty$, which completes the proof of the lemma. □

Let us denote simply by ω_n the operator $\omega_{\mathrm{d}} \oplus \omega_{\mathrm{c}}(([k]_{\nu_n})$. Since $[\omega_n, \pi_n^* \pi_n] = 0$, we have

$$H_{0,n} = U_n\big(\mathrm{d}\Gamma(\omega_n|_{\mathfrak{h}_n}) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega_n|_{\mathfrak{h}_n^\perp})\big)U_n^*\,,$$

where $U_n : \Gamma(\mathfrak{h}_n) \otimes \Gamma(\mathfrak{h}_n^\perp) \to \Gamma(\mathfrak{h})$ is the exponential map. This implies that

$$U_n^* H_n U_n = \hat{H}_n \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega_n|_{\mathfrak{h}_n^\perp})\,, \quad U_n^* N_n^\tau U_n = \hat{N}_n^\tau \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega_n^\tau|_{\mathfrak{h}_n^\perp})\,,$$

for $\hat{H}_n = \mathrm{d}\Gamma(\omega_n|_{\mathfrak{h}_n}) + V_n$, $\hat{N}_n^\tau = \mathrm{d}\Gamma(\omega_n^\tau|_{\mathfrak{h}_n})$.

Proposition 8.5. *Assume hypotheses* (*BMi*) *for* $i = 1, 2, 3$. *Set for* $J = \{1, \ldots, , s\} \subset \mathbb{N}$ *and* $K_i \in I \sqcup \mathbb{R}$*:*

$$V_n^J := \mathrm{ad}_{a(K_1)} \ldots \mathrm{ad}_{a(K_s)} V_n\,.$$

Then there exists $b, c > 0$ *such that for all* $\lambda_1, \lambda_2 < -b$

$$\|(H_n - \lambda_2)^{-\frac{1}{2}} V_n^J (H_n - \lambda_1)^{-\frac{1}{2}}\| \le c \prod_1^s F(K_i)\,,$$

where $F : I \sqcup \mathbb{R} \to \mathbb{R}^+$ *satisfies for each* $\delta > 0$*:*

$$\int_{I \sqcup \mathbb{R}} |F(x, K)|^2 \omega(K)^{-\delta} \mathrm{d}K \le C\,.$$

Proof. We have using (8.7):

$$V_n = \int g(x) : P\big(x, \varphi_n(x)\big) : \mathrm{d}x\,, \tag{8.10}$$

where

$$:\varphi_n(x)^p := \sum_{r=0}^{p} \binom{p}{r} \prod_{1}^{r} a^*(\pi_n^*\pi_n W\omega^{-\frac{1}{2}}\delta_x) \prod_{r+1}^{p} a(\pi_n^*\pi_n W\omega^{-\frac{1}{2}}\delta_x)\,. \tag{8.11}$$

We note that

$$\pi_n^*\pi_n = \sum_{|\delta|\le\kappa_n} |e_\delta\rangle\langle e_\delta|\,,$$

which yields

$$\begin{aligned} &:\varphi_n(x)^p: \\ &\quad = \sum_{r=0}^{p} \binom{p}{r} \sum_{\delta_1,\dots,\delta_p\in T_n} \prod_{1}^{r} a^*(\delta_i) \prod_{r+1}^{p} a(\delta_i) \prod_{1}^{r} m_n(x,\delta_i) \prod_{r+1}^{p} \overline{m}_n(x,\delta_i)\,, \end{aligned} \tag{8.12}$$

where

$$m_n(x,\delta) = (e_\delta|W\delta_x)_{\mathfrak{h}}\,.$$

Let for $k\in\mathbb{R}$:

$$C_n(k) := \left[[k]_{\nu_n} - \frac{1}{2}\nu_n^{-1}, [k]_{\nu_n} + \frac{1}{2}\nu_n^{-1}\right],$$

be the cell of Γ_n centered at $[k]_{\nu_n}$. Using (8.3) we get:

$$m_n(x,\delta) := \begin{cases} \nu_n^{\frac{1}{2}} \int_{C_n(\gamma)} (k^2+m_\infty^2)^{-\frac{1}{4}} \overline{\psi}(x,k)\mathrm{d}k & \text{if} \quad \delta=\gamma\in\Gamma_n\,, \\ (\lambda_l+m_\infty^2)^{-\frac{1}{4}}\psi_l(x)\,, \quad \text{if} \quad \delta = l\in I\,. \end{cases} \tag{8.13}$$

Then as in [2, 11], we obtain that

$$V_n^J = \int_{\mathbb{R}} g(x) \prod_{1}^{s} r_n(x,K_i) :P^{(s)}\big(x,\varphi_n(x)\big): \mathrm{d}x\,,$$

where $P^{(s)}(x,\lambda) = (\frac{\mathrm{d}}{\mathrm{d}\lambda})^s P(x,\lambda)$ and

$$r_n(x,K) = \begin{cases} \nu_n \int_{C_n(k)} \omega(k')^{-\frac{1}{2}} \psi(x,k')\mathrm{d}k' & \text{if} \quad K=k\in\mathbb{R}\,, \\ \psi_l(x) \quad \text{if} \quad K=l\in I\,. \end{cases}$$

We note that

$$V_n^J = \Pi_n \int_{\mathbb{R}} g(x) :R_n\big(x,K_1,\dots,K_s,\varphi(x)\big): \mathrm{d}x\,, \tag{8.14}$$

for

$$R_n(x,K_1,\dots,K_s,\lambda) = P^{(s)}(x,\lambda)\prod_{1}^{s} r_n(x,K_i)\,.$$

Since assumptions $(BM1)$, $(BM2)$ are satisfied, we know that:

$$\begin{aligned} |\psi(x,k)| &\le CM(x)\,, \quad \text{uniformly for} \quad x,k\in\mathbb{R}\,, \\ |\psi_l(x)| &\le C\epsilon_l M(x)\,, \quad \text{uniformly for} \quad x\in\mathbb{R}\,, \quad l\in I\,, \end{aligned}$$

where $\sum_{l\in I}\epsilon_l^2 < \infty$.

Let us now prove corresponding bounds on the functions $r_n(x,K)$. We consider first the case $K = l \in I$: we have:

$$|r_n(x,l)| \leq C\epsilon_l M(x)\,, \quad \text{uniformly in} \quad x,l\,. \tag{8.15}$$

If $K = k \in \mathbb{R}$ we get:

$$\omega(k')^{-\frac{1}{2}}|\psi(x,k')| \leq C\omega(k)^{-\frac{1}{2}}M(x)\,, \quad \text{uniformly for} \quad n \in \mathbb{N}\,, \quad k' \in C_n(k)\,, \quad x \in \mathbb{R}\,,$$

which yields:

$$|r_n(x,k)| \leq C\omega(k)^{-\frac{1}{2}}M(x)\,, \quad \text{uniformly for} \quad n \in \mathbb{N}\,, \quad k,x \in \mathbb{R}\,. \tag{8.16}$$

If we set:

$$F(K) = \begin{cases} \omega(k)^{-\frac{1}{2}} \quad \text{if} \quad K = k \in \mathbb{R}\,, \\ \epsilon_l \quad \text{if} \quad K = l \in I\,, \end{cases} \tag{8.17}$$

and collect (8.15), (8.16) we get:

$$|r_n(x,K)| \leq CF(K)M(x)\,, \quad \text{uniformly for} \quad n \in \mathbb{N}\,, \quad K \in I \sqcup \mathbb{R}\,, \quad x \in \mathbb{R}\,. \tag{8.18}$$

We note that by condition ($BM3$), we have:

$$ga_pM^s \in L^2\,, \quad g(a_pM^s)^{\frac{2n}{2n-p+s}} \in L^1\,, \quad 0 \leq s \leq p \leq 2n-1\,.$$

If we apply the arguments in Proposition 7.2 to the polynomial

$$Q_n(x,K_1,\dots,K_s,\lambda) = P^{(s)}(x,\lambda)\prod_1^s F(K_i)^{-1}r_n(x,K_i)\,,$$

using the bound (8.18), we obtain that

$$\mathrm{e}^{-t(V \pm W_n)} \quad \text{is uniformly bounded in} \quad L^1(Q)\,, \tag{8.19}$$

for

$$W_n(K_1,\dots K_s) = \int g(x) :Q_n\big(x,K_1,\dots,K_s,\varphi(x)\big): \mathrm{d}x\,.$$

By (8.9) ii) , this implies that

$$\mathrm{e}^{-t(V_n \pm \Pi_n W_n)} \quad \text{is uniformly bounded in} \quad L^1(Q)\,.$$

Applying then Theorem 2.7 to $a = \omega_n$ and using (8.19) we get that there exists $C > 0$ such that

$$\pm V_n^J \leq \prod_1^s F(K_i)(H_n + C)\,, \quad \text{uniformly in} \quad n\,.$$

To complete the proof of the proposition it remains to check that for each $\delta > 0$

$$\int_{I \sqcup \mathbb{R}} F(K)^2\omega(K)^{-\delta}\mathrm{d}K < \infty\,,$$

which follows from (8.17) since $\sum_{l\in I}\epsilon_l^2 < \infty$. □

Proof of Theorem 8.1. We follow the proof in [11]. This proof consists in first proving higher order estimates for the cutoff Hamiltonians H_n and N_n^τ, with constants uniform in n. The corresponding estimates for the Hamiltonians without cutoffs are then obtained by the principle of cutoff independence [11, Proposition 4.1]. The convergence results needed to apply [11, Proposition 4.1] are proved in Lemma 8.4. The estimates for the cutoff Hamiltonians rely on three kinds of intermediate results:

The first ([11, Lemma 4.2], [11, Corollary 4.3]) consists of identities expressing expectation values of (powers of) N^τ in terms of Wick monomials. These identities carry over directly to our case, replacing $\mathbb{R}$ by $I \sqcup \mathbb{R}$, $a^\sharp(k)$ by $a^\sharp(K)$ and the mesure $\mathrm{d}k$ by $\mathrm{d}K$.

The second [11, Proposition 4.5] is the generalized pull through formula which also carries over to our case. The last is the bound in [11, Lemma 4.4] which is replaced in our case by Proposition 8.5. Carefully looking at the proof of the higher order estimates for the cutoff Hamiltonians in [11, Theorem 4.7] and [11, Corollary 4.8] we see that it relies on the fact that the

$$\int_{I \sqcup \mathbb{R}} F(K)^2 \omega(K)^{-\delta} \mathrm{d}K < \infty \,,$$

(in [11] $F(K)$ equals simply $\omega(k)^{-\frac{1}{2}}$), which is checked in Proposition 8.5. This completes the proof of Theorem 8.1. □

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Appendix A.

In this section we will give sufficient conditions on the functions a, c in the definition of ω for conditions $(BM1)$, $(BM2)$ to hold.

A.1. Sufficient conditions for $(BM1)$

Proposition A.1. *Let $h = Da(x)D + c(x)$ be as in Theorem* 3.1*. Then:*

i) *condition $(BM1)$ is satisfied for $M(x) = \langle x \rangle^\alpha$ for $\alpha > \frac{1}{2}$.*
ii) *if h has a finite number of eigenvalues, condition $(BM1)$ is satisfied for $M(x) = 1$.*

Proof. ii) is obvious. To prove i) we take an orthonormal basis $\{\psi_l\}_{l \in I}$ of the point spectrum subspace of h, and set $u_l = \langle D \rangle^s \langle x \rangle^{-\alpha} \psi_l$ for some $s > \frac{1}{2}$. By Sobolev's theorem we have

$$\| \langle x \rangle^{-\alpha} \psi_l \|_\infty^2 \leq C \| u_l \|_2^2 \,, \quad \text{uniformly in} \quad l \in I \,.$$

Next

$$\begin{aligned}\sum_{l\in I}\|u_l\|_2^2 &= \mathrm{Tr}\sum_{l\in I}|u_l)(u_l| \\ &= \mathrm{Tr}\langle D\rangle^s\langle x\rangle^{-\alpha}\sum_{l\in I}|\psi_l)(\psi_l|\langle x\rangle^{-\alpha}\langle D\rangle^s \\ &= \mathrm{Tr}\langle D\rangle^s\langle x\rangle^{-\alpha}\mathbb{1}_{]-\infty,m_\infty^2]}(h)\langle x\rangle^{-\alpha}\langle D\rangle^s\,.\end{aligned}$$

We use then that $\mathbb{1}_{]-\infty,m_\infty^2]}(h)\langle D\rangle^m$ is bounded for all $m\in\mathbb{N}$ by elliptic regularity, which implies that $\mathbb{1}_{]-\infty,m_\infty^2]}(h)\langle x\rangle^{-\alpha}\langle D\rangle^s$ is Hilbert–Schmidt if $\alpha>\frac{1}{2}$. This completes the proof of the proposition. □

A.2. Generalized eigenfunctions

In this subsection we show that if $h = Da(x)D + c(x)$ is a second order differential operator as in Theorem 3.1 satisfying $(BM2)$, then the generalized eigenfunctions $\psi(x,k)$ can be chosen to satisfy additionally the following reality condition:

$$\overline{\psi}(x,k)=\psi(x,-k)\,,\quad k \text{ a.e.}$$

Lemma A.2. *Assume that the family $\{\phi(\,\cdot\,,k)\}_{k\in\mathbb{R}}$ satisfies assumption $(BM2)$. Then there exists a family $\{\psi(\,\cdot\,,k)\}_{k\in\mathbb{R}}$ of generalized eigenfunctions of h satisfying $(BM2)$ and additionally:*

$$\overline{\psi}(x,k)=\psi(x,-k)\,,\quad k \text{ a.e.} \tag{A.1}$$

Let $\{\phi(x,k)\}_{k\in\mathbb{R}}$ be a basis of generalized eigenfunctions for h. To such a family one can associate a unitary map:

$$W_\phi:\mathbb{1}_{[m_\infty^2,+\infty[}(h)L^2(\mathbb{R},\mathrm{d}x)\to L^2(\mathbb{R},\mathrm{d}x)\,,$$

defined by

$$W_\phi u(x)=(2\pi)^{-1}\int\!\!\int \mathrm{e}^{\mathrm{i}kx}\overline{\phi}(y,k)u(y)\mathrm{d}y\;\mathrm{d}k\,, \tag{A.2}$$

which satisfies

$$W_\phi h=(D^2+m_\infty^2)W_\phi\,.$$

Note that if ψ satisfy (A.1) and W_ψ is defined as in (A.2), then W_ψ is a *real* operator i.e.

$$\overline{W_\psi u}=W_\psi\overline{u}\,,\quad u\in L^2(\mathbb{R})\,.$$

Proof. Let us define the unitary operator $\Omega: L^2(\mathbb{R},\mathrm{d}x)\to L^2(\mathbb{R}^+,\mathrm{d}k)\otimes\mathbb{C}^2$ obtained from W_ϕ by Fourier transform:

$$\Omega u(k)=(2\pi)^{-1}\Big(\big(\phi(\,\cdot\,,k)|u\big),\big(\phi(\,\cdot\,,-k)|u\big)\Big)\,,$$

satisfying $\Omega h=(k^2+m_\infty^2)\otimes\mathbb{1}_{\mathbb{C}^2}\Omega$.

Set

$$\tilde{\phi}(x,k):=\overline{\phi}(x,-k)\quad x\,,\quad k\in\mathbb{R}\,. \tag{A.3}$$

Clearly $\{\tilde{\phi}(\,\cdot\,,k)\}_{k\in\mathbb{R}}$ is a family of generalized eigenfunctions of h. Therefore we can introduce the unitary map

$$\tilde{\Omega}u(k) = (2\pi)^{-1}\Big(\big(\tilde{\phi}(\,\cdot\,,k)|u\big), \big(\tilde{\phi}(\,\cdot\,,-k)|u\big)\Big)\,.$$

If $S = \Omega\tilde{\Omega}^{-1}$, then S commutes with $(k^2 + m_\infty^2)\otimes \mathbb{1}_{\mathbb{C}^2}$ and is unitary, so:

$$S = \int_{\mathbb{R}^+}^{\oplus} S(k)\mathrm{d}k\,,$$

for $S(k) \in U(\mathbb{C}^2)$. Using that $(\overline{\phi}(x,k), \overline{\phi}(x,-k)) = \Omega\delta_x$ for $x \in \mathbb{R}$, the similar identity for $\tilde{\phi}$ and (A.3), we obtain that

$$\big(\overline{\phi}(x,k), \overline{\phi}(x,-k)\big) = S(k)T\big(\phi(x,k), \phi(x,-k)\big)\,, \quad x\in\mathbb{R}\,, \quad k>0\,, \tag{A.4}$$

where $T(z_1,z_2) = (z_2,z_1)$. Iterating this formula we obtain the identity:

$$T\overline{S}T = S^{-1}\,. \tag{A.5}$$

Let us find the generalized eigenfunctions $\psi(x,k)$ under the form

$$\big(\psi(x,k), \psi(x,-k)\big) = A(k)\big(\phi(x,k), \phi(x,-k)\big) \quad x\in\mathbb{R}\,, \quad k>0\,.$$

Clearly $\{\psi(\,\cdot\,,k)\}_{k\in\mathbb{R}}$ will be a basis of generalized eigenfunctions of h as soon as $A(k) \in U(\mathbb{C}^2)$. Using (A.4) we see that it will satisfy (A.1) if

$$\overline{A}(k) = TA(k)\overline{S}(k)T\,. \tag{A.6}$$

To solve (A.6), we deduce first from (A.5) that

$$T\overline{S}^nT = S^{-n}\,, \quad n\in\mathbb{N}\,.$$

Therefore $Tg(\overline{S})T = g(S^{-1})$ if g is a polynomial. By the standard approximation argument and the spectral theorem for unitary operators this extends to all measurable functions on the unit circle.

For $z = \mathrm{e}^{\mathrm{i}\theta}$, $-\pi < \theta \leq \pi$, we set $z^\alpha = \mathrm{e}^{i\alpha\theta}$. Setting $A(k) = (\overline{S})^{-\frac{1}{2}}(k)$, we get

$$TA\overline{S}T = T\overline{S}^{\frac{1}{2}}T = S^{-\frac{1}{2}}\,.$$

Since $\overline{z}^{-\frac{1}{2}} = \overline{z^{-\frac{1}{2}}}$, we obtain that $S^{-\frac{1}{2}}(k) = \overline{A}(k)$. Therefore $A(k)$ satisfies (A.6). Moreover since $z^{-\frac{1}{2}}$ preserves the unit circle, $A(k)$ is unitary. Therefore the family $\{\psi(\,\cdot\,,k)\}_{k\in\mathbb{R}}$ is a basis of generalized eingenfunctions of h. Moreover since the matrix $A(k)$ is unitary, all entries have modulus less than 1, which implies that if $\{\phi(\,\cdot\,,k)\}_{k\in\mathbb{R}}$ satisfies $(BM2)$, so does $\{\psi(\,\cdot\,,k)\}_{k\in\mathbb{R}}$. □

A.3. Reduction to the case of the constant metric

We show in this subsection that in order to verify condition $(BM2)$ we can reduce ourselves to the case $a(x)\equiv 1$. We have then

$$h = D^2 + V + m_\infty^2\,, \quad \text{for} \quad V(x) = c(x) - m_\infty^2 \in S^{-\mu}\,,$$

which will allow to use standard results on generalized eigenfunctions for Schrödinger operators in one dimension.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism with $\psi' > 0$. We denote by ψ^{-1} the inverse of ψ. To ψ we associate the unitary map $T_\psi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$T_\psi u(x) := \psi'(x)^{\frac{1}{2}} u \circ \psi(x) .$$

Lemma A.3. *Let a, c satisfying* (3.1) *and set $g = a^{\frac{1}{2}}$ and $\psi = \phi^{-1}$ for*

$$\phi(x) = \int_0^x \frac{1}{g(s)}\, ds .$$

Then

i)

$$Da(x)D + c(x) = T_\psi^* \big(D^2 + \tilde{c}(x)\big) T_\psi ,$$

where

$$\tilde{c}(x) = c \circ \psi(x) + \left(\frac{1}{4}(g')^2 + \frac{1}{2} g g''\right) \circ \psi(x) .$$

ii) $\tilde{c} - m_\infty^2 \in S^{-\mu}$.

iii) *if $D^2 + \tilde{c}(x)$ satisfies* $(BM1)$, $(BM2)$ *for a weight function $\tilde{M}$, then $Da(x)D + c(x)$ satisfies* $(BM1)$, $(BM2)$ *for $M(x) = ((\psi')^{-\frac{1}{2}} \tilde{M}) \circ \psi^{-1}(x)$. If $\tilde{M}(x) = \langle x \rangle^\alpha$, then $M(x) \simeq \langle x \rangle^\alpha$.*

Proof. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism as above. We have

$$\partial_x T_\psi u = T_\psi \left(\frac{1}{2} \frac{\psi''}{\psi'} \circ \psi^{-1} u + \psi' \circ \psi^{-1} \partial_x \right) u .$$

Choosing ψ as in the lemma we get $g(x) = \psi' \circ \psi^{-1}(x)$ and

$$\partial_x T = T \left(g(x) \partial_x + \frac{1}{2} g'(x) \right) =: TA .$$

This yields

$$\begin{aligned} -\partial_x a(x) \partial_x &= \left(A^* - \frac{1}{2} g'\right)\left(A - \frac{1}{2} g'\right) \\ &= A^* A - \frac{1}{2}(A^* g' + g' A) + \frac{1}{4}(g')^2 \\ &= A^* A + \frac{1}{4}(g')^2 + \frac{1}{2} g g'' . \end{aligned}$$

This easily implies the first statement of the lemma. Next from (3.1) we get that $g - 1 \in S^{-\mu}$ hence $\phi(x) - x \in S^{1-\mu}$ from which $\psi(x) - x \in S^{1-\mu}$ follows. This implies that $(\frac{1}{4}(g')^2 + \frac{1}{2} g g'') \circ \psi \in S^{-2-\mu}$ and $c \circ \psi - m_\infty^2 \in S^{-\mu}$. Statement iii) is obvious. $\square$

Appendix B.

In this section we recall some results about generalized eigenfunctions for one-dimensional Schrödinger operators, taken from [16, 17]. For the reader's convenience, we will sketch some of the proofs. These results are used to obtain some sufficient conditions for (*BM*2). We saw in Subsection A.3 that we can reduce ourselves to considering a Schrödinger operator:

$$h = D^2 + V(x) + m_\infty^2 , \quad V \in S^{-\mu} \quad \text{for} \quad \mu > 0 .$$

It turns out that condition (*BM*2) is really a condition on the behavior of generalized eigenfunctions $\psi(x,k)$ for k near 0. In this respect the potentials fall naturally into two classes, depending on whether $\mu > 2$ or $\mu \le 2$.

This distinction is also relevant to condition (*BM*1). In fact by the Kato–Agmon–Simon theorem (see [10, Theorem XIII. 58]) if $V \in S^{-\mu}$ for $\mu > 0$ h has no strictly positive eigenvalues. As is well known h has a finite number of negative eigenvalues if $\mu > 2$. Therefore condition (*BM1*) is always satisfied for $M(x) \equiv 1$ if $\mu > 2$.

Results of Subsections B.1, B.2, B.3 are standard results. We used the reference [16]. Results of Subsections B.4, B.5, B.6 are easy adaptations from those in [17].

For $-\pi \le a < b \le \pi$, we denote by $\mathrm{Arg}]a,b[$ the open sector $\{z \in \mathbb{C} | \ a < \arg z < b\}$. The corresponding closed sector (with 0 excluded) will be denoted by $\mathrm{Arg}[a,b]$. For $\alpha \in \mathbb{R}$ the function z^α is defined by $(r\mathrm{e}^{\mathrm{i}\theta})^\alpha = r^\alpha e^{\mathrm{i}\alpha\theta}$, for $-\pi < \theta \le \pi$.

B.1. Jost solutions for quickly decreasing potentials

For two solutions f, g of the equation

$$-u'' + Vu = \zeta^2 u ,$$

the Wronskian $W(f,g) = f'(x)g(x) - f(x)g'(x)$ is independent on x. We start by recalling a well-known fact about existence of Jost solutions.

Proposition B.1. *Assume $V \in S^{-\mu}(\mathbb{R})$ for $\mu > 2$. Then for any $\zeta \in \mathrm{Arg}[0,\pi]$ there exist unique solutions $\theta_\pm(x,\zeta)$ of*

$$-u'' + Vu = \zeta^2 u ,$$

with asymptotics

$$\theta_\pm(x,\zeta) = \mathrm{e}^{\pm\mathrm{i}\zeta x}\big(1 + o(1)\big) , \quad \theta'_\pm(x,\zeta) = \pm\mathrm{i}\zeta\mathrm{e}^{\pm\mathrm{i}\zeta x}\big(1 + o(1)\big)$$

when $x \to \pm\infty$. They satisfy the estimates

$$|\theta_\pm(x,\zeta) - \mathrm{e}^{\pm\mathrm{i}\zeta x}| \le \mathrm{e}^{\mp\mathrm{Im}\zeta x} C\langle x\rangle^{-\mu+1} ,$$

uniformly for $\pm x \ge 0$ and $\zeta \in \mathrm{Arg}[0,\pi]$.

Moreover one has:

$$\overline{\theta_\pm}(x,\zeta) = \theta_\pm(x,-\overline{\zeta}) .$$

Proof. Uniqueness of $\theta_\pm$ is obvious since the Wronskian of two solutions vanishes at $\pm\infty$. We look for $\theta_\pm(x,\zeta)$ as solutions of the Volterra equations:

$$\theta_\pm(x,\zeta) = \mathrm{e}^{\pm\mathrm{i}\zeta x} + K_\pm\theta_\pm(x,\zeta)\,, \tag{B.1}$$

where:

$$\begin{aligned} K_+(\zeta)u(x) &= \int_x^{+\infty} \zeta^{-1}\sin\big(\zeta(y-x)\big)V(y)u(y)\mathrm{d}y\,,\\ K_-(\zeta)u(x) &= \int_{-\infty}^{x} \zeta^{-1}\sin\big(\zeta(x-y)\big)V(y)u(y)\mathrm{d}y\,. \end{aligned}$$

Using the bound

$$\big|\zeta^{-1}\sin\big(\zeta(y-x)\big)\mathrm{e}^{-\mathrm{Im}\zeta y}\big| \le Cy\mathrm{e}^{-\mathrm{Im}\zeta x}\,, \quad 0\le x\le y\,,$$

we obtain that

$$\big|\big(K_+(\zeta)\big)^n u(x)\big| \le \mathrm{e}^{-\mathrm{Im}\zeta x}(n!)^{-1}\left(C\int_x^{+\infty} y|V(y)|\mathrm{d}y\right)^n\,, \quad \text{for} \quad x\ge 0\,,$$

which gives the estimate

$$|\theta_+(x,\zeta) - \mathrm{e}^{\mathrm{i}\zeta\,\cdot\,x}| \le \mathrm{e}^{-\mathrm{Im}\zeta x}\big(\mathrm{e}^{C\int_x^{+\infty}|y||V(y)|\mathrm{d}y} - 1\big)\,,$$

proving the desired bound for $\theta_+(x,\zeta)$. The case of $\theta_-(x,\zeta)$ is treated similarly. The last identity follows from uniqueness. □

We recall additional identities between Jost solutions $\theta_\pm(\,\cdot\,,\zeta)$ if $\zeta = k > 0$. We first set:

$$w(k) := W\big(\theta_+(\,\cdot\,,k), \theta_-(\,\cdot\,,k)\big)\,.$$

Next by computing the Wronskian below at $\pm\infty$, we get that:

$$W\big(\theta_\pm(\,\cdot\,,k), \theta_\pm(\,\cdot\,,-k)\big) = \pm 2\mathrm{i}k\,.$$

Clearly

$$\begin{aligned} \theta_-(x,k) &= m_{++}(k)\theta_+(x,k) + m_{+-}(k)\theta_+(x,-k)\,,\\ \theta_+(x,k) &= m_{--}(k)\theta_-(x,k) + m_{-+}(k)\theta_-(x,-k)\,. \end{aligned} \tag{B.2}$$

We set

$$m(k) := (2\mathrm{i}k)^{-1}w(k)\,. \tag{B.3}$$

We can express the coefficients in (B.2) using Wronskians and get

$$\begin{aligned} m_{+-}(k) = m_{-+}(k) &= (2\mathrm{i}k)^{-1}W\big(\theta_+(\,\cdot\,,k)\theta_-(\,\cdot\,,k)\big) = m(k)\,,\\ m_{++}(k) &= -(2\mathrm{i}k)^{-1}W\big(\theta_+(\,\cdot\,,-k), \theta_-(\,\cdot\,,k)\big)\,,\\ m_{--}(k) &= -(2\mathrm{i}k)^{-1}W\big(\theta_+(\,\cdot\,,k), \theta_-(\,\cdot\,,-k)\big)\,. \end{aligned} \tag{B.4}$$

Using the identity $\theta_\pm(x,-k) = \overline{\theta}_\pm(x,k)$ and iterating the identities (B.2), we obtain

$$\begin{aligned} m(-k) = \overline{m}(-k)\,, &\quad \overline{m}_{++}(k) = -m_{--}(k)\,,\\ |m(k)|^2 = 1 + |m_{++}(k)|^2 &= 1 + |m_{--}(k)|^2\,. \end{aligned} \tag{B.5}$$

B.2. Resolvent and spectral family

Proposition B.2. *The family* $\{\psi(\,\cdot\,,k)\}$ *defined by*

$$\psi(x,k) := \begin{cases} m(k)^{-1}\theta_+(x,k) & k>0\,, \\ m(-k)^{-1}\theta_-(x,-k) & k<0 \end{cases} \tag{B.6}$$

is a family of generalized eigenfunctions of h.

Proof. Since $\theta_\pm(\,\cdot\,,\zeta)\in L^2(\mathbb{R}^\pm)$ for $\mathrm{Im}\zeta>0$, we obtain by the standard argument that the resolvent $(h-z)^{-1}$ has kernel

$$R(x,y,z) = \begin{cases} -w(\zeta)^{-1}\theta_+(x,\zeta)\theta_-(y,\zeta)\,, & y\le x\,, \\ -w(\zeta)^{-1}\theta_-(x,\zeta)\theta_+(y,\zeta)\,, & x\le y\,, \end{cases}$$

for $\zeta^2=z$, $\mathrm{Im}\zeta>0$ and

$$w(\zeta) = W\big(\theta_+(\,\cdot\,,\zeta),\theta_-(\,\cdot\,,\zeta)\big)\,.$$

The zeroes of w lie on $\mathrm{i}\mathbb{R}^+$ and correspond to negative eigenvalues of h.

If $E(\lambda)=\mathbb{1}_{]-\infty,\lambda]}(h)$, then from

$$(2\mathrm{i}\pi)\frac{\mathrm{d}E}{\mathrm{d}\lambda}(\lambda) = R(\lambda+\mathrm{i}0)-R(\lambda-\mathrm{i}0)\,.$$

We obtain that for $\lambda>0$ $\frac{\mathrm{d}E}{\mathrm{d}\lambda}(\lambda)$ has a kernel satisfying:

$$\begin{aligned} 4\pi k\frac{\mathrm{d}E}{\mathrm{d}\lambda}(x,y,\lambda) &= m(k)^{-1}\theta_+(x,k)\theta_-(y,-k) \\ &\quad + m(-k)^{-1}\theta_+(x,-k)\theta_-(y,k)\,, \quad \text{for} \quad y\le x\,, \end{aligned}$$

where $k^2=\lambda$. Note that $\frac{\mathrm{d}E}{\mathrm{d}\lambda}(\lambda)$ is both real and self-adjoint hence $\frac{\mathrm{d}E}{\mathrm{d}\lambda}(x,y,\lambda) = \frac{\mathrm{d}E}{\mathrm{d}\lambda}(y,x,\lambda)$.

Using the identities (B.2) and (B.5), we obtain that

$$4\pi k\frac{\mathrm{d}E}{\mathrm{d}\lambda}(x,y,\lambda) = |m(k)|^{-2}\big(\theta_+(x,k)\theta_+(y,-k)+\theta_-(x,k)\theta_-(y,-k)\big) \tag{B.7}$$

for $k^2=\lambda$. Setting

$$\psi_\pm(x,k) := m(k)^{-1}\theta_\pm(x,k) \quad k>0\,, \tag{B.8}$$

we obtain

$$4\pi k\frac{\mathrm{d}E}{\mathrm{d}\lambda}(x,y,\lambda) = \psi_+(x,k)\overline{\psi}_+(y,k)+\psi_-(x,k)\overline{\psi}_-(y,k)\,, \tag{B.9}$$

for $k^2=\lambda$, which shows that $\{\psi(\,\cdot\,,k)\}_{k\in\mathbb{R}}$ defined in (B.6) is a family of generalized eigenfunctions of h. $\square$

B.3. Condition $(BM2)$ for quickly decreasing potentials

Let us now consider in more details the Volterra integral equations (B.1) for $\zeta = k > 0$. Let $F_\pm$ be the Banach space of C^1 functions on $\mathbb{R}^\pm$ bounded with bounded derivatives equipped with the obvious norm.

The operators $(\mathbb{1} - K_\pm(k))^{-1}$ are bounded on $F_\pm$ and $]0, +\infty[\ni k \mapsto (\mathbb{1} - K_\pm(k))^{-1} \in B(F_\pm)$ is norm continuous. It follows that $k \to \theta_\pm(\cdot\,, k) \in F_\pm$ is continuous on $]0, +\infty[$, and hence $w(k)$ is continuous on $]0, +\infty[$.

Moreover when $k \to 0$, $(\mathbb{1} - K_\pm(k))^{-1}$ converges in $B(F_\pm)$ to $(\mathbb{1} - K_\pm(0))^{-1}$, where

$$
\begin{aligned}
K_+(0)u(x) &= \int_0^{+\infty} (y - x)V(y)u(y)\mathrm{d}y\,, \\
K_-(0)u(x) &= \int_{-\infty}^{x} (x - y)V(y)u(y)\mathrm{d}y\,.
\end{aligned}
$$

Therefore

$$\lim_{k\to 0} w(k) =: w(0) \text{ exists}$$

and $w(0) = 0$ iff there exists a solution u of

$$-u'' + Vu = 0\,,$$

with asymptotics:

$$u(x) \to u_\pm\,, \quad u'(x) \to 0 \quad \text{for} \quad x \to \pm\infty\,, \quad u_\pm \neq 0\,.$$

Such a solution is called a *zero energy resonance* for h. Recall that condition $(BM2')$ is introduced in Remark 4.2.

Proposition B.3. *Assume that $V \in S^{-\mu}$ for $\mu > 2$. Then:*

1) *if h has no zero energy resonance, then h satisfies $(BM2)$ for $M(x) \equiv 1$.*
2) *if h has a zero energy resonance and $|w(k)| \geq C|k|^{3/2-\epsilon}$ in $|k| \leq 1$ for some $\epsilon > 0$ then h satisfies $(BM2')$ for $M(x) \equiv 1$.*

Remark B.4. Assume $V \in S^{-\mu}$ for $\mu > 3$. Then if h has a resonance, $|w(k)| \geq C|k|$ (see [16, Prop.7.13]).

Proof. For $k > 0$ we deduce from (B.2) that:

$$
\psi(x, k) = \begin{cases} \frac{1}{m(k)}\theta_+(x, k)\,, & x \geq 0\,, \\ \frac{m_{--}(k)}{m(k)}\theta_-(x, k) + \theta_-(x, -k) & x \leq 0 \end{cases}
$$

and

$$
\psi(x, -k) = \begin{cases} \frac{1}{m(k)}\theta_-(x, k)\,, & x \leq 0\,, \\ \frac{m_{++}(k)}{m(k)}\theta_+(x, k) + \theta_+(x, -k) & x \geq 0\,. \end{cases}
$$

By Proposition A.2 the functions $\theta_\pm(x, k)$ are uniformly bounded in $k > 0$ and $x \in \mathbb{R}^\pm$. Moreover by (B.5)

$$\left|\frac{m_{++}(k)}{m(k)}\right| \leq 1\,, \quad \left|\frac{m_{--}(k)}{m(k)}\right| \leq 1\,.$$

Therefore it suffices to bound $m(k)^{-1}$. Using the integral equations (B.1), we obtain that:

$$\theta_{\pm}(0,k) = 1 + O(k^{-1}), \quad \theta'_{\pm}(0,k) = \pm \mathrm{i}k + O(1)$$

when $k \to +\infty$. Therefore $w(k) = 2\mathrm{i}k(1 + O(k^{-1}))$ and $m(k)^{-1}$ is bounded near $+\infty$. If $w(0) = \lim_{k\to 0} w(k) \neq 0$, then $\lim_{k\to 0} m(k)^{-1} = 0$ and hence $m(k)^{-1}$ is uniformly bounded on $]0, +\infty[$. If $|w(k)| \geq C|k|^{-3/2+\epsilon}$ for $|k| \leq 1$, we get instead that $|m(k)|^{-1} \leq C|k|^{-\frac{1}{2}+\epsilon}$ for $|k| \leq 1$. This completes the proof of the theorem. □

B.4. Quasiclassical solutions for slowly decreasing potentials

Let $V \in S^{-\mu}$ for $0 < \mu \leq 2$. For $\mathrm{Im}\zeta \geq 0$, we set

$$F(x,\zeta) := \big(V(x) - \zeta^2\big)^{\frac{1}{2}},$$

where $z^{\frac{1}{2}}$ is defined as in the beginning of Section B. We see that $F(x,\zeta)$ is holomorphic in ζ in the two sectors $\mathrm{Arg}]0,\pi/2[$ and $\mathrm{Arg}]\pi/2,\pi[$ and C^∞ in x if ζ belongs to the above sectors. It is continuous in ζ in the closed sectors $\mathrm{Arg}[0,\pi/2]$ and $\mathrm{Arg}[\pi/2,\pi]$ but may not be continuous across $\mathrm{Arg}\zeta = \pi/2$ depending on the value of $V(x)$.

Note that $\overline{(z^{\frac{1}{2}})} = \overline{z}^{\frac{1}{2}}$ if $\mathrm{Arg}z \neq \pi$, which implies that

$$\overline{F}(x,\zeta) = F(x,-\overline{\zeta}), \quad \zeta \in \mathrm{Arg}]0,\pi/2[\cup\mathrm{Arg}]\pi/2,\pi[, \quad \overline{F}(x,k) = F(x,-k) \quad k > 0. \tag{B.10}$$

We set also

$$S(a,x,\zeta) := \int_a^x F(y,\zeta)\mathrm{d}y, \quad a \in \mathbb{R}. \tag{B.11}$$

Proposition B.5. *For $\zeta \in \mathrm{Arg}[0,\pi]$, $\zeta \neq 0$ there exist unique solutions $\eta_{\pm}(x,\zeta)$ of*

$$-u'' + Vu = \zeta^2 u, \quad (E)$$

with asymptotics

$$\begin{aligned}
\eta_+(x,\zeta) &= F(x,\zeta)^{-\frac{1}{2}}\mathrm{e}^{-S(0,x,\zeta)}\big(1 + o(1)\big),\\
\eta'_+(x,\zeta) &= -F(x,\zeta)^{\frac{1}{2}}\mathrm{e}^{-S(0,x,\zeta)}\big(1 + o(1)\big), \quad x \to +\infty,\\
\eta_-(x,\zeta) &= F(x,\zeta)^{-\frac{1}{2}}\mathrm{e}^{S(0,x,\zeta)}\big(1 + o(1)\big),\\
\eta'_-(x,\zeta) &= F(x,\zeta)^{\frac{1}{2}}\mathrm{e}^{S(0,x,\zeta)}\big(1 + o(1)\big), \quad x \to -\infty.
\end{aligned}$$

We have

$$\overline{\eta}_{\pm}(x,\zeta) = \eta_{\pm}(x,-\overline{\zeta}).$$

For $\epsilon > 0$ let $R(\epsilon)$ be such that $|V(y)| \leq \epsilon/2$ for $|y| \geq R(\epsilon)$. Then the following estimates are valid:

$$|\eta_{\pm}(x,\zeta)| \leq C(\epsilon)|\zeta|^{-\frac{1}{2}}, \quad |\eta'_{\pm}(x,\zeta)| \leq C(\epsilon)|\zeta|^{\frac{1}{2}},$$
$$\textit{uniformly in} \quad \pm x \geq \pm R(\epsilon), \quad |k| \geq \epsilon. \tag{B.12}$$

Proof. We follow [17] and treat only the case of η_+. For $|\zeta| \geq \epsilon$ and $x \geq R(\epsilon)$, we look for $(\eta_+(x,\zeta), \eta'_+(x,\zeta))$ of the form

$$\begin{aligned}\eta_+(x,\zeta) &= F(x,\zeta)^{-\frac{1}{2}} \mathrm{e}^{-S(0,x,\zeta)} \big(u_1(x,\zeta) + u_2(x,\zeta)\big)\,,\\ \eta'_+(x,\zeta) &= F(x,\zeta)^{-\frac{1}{2}} \mathrm{e}^{-S(0,x,\zeta)} \Big(\big(F(x,\zeta) - V'(x)(4F^2)^{-1}\big) u_1(x,\zeta) \\ &\qquad - \big(F(x,\zeta) + V'(x)(4F^2)^{-1}\big) u_2(x,\zeta)\Big)\,.\end{aligned} \tag{B.13}$$

We find that $(u_1(\,\cdot\,,\zeta), u_2(x,\zeta))$ has to satisfy the following Volterra equation:

$$\begin{aligned}u_1(x,\zeta) &= -\int_x^{+\infty} \mathrm{e}^{-2S(x,y,\zeta)} M(y,\zeta)\big(u_1(y,\zeta) + u_2(y,\zeta)\big)\mathrm{d}y\,,\\ u_2(x,\zeta) &= 1 + \int_x^{+\infty} M(y,\zeta)\big(u_1(y,\zeta) + u_2(y,\zeta)\big)\mathrm{d}y\,,\end{aligned} \tag{B.14}$$

for

$$M(x,\zeta) = (32)^{-1}\big(4V''(x)F(x,\zeta)^{-3} - 5(V')^2(x)F(x,\zeta)^{-5}\big)\,.$$

Uniformly for $|\zeta| \geq \epsilon$ and $y \geq x \geq R(\epsilon)$, we have:

$$|M(x,\zeta)| \leq C(\epsilon)\langle x\rangle^{-2-\mu}\,, \quad |\mathrm{e}^{-S(x,y,\zeta)}| \leq 1\,.$$

The equation (B.14) can be solved by iteration and we obtain as in the proof of Proposition B.1 that:

$$\begin{aligned}|u_1(x,\zeta)| &\leq (\mathrm{e}^{C(\epsilon)\langle x\rangle^{-1-\mu}} - 1)\,,\\ |u_2(x,\zeta)| &\leq \mathrm{e}^{C(\epsilon)\langle x\rangle^{-1-\mu}}\,,\end{aligned}$$

uniformly for $|\zeta| \geq \epsilon$ and $y \geq x \geq R(\epsilon)$. Since

$$C_1(\epsilon)|\zeta|^{\frac{1}{2}} \leq |F(x,\zeta)| \leq C_2(\epsilon)|\zeta|^{\frac{1}{2}}\,,$$

we obtain the desired bounds on $\eta_+(\,\cdot\,,\zeta)$, $\eta'_+(\,\cdot\,,\zeta)$.

To prove uniqueness of $\eta_+(\,\cdot\,,\zeta)$, we verify that the Wronskian of two solutions computed at $x = +\infty$ vanishes. The fact that $\overline{\eta}_\pm(\,\cdot\,,\zeta) = \eta_\pm(\,\cdot\,,-\overline{\zeta})$ follows from (B.10). □

As in Subsection B.1, we compute some Wronskians.

Lemma B.6. *For $|k| \geq \epsilon$ we have:*

$$\begin{aligned}W\big(\eta_+(\,\cdot\,,k), \eta_+(\,\cdot\,,-k)\big) &= 2\mathrm{isgn}(k)\mathrm{e}^{-2\mathrm{Re}\int_0^{R(\epsilon)}(V(y)-k^2)^{\frac{1}{2}}\mathrm{d}y}\,,\\ W\big(\eta_-(\,\cdot\,,k), \eta_-(\,\cdot\,,-k)\big) &= -2\mathrm{isgn}(k)\mathrm{e}^{-2\mathrm{Re}\int_{-R(\epsilon)}^0(V(y)-k^2)^{\frac{1}{2}}\mathrm{d}y}\,.\end{aligned}$$

Proof. From Proposition B.5 we obtain that:

$$\eta_+(x,k) \sim (-\mathrm{i}k)^{-\frac{1}{2}}\mathrm{e}^{-S(0,x,k)}\,, \quad \eta_+(x,k) \sim -(-\mathrm{i}k)^{\frac{1}{2}}\mathrm{e}^{-S(0,x,k)}\,, \quad x \to +\infty\,.$$

Using that

$$(\mathrm{i}k)^{\frac{1}{2}}(-\mathrm{i}k)^{-\frac{1}{2}} = \mathrm{i}\,, \quad (-\mathrm{i}k)^{\frac{1}{2}}(\;ik)^{-\frac{1}{2}} = -\mathrm{i} \quad \text{for} \quad k > 0\,,$$

we obtain

$$\eta_+'(x,k)\eta_+(x,-k) - \eta_+(x,k)\eta_+'(x,-k) \sim 2\mathrm{i}\mathrm{e}^{-2\mathrm{Re}S(0,x,k)}\,, \quad x \to +\infty\,. \tag{B.15}$$

If $k \geq \epsilon$ and $y \geq R(\epsilon)$ we have $V(y) - k^2 < 0$ so $\mathrm{Re}(V(y) - k^2) = 0$. Letting $x \to +\infty$ in (B.14) we obtain the first identity for $k > 0$ and replacing then k by $-k$ for all $k \neq 0$. The proof of the second identity is similar, using instead

$$\eta_-(x,k) \sim (-\mathrm{i}k)^{-\frac{1}{2}}\mathrm{e}^{S(0,x,k)}\,, \quad \eta_-(x,k) \sim (-\mathrm{i}k)^{\frac{1}{2}}\mathrm{e}^{S(0,x,k)}\,, \quad x \to -\infty\,. \qquad \square$$

Proposition B.7. *Set for* $|k| \geq \epsilon$*:*

$$\begin{aligned} \theta_+(x,k) &:= \eta_+(x,k)|k|^{\frac{1}{2}}\mathrm{e}^{\mathrm{Re}\int_0^{R(\epsilon)}(V(y)-k^2)\mathrm{d}y}\,, \\ \theta_-(x,k) &:= \eta_-(x,k)|k|^{\frac{1}{2}}\mathrm{e}^{\mathrm{Re}\int_{-R(\epsilon)}^{0}(V(y)-k^2)\mathrm{d}y}\,. \end{aligned} \tag{B.16}$$

Then we have:

$$\overline{\theta_\pm}(x,k) = \theta_\pm(x,-k) \quad W\big(\theta_\pm(\,\cdot\,,k), \theta_\pm(\,\cdot\,,-k)\big) = \pm 2\mathrm{i}k\,,$$

and

$$|\theta_\pm(x,k)| \leq C_\epsilon\,, \quad |\theta_\pm'(x,k)| \leq C_\epsilon|k|\,, \quad \textit{uniformly for} \quad |k| \geq \epsilon\,, \quad \pm x \geq 0\,.$$

Proof. The first statement follows from Lemma B.6. To prove the second statement we use Proposition B.5. In fact by (B.12), the bounds in the second statement are valid uniformly for $|k| \geq \epsilon$ and $\pm x \geq \pm R(\epsilon)$. Let us first fix $C \gg 1$ such that for $\epsilon \geq C \gg 1$, we have $R(\epsilon) = 0$. Hence the bounds in the second statement are valid uniformly for $|k| \geq C$ and $\pm x \geq 0$.

It remains to check the bounds uniformly for $\epsilon \leq |k| \leq C$ and $\pm x \in [0, R(\epsilon)]$. We have

$$\big|\theta_\pm\big(\pm R(\epsilon), k\big)\big| + \big|\theta_\pm'\big(\pm R(\epsilon), k\big)\big| \leq C(\epsilon)\,.$$

Writing the differential equation satisfied by $\theta_\pm$ as a first order system, we see that this bound extends to $\pm x \in [0, R(\epsilon)]$ uniformly for $\epsilon \leq |k| \leq C$. $\square$

B.5. Resolvent and spectral family

Proposition B.8. *We set as in Subsection* B.1*:*

$$w(k) := W\big(\theta_+(\,\cdot\,,k), \theta_-(\,\cdot\,,k)\big)\,, \quad m(k) := (2\mathrm{i}k)^{-1}w(k)\,.$$

The family $\{\psi(\,\cdot\,,k)\}$ *defined by*

$$\psi(x,k) := \begin{cases} m(k)^{-1}\theta_+(x,k) & k > 0\,, \\ m(-k)^{-1}\theta_-(x,-k) & k < 0 \end{cases} \tag{B.17}$$

is a family of generalized eigenfunctions of h *in* $|k| \geq \epsilon$.

Proof. As in Subsection B.2, we can since $\eta_\pm(\,\cdot\,,\zeta) \in L^2(\mathbb{R}^\pm)$ for $\mathrm{Im}\zeta > 0$ write the kernel of $(h-z)^{-1}$ as:

$$R(x,y,z) = \begin{cases} -r(\zeta)^{-1}\eta_+(x,\zeta)\eta_-(y,\zeta)\,, & y \leq x\,, \\ -r(\zeta)^{-1}\eta_-(x,\zeta)\eta_+(y,\zeta)\,, & x \leq y\,. \end{cases}$$

for $\zeta^2 = z$, $\mathrm{Im}\zeta > 0$ and

$$r(\zeta) = W\big(\eta_+(\,\cdot\,,\zeta), \eta_-(\,\cdot\,,\zeta)\big)\,.$$

The zeroes of w lie on $\mathrm{i}\mathbb{R}^+$ and correspond to negative eigenvalues of h. We write the kernel of the spectral family $\frac{\mathrm{d}E}{\mathrm{d}\lambda}(x, y, \lambda)$ using the functions $\theta_\pm(x, \pm k)$. Using (B.16) we obtain:

$$\begin{aligned} 4\pi k \frac{\mathrm{d}E}{\mathrm{d}\lambda}(x, y, \lambda) &= m(k)^{-1}\theta_+(x, k)\theta_-(y, -k) \\ &\quad + m(-k)^{-1}\theta_+(x, -k)\theta_-(y, k)\,, \quad \text{for} \quad y \le x, \end{aligned}$$

where $k^2 = \lambda$ and

By Proposition B.7 the algebraic identities used in the proof of Proposition B.2 are satisfied by $\theta_\pm(\,\cdot\,, k)$. Repeating the above proof we obtain the proposition. □

B.6. Bounds on generalized eigenfunctions away from $k = 0$

The following result shows that generalized eigenfunctions are always uniformly bounded in $|k| \ge \epsilon$ for $\epsilon > 0$.

Proposition B.9. *Assume $V \in S^{-\mu}$ for $\mu > 0$. Then for $\{\psi(x, k)\}_{k\in\mathbb{R}}$ defined in* (B.17) *one has for all $\epsilon > 0$:*

$$\|\psi(\,\cdot\,, k)\|_\infty \le C_\epsilon \quad \textit{uniformly for} \quad |k| \ge \epsilon\,.$$

Proof. Arguing as in the proof of Theorem B.3 it suffices by Proposition B.7 to verify that

$$|m(k)|^{-1} = \frac{2|k|}{w(k)}\,,$$

is uniformly bounded for $|k| \ge \epsilon$.

We first claim that $w(k)$ is a continuous function of k in $|k| \ge \epsilon$. In fact writing the Volterra integral equation (B.14) as a fixed point equation in an appropriate Banach space of continuous functions, we see that for a fixed $x \ge R(\epsilon)$, $u_1(x, k)$ and $u_2(x, k)$ are continuous functions of k in $|k| \ge \epsilon$. The same holds for $\eta_+(x, k)$, $\eta'_+(x, k)$. Using the differential equation satisfied by $\eta_+(\,\cdot\,, k)$, we see that $k \mapsto (\eta_+(0, k), \eta'_+(0, k))$ is continuous in k. Using the same argument for $\eta_-(\,\cdot\,, k)$, we obtain the continuity of $w(k)$ in $|k| \ge \epsilon$. We note that $w(k)$ does not vanish in $|k| \ge \epsilon$ since $w(k) = 0$ would imply that k^2 is an eigenvalue of h which is impossible if $V \in S^{-\mu}$.

Therefore $|m(k)|^{-1}$ is locally bounded in $|k| \ge \epsilon$. It remains to bound $|m(k)|^{-1}$ near infinity. We use the notation in the proof of Proposition B.5. Let us pick $C \gg 1$ such that $R(C) = 0$. Then for $k \ge C$ we have:

$$F(x, k) = -\mathrm{i}k\Big(1 + 0\big(\langle x\rangle^{-\mu}|k|^{-2}\big)\Big)\,, \quad M(x, k) = O\big(\langle x\rangle^{-2-\mu}|k|^{-3}\big)\,.$$

Using the fact that u_1, u_2 are uniformly bounded in $x \ge 0$ and $k \ge C$, we obtain from (B.14) that

$$u_1(0, k) = O(|k|^{-3})\,, \quad u_2(0, k) = 1 + O(|k|^{-3})\,,$$

which yields

$$\eta_+(0,k) = (-\mathrm{i}k)^{-\frac{1}{2}}\big(1 + O(|k|^{-2})\big)\,, \quad \eta'_+(0,k) = -(-\mathrm{i}k)^{\frac{1}{2}}\big(1 + O(|k|^{-2})\big)\,.$$

The same argument gives

$$\eta_-(0,k) = (-\mathrm{i}k)^{-\frac{1}{2}}\big(1 + O(|k|^{-2})\big)\,, \quad \eta'_-(0,k) = (-\mathrm{i}k)^{\frac{1}{2}}\big(1 + O(|k|^{-2})\big)\,,$$

and hence $W(\eta_+(\,\cdot\,,k),\eta_-(\,\cdot\,,k)) = -2+0(|k|^{-2})$. Using that $\theta_\pm(x,k) = \eta_\pm(x,k)|k|^{\frac{1}{2}}$ for $k \geq C$, we obtain that $|w|(k) \sim 2|k|$ when $k \to \infty$, which shows that $|m|^{-1}(k)$ is uniformly bounded near infinity. □

B.7. Condition $(BM2)$ for slowly decreasing potentials

In this subsection we give some classes of slowly decreasing potentials for which condition $(BM2)$ holds.

As in Subsection B.3 the possible existence of *zero energy resonances* has to be taken into account. For quickly decreasing potentials, the definition of zero energy resonances is connected with the integral equation (B.1) for $\zeta = 0$. For slowly decreasing potentials, we have to consider instead the integral equations (B.14). This leads to the following definition:

Assume that $v \in S^{-\mu}$ for $0 < \mu < 2$ is such that $|V(x)| \geq c\langle x\rangle^{-\mu}$ for $|x|$ large enough. We will say that h has a *zero energy resonance* if there exists a solution of

$$-u'' + Vu = 0\,,$$

with asymptotics:

$$u(x) = u_\pm V(x)^{-1/4}\mathrm{e}^{\mp\int_0^x (V(s))^{\frac{1}{2}}\mathrm{d}s}\big(1 + o(1)\big)\,, \quad x \to \pm\infty\,,$$
$$u'(x) = \mp u_\pm V(x)^{1/4}\mathrm{e}^{\pm\int_R^x (V(s))^{\frac{1}{2}}\mathrm{d}s}\big(1 + o(1)\big)\,, \quad x \to \pm\infty\,,$$

for constants $u_\pm \neq 0$.

Potentials negative near infinity

We consider first the case of potentials which are *negative* near infinity. We assume that $V \in S^{-\mu}$ for $0 < \mu < 2$ and:

$$V(x) \leq -c\langle x\rangle^{-\mu} \quad \text{in} \quad |x| \geq R\,, \quad \text{for some} \quad c, R > 0\,. \tag{B.18}$$

Proposition B.10. *Assume that $V \in S^{-\mu}$ for $0 < \mu < 2$ satisfies* (B.18) *and has no zero energy resonance. Then condition* $(BM2)$ *holds for $M(x) = \langle x\rangle^{\mu/4}$.*

Proof. By Proposition B.9 it suffices to consider the region $|k| \leq 1$. We fix R as (B.18) and define the functions $\eta_\pm(x,k)$ using the phase $S(\pm R, x, \zeta)$. We will consider only the + case. We first claim that

$$\theta_+(x,k) \in O\big(\langle x\rangle^{\mu/4}\big)\,, \quad \theta'_+(x,k) \in O(1)\,, \qquad \text{uniformly in} \quad x \geq -R\,, \quad |k| \leq 1\,. \tag{B.19}$$

Clearly it suffices to prove the statement in $x \geq R$, since we can extend the bound to $[-R, R]$ using the differential equation satisfied by θ_+. Let us prove (B.19). We will simply write "$f(x,k) \in O(\langle x\rangle^\epsilon)$" for "$f(x,k) \in O(\langle x\rangle^\epsilon)$ uniformly in $x \geq R$, $|k| \leq 1$".

The function $F(x,k)$ is smooth in $|x| \geq R$ and one has $|F(x,k)| \geq c\langle x\rangle^{-\mu/2}$. This implies that $M(x,k) \in O(\langle x\rangle^{-2+\mu/2})$, from which we get

$$u_1(x,k) \in O\big(\langle x\rangle^{-1+\mu/2}\big)\,, \quad u_2(x,k) \in O(1),$$

and

$$\eta_+(x,k) \in O\big(\langle x\rangle^{\mu/4}\big)\,, \quad \eta_+'(x,k) \in O(1)\,.$$

This proves (B.19). Next as in Subsection B.3, we can set $U = (u_1, u_2 - 1)$ and consider the equations (B.14) as a fixed point equation:

$$\big(\mathbb{1} + T(k)\big)U = F\,,$$

in the Banach space

$$\mathcal{B} = \left\{ U = (v_1, v_2)\,|v_i \text{ continuous}, \quad \sup_{[R,+\infty[} |\langle x\rangle^{1-\mu/2} v_i(x)| < \infty \right\}.$$

For R large enough, $\|T(k)\| < \frac{1}{2}$ uniformly in $|k| \leq 1$ and $k \mapsto T(k)$ is norm continuous. It follows that $k \mapsto U(k) \in \mathcal{B}$ is continuous up to $k = 0$. Therefore $(u_1(\,\cdot\,,k), u_2(\,\cdot\,,k) - 1)$ has a limit $(u_1(\,\cdot\,,0), u_2(\,\cdot\,,0) - 1)$ in $\mathcal{B}$ when $k \to 0$.

This implies also that $(\eta_+(\,\cdot\,,k), \eta_+'(\,\cdot\,,k))$ converges locally uniformly when $k \to 0$ to the pair $(\eta_+(\,\cdot\,,0), \eta_+'(\,\cdot\,,0))$ obtained from $(u_1(\,\cdot\,,0), u_2(\,\cdot\,,0))$ by formula (B.13) for $k = 0$.

We see that $\eta_+(x,0)$ is a solution of

$$-u'' + V(x)u = 0\,,$$

with asymptotics:

$$\begin{aligned} \eta_+(x,0) &= V(x)^{-1/4} \mathrm{e}^{\mathrm{i}\int_R^x (-V(s))^{\frac{1}{2}}\,\mathrm{d}s}\big(1 + o(1)\big)\,, \\ \eta_+'(x,0) &= -V(x)^{1/4} \mathrm{e}^{\mathrm{i}\int_R^x (-V(s))^{\frac{1}{2}}\,\mathrm{d}s}\big(1 + o(1)\big)\,, \end{aligned} \qquad x \to +\infty\,.$$

By the convergence result above (and its analog for $\eta_-(\,\cdot\,,k)$), we also see that

$$\lim_{k\to 0} m(k) =: m(0) = cW\big(\eta_+(\,\cdot\,,0), \eta_-(\,\cdot\,,0)\big)\,,$$

for some $c \neq 0$. Clearly $m(0) = 0$ iff h admits a zero energy resonance. Using (B.17), (B.19) and Proposition B.9, we obtain then that

$$|\psi(x,k)| \leq C\langle x\rangle^{\mu/4}\,, \quad \text{uniformly for} \quad x \in \mathbb{R}\,, \quad k \in \mathbb{R}\,,$$

which completes the proof of the proposition. $\square$

Potentials positive near infinity

Let us now consider the case of potentials which are *positive near infinity*. The following lemma is shown in [17, Theorem 4].

Lemma B.11. *Assume that* $V \in S^{-\mu}$ *for* $0 < \mu < 2$ *is* positive near infinity*, more precisely:*

$$V(x) \sim q_0|x|^{-\mu}\,, \quad x \to \infty\,, \quad q_0 > 0\,. \tag{B.20}$$

Then there exists unique solutions $\eta_\pm(x,0)$ *of*

$$-u'' + Vu = 0\,,$$

with asymptotics:

$$\eta_+(x,0) \sim V(x)^{-\frac{1}{4}}\mathrm{e}^{-\int_a^x (V(y))^{\frac{1}{2}}\mathrm{d}y}\,, \quad \eta'_+(x,0) \sim -V(x)^{\frac{1}{4}}\mathrm{e}^{-\int_a^x (V(y))^{\frac{1}{2}}\mathrm{d}y}\,, \quad x \to +\infty\,,$$

$$\eta_-(x,0) \sim V(x)^{-\frac{1}{4}}\mathrm{e}^{-\int_a^x (V(y))^{\frac{1}{2}}\mathrm{d}y}\,, \quad \eta'_-(x,0) \sim V(x)^{\frac{1}{4}}\mathrm{e}^{-\int_{-a}^x (V(y))^{\frac{1}{2}}\mathrm{d}y}\,, \quad x \to -\infty\,,$$

where $a \gg 1$ *is such that* $V(x) > 0$ *in* $|x| \geq a$.

Lemma B.12. *Assume in addition to* (B.20) *that there exists* $\theta, R > 0$ *such that* V *extends holomorphically to* $D(R,\theta) = \{z \in \mathbb{C} \,|\, |z| > R\,, \quad |\mathrm{Arg}z| < \theta\}$ *and satisfies*

$$|V(z)| \leq C(1+|z|)^{-\mu}\,, \quad z \in D(R,\theta)\,.$$

Then for any $\pm x \geq R$, $(\eta_\pm(x,s), \eta'_\pm(x,s))$ *converges to* $(\eta_\pm(x,0), \eta'_\pm(x,0))$ *when* $s \to 0$.

Proof. We check that the assumptions of [17, Theorem 7] are satisfied. We consider the two parts $D^\pm(R,\theta) = D(R,\theta) \cap \{\pm\mathrm{Re}z > 0\}$ of $D(R,\theta)$ and set $z' = \log(\pm z)$ for $z \in D^\pm(R,\theta)$. Applying Hadamard three lines theorem to $F(z') = V(\mathrm{e}^{z'})(\pm\mathrm{e}^{\mu z'}) - q_0$, we obtain

$$V(z) \sim q_0(\pm z)^{-\mu} \quad \text{when} \quad |z| \to +\infty\,,$$

uniformly in $D(R,\theta_0) \cap \{\pm\mathrm{Re}z > 0\}$ for all $0 < \theta_0 < \theta$. Similarly it follows from Cauchy's inequalities that

$$|\partial_z^k V(z)| \leq C(1+|z|)^{-k-\mu}\,, \quad z \in D(R,\theta_0)\,.$$

Applying then [17, Theorem 7], we obtain the lemma. □

Lemma B.13. *The functions* $\eta_\pm(x,k)$ *are uniformly bounded for* $|x| \leq R$, $|k| \leq 1$.

Proof. We consider only the case of $\eta_+(x,k)$. Let $\phi_0(x,k)$, $\phi_1(x,k)$ the two regular solutions of (E) with boundary conditions:

$$\phi_0(0,k) = 1\,, \quad \phi'_0(0,k) = 0\,, \quad \phi_1(0,k) = 0\,, \quad \phi'_1(0,k) = 1\,.$$

Clearly $\phi_i(x,k)$, $\phi'_i(x,k)$ are uniformly bounded and continuous in $\{(x,k) |\, |x| \leq R, |k| \leq C\}$. We have

$$\eta_+(\,\cdot\,,k) = a_1(k)\phi_0(\,\cdot\,,k) + a_0(k)\phi_1(\,\cdot\,,k)\,, \quad \text{for} \quad a_i(k) = W\big(\eta_+(\,\cdot\,,k), \phi_i(\,\cdot\,,k)\big)\,.$$

By Lemma B.12, $a_i(k)$ converges to $a_i(0)$ when $k \to 0$. □

Proposition B.14. *Assume that $V \in S^{-\mu}$ for $0 < \mu < 2$ satisfies the hypotheses of Lemma* B.12 *and has no zero energy resonance. Then for each $R > 0$ condition* ($BM2$) *is satisfied for*

$$M(x) = \begin{cases} 1 & \text{for} \quad |x| \le R\,, \\ +\infty & \text{for} \quad |x| > R\,. \end{cases}$$

We refer the reader to Remark 4.1 for the meaning of ($BM2$) if M takes its values in $[0, +\infty]$.

Proof. The uniform boundedness of $\eta_\pm(x,k)$ and hence of $\theta_\pm(x,k)$ for $|x| \le R$ and $|k| \le 1$ is shown in Lemma B.13. We have to show that $\lim_{k\to 0} m(k) =: m(0) \ne 0$. The limit exists and equals $cW(\eta_+(\,\cdot\,,0), \eta_-(\,\cdot\,,0))$ for some $c \ne 0$ by Lemma B.12. By Lemma B.11 $m(0) = 0$ iff h has a zero energy resonance. □

Appendix C.

C.1. Proof of Proposition 2.10

We forget the superscript w to simplify notation. We recall the following characterization of $\mathrm{Op}(S^{p,m})$ known as the *Beals criterion*:

$M \in \mathrm{Op}(S^{p,m})$ iff $M : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ and

$$\langle D\rangle^{-p+|\alpha|}\langle x\rangle^{-m+|\beta|}\mathrm{ad}_x^\alpha \mathrm{ad}_D^\beta M \quad \text{is bounded on} \quad L^2(\mathbb{R}^d)\,, \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}^d\,. \tag{C.1}$$

The topology given by the norms of the multicommutators with $\mathrm{Op}(m)$ in (C.1) is the same as the original topology on $S^{p,m}$.

We will need also similar objects for symbols and operators depending on a real parameter $s \ge 0$. We say that $m(s,x,\xi)$ belongs to $S^{p,m,k}$ if

$$|\partial_x^\alpha \partial_k^\beta m(s,x,k)| \le C_{\alpha,\beta}\big(\langle k\rangle^2 + \langle s\rangle\big)^k \langle k\rangle^{-p+|\alpha|}\langle x\rangle^{-m+|\beta|}\,, \quad \alpha, \beta \in \mathbb{N}^d\,,$$

uniformly for $s \ge 0$. By the result recalled above, we see that $M(s) \in \mathrm{Op}(S^{p,m,k})$ iff $M(s) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ and

$$\big(\langle D\rangle^2 + \langle s\rangle\big)^{-k}\langle D\rangle^{-p+|\alpha|}\langle x\rangle^{-m+|\beta|}\mathrm{ad}_x^\alpha \mathrm{ad}_D^\beta M(s) \quad \text{is bounded on} \quad L^2(\mathbb{R}^d)\,, \tag{C.2}$$

uniformly for $s \ge 0$.

Let us now prove Proposition 2.10. By elliptic regularity, we know that h is self-adjoint and bounded below on $H^2(\mathbb{R}^d)$ and $(h+s)^{-1}$ preserves $\mathcal{S}(\mathbb{R}^d)$. Computing multicommutators $\mathrm{ad}_x^\alpha \mathrm{ad}_D^\beta (h+s)^{-1}$ on $\mathcal{S}(\mathbb{R}^d)$, we first see by induction on α, β that $(\langle D\rangle^2 + \langle s\rangle)\langle x\rangle^\alpha \langle D\rangle^\beta (h+s)^{-1}\langle x\rangle^{-\alpha}\langle D\rangle^{-\beta} \in O(1)$, uniformly in $s \ge 0$.

The same computations show then that $(\langle D\rangle^2 + \langle s\rangle)\langle D\rangle^{|\alpha|}\langle x\rangle^{|\beta|}\mathrm{ad}_x^\alpha \mathrm{ad}_D^\beta (h+s)^{-1}$ is uniformly bounded on $L^2(\mathbb{R}^d)$, which by the Beals criterion show that

$$(h+s)^{-1} \in \mathrm{Op}(S^{0,0,-2})\,. \tag{C.3}$$

Using the formula

$$\lambda^{-\alpha} = c_\alpha \int_0^{+\infty} s^{-\alpha}(\lambda+s)^{-1}\mathrm{d}s\,, \quad \text{for} \quad \lambda \geq 0\,, \quad \alpha \in]0,1[\tag{C.4}$$

we obtain that $h^{-\alpha} \in \mathrm{Op}(S^{-2\alpha,0})$ for $\alpha \in]0,1[$. Using also that $h^n \in \mathrm{Op}(S^{2n,0})$ for integer n, we obtain ii).

C.2. Proof of Proposition 2.11

Let us first prove i). We use the notation in Subsection C.1. Set $T(s) = \mathrm{Op}((b+s)^{-1})$. By pdo calculus and (C.3), we get that

$$(h+s)T(s) - \mathbb{1} \in \mathrm{Op}(S^{0,-1-\mu,-\frac{1}{2}})$$

hence

$$(h+s)^{-1} - T(s) \in \mathrm{Op}(S^{0,-1-\mu,-3/2})\,.$$

Using (C.4) for $\epsilon = \frac{1}{2}$ this implies that

$$h^{-\frac{1}{2}} - \mathrm{Op}(b^{-\frac{1}{2}}) \in \mathrm{Op}(S^{-2,-1-\mu})\,.$$

Next we write using again pdo calculus:

$$h^{\frac{1}{2}} = hh^{-\frac{1}{2}} = h\mathrm{Op}(b^{-\frac{1}{2}}) + \mathrm{Op}(S^{0,-1-\mu}) = \mathrm{Op}(b^{\frac{1}{2}}) + \mathrm{Op}(S^{0,-1-\mu})\,,$$

which proves i). Let us now prove ii). By Proposition 2.10, we know that

$$\omega = \mathrm{Op}(c)\,, \quad \text{for} \quad c - b^{\frac{1}{2}} \in S^{0,-1-\mu}\,,$$

where b is defined in Proposition 2.10. By pseudodifferential calculus, we obtain that:

$$\Big[\omega, \mathrm{i}\big[\omega, \mathrm{i}\langle x\rangle\big]\Big] = \mathrm{Op}\Big(\Big\{c, \big\{c, \langle x\rangle\big\}\Big\}\Big) + \mathrm{Op}(S^{0,-2})\,.$$

Since $c - \langle k\rangle \in S^{1,-\mu}$, we get:

$$\Big\{c, \big\{c, \langle x\rangle\big\}\Big\} = \Big\{\langle k\rangle, \big\{\langle k\rangle, \langle x\rangle\big\}\Big\} + S^{0,-1-\mu} = \langle x\rangle^{-1}\left(\frac{\xi^2}{\langle k\rangle^2} - \frac{(\xi|x)^2}{\langle k\rangle^2\langle x\rangle^2}\right) + S^{0,-1-\mu}\,.$$

We pick $0 < \epsilon \ll 1$ and write:

$$\begin{aligned}\left(\frac{\xi^2}{\langle k\rangle^2} - \frac{(\xi|x)^2}{\langle k\rangle^2\langle x\rangle^2}\right) &= d^2(x,\xi) - \langle x\rangle^{-2\epsilon}\,, \quad \text{for} \quad d(x,\xi) \\ &= \left(\frac{\xi^2}{\langle k\rangle^2} - \frac{(\xi|x)^2}{\langle k\rangle^2\langle x\rangle^2} + \langle x\rangle^{-2\epsilon}\right)^{\frac{1}{2}}\,.\end{aligned}$$

Using that $d^2 \in S^{0,0}$ and $d^2 \geq \langle x\rangle^{-2\epsilon}$, we see easily that $d \in S^{0,0}_\epsilon$, hence $\langle x\rangle^{-\frac{1}{2}}d \in S^{0,-\frac{1}{2}}_\epsilon$.

Using again (2.6), we get:

$$\mathrm{Op}\big(\langle x\rangle^{-1}d^2\big) = \mathrm{Op}\big(\langle x\rangle^{-\frac{1}{2}}d\big)^2 + \mathrm{Op}(S^{0,-3+4\epsilon}_\epsilon)\,.$$

Choosing $\epsilon > 0$ small enough and setting $\gamma = \langle x\rangle^{-\frac{1}{2}}d$, we obtain the proposition. $\square$

C.3. A technical lemma

Lemma C.1. *Let* $h = Da(x)D + c(x)$ *for* a, c *as in* (3.1), $h_\infty = D^2 + m_\infty^2$. *Set* $\omega = h^{\frac{1}{2}}$, $\omega_\infty = h_\infty^{\frac{1}{2}}$. *Let* $\chi \in C_0^\infty(\mathbb{R})$ *and* $F \in C^\infty(\mathbb{R})$ *with* $F \equiv 0$ *near* 0 *and* $F \equiv 1$ *near* ∞. *Then for* C *large enough*

$$\chi\left(\frac{\omega}{\kappa}\right) F\left(\frac{\omega_\infty}{C\kappa}\right)\omega_\infty \in O(1)\,.$$

Proof. We know from Proposition 2.10 ii) that ω and ω_∞ and hence $[\omega, \omega_\infty]$ belong to $\mathrm{Op}^{\mathrm{w}}(S^{1,0})$. Using formula (2.1), we deduce from this fact that for $\chi \in C_0^\infty(\mathbb{R})$:

$$\left[\chi\left(\frac{\omega}{\kappa}\right), \omega_\infty\right] \in O(1)\,. \tag{C.5}$$

We take $\chi_1 \in C_0^\infty(\mathbb{R})$ with $\chi_1\chi = \chi$ and set

$$\tilde{\omega}_\infty = \chi_1\left(\frac{\omega}{\kappa}\right)\omega_\infty\chi_1\left(\frac{\omega}{\kappa}\right).$$

We first see that

$$\chi\left(\frac{\omega}{\kappa}\right)(\omega_\infty - \tilde{\omega}_\infty) = \left[\chi\left(\frac{\omega}{\kappa}\right), \omega_\infty\right](1-\chi_1)\left(\frac{\omega}{\kappa}\right) \in O(1)\,, \tag{C.6}$$

by (C.5). We claim also that for $F \in C_0^\infty(\mathbb{R})$:

$$\chi\left(\frac{\omega}{\kappa}\right)\left(F\left(\frac{\omega_\infty}{\kappa}\right) - F\left(\frac{\tilde{\omega}_\infty}{\kappa}\right)\right) \in O(\kappa^{-1})\,. \tag{C.7}$$

In fact we write using (2.1):

$$\begin{aligned}
&\chi\left(\frac{\omega}{\kappa}\right)\left(F\left(\frac{\omega_\infty}{\kappa}\right) - F\left(\frac{\tilde{\omega}_\infty}{\kappa}\right)\right)\\
&\quad= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{F}(z)\chi\left(\frac{\omega}{\kappa}\right)\left(z - \frac{\omega_\infty}{\kappa}\right)^{-1}\kappa^{-1}(\omega_\infty - \tilde{\omega}_\infty)\left(z - \frac{\tilde{\omega}_\infty}{\kappa}\right)^{-1} \mathrm{d}z \wedge \mathrm{d}\,\overline{z}\\
&\quad= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{F}(z)\chi\left(\frac{\omega}{\kappa}\right)\left(z - \frac{\omega_\infty}{\kappa}\right)^{-1}\kappa^{-1}\chi\left(\frac{\omega}{\kappa}\right)(\omega_\infty - \tilde{\omega}_\infty)\\
&\qquad\times\left(z - \frac{\tilde{\omega}_\infty}{\kappa}\right)^{-1} \mathrm{d}z \wedge \mathrm{d}\,\overline{z}\\
&\qquad+ \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{F}(z)\left(z - \frac{\omega_\infty}{\kappa}\right)^{-1}\kappa^{-1}\left[\chi\left(\frac{\omega}{\kappa}\right), \omega_\infty\right]\left(z - \frac{\omega_\infty}{\kappa}\right)^{-1}\\
&\qquad\times\kappa^{-1}(\omega_\infty - \tilde{\omega}_\infty)\left(z - \frac{\tilde{\omega}_\infty}{\kappa}\right)^{-1} \mathrm{d}z \wedge \mathrm{d}\,\overline{z}\,.
\end{aligned}$$

This is easily seen to be $O(\kappa^{-1})$ using the fact that $(z-a)^{-1}$, $a(z-a)^{-1}$ are $O(|\mathrm{Im} z|^{-1})$ for $z \in \operatorname{supp}\tilde{F}$.

We note then that

$$\tilde{\omega}_\infty \le c_1\kappa\,,$$

for some $c_1 > 0$ since $\omega_\infty \le c_0\omega$. Hence if $G(s) = F(C^{-1}s)$ for F as in the lemma and C is large enough, we have $G(\frac{\tilde{\omega}_\infty}{\kappa}) = 0$. Applying then (C.7) to $F = 1 - G$, we obtain that

$$\chi\left(\frac{\omega}{\kappa}\right) G\left(\frac{\omega_\infty}{\kappa}\right) \in O(\kappa^{-1})\,. \tag{C.8}$$

We write:

$$\begin{aligned}\chi\left(\frac{\omega}{\kappa}\right) G\left(\frac{\omega_\infty}{\kappa}\right) \omega_\infty &= \omega_\infty \chi\left(\frac{\omega}{\kappa}\right) G\left(\frac{\omega_\infty}{\kappa}\right) + \left[\chi\left(\frac{\omega}{\kappa}\right), \omega_\infty\right] G\left(\frac{\omega_\infty}{\kappa}\right) \\ &= \omega_\infty \omega^{-1} \omega \chi\left(\frac{\omega}{\kappa}\right) G\left(\frac{\omega_\infty}{\kappa}\right) + \left[\chi\left(\frac{\omega}{\kappa}\right), \omega_\infty\right] G\left(\frac{\omega_\infty}{\kappa}\right).\end{aligned}$$

The first term in the last line is $O(1)$ using (C.8), the second also using (C.5). This completes the proof of the Lemma. □

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