Scattering theory of infrared divergent Pauli-Fierz Hamiltonians

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November 2003

Abstract

We consider in this paper the scattering theory of infrared divergent massless Pauli-Fierz Hamiltonians. We show that the CCR representations obtained from the asymptotic field contain so-called *coherent sectors* describing an infinite number of asymptotically free bosons. We formulate some conjectures leading to mathematically well defined notion of *inclusive and non-inclusive scattering cross-sections* for Pauli-Fierz Hamiltonians. Finally we give a general description of the scattering theory of QFT models in the presence of coherent sectors for the asymptotic CCR representations.

Acknowledgments. Both authors were partly supported by the NATO Grant PST.CLG.979341 and by the Postdoctoral Training Program HPRN-CT-2002-0277. The research of J. D. was also partly supported by the Komitet Badań Naukowych (the grants SPUB127 and 2 P03A 027 25). A part of this work was done during his visit to Aarhus University supported by MaPhySto funded by the Danish National Research Foundation.

1 Introduction

The main motivation for this paper is our desire to gain some rigorous understanding of the *infrared problem* in quantum field theory, in particular in QED. This is not an easy task, since we even do not know how to construct rigorously realistic models of QED.

Some authors tried to analyze the infrared problem in the axiomatic framework of local quantum theory. Considerable progress in this direction has been achieved [FMS, Bu]. We will not, however, discuss these results, often deep and interesting.

The infrared problem is not restricted, however, to local quantum theory. Some of its aspects persist even in various simplified models derived from QED, which have ultraviolet cutoffs or treat a part of the system in a classical way. In our paper we consider a class of such models. These models are quite far from the "true QED" or from the axioms of local quantum theory. Yet, we will see that their infrared problem is quite nontrivial. Besides, unlike QED, these models can be rigorously defined.

Infrared problem, both in "true QED" and in various simplified models appears mostly if we try to compute scattering amplitudes. Thus it is primarily a symptom of a pathological scattering theory.

QED is a theory of charged particles interacting with photons. Correspondingly, it has two distinct kinds of the infrared problem: the first kind involves the dynamics of charged particles and the second involves photons. In the following two subsections we would like to make some comments about these two kinds of the infrared problem of QED, focusing mostly on various simplified models.

1.1 Infrared problem for charged particles

Let us shortly discuss the first kind. Scattering of charged particles is made difficult by the long-range nature of their interaction. To partly understand this phenomenon, let us fix the Coulomb gauge in QED, drop photons and use the non-relativistic approximation. Then QED becomes a theory of charged particles whose dynamics is described by the many body Schrödinger Hamiltonian with Coulomb interactions. As is well known, the usual scattering theory breaks down for such systems. Naive rules for computing scattering amplitudes in terms of Feynman diagrams presuppose that we want to construct the usual wave and scattering operators, which do not exist because Coulomb potentials are long-range. Therefore, we get meaningless divergent expressions.

It is well understood how to cure this problem, at least in the context of many body Schrödinger Hamiltonians. Two approaches are possible:

(1) One can compute only scattering cross-sections, staying away from ill-defined wave and scattering operators. The standard way is to approximate Coulomb interaction by the Yukawa interaction of mass $m > 0$, which is short range, compute the cross-sections and take the $m \to 0$ limit. This is the approach found in most textbooks on quantum mechanics.

(2) One can introduce *modified wave and scattering operators*. From the conceptual point of view it is a more satisfactory approach—it gives deeper insight into the problem. The mathematics of this approach is very interesting and nowadays well understood (see e.g. [DG1]). On the other hand, it is more complicated computationally than the first approach and uses non-canonical objects: the modified wave and scattering operators depend on the choice of the so-called modifier.

Apart from the remarks above, in our paper we will not touch this aspect of the infrared problem.

1.2 Infrared problem for photons

Let us now discuss the photonic aspect of the infrared problem. In our discussion we will consider both the perturbative QED and various simplified models such as Pauli-Fierz Hamiltonians.

If one tries to compute scattering cross-sections involving states with a finite number of asymptotic photons, one often obtains infrared divergent integrals. After an appropriate renormalization, one obtains scattering cross-sections equal to zero. This is usually interpreted by saying that "the vacuum escapes from the physical Hilbert space" and that "all states contain an infinite number of soft photons".

In the literature one can find 4 approaches to cure this problem in QED-like theories that make possible computing physically meaningful cross-sections.

(1) One can restrict oneself to the so-called *inclusive cross-sections*, which take into account all possible "soft photon states" below a certain energy $\epsilon > 0$. The philosophy behind this prescription is: do not attempt to compute or even ask about the existence of the wave and scattering operators—try to compute scattering cross-sections relevant for realistic experiments. This point of view is most common in standard textbooks [JR] and can be traced back to [BN] (see also [YFS]).

(2) Naive rules for computing scattering amplitudes in terms of Feynman diagrams presuppose that the asymptotic fields form a Fock CCR representation. This assumption can be wrong because of the infrared problem. To eliminate this difficulty, one can treat seriously non-Fock representations. One class of non-Fock representations is especially easy to handle—the so-called *coherent representations*. One can define wave and scattering operators between coherent sectors, and also asymptotic Hamiltonians. Scattering theory is somewhat less intuitive than in the case of Fock representations, but it is naturally defined and not much more difficult.

    This approach can be traced back to Kibble [Ki]. We regard it as the most satisfactory approach to the infrared problem. It provides an appropriate framework for the infrared problem in the case of exactly solvable van Hove Hamiltonians [De]. In our paper we will argue that this approach works also well in the case of Pauli-Fierz Hamiltonians, although one cannot rule out the appearance of other types of CCR representations besides the coherent ones.

    In order for this approach to be meaningful, one needs to use a certain version of the so-called *LSZ approach*, that means, one needs to construct the *asymptotic fields*. This requires some, usually mild, assumptions on the interaction of the "short range" type. This is the main weakness of this approach.

(3) One can keep the formal expression for the Hamiltonian and change the CCR representation. This amounts to a change of the underlying Hilbert space and of the Hamiltonian. The new Hamiltonian is sometimes called the renormalized Hamiltonian. The main requirement for the renormalized Hamiltonian is to have a ground state, which implies that the representation of its asymptotic fields contains a Fock sector.

    Shifting the asymptotic CCR representations can always be done in the case of exactly solvable van Hove Hamiltonians. In the case of Pauli-Fierz Hamiltonians it seems possible only under some special assumptions on the interaction, such as Assumption 2.D (the possibility to split the interaction into a scalar part and a regular part).

    One can criticize this approach in two separate points.

First of all, as we mentioned above, we need special assumptions to make this approach work. One can argue that Approach (2) is more general and does not need these assumptions.

Secondly, in general there is a large degree of arbitrariness in how to shift the Hamiltonian. Therefore, the renormalized Hamiltonian is not a canonical object.

One can try to give a justification of this approach by using C^*-algebras—Approach (4). The passage from the initial to a renormalized Hamiltonian would correspond to a change of a representation of the given C^*-algebraic system.

If applicable, this approach is very useful. In recent literature it was applied in [Ar] and [HHS]. It will be also an important tool in our paper.

(4) Sometimes one can describe a quantum system in terms of a dynamics on a C^*-algebra [FNV, BR]. This algebra may have many inequivalent representations. In some of them the dynamics may be generated by a Hamiltonian with a ground state, so that the infrared problem disappears.

This approach can be used to justify Approach (3). One can say that the initial Hilbert space is just one of many representations of the C^*-algebra and one needs to go to a different representation, where the representation of asymptotic fields has a Fock sector.

It seems that this approach is inadequate for Pauli-Fierz systems unless one makes some very special assumptions on the interaction. In general it is difficult (probably impossible) to find a physically motivated C^*-algebra which is preserved by the dynamics.

In our paper we will discuss in detail Approach (2) to the infrared problem in the context of Pauli-Fierz Hamiltonians. Approach (3) will play an important role, but it will be treated as a tool in the study of Approach (2). We will also discuss Approach (1).

1.3 Scattering theory for Pauli-Fierz Hamiltonians

There exist a number of simplified models that can be used to test some of the photonic aspects of QED. Probably the simplest are quadratic bosonic Hamiltonians with a linear perturbation. In [Sch] such Hamiltonians are called *van Hove Hamiltonians*, and we will use this name. They are exactly solvable and one can study their scattering theory in full detail [De]. A typical van Hove Hamiltonian can be written in the form:

$$\int \left(a^*(k) + \frac{\overline{z}(k)}{\omega(k)}\right) \omega(k) \left(a(k) + \frac{z(k)}{\omega(k)}\right) \mathrm{d}k, \tag{1.1}$$

where $z(k)$ is some given function and $\omega(k)$ is the dispersion relation, e.g. $\omega(k) = |k|$. Note that if we consider QED with prescribed classical charges, then we obtain a van Hove Hamiltonian

In our paper we consider the so-called *abstract Pauli-Fierz Hamiltonians*. They can also be used to understand interaction of photons with matter, but are more difficult and rich than the van Hove Hamiltonians. They are not exactly solvable and their mathematical understanding is far from complete. They are a caricature of QED with charged particles confined in an infinite well.

Consider the Hilbert space $\mathcal{K}\otimes\Gamma_{\rm s}(L^2(\mathbb{R}^d))$, where the Hilbert space $\mathcal{K}$ describes the confined charged particles and $\Gamma_{\rm s}(L^2(\mathbb{R}^d))$ is a bosonic Fock space. Following the terminology of [DG2, DJ, Ge1], an operator of the form

$$\begin{aligned} H \ &:= K\otimes 1\!\!1 + 1\!\!1\otimes \int \omega(k)a^*(k)a(k){\rm d}k \\ &+\int v(k)\otimes a^*(k){\rm d}k + \int v^*(k)\otimes a(k){\rm d}k \end{aligned}$$

will be called a *Pauli-Fierz Hamiltonian*. For simplicity, in our paper charged particles are described by an abstract Hamiltonian K and their confinement is expressed by the condition that K has a compact resolvent.

Let us sketch the main ideas of scattering theory for Pauli-Fierz Hamiltonians. We follow the formalism of [DG2, DG3], which can be traced back to much earlier work, such as [HK]. In the introduction we will not aim at the mathematical precision, for instance we will freely use the operator valued measures $a^{(*)}(k)$ and we will not precise the type of the limits involved in our statements. All the rigorous details will be provided in subsequent sections.

Under appropriate assumptions one can show the existence of the following limits:

$$a^{*\pm}(k) := \lim_{t\to\infty} {\rm e}^{{\rm i}tH}{\rm e}^{-{\rm i}t\omega(k)}a^*(k){\rm e}^{-{\rm i}tH}, \quad a^{\pm}(k) := \lim_{t\to\infty} {\rm e}^{{\rm i}tH}{\rm e}^{{\rm i}t\omega(k)}a(k){\rm e}^{-{\rm i}tH}.$$

We will call $a^{*\pm}(k)$ and $a^{\pm}(k)$ the *asymptotic creation/annihilation operators*. (If we want to be more precise, then we will say outgoing/incoming creation/annihilation operators). Note that they form covariant CCR representations:

$$\begin{aligned} &[a^{\pm}(k_1), a^{\pm}(k_2)] = 0, \quad [a^{\pm *}(k_1), a^{\pm *}(k_2)] = 0, \quad [a^{\pm}(k_1), a^{\pm *}(k_2)] = \delta(k_1-k_2), \\ &{\rm e}^{{\rm i}tH}a^{*\pm}(k){\rm e}^{-{\rm i}tH} = {\rm e}^{{\rm i}t\omega(k)}a^{\pm *}(k), \quad {\rm e}^{{\rm i}tH}a^{\pm}(k){\rm e}^{-{\rm i}tH} = {\rm e}^{-{\rm i}t\omega(k)}a^{\pm *}(k). \end{aligned}$$

We define $\mathcal{K}_0^{\pm}$ to be the space of $\Psi\in\mathcal{H}$ satisfying

$$a^{\pm}(k)\Psi = 0, \quad k\in\mathbb{R}^d.$$

Elements of $\mathcal{K}_0^{\pm}$ will be called *asymptotic vacua*. The Fock sectors of the asymptotic space are defined as

$$\mathcal{H}_0^{\pm} := \mathcal{K}_0^{\pm}\otimes\Gamma_{\rm s}(L^2(\mathbb{R}^d)).$$

The wave operators in the Fock sector are defined as linear maps $\Omega_0^{\pm}:\mathcal{H}_0^{\pm}\to\mathcal{H}$ satisfying

$$\Omega_0^{\pm}\ \Psi\otimes a^*(k_1)\cdots a^*(k_n)\Omega := a^{\pm *}(k_1)\cdots a^{\pm *}(k_n)\Psi, \quad \Psi\in\mathcal{K}_0^{\pm},$$

(Ω denotes the vacuum in the Fock space. The same letter decorated by the superscript $+$ or $-$ denotes the appropriate wave operator). We also introduce the Hamiltonian of the asymptotic vacua

$$K_0^{\pm} := H\Big|_{\mathcal{K}_0^{\pm}},$$

and the full asymptotic Hamiltonian:

$$H_0^{\pm} := K_0^{\pm}\otimes 1\!\!1 + 1\!\!1\otimes\int \omega(k)a^*(k)a(k){\rm d}k.$$

Now the following is true:

(1) $\Omega_0^\pm$ are isometric;

(2) $\Omega_0^\pm\ 1\!\!1\otimes a(k) = a^\pm(k)\Omega_0^\pm$,
$\Omega_0^\pm\ 1\!\!1\otimes a^*(k) = a^{\pm *}(k)\Omega_0^\pm$;

(3) $\mathcal{K}_0^\pm$ contains all eigenvectors of H;

(4) $\Omega_0^\pm H_0^\pm = H\Omega_0^\pm$.

One can formulate two desirable properties, called sometimes jointly the asymptotic completeness [DG2, DG3]:

- **The operators $\Omega_0^\pm$ are unitary**, in other words, **the asymptotic CCR representations are Fock**.
- **All asymptotic vacua are linear combinations of bound states of H.**

For massive bosons, (e.g. if $\omega(k) = \sqrt{k^2+m^2}$ with $m>0$), under quite weak assumptions one can show that both above properties are true [HK, DG2, DG3]. If $m=0$, little is known about these two properties except for the case of van Hove Hamiltonians [De]. Typically, the breakdown of the above properties is closely related to the infrared problem.

Note that the conventional scattering theory starts from a given pair of operators: the full Hamiltonian H and the free Hamiltonian H_0 and then proceeds to construct wave operators by considering the limit (in appropriate sense) of $\mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}$ as t goes to $\pm\infty$. The formalism of scattering theory that we described above differs substantially from the conventional one. Instead of the "free Hamiltonian" we have the asymptotic Hamiltonians $H_0^\pm$. The Hamiltonians $H_0^\pm$ are simpler than the full Hamiltonian H: they have the form of a "free Pauli-Fierz Hamiltonian". Nevertheless, they are not given a priori—they are constructed from H.

If $\Omega_0^\pm$ is not unitary, then the asymptotic fields have some non-Fock sectors. It may even happen that there are no nonzero asymptotic vacua, so that there are no asymptotic Fock sectors at all. This motivates us to give a description of scattering theory in the presence of non-Fock sectors.

Among non-Fock sectors the most manageable ones are the so-called *coherent sectors*. Our paper is to a large extent devoted to the description of scattering theory in their presence.

Let $\mathbb{R}^d \ni k \mapsto g(k)$ be a complex function. We define $\mathcal{K}_g^\pm$ to be the space of $\Psi\in\mathcal{H}$ satisfying

$$a^\pm(k)\Psi = \sqrt{2}g(k)\Psi, \quad k\in\mathbb{R}^d.$$

The elements of $\mathcal{K}_g^\pm$ will be called asymptotic g-coherent vectors. The asymptotic g-coherent space is defined as

$$\mathcal{H}_g^\pm := \mathcal{K}_g^\pm \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)).$$

The g-coherent wave operator is the linear map $\Omega_g^\pm : \mathcal{H}_g \to \mathcal{H}$ defined as

$$\Omega_g^\pm\ \Psi\otimes a^*(k_1)\cdots a^*(k_n)\Omega := (a^{\pm *}(k_1) - \sqrt{2}\overline{g}(k_1))\cdots(a^{\pm *}(k_n) - \sqrt{2}\overline{g}(k_n))\Psi, \quad \Psi\in\mathcal{K}_0^\pm.$$

We define the asymptotic Hamiltonian in the g-coherent sector as $H_g^\pm := \Omega_g^{\pm *}H\Omega_g^\pm$. The following can be easily shown:

(1) $\Omega_g^\pm$ are isometric;

(2) $\Omega_g^\pm \ \mathbb{1}\otimes a(k) = (a^\pm(k) - \sqrt{2}g(k))\Omega_g^\pm$,
$\Omega_g^\pm \ \mathbb{1}\otimes a^*(k) = (a^{\pm *}(k) - \sqrt{2}\overline{g}(k))\Omega_g^\pm$;

(3) $\Omega_g^\pm H_g^\pm = H\Omega_g^\pm$.

(4) There exists a decomposition

$$H_g^\pm = K_g^\pm \otimes \mathbb{1} + \mathbb{1}\otimes \int \left(a^*(k) + \sqrt{2}\overline{g}(k)\right)\omega(k)\left(a(k) + \sqrt{2}g(k)\right)\mathrm{d}k \tag{1.2}$$

(5) If g_1 and g_2 differ by a square integrable function, then the ranges of $\Omega_{g_1}^\pm$ and $\Omega_{g_2}^\pm$ coincide.

Note that the second term on the right of (1.2) is a van Hove Hamiltonian. If g is not square integrable then the asymptotic CCR representations on the range of $\Omega_g^\pm$ are non-Fock and the asymptotic Hamiltonians do not have a ground state—nevertheless, we have well defined wave operators that can be used to compute scattering cross-sections.

We are not aware of a full description of the above formalism in the literature, although some of its elements may belong to the so-called folklore. In particular, the fact that the asymptotic Hamiltonians have the form given in the equation (1.2) is quite interesting and not obvious.

1.4 Renormalized Hamiltonian and dressing operator

The main new "analytical" result of the paper is the proof of the existence of a nontrivial non-Fock coherent sector for asymptotic fields in a certain nontrivial class of Pauli-Fierz Hamiltonians. The most important additional assumption that we need to get this result is the possibility to split the interaction into two parts: an infrared divergent scalar part and an infrared convergent matrix part.

Using this assumption we can define the *renormalized Hamiltonian* H_{ren}. On the formal level the so-called renormalized Hamiltonian is unitarily equivalent to the initial Hamiltonian H:

$$H_{\mathrm{ren}} = \mathbb{1}\otimes W(-\mathrm{i}g)\, H_{\mathrm{ren}}\, \mathbb{1}\otimes W(\mathrm{i}g), \tag{1.3}$$

where $W(\mathrm{i}g)$ is formally a Weyl operator. Strictly speaking, however, $W(\mathrm{i}g)$ is not well defined, since g is not square integrable. Still, H_{ren} turns out to be a correctly defined Pauli-Fierz operator. Moreover, with an appropriate choice of g, H_{ren} has a mild infrared singularity, so that one can apply the results of [Ge1], which imply that H_{ren} possesses a ground state.

Under appropriate assumptions, one can show that for both H and H_{ren} one can define asymptotic fields. Besides, one can define the so-called *dressing operators* $U^\pm$. The dressing operators are some kind of unitary intertwiners between the objects related to H_{ren} and H. They do not intertwine, however, in the usual meaning of this word: it is not true that $HU^\pm = U^\pm H_{\mathrm{ren}}$. The action of $U^\pm$ gives some sort of a translation in phase space by g. In particular, $U^\pm$ map coherent sectors of the asymptotic fields of H_{ren} onto the coherent asymptotic sectors of H shifting them by g. In particular, they map the Fock sector of the asymptotic CCR representation for H_{ren} onto the g-coherent sector of the CCR representation for H, which is non-Fock. But we know that H_{ren} has a ground state. Hence it has nontrivial Fock asymptotic sectors. Therefore, H has nontrivial g-coherent asymptotic sectors.

According to Approach (3) described above one could discard H in favor of H_{ren}, and treat H_{ren} as the physical Hamiltonian. After this replacement, the asymptotic fields have Fock sectors, where the infrared problem is avoided. We, however, prefer the (more canonical and general) Approach (2), which treats H as the basic physical Hamiltonian and H_{ren} as a technical tool used to prove certain properties of scattering for H.

1.5 Comparison with literature

It is difficult to compare our results with the literature, since a large part of it is non-rigorous and a variety of models are studied.

Perhaps one of the oldest examples of "infrared renormalization" can be found in a paper of Pauli and Fierz [PF] devoted to non-relativistic QED. In that paper one can find what is nowadays often called "the Pauli-Fierz transformation", which can be used to make the Hamiltonian of non-relativistic QED less singular.

Blanchard considered scattering for the Hamiltonian of QED in the dipole approximation perturbed by a short range potential [Bl]. He showed that it is possible to construct wave operators if one replaces the original Hamiltonian by an appropriately renormalized one. Note that Blanchard's Hamiltonian is different from ours. In particular, in his case one can define usual wave operators and the formalism of asymptotic fields is not necessary, unlike in the case of our Hamiltonian.

Faddeev and Kulish made an interesting attempt to define wave and scattering operators for the full QED, taking into account both the long-range nature of the interaction between charged particles and the emergence of non-Fock representations of photons [KF]. Their work was not completely rigorous.

The infrared problem for the so-called *Nelson model* in the one-electron sector was studied by Fröhlich in [F], and more recently by Pizzo [Pi]. In these papers one can find an operator essentially equivalent to our dressing operators $U^{\pm}$. Fröhlich and Pizzo consider translation invariant models, which introduces additional complications in their analysis. A complete construction of dressed one electron states is not achieved in [F], (some parts of the construction relied on physically reasonable but conjectural assumptions). A complete construction was recently given by Pizzo [Pi].

Examples of the infrared renormalization, similar to the one in (1.3), can be found in [Ar, HHS].

Our paper can be considered to be a sequel to a number of papers devoted to scattering in quantum field theory [HK, DG2, DG3, Ge2, FGS]. All of these paper, except for [Ge2], are devoted to massive fields, which are not subject to the infrared problem,

When comparing the literature on models related to ours one should make a distinction between translation invariant models, such as those considered in [KF, F, Pi, FGS], and the models where the perturbation is localized and thus the translation invariance is broken, such as [HK, DG2, DG3, Ge2] and this paper. Translation invariant models are more difficult and rigorous results about them are scarce. The fact that we restrict ourselves to a confined system without translation invariance enables us to give a more transparent and thorough analysis of the scattering theory in presence of the infrared divergences.

1.6 Organization of the paper

Our paper can be divided into two parts. The first consists of Section 2, where we describe the main results of our paper. We introduce a certain class of abstract Pauli-Fierz Hamiltonians. We recall and partly extend basic results on the existence of asymptotic fields [DG2], [Ge2] and on the existence and non-existence of ground states [Ge1]. The asymptotic fields may have non-Fock sectors. We concentrate our attention on the so-called coherent sectors. We show how to define wave operators, scattering operators and asymptotic Hamiltonians for coherent sectors. We demonstrate that they are not much more difficult than the usual Fock sectors, and thus we explain how one can overcome the conceptual problems caused by the infrared problem. We show the existence of non-Fock sectors for a class of Pauli-Fierz Hamiltonians, that includes a certain class of Nelson Hamiltonians.

We end Section 2 with a discussion of inclusive cross-sections in our model. Let us stress that, in principle, by using our formalism one can describe predictions for experiments that measure "soft components of the system" and one does not need to restrict oneself to inclusive cross-sections. One can argue, however, that in realistic experiments the soft background should be irrelevant and measurable quantities should depend only on the "hard components". We discuss how to define such inclusive cross-sections and state some physically motivated conjectures about them.

The remaining part of our paper is somewhat more mathematical. It contains a systematic exposition of various elements of mathematical formalism used in Section 2. Some of them are presented in a more general context and proved in bigger generality. Let us stress that Sections 3, 4, 5, 6, 7 and the Appendix can be read independently of Section 2.

In Section 3 we study general CCR representations. A particular attention is devoted to the so-called *coherent representations*. These representations are obtained by translating the Fock representation by an antilinear functional. If the functional is not continuous, then this representation is not unitarily equivalent to the Fock representation.

In Section 4 we study the so-called *covariant CCR representations*. They are CCR representations equipped with a dynamics, which is implemented both on the level of the full space and of the 1-particle space. We show how to describe covariant representations in a coherent sector. It turns out that in every coherent sector the dynamics has a certain natural decomposition, one part of which is given by a quadratic Hamiltonian perturbed by a linear one (a van Hove Hamiltonian). In our opinion this is quite an interesting and hitherto unknown fact.

Covariant CCR representations arise naturally in scattering theory of certain quantum systems. Based on the ideas of the LSZ formalism, such representations were constructed and studied e.g. in [HK], and more recently in [DG2], [DG3] and [Ge2]. In Section 5 we study such representations in an abstract context. One of them describes the observables in the distant past—the *incoming representation* $W^{-}(\cdot)$, the other describes the observables in the distant future—the *outgoing representation* $W^{+}(\cdot)$. Collectively, they are called *asymptotic representations*. We show in particular that eigenvectors of the Hamiltonian are always vacua of both asymptotic representations and thus give rise to nontrivial Fock sectors.

Note that the material of Sections 3, 4 and 5 is rather basic and mostly belongs to the folklore (although our presentation has some points which are new). Section 6 is more special: here we introduce the so-called *dressing operator* between two CCR representations.

In Section 7 we introduce a relatively general class of Pauli-Fierz Hamiltonians. For these Hamiltonians, under some relatively mild assumptions on the interaction, asymptotic CCR representations exist and one can apply the formalism developed in the previous sections. One can also introduce the renormalized Hamiltonian and the dressing operators.

2 Overview of main results and some open problems

In this section we describe most of main results of our paper in a somewhat simplified form. We also discuss some aspects of the physical content of our mathematical constructions. We formulate some open mathematical problems inspired by physical considerations.

Let us make some remarks about our notation. If A is an operator, then $\mathrm{Dom}A$, $\mathrm{Ran}A$ and $\mathrm{sp}A$ denote its domain, range and spectrum. If A is self-adjoint and Θ a Borel subset of $\mathbb{R}$, then $1\!\!1_\Theta(A)$ denotes the spectral projection of A onto Θ. We also write $\langle x\rangle$ for $(1+x^2)^{1/2}$.

2.1 Pauli-Fierz Hamiltonians

Suppose that $\mathcal{K}$ is a separable Hilbert space representing the degrees of freedom of the atomic system. Let K be a positive operator on $\mathcal{K}$—the Hamiltonian of the atomic system. We will sometimes use

Assumption 2.A

$$(K+\mathrm{i})^{-1} \textit{ is compact on } \mathcal{K}.$$

The physical interpretation of this assumption is that the small system is confined.

Let $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$ be the 1−particle Hilbert space in the momentum representation and let $\Gamma_{\mathrm{s}}(\mathfrak{h})$ be the bosonic Fock space over $\mathfrak{h}$, representing the field degrees of freedom. Ω will stand for the vacuum in $\Gamma_{\mathrm{s}}(\mathfrak{h})$. We will denote by k the momentum operator of multiplication by k on $L^2(\mathbb{R}^d, \mathrm{d}k)$. Let

$$\omega := |k|$$

be the dispersion relation. For $f \in \mathfrak{h}$ the operators of creation and annihilation of f are denoted by

$$\int f(k)a^*(k)\mathrm{d}k, \qquad \int \overline{f}(k)a(k)\mathrm{d}k.$$

The Hamiltonian describing the field is equal to

$$\mathrm{d}\Gamma(\omega) = \int \omega(k)a^*(k)a(k)\mathrm{d}k.$$

(See e.g. [BR, vol. II] or [DG2, DG3] for basic concepts related to the second quantization).

Assumption 2.B *The interaction between the atom and the boson field is described with a coupling function v*

$$\mathbb{R}^d \ni k \mapsto v(k),$$

such that for a.e. $k \in \mathbb{R}^d$*,* $v(k)$ *is a bounded operator from* $\mathrm{Dom}(K^{1/2}$ *into* $\mathcal{K}$*. We will assume:*

$$\begin{aligned}
&\textit{for a.e.} \quad k \in \mathbb{R}^d, \quad v(k)(K+1)^{-\frac{1}{2}} \in B(\mathcal{K}), \\
&\forall\, \Psi_1, \Psi_2 \in \mathcal{K}, \quad k \mapsto (\Psi_2, v(k)(K+1)^{-\frac{1}{2}}\Psi_1) \textit{ is measurable,} \\
&\limsup_{R\to\infty} \int (1+\omega(k)^{-1})\|v(k)(K+R)^{-\frac{1}{2}}\|^2 \mathrm{d}k < 1/2.
\end{aligned}$$

Note that the functions $k \mapsto \|v(k)(K+R)^{-\frac{1}{2}}\|$ is measurable (see for example [Ge2, Appendix]), and hence the last condition in Assumption 2.B has a meaning.

We set

$$H_0 = K\otimes\mathbb{1} + \mathbb{1}\otimes \int \omega(k)a^*(k)a(k)\mathrm{d}k,$$

$$H = H_0 + \int v(k) \otimes a^*(k)\mathrm{d}k + \int v^*(k) \otimes a(k)\mathrm{d}k.$$

H_0 is called the *free Pauli-Fierz Hamiltonian* and H *the full Pauli-Fierz Hamiltonian*.

One can easily show that

Theorem 2.1 *Under Assumptions 2.A and 2.B, the operator* H *is self-adjoint and bounded from below with the form domain* $\mathrm{Dom}(H_0^{1/2})$*.*

2.2 The confined massless Nelson model

In this subsection we describe one of the main examples of Pauli-Fierz Hamiltonians. It is a model describing a confined atom interacting with a field of scalar bosons. A similar model (without the ultraviolet cut-off) was studied in a well known paper by Nelson [Ne]. Hence, in a part of the mathematical literature it is called the *Nelson model* (see [A], [Ar], [LMS]). To be more precise, the model that we will consider can be called the *confined massless ultraviolet cut-off Nelson model*.

We will prove that a large class of such models satisfies all the assumptions of this section. Thus Lemma 2.2 means that all the results presented in in Sections 2 and 7 apply to this class. In particular, their asymptotic CCR representations contain a non-Fock coherent sector.

The atom is described with the Hilbert space

$$\mathcal{K} := L^2(\mathbb{R}^{3P}, \mathrm{dx}),$$

where $\mathrm{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_P)$, x_i is the position of particle i, and the Hamiltonian:

$$K := \sum_{i=1}^{P} \frac{-1}{2m_i}\Delta_i + \sum_{i<j} V_{ij}(\mathrm{x}_i - \mathrm{x}_j) + W(\mathrm{x}_1, \ldots, \mathrm{x}_P),$$

where m_i is the mass of particle i, V_{ij} is the interaction potential between particles i and j and W is an external confining potential.

We will assume

$$(H0)\quad \begin{array}{l} V_{ij} \text{ is } \Delta- \text{ bounded with relative bound } 0, \\ W \in L^2_{\mathrm{loc}}(\mathbb{R}^{3P}), \quad W(\mathrm{x}) \geq c_0|\mathrm{x}|^{2\alpha} - c_1, \quad c_0 > 0,\ \alpha > 0. \end{array}$$

It follows from *(H0)* that K is symmetric and bounded below on $C_0^\infty(\mathbb{R}^{3P})$. We still denote by K its Friedrichs extension. Moreover we have $\mathrm{Dom}((K+b)^{\frac{1}{2}}) \subset H^1(\mathbb{R}^{3P}) \cap \mathrm{Dom}(|\mathrm{x}|^\alpha)$, which implies that

$$|\mathrm{x}|^\alpha (K+b)^{-\frac{1}{2}} \text{ is bounded.} \tag{2.1}$$

Note also that *(H0)* implies that K has compact resolvent on $L^2(\mathbb{R}^{3P})$.

The one-particle space for bosons is

$$\mathfrak{h} := L^2(\mathbb{R}^3, \mathrm{d}k),$$

where the observable k is the boson momentum. and the one-particle energy is $\omega(k) = |k|$.

The interaction is given by the operator $\mathbb{R}^3 \ni k \mapsto v(k) \in \mathcal{B}(\mathcal{K})$, where $v(k)$ is a multiplication operator on $L^2(\mathbb{R}^{3P}, \mathrm{dx})$ equal to

$$v(k, \mathrm{x}) = \frac{1}{\sqrt{2}} \sum_{j=1}^{P} \frac{\chi(|k|)}{|k|^{\frac{1}{2}}} \mathrm{e}^{-\mathrm{i}k\cdot \mathrm{x}_j}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a real, even function such that $\chi \equiv 1$ near 0. The function χ plays the role of an ultraviolet cutoff.

Lemma 2.2 *If hypothesis (H0) holds for $\alpha > 1$, the confined Nelson model satisfies assumptions 2.A, 2.B and 2.C, 2.D, 2.E, 2.F below, where in Assumptions 2.D and 2.F we set*

$$z(k) = \frac{P}{\sqrt{2}} \frac{\chi(|k|)}{|k|^{\frac{1}{2}}}, \qquad v_{\mathrm{ren}}(k, \mathrm{x}) = \frac{1}{\sqrt{2}} \sum_{j=1}^{P} \frac{\chi(|k|)}{|k|^{\frac{1}{2}}} (\mathrm{e}^{-\mathrm{i}k\cdot \mathrm{x}_j} - 1).$$

Proof. We already know that Assumption 2.A is true.

We have $|v(k, \mathrm{x})| \leq C|k|^{-1/2}$. Therefore,

$$\|v(k)\| \in L^2(\mathbb{R}^3, (1 + |k|^{-1})\mathrm{d}k),$$

and hence Assumptions 2.B and 2.E are satisfied.

We will now show that Assumption 2.C holds with $\mathfrak{g} := C_0^\infty(\mathbb{R}^3 \backslash \{0\})$. Let $h \in \mathfrak{g}$. Define

$$\begin{aligned} m_{j,t}(\mathrm{x}) \quad &:= \textstyle\int \overline{h}(k) \mathrm{e}^{\mathrm{i}t|k|} v(k, \mathrm{x}_j) \mathrm{d}k + \mathrm{cc} \\ &= \textstyle\int \mathrm{e}^{\mathrm{i}(t|k| - \mathrm{x}_j \cdot k)} \frac{\overline{h}(k)\chi(k)}{|k|^{1/2}} \mathrm{d}k + \mathrm{cc}. \end{aligned}$$

(The symbol cc denotes the complex conjugate). We can write

$$\begin{aligned} m_{j,t}(\mathrm{x})(1+K)^{-1/2} \quad &= m_{j,t}(\mathrm{x}) \mathbb{1}_{[0, \frac{t}{2}]}(|\mathrm{x}|)(1+K)^{-1/2} \\ &+ m_{j,t}(\mathrm{x}) \mathbb{1}_{]\frac{t}{2}, \infty[}(|\mathrm{x}|) \langle \mathrm{x} \rangle^{-\alpha} \langle \mathrm{x} \rangle^{\alpha} (1+K)^{-1/2}. \end{aligned} \tag{2.2}$$

Since by (2.1) $|\mathrm{x}|^\alpha(K+1)^{-\frac{1}{2}}$ is bounded, the second term is $O(t^{-\alpha})$, hence integrable.

To deal with the first term, note that the function $\overline{h}(k)\frac{\chi(|k|)}{|k|^{\frac{1}{2}}}$ is in $C_0^\infty(\mathbb{R}^3\backslash\{0\})$. Because of the cutoff function, the phase $t|k|-k\cdot\mathrm{x}_j$ is smooth without stationary points on $|\mathrm{x}|<t/2$. Using the non-stationary phase method, we obtain that the second term of (2.2) is $O(t^{-\infty})$. Hence for $\alpha>1$, Assumption 2.C is satisfied.

Consider now Assumption 2.D. We note that

$$|\mathrm{e}^{-\mathrm{i}k\cdot\mathrm{x}_j}-1|\leq|k|\,|\mathrm{x}_j|. \tag{2.3}$$

Hence

$$\|v_{\mathrm{ren}}(k)\langle\mathrm{x}\rangle^{-1}\|\leq C|k|^{1/2},$$

which implies $\|v_{\mathrm{ren}}(k)(1+K)^{-1/2}\|\in L^2(\mathbb{R}^3,|k|^{-2}\mathrm{d}k)$. This proves Assumption 2.D.

Finally, we prove Assumption 2.F. We set

$$\begin{aligned}
m_{j,t}(\mathrm{x}_j) &:= \int g(k)\mathrm{e}^{\mathrm{i}t|k|}v_{\mathrm{ren}}(k,\mathrm{x}_j)\mathrm{d}k+\mathrm{cc}\\
&= P\sqrt{2}\int\frac{\chi(|k|)^2}{k^2}\big(\cos(t|k|-\mathrm{x}_j\cdot k)-\cos t|k|\big)\mathrm{d}k.
\end{aligned}$$

We go to spherical coordinates (r,θ,ϕ), $r\in\mathbb{R}^+$, $\theta\in[0,\pi]$, $\phi\in[0,2\pi]$, and get:

$$\begin{aligned}
m_{j,t}(\mathrm{x}_j) :=\ & P\sqrt{2}\int_0^\infty\int_0^\pi\int_0^{2\pi}\chi(r)^2\big(\cos(tr-|\mathrm{x}_j|\cos\theta r)-\cos tr\big)\mathrm{d}r\mathrm{d}\cos\theta\mathrm{d}\phi\\
&= P\sqrt{2}2\pi\int_0^\infty\chi(r)^2\Big(\tfrac{\sin(tr+|\mathrm{x}_j|r)-\sin(tr-|\mathrm{x}_j|r)}{|\mathrm{x}_j|r}-2\cos tr\Big)\mathrm{d}r\\
&= P\sqrt{2}2\pi\int_{-\infty}^\infty\chi(r)^2\cos tr\Big(\tfrac{\sin(|\mathrm{x}_j||r|)}{|\mathrm{x}_j||r|}-1\Big)\mathrm{d}r\\
&= O(t^{-n}\langle\mathrm{x}_j\rangle^n).
\end{aligned}$$

for any $n\in\mathbb{N}$, where in the last step we integrated by parts. By interpolation we actually can replace n with any positive real α. Thus we see that

$$\begin{aligned}
&m_{j,t}(\mathrm{x}_j)(1+K)^{-1/2}\\
&= m_{j,t}(\mathrm{x}_j)\langle\mathrm{x}\rangle^{-\alpha}\langle\mathrm{x}\rangle^{\alpha}(1+K)^{-1/2}=O(t^{-\alpha}),
\end{aligned}$$

which ends the proof of Assumption 2.F. □

2.3 Asymptotic fields

For $h\in\mathfrak{h}$ we define the field and the Weyl operators

$$\phi(h):=\frac{1}{\sqrt{2}}\int\big(h(k)a^*(k)+\overline{h}(k)a(k)\big)\mathrm{d}k,\quad W(f):=e^{\mathrm{i}\phi(h)}.$$

Let

$$\mathfrak{h}_1:=\left\{h\in\mathfrak{h}\ |\ \int(1+\omega(k)^{-1})|h(k)|^2\mathrm{d}k<\infty\right\}=\mathrm{Dom}(\omega^{-1/2}),$$

with the norm $\|h\|_{\mathfrak{h}_1}:=\|(1+\omega^{-1})^{1/2}h\|_{\mathfrak{h}}$.

The following assumption can be called the *short range condition*.

Assumption 2.C *There exists a dense subspace* $\mathfrak{g} \subset \mathfrak{h}_1 \cap \mathrm{Dom}(\omega^{1/2})$ *such that for* $h \in \mathfrak{g}$*,*

$$\int_0^\infty \left\| \int \left(\mathrm{e}^{\mathrm{i}t\omega(k)} \overline{h}(k) v(k) + v^*(k) \mathrm{e}^{-\mathrm{i}t\omega(k)} h(k) \right) (1+K)^{-1/2} \mathrm{d}k \right\|_{B(\mathcal{K})} \mathrm{d}t < \infty.$$

Theorem 2.3 *Suppose that assumptions 2.B and 2.C hold. Then:*

(1) *For all* $h \in \mathfrak{h}_1$*, there exist*

$$W^\pm(h) := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} 1\!\!1_{\mathcal{K}} \otimes W(\mathrm{e}^{-\mathrm{i}t\omega} h) \mathrm{e}^{-\mathrm{i}tH}. \tag{2.4}$$

(2)

$$W^\pm(h_1) W^\pm(h_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h_1|h_2)} W^\pm(h_1 + h_2), \quad h_1, h_2 \in \mathfrak{h}_1,$$
$$\mathbb{R} \ni t \mapsto W^\pm(th) \quad \textit{is strongly continuous}, \quad h \in \mathfrak{h}_1;$$

in other words,

$$\mathfrak{h}_1 \ni h \mapsto W^\pm(h) \tag{2.5}$$

are regular CCR representations (see Section 3).

(3)

$$\mathrm{e}^{\mathrm{i}tH} W^\pm(h) \mathrm{e}^{-\mathrm{i}tH} = W^\pm(\mathrm{e}^{\mathrm{i}t\omega} h), \quad h \in \mathfrak{h}_1,$$

in other words, $(W^\pm, \omega, H)$ *are covariant CCR representations (see Section 4).*

(4) *If* $H\Psi = E\Psi$*, then*

$$(\Psi | W^\pm(h) \Psi) = \mathrm{e}^{-\|h\|^2/4} \|\Psi\|^2, \tag{2.6}$$

in other words, eigenvectors of H *are vacua for (2.5) (see Section 3) and Theorem 5.2).*

The above theorem is a simplified version of Theorem 7.4 proved later in our paper.

It is convenient to introduce the following notation. $\mathcal{H}_{\mathrm{p}}(H)$ will denote the closure of the span of eigenvectors of H. The set of vacua for (2.5), i.e. the set of $\Phi \in \mathcal{H}$ satisfying (2.6) is denoted by $\mathcal{K}_0^\pm$. Note that $\mathcal{K}_0^\pm$ is a closed subspace of $\mathcal{H}$. By Theorem 2.3 (4), $\mathcal{K}_0^\pm$ contains $\mathcal{H}_{\mathrm{p}}(H)$.

The closure of the span of vectors $W(h)\Phi$ with $h \in \mathfrak{h}_1$, $\Phi \in \mathcal{K}_0^\pm$ will be denoted by $\mathcal{H}_{[0]}^\pm$. It is the largest subspace of $\mathcal{H}$ on which (2.5) is equivalent to the Fock representation.

Let us state the following conjecture:

Conjecture 2.4 *Suppose Assumptions 2.A, 2.B and 2.C hold. Assume also*

$$\int \frac{\|v(k)\|^2}{k^2} \mathrm{d}k < \infty. \tag{2.7}$$

Then

(1) $\mathcal{H}_{[0]}^\pm = \mathcal{H}$*, in other words, the asymptotic representations are multiples of the Fock representation.*

(2) $\mathcal{K}_0^{\pm} = \mathcal{H}_{\mathrm{p}}(H)$, *in other words, all the asymptotic vacua are linear combinations of eigenstates of* H.

There are two situations when we can prove the above conjecture.

If $\dim \mathcal{K} = 1$, then the Hamiltonian H is the exactly solvable van Hove Hamiltonian and the conjecture follows by explicit computations, see e.g. [De].

If $v(k) = 0$ in a neighborhood of zero, then the problem reduces to the case with a positive mass. Conjecture 2.4 (1) follows then from the arguments due to Hoegh-Krohn [HK] described in [DG2], see also a different proof in [DG3]. Conjecture 2.4 (2) follows then from [DG2], see also a somewhat simpler proof given in [DG3].

Note that the power $|k|^{-2}$ in (2.7) is natural, since it is suggested by the exactly solvable case. However, we do not know how to prove our conjecture even under much stronger assumptions, e.g. if for any N

$$\int \frac{\|v(k)\|^2}{k^N} \mathrm{d}k < \infty.$$

2.4 Existence and nonexistence of a ground state

The following assumption will be very important in the sequel:

Assumption 2.D $v(k)$ *can be split as*

$$\begin{aligned}
&v(k) = z(k)\mathbb{1}_{\mathcal{K}} + v_{\mathrm{ren}}(k), \ \ \textit{where} \\
&z(k) \in \mathbb{C}, \qquad v_{\mathrm{ren}}(k) \in B(\mathrm{Dom}(K^{\frac{1}{2}}), \mathcal{K}), \\
&\textstyle\int (1 + \omega(k)^{-1})|z(k)|^2 \mathrm{d}k < \infty, \\
&\textstyle\int \omega(k)^{-2} \|v_{\mathrm{ren}}(k)(K+1)^{-1/2}\|^2 \mathrm{d}k < \infty.
\end{aligned}$$

In order to use the results of [Ge1] we will also need the following (probably unnecessary) assumption, which is stronger than Assumption 2.B:

Assumption 2.E

$$\begin{aligned}
&\textit{for a.e.} \quad k \in \mathbb{R}^d, \quad v(k)(K+1)^{-\frac{1}{2}},\ (K+1)^{-\frac{1}{2}} v(k) \in B(\mathcal{K}), \\
&\forall \Psi_1, \Psi_2 \in \mathcal{K}, \quad k \mapsto (\Psi_2, (K+1)^{-\frac{1}{2}} v(k) \Psi_1) \ \textit{and} \ k \mapsto (\Psi_2, v(k)(K+1)^{-\frac{1}{2}} \Psi_1) \ \textit{are measurable}, \\
&\lim_{R \to \infty} \textstyle\int (1 + \omega(k)^{-1}) \big(\|v(k)(K+R)^{-\frac{1}{2}}\|^2 + \|(K+R)^{-\frac{1}{2}} v(k)\|^2 \big) \mathrm{d}k = 0.
\end{aligned}$$

Theorem 2.5 *Assume Hypotheses 2.D and 2.E. Then:*

(1) *if Assumption 2.A holds and* $\int \omega(k)^{-2}|z(k)|^2 \mathrm{d}k < \infty$, *then* $\inf \mathrm{sp}(H)$ *is an eigenvalue.*

(2) *if* $\inf \mathrm{sp}(H)$ *is an eigenvalue, then* $\int \omega(k)^{-2}|z(k)|^2 \mathrm{d}k < \infty$.

In particular under Assumption 2.A, the existence of a ground state is *equivalent* to the condition

$$\int \omega(k)^{-2}|z(k)|^2 \mathrm{d}k < \infty.$$

Proof. Part (1) has been shown in [Ge1, Thm. 1]. Let us prove part (2) by contradiction. Assume that

$$\int \omega(k)^{-2}|z(k)|^2 \mathrm{d}k = \infty,$$

and let $\Psi_0 \in \mathcal{H}$ be a ground state of H. The following pull-through formula is valid (see e.g. [Ge1, Sect. III.4]):

$$\begin{array}{rl} & (H+\omega(k)-z)^{-1}a(k)\Psi \\ = & a(k)(H-z)^{-1}\Psi + (H+\omega(k)-z)^{-1}v(k)(H-z)^{-1}\Psi, \ \Psi \in \mathcal{H}, \end{array} \tag{2.8}$$

as an identity on $L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash\{0\}, \mathrm{d}k; \mathcal{H})$. Applying this identity to Ψ_0, we obtain

$$a(k)\Psi_0 = (E - H - \omega(k))^{-1}v(k)\Psi_0,$$

as an identity on $L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash\{0\}, \mathrm{d}k; \mathcal{H})$. Hence

$$a(k)\Psi_0 = \frac{z(k)}{\omega(k)}\Psi_0 + (E - H - \omega(k))^{-1}v_{\mathrm{ren}}(k)\Psi_0.$$

Let

$$r(k) := a(k)\Psi_0 - \frac{z(k)}{\omega(k)}\Psi_0 = (E - H - \omega(k))^{-1}v_{\mathrm{ren}}(k)\Psi_0.$$

We have

$$\|r(k)\| \leq c\frac{1}{\omega(k)}\|v_{\mathrm{ren}}(k)(K+1)^{-\frac{1}{2}}\|.$$

Hence, by the last condition of Assumption 2.D, $r \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H})$. Since $\frac{z}{\omega} \notin \mathfrak{h}$, applying Lemma 2.6 below we obtain $\Psi_0 = 0$, which is a contradiction. $\Box$

Lemma 2.6 *Let $\Psi \in \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$ such that*

$$\int \|(a(k) - g(k))\Psi\|^2 \mathrm{d}k < \infty, \tag{2.9}$$

where $k \mapsto g(k) \in \mathbb{C}$ is measurable and

$$\int |g(k)|^2 \mathrm{d}k = \infty.$$

Then $\Psi = 0$.

Proof. We write

$$\Psi = (\Psi_0, \Psi_1, \cdots, \Psi_n, \cdots)$$

where $\Psi_n \in \otimes_{\mathrm{s}}^n \mathfrak{h}$. From (2.9) we obtain since $g \notin L^2(\mathbb{R}^d)$ and $\Psi_n \in \otimes^n L^2(\mathbb{R}^d)$

$$(n+1)^{\frac{1}{2}}\Psi_{n+1}(k, k_1, \ldots, k_n) - g(k)\Psi_n(k_1, \ldots, k_n) \in \otimes^{n+1}L^2(\mathbb{R}^d),$$

which implies that $\Psi_n = 0$. Hence $\Psi = 0$. $\Box$

2.5 Existence of non-Fock sectors for asymptotic fields

Set

$$g(k) := \sqrt{2}\omega^{-1}(k)z(k). \tag{2.10}$$

Let us introduce the following assumption:

Assumption 2.F

$$\int_0^\infty \left\| \int \left(\mathrm{e}^{\mathrm{i}t\omega(k)}\overline{g}(k)v_{\mathrm{ren}}(k) + v_{\mathrm{ren}}^*(k)\mathrm{e}^{-\mathrm{i}t\omega(k)}g(k)\right)(1+K)^{-1/2}\mathrm{d}k \right\| \mathrm{d}t < \infty.$$

Theorem 2.7 *Assume Hypotheses 2.A,2.C, 2.D, 2.E and 2.F. Then there exists a nonzero vector $\Phi \in \mathcal{H}$ such that*

$$(\Phi|W^{\pm}(h)\Phi) = \|\Phi\|^2 \mathrm{e}^{\mathrm{iRe}(h|g)}\mathrm{e}^{-\|h\|^2/4}. \tag{2.11}$$

In particular, if $g \notin L^2$, then the CCR representations (2.5) have non-Fock coherent sectors.

Let us introduce the following notation. The set of vectors Φ satisfying (2.11) will be denoted by $\mathcal{K}_g^{\pm}$. Such vectors will be called g-coherent vectors for (2.5) (see Section 3). They form a closed subspace of $\mathcal{H}$.

The closure of the span of vectors $W(h)\Phi$ with $h \in \mathfrak{h}_1$, $\Phi \in \mathcal{K}_g^{\pm}$, will be denoted by $\mathcal{H}_{[g]}^{\pm}$. It is the largest subspace of $\mathcal{H}$ on which (2.5) is equivalent to the so-called *g-coherent representation*.

Conjecture 2.8 *Under the hypotheses of Theorem 2.7, $\mathcal{H}_{[g]} = \mathcal{H}$, in other words, the representation of asymptotic fields is equivalent to a multiple of the g-coherent representation.*

Note that given the methods of the proof of Theorem 2.7, Conjecture 2.8 essentially follows from Conjecture 2.4 (1).

2.6 Renormalized Hamiltonian

In this and the next subsection we will describe the main ideas of the proof of Theorem 2.7. One of them is the use of the so-called renormalized Hamiltonian. It is defined as

$$H_{\mathrm{ren}} := K_{\mathrm{ren}} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \int (v_{\mathrm{ren}}(k)a^*(k) + v_{\mathrm{ren}}^*(k)a(k))\mathrm{d}k,$$

where

$$K_{\mathrm{ren}} \quad := K - \int \left(\tfrac{|z(k)|^2}{\omega(k)} + \tfrac{\overline{z}(k)v_{\mathrm{ren}}(k)}{\omega(k)} + \tfrac{v_{\mathrm{ren}}^*(k)z(k)}{\omega(k)}\right)\mathrm{d}k.$$

Note that Assumptions 2.A, 2.D, 2.E for H imply Assumptions 2.A, 2.D and 2.E for H_{ren} with $z_{\mathrm{ren}} = 0$. Therefore, by the result of [Ge1] quoted in Theorem 2.5 (1), H_{ren} has a ground state.

Suppose Assumptions 2.C and 2.F hold as well. Then, by Theorem 7.5, we can define asymptotic fields for H_{ren}

$$W_{\mathrm{ren}}^{\pm}(h) := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}\mathbb{1}\otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}.$$

Clearly, $W_{\mathrm{ren}}^{\pm}$ satisfy the obvious analog of Theorem 2.3. The ground state of H_{ren} is a vacuum for the renormalized asymptotic fields.

Remark 2.9 *Note that if $g \in \mathfrak{h}$, then*

$$H = W(\mathrm{i}g)H_{\mathrm{ren}}W(-\mathrm{i}g). \tag{2.12}$$

If $g \notin \mathfrak{h}$, then $W(\pm \mathrm{i}g)$ is not well defined. Still, we can use (2.12) on a formal level. To make it rigorous we can proceed in a variety of ways. We can choose a sequence of approximations of g

$$g_\sigma := g 1\!\!1_{[\sigma,\infty[}(\omega), \quad 0 < \sigma < 1.$$

Then it is easy to show that

$$(\mathrm{i} + H_{\mathrm{ren}})^{-1} = \mathrm{s-}\lim_{\sigma \searrow 0} \big(\mathrm{i} + W(\mathrm{i}g_\sigma)HW(\mathrm{i}g_\sigma)\big)^{-1}.$$

2.7 Dressing operators

Clearly, $\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)$ is well defined and $(1 - \mathrm{e}^{-\mathrm{i}t\omega})g \in \mathfrak{h}$. Therefore the following definition makes sense:

$$U(t) := \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}\mathrm{e}^{\mathrm{i}tH}W(\mathrm{i}(1 - \mathrm{e}^{-\mathrm{i}t\omega})g)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}.$$

Theorem 2.10 *Under Assumptions 2.B, 2.D and 2.F, there exists $U^\pm := \mathrm{s-}\lim_{t\to\pm\infty} U(t)$. $\mathrm{s-}\lim_{t\to\pm\infty} U(t)^*$ also exists and equals $U^{\pm *}$.*

The above theorem will be proved under more general conditions later as Theorem 7.5.

The operators $U^\pm$ will be called the *dressing operators*. They have the following properties:

Theorem 2.11 *Suppose Assumptions 2.B, 2.C, 2.D and 2.F are true. Then, for $h \in \mathfrak{h}_1$, we have*

$$\begin{aligned}
W^\pm(h)U^\pm &= U^\pm W^\pm_{\mathrm{ren}}(h)\mathrm{e}^{\mathrm{i}\mathrm{Re}(h,g)},\\
\mathrm{e}^{\mathrm{i}tH}U^\pm\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} &= U^\pm W^\pm_{\mathrm{ren}}(\mathrm{i}(1 - \mathrm{e}^{\mathrm{i}t\omega})g)\mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}\\
&= W^\pm(\mathrm{i}(1 - \mathrm{e}^{\mathrm{i}t\omega})g)U^\pm\mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}.
\end{aligned}$$

Therefore, $U^\pm$ maps $\mathcal{K}^\pm_{0,\mathrm{ren}}$ onto $\mathcal{K}^\pm_g$.

The above properties of dressing operators are proved in Section 6.

2.8 Wave operators

We define the *g-coherent asymptotic space* as

$$\mathcal{H}^\pm_g := \mathcal{K}^\pm_g \otimes \Gamma_{\mathrm{s}}(\mathfrak{h}).$$

It is easy to show that there exists a unique linear operator $\Omega^\pm_g : \mathcal{H}^\pm_g \to \mathcal{H}$ such that

$$\Omega^\pm_g\ \Phi\otimes W(h)\Omega = \mathrm{e}^{-\mathrm{i}\mathrm{Re}(h|g)}W^\pm(h)\Phi, \quad \Phi \in \mathcal{K}^\pm_g, \quad h \in \mathfrak{h}_1.$$

The operator $\Omega_g^\pm$ is isometric and its range equals $\mathcal{H}_{[g]}^\pm$. It will be called the *g-coherent wave operator*. (Note that Ω, without any superscripts, still denotes the vacuum in a Fock space).

The *g-coherent asymptotic Hamiltonian* is defined as

$$H_g^\pm := \Omega_g^{\pm *} H \Omega_g^\pm.$$

Clearly, $H_g^\pm$ is a self-adjoint operator on $\mathcal{H}_g^\pm$ satisfying

$$\Omega_g^\pm H_g^\pm = H \Omega_g^\pm.$$

What is a little less obvious is the following decomposition of $H_g^\pm$, proved in Theorem 4.5:

$$H_g^\pm = K_g^\pm \otimes \mathbb{1} + \mathbb{1} \otimes \int \left(a^*(k) + \frac{z(k)}{\omega(k)} \right) \omega(k) \left(a(k) + \frac{\overline{z(k)}}{\omega(k)} \right) \mathrm{d}k \tag{2.13}$$

Thus, in particular, the asymptotic Hamiltonians $H_g^\pm$ do not have ground states.

Note that the subspaces $\mathcal{H}_{[g]}^\pm$ are invariant with respect to $W^\pm(h)$ and H. They depend only on the equivalence class $[g]$ of g in $\mathfrak{h}_1^*/\mathfrak{h}$, where $\mathfrak{h}_1^*$ denotes the space of all antilinear functionals on $\mathfrak{h}_1$ (see Theorem 3.7 (1)).

If one introduces

$$H_{[g]}^\pm = H\Big|_{\mathcal{H}_{[g]}^\pm} = \Omega_g^\pm H_g^\pm \Omega_g^{\pm *},$$

then again $H_{[g]}^\pm$ depends only on $[g]$.

If Conjecture 2.8 is true then $\mathcal{H} = \mathcal{H}_{[g]}^\pm$ and $H = H_{[g]}^\pm$.

2.9 Scattering operator

We can define the *scattering operator* for the $g - g$ channel as

$$S_{gg} := \Omega_g^{+*} \Omega_g^-.$$

It is unitary iff $\mathcal{H}_{[g]}^- = \mathcal{H}_{[g]}^+$.

Suppose that we prepare a state in a distant past inside the incoming g-coherent sector. We can describe it by a density matrix (a positive operator of trace 1) ρ on $\mathcal{H}_g^-$.

Suppose that we make a measurement in a distant future in the outgoing g-coherent sector. We can describe it by an observable (a self-adjoint operator) A on $\mathcal{H}_g^+$.

The expectation value of the experiment is given by the trace

$$\mathrm{Tr} S_{gg} \rho S_{gg}^* A. \tag{2.14}$$

Note that there is no infrared problem in the formula above. In principle, one has a well defined procedure to compute the expectation value of an experiment involving any initial state and any final observable—there is no need to restrict oneself to "inclusive cross-sections".

The infrared problem manifests itself in the non-canonical choice of the functional g. In fact, g is not determined by the Hamiltonian H itself. One can argue that in a realistic experiment all the quantities depending on the choice of g are unmeasurable (or at least are much more difficult to measure). This is quite similar to long-range scattering for Schrödinger operators, where the modified scattering operator depends on a non-canonical modifier and one usually assumes that measurable quantities are independent of its choice. In the remaining part of this section we will analyze scattering of infrared singular Pauli-Fierz Hamiltonians and point out quantities that are likely to be physically relevant.

Let us note a certain discrepancy between mathematics and physics of the problem. In the construction of wave and scattering operators the past is treated in the same way as the future. Thus mathematics of scattering theory is in some sense symmetric with respect to time reversal. This is not the case for the formula (2.14), which gives physical interpretation of the scattering operator: the past is represented by a density matrix whereas the future by an arbitrary selfadjoint operator. This asymmetry between past and future will be even more pronounced in the next subsections, where we discuss inclusive cross-sections. It will be clear which observables should be considered in the future, it will be less clear which initial states should be taken into account.

2.10 Soft and hard photons

Let $\epsilon \geq 0$. Define

$$\mathfrak{h}_{\leq\epsilon} := \mathrm{Ran}\mathbb{1}_{[0,\epsilon]}(\omega), \quad \mathfrak{h}_{>\epsilon} := \mathrm{Ran}\mathbb{1}_{]\epsilon,\infty[}(\omega),$$

so that $\mathfrak{h} = \mathfrak{h}_{\leq\epsilon} \oplus \mathfrak{h}_{>\epsilon}$. Clearly, we can make the identification

$$\mathcal{H}_g^{\pm} \simeq \mathcal{H}_{g,\leq\epsilon}^{\pm} \otimes \Gamma_{\mathrm{s}}(\mathfrak{h}_{>\epsilon}), \tag{2.15}$$

where $\mathcal{H}_{g,\leq\epsilon}^{\pm} := \mathcal{K}_g^{\pm} \otimes \Gamma_{\mathrm{s}}(\mathfrak{h}_{\leq\epsilon})$.

Let us make an additional assumption

$$\mathbb{1}_{]\epsilon,+\infty[}(\omega) g = 0. \tag{2.16}$$

Since g is given in terms of z by the equality (2.10), this is equivalent to $\mathbb{1}_{]\epsilon,+\infty[}(\omega) z = 0$, which we can always assume. By this assumption, the asymptotic Hamiltonian can be written as

$$\begin{aligned} H_g^{\pm} &= K_g^{\pm} \otimes \mathbb{1} + \mathbb{1} \otimes \int\limits_{\omega<\epsilon} \left(a^*(k) + \tfrac{z(k)}{\omega(k)}\right) \omega(k) \left(a(k) + \tfrac{\overline{z(k)}}{\omega(k)}\right) \mathrm{d}k \\ &+ \mathbb{1} \otimes \int\limits_{\omega\geq\epsilon} \omega(k) a^*(k) a(k) \mathrm{d}k. \end{aligned}$$

Therefore, with respect to the decomposition (2.15), the asymptotic Hamiltonians can be written as

$$H_g^{\pm} = H_{g,\leq\epsilon}^{\pm} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega_{>\epsilon}),$$

where $\omega_{>\epsilon} = \omega \mathbb{1}_{]\epsilon,\infty[}(\omega)$.

One can ask whether the decomposition into soft and hard components is sensitive to the choice of g. Introduce the soft Hamiltonian

$$H_{[g],\leq\epsilon}^{\pm} := \Omega_g^{\pm}\ H_{g,\leq\epsilon}^{\pm} \otimes \mathbb{1}\ \Omega_g^{\pm *},$$

and the hard Hamiltonian

$$H^{\pm}_{[g],>\epsilon} := \Omega^{\pm}_{g}\ 1\!\!1\otimes \mathrm{d}\Gamma(\omega_{>\epsilon})\ \Omega^{\pm *}_{g}.$$

We have

$$H^{\pm}_{[g]} = H^{\pm}_{[g],\leq\epsilon} + H^{\pm}_{[g],>\epsilon} \tag{2.17}$$

and the Hamiltonians in (2.17) depend only on $[g]$.

A similar question can be asked concerning the observables. On the level of asymptotic spaces we have clearly

$$B(\mathcal{H}^{\pm}_{g}) \simeq B(\mathcal{H}^{\pm}_{g,\leq\epsilon}) \otimes B(\mathcal{H}^{\pm}_{g,>\epsilon}). \tag{2.18}$$

Denote the range of the homomorphism

$$B(\mathcal{H}^{\pm}_{g}) \ni A \mapsto \Omega^{\pm}_{g}\ A\Omega^{\pm *}_{g} \in B(\mathcal{H})$$

by $\mathfrak{A}_{[g]}$. $\mathfrak{A}_{[g]}$ depends only on $[g]$ and is equal to $B(\mathcal{H}^{\pm}_{[g]})$. Inside $\mathfrak{A}_{[g]}$ we can distinguish the "algebra of soft observables"

$$\mathfrak{A}^{\pm}_{[g],\leq\epsilon} := \Omega^{\pm}_{g}\ B(\mathcal{H}^{\pm}_{g,\leq\epsilon})\otimes 1\!\!1\ \Omega^{\pm *}_{g}, \tag{2.19}$$

and the "algebra of hard observables"

$$\mathfrak{A}^{\pm}_{[g],>\epsilon} := \Omega^{\pm}_{g}\ 1\!\!1\otimes B(\mathcal{H}^{\pm}_{g,>\epsilon})\ \Omega^{\pm *}_{g}. \tag{2.20}$$

(2.19) and (2.20) depend only on $[g]$. The hard observables are even more independent of g. The automorphism

$$B(\Gamma_{\mathrm{s}}(\mathfrak{h}_{>\epsilon})) \ni A_{>\epsilon} \mapsto \Omega^{\pm}_{g}\ 1\!\!1\otimes A_{>\epsilon}\ \Omega^{\pm *}_{g} \in \mathfrak{A}^{\pm}_{[g],>\epsilon}$$

depends only on $[g]$ if we assume (2.16).

2.11 Inclusive cross-sections

To simplify the discussion, we will assume in what follows that

$$\mathcal{H} = \mathcal{H}^{+}_{[g]} = \mathcal{H}^{-}_{[g]}. \tag{2.21}$$

In what follows we will drop the subscripts g wherever possible, thus we will write $\Omega^{\pm}$, $\mathcal{H}^{\pm}$, $H^{\pm}$, $H^{\pm}_{\leq\epsilon}$, etc. instead of $\Omega^{\pm}_{g}$, $\mathcal{H}^{\pm}_{g}$, $H^{\pm}_{g}$, $H^{\pm}_{g,\leq\epsilon}$, etc.

Set $E := \inf H$. Clearly,

$$E = \inf H^{-} = \inf H^{+} = \inf H^{-}_{\leq\gamma} = \inf H^{+}_{\leq\gamma}$$

for any $\gamma > 0$.

Note that by the assumption (2.21), the wave operators $\Omega^{\pm}$ are unitary from $\mathcal{H}^{\pm}$ to $\mathcal{H}$ and the scattering operator $S = \Omega^{+*}\Omega^{-}$ is unitary from $\mathcal{H}^{-}$ to $\mathcal{H}^{+}$.

Suppose now that the experimentalist can only control the components of the system above the threshold ϵ. In particular, since the functional g depends on the soft components, the quantities that depend on g are not measurable.

The quantum description of an experiment has two aspects: preparation of the incoming state and measurement of the outgoing observable. It is easy to say which observables can in principle be measured by the experimentalist. They are the observables in the hard algebra $\mathfrak{A}^+_{>\epsilon}$, that means the observables of the form $\Omega^+ \ 1\!\!1 \otimes A_{>\epsilon} \ \Omega^{+*}$, where $A_{>\epsilon} \in B(\Gamma_{\rm s}(\mathfrak{h}_{>\epsilon}))$.

It is more difficult to say which incoming states the experimentalist can prepare. Recall that $\mathcal{H}^- = \mathcal{H}^-_{\leq\epsilon} \otimes \mathcal{H}^-_{>\epsilon}$. Thus we can introduce the partial trace wrt $\mathcal{H}^-_{\leq\epsilon}$, denoted

$$l^1(\mathcal{H}^-) \ni \rho \mapsto \mathrm{Tr}^-_{\leq\epsilon} \ \rho \in l^1(\mathcal{H}^-_{>\epsilon}),$$

where $l^1(\mathcal{H})$ denotes the space of trace class operators on a Hilbert space $\mathcal{H}$. In particular, if ρ is a density matrix on $\mathcal{H}^-$, then $\mathrm{Tr}^-_{\leq\epsilon} \ \rho$ is a density matrix on $\mathcal{H}^-_{>\epsilon}$.

We assume that the initial state of the system is described by a density matrix ρ on $\mathcal{H}^-$. We also suppose that the experimentalist does not have full information about ρ and is able to control only $\mathrm{Tr}^-_{\leq\epsilon}\rho$. More precisely, for a given density matrix $\rho_{>\epsilon}$ on $\mathcal{H}^-_{>\epsilon}$, while preparing his experiment, he can make sure that

$$\mathrm{Tr}^-_{\leq\epsilon} \ \rho = \rho_{>\epsilon}. \tag{2.22}$$

Of course, there are many density matrices ρ satisfying (2.22). The choice of ρ should be determined by physics. Let us suppose that the experiment is conducted at a low temperature, so that everything tends to have the lowest possible energy.

Suppose for a moment that the infrared problem is absent in the sense that the Hamiltonian H, hence also $H^\pm$ and $H^\pm_{\leq\epsilon}$, has a non-degenerate ground state. Then it is natural to assume that the incoming density matrix equals

$$1\!\!1_E(H^-_{\leq\epsilon}) \otimes \rho_{>\epsilon}.$$

(recall that $\inf \mathrm{sp}(H^-_{\leq\epsilon}) = E$, and hence $1\!\!1_E(H^-_{\leq\epsilon})$ denotes the spectral projection onto the ground state of $H^-_{\leq\epsilon}$). Thus one can argue that if the experimentalist prepared the hard part of the incoming state as $\rho_{>\epsilon}$ and measures the observable A, then the expectation value of the measurement (which we will somewhat imprecisely call the cross-section) will be

$$\mathrm{Tr} \ S \ 1\!\!1_E(H^-_{\leq\epsilon}) \otimes \rho_{>\epsilon} \ S^* \ A. \tag{2.23}$$

If we have an infra-red problem—if H has no ground state at all or even if its ground state is degenerate—then it is not clear which ρ satisfying (2.22) should be taken. We can argue that ρ should satisfy

$$1\!\!1_{[E,E+\delta]}(H^-_{\leq\epsilon}) \otimes 1\!\!1 \ \rho = \rho$$

for some small $\delta > 0$. Of course this does not fix the choice of ρ either.

Motivated by these considerations, if $\delta > 0$, $\rho_{>\epsilon}$ is a density matrix on $\mathcal{H}^-_{>\epsilon}$ and A is observable on $\mathcal{H}^+$, we define

$$\begin{aligned} \mathrm{Cross}_\delta(\rho_{>\epsilon}, A) \ &:= \big\{ \mathrm{Tr} \rho S^* A S \ : \rho \ \text{ is a density matrix on } \mathcal{H}^-, \\ &\qquad 1\!\!1_{[E,E+\delta]}(H^-_{\leq\epsilon}) \otimes 1\!\!1 \ \rho = \rho, \mathrm{Tr}^-_{\leq\epsilon} \ \rho = \rho_{>\epsilon} \big\}. \end{aligned}$$

This is the set of all possible cross-sections compatible with the pair $(\rho_{>\epsilon}, A)$ under the assumption that the soft part of the initial state has the excess energy below δ. Clearly, $\mathrm{Cross}_\delta(\rho_{>\epsilon}, A)$ is a family of nonempty intervals in $[-\|A\|, \|A\|]$ decreasing as $\delta \searrow 0$.

It would be interesting to investigate whether a large class of Pauli-Fierz Hamiltonians has the following property:

Property P.a *The Pauli-Fierz Hamiltonian H has the property of the* **continuity of cross-sections at the bottom of spectrum** *iff for any $\rho_{>\epsilon}$ and A,*

$$\bigcap_{0<\delta<\epsilon} \mathrm{Cross}_\delta(\rho_{>\epsilon}, A)^{\mathrm{cl}} \tag{2.24}$$

is a single point. (The superscript cl *denotes the closure of a set).*

If Property P.a holds, then the number given by (2.24) can be viewed as the cross-section for the experiment described by $\rho_{>\epsilon}$ and A. Note that if H has a non-degenerate ground state, then (2.24) contains the number (2.23).

Clearly (2.24) depends on the choice of g within its equivalence class, hence one can argue that in such a case it does not correspond to a physical experiment. If one assumes that the observable is of the form $A = \mathbb{1} \otimes A_{>\epsilon}$ with $A_{>\epsilon}$ an observable on $\Gamma_{\mathrm{s}}(\mathfrak{h}_{>\epsilon})$, then

$$\mathrm{Cross}_\delta(\rho_{>\epsilon}, \mathbb{1} \otimes A_{>\epsilon}) \tag{2.25}$$

does not depend on the choice of g satisfying (2.16), using the covariance properties shown in Subsection 7.5. (2.25) is the set of possible inclusive cross-sections compatible with the pair $(\rho_{>\epsilon}, A_{>\epsilon})$. One can introduce a property weaker than (P.a):

Property P.b *The Pauli-Fierz Hamiltonian H has the property of the* **continuity of inclusive cross-sections at the bottom of spectrum** *iff for any $\rho_{>\epsilon}$ and $A_{>\epsilon}$,*

$$\bigcap_{0<\delta<\epsilon} \mathrm{Cross}_\delta(\rho_{>\epsilon}, \mathbb{1} \otimes A_{>\epsilon})^{\mathrm{cl}}$$

is a single point.

If Property P.a is true then the theory based on the Pauli-Fierz Hamiltonian H has quite a strong predictive power. The experimentalist does not have to worry about preparing precisely the soft part of the initial state; it is enough if its soft part is sufficiently low energetic. Then he can measure all observables he likes, even those involving soft modes. The theory will give well defined cross-sections for his experiments.

If the experimentalist measures only hard components of the final state, then it is sufficient that Property P.b holds to have well defined cross-sections for all experiments.

Note that the stronger Property P.a is true in the case of the exactly solvable van Hove Hamiltonian, where the scattering operator is equal to identity.

2.12 Insensitivity to soft background

One could argue, however, that Properties P.a and P.b are too modest and do not correspond to realistic physical situations. It may be unjustified to expect that the soft modes of the radiation will dissipate their energy while the experimentalist prepares the experiment. Nevertheless, one can hope that soft modes should not influence the outcome of measurement too much provided that their energy is reasonably bounded. This intuition leads to yet another conjecture.

In order to state it, we introduce a new definition. Let $\delta > 0$ and $0 < \gamma \leq \epsilon$. Suppose that the experimentalist can control the incoming states up to the modes of energy γ. He can make sure that there are no photons of energy in $[\gamma, \epsilon]$—the system is in the lowest possible energetic state for the modes of energy in this energy range. This means that

$$\mathrm{Tr}^{-}_{\leq\gamma}\, \rho = |W(-\mathrm{i}g_{[\gamma,\epsilon]})\Omega)(W(-\mathrm{i}g_{[\gamma,\epsilon]})\Omega| \otimes \rho_{>\epsilon}. \tag{2.26}$$

Here

$$g_{[\gamma,\epsilon]} = 1\!\!1_{[\gamma,\epsilon]}(\omega)g,$$

and $|W(-\mathrm{i}g_{[\gamma,\epsilon]})\Omega)(W(-\mathrm{i}g_{[\gamma,\epsilon]})\Omega|$ denotes the orthogonal projection onto the coherent vector $W(-\mathrm{i}g_{[\gamma,\epsilon]})\Omega$.

Suppose also that the experimentalist can guarantee that the soft modes have the excess of the energy below $\delta > 0$, which however does not have to be very small. This means that

$$1\!\!1_{[E,E+\delta]}(H^{-}_{\leq\gamma})\otimes 1\!\!1\ \rho = \rho.$$

Note that by (2.26) this is equivalent to

$$1\!\!1_{[E,E+\delta]}(H^{-}_{\leq\epsilon})\otimes 1\!\!1\ \rho = \rho.$$

Cross-sections compatible with this information are given by the set

$$\begin{aligned}\mathrm{Cross}_{\delta,\gamma}(\rho_{>\epsilon}, A) \quad &:= \big\{\mathrm{Tr}\rho S^{*}AS \ : \rho \ \text{ is a density matrix on } \mathcal{H}^{-}, \\ &1\!\!1_{[E,E+\delta]}(H^{-}_{\leq\gamma})\otimes 1\!\!1\ \rho = \rho, \mathrm{Tr}^{-}_{\leq\gamma}\ \rho = |W(-\mathrm{i}g_{[\gamma,\epsilon]})\Omega)(W(-\mathrm{i}g_{[\gamma,\epsilon]})\Omega|\otimes\rho_{>\epsilon}\big\}.\end{aligned}$$

Clearly $\mathrm{Cross}_{\delta,\gamma}(\rho_{>\epsilon}, A)$ decrease if δ or γ decrease. Moreover if $\delta < \gamma$, then

$$\mathrm{Cross}_{\delta,\gamma}(\rho_{>\epsilon}, A) = \mathrm{Cross}_{\delta}(\rho_{>\epsilon}, A).$$

If $A_{>\epsilon}$ is as above, then

$$\mathrm{Cross}_{\delta,\gamma}(\rho_{>\epsilon}, 1\!\!1 \otimes A_{>\epsilon}) \tag{2.27}$$

does not depend on the choice of g satisfying the condition (2.16).

Let $[0, \epsilon] \ni \gamma \mapsto \delta(\gamma)$ be a function with values in positive real numbers. One could expect that a large class of Pauli-Fierz Hamiltonians satisfy the following property for $\delta(\gamma)$ such that $\lim_{\gamma\to 0} \frac{\delta(\gamma)}{\gamma} = +\infty$:

Property P.c *A Pauli-Fierz Hamiltonian H has the property of* **δ-insensitivity of inclusive cross-sections to soft background** *iff the following is true. Let $\rho_{>\epsilon}$ and $A_{>\epsilon}$ be as above. Then*

$$\bigcap_{0<\gamma<\epsilon} \mathrm{Cross}_{\delta(\gamma),\gamma}(\rho_{>\epsilon}, 1\!\!1\otimes A_{>\epsilon})^{\mathrm{cl}}$$

is a single point.

Note that the van Hove Hamiltonians have Property P.c with $\delta(\gamma)=\infty$—soft modes and hard modes are completely decoupled.

3 Canonical commutation relations

Here begins the second part of this paper, consisting of Sections 3-7 and Appendix, which is more mathematical than the previous section. In this part we develop systematically various elements of mathematical formalism useful in the study of infrared problem. In particular we prove most of the statements described in Section 2.

Let us stress that this and the following sections can be read independently of Section 2 and of the introduction.

In this section we collect basic constructions and facts concerning CCR representations [BR], [BSZ], [DG3], concentrating especially on the so-called *coherent representations*. The notation that we develop here will be used throughout the paper. Note in particular that in the applications that will start with Section 5, the superscript π will be replaced by the superscript $-$ or $+$ corresponding to the incoming or outgoing representation.

3.1 CCR Representations

Let $\mathfrak{g}$ be a complex vector space with a scalar product $(\cdot|\cdot)$ antilinear wrt the first argument. Let $\mathcal{H}$ be a Hilbert space. Let $U(\mathcal{H})$ denote the set of unitary operators on $\mathcal{H}$. Recall that

$$\mathfrak{g}\ni h\mapsto W^\pi(h)\in U(\mathcal{H}) \tag{3.1}$$

is a *CCR representation* over $\mathfrak{g}$ in $\mathcal{H}$ if

$$W^\pi(h_1)W^\pi(h_2)=\mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h_1|h_2)}W^\pi(h_1+h_2),\quad h_1,h_2\in\mathfrak{g}.$$

We say that a vector $\Psi\in\mathcal{H}$ is *regular* if

$$\mathbb{R}\ni t\mapsto W^\pi(th)\Psi,\quad h\in\mathfrak{g}$$

is continuous. Let $\mathcal{H}^\pi_{\mathrm{reg}}$ be the set of regular vectors—the regular sector of (3.1). It is easy to see that $\mathcal{H}^\pi_{\mathrm{reg}}$ is a closed subspace of $\mathcal{H}$ invariant under (3.1). We say that (3.1) is *regular* if $\mathcal{H}=\mathcal{H}^\pi_{\mathrm{reg}}$. The field operator associated to the representation π and $h\in\mathfrak{g}$ is the self-adjoint operator defined as follows: $\Psi\in\mathrm{Dom}(\phi^\pi(h))$ iff there exists

$$\phi^\pi(h)\Psi=\frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t}W^\pi(th)\Psi\Big|_{t=0}.$$

Clearly, $\mathrm{Dom}(\phi^\pi(h))$ is contained and dense in $\mathcal{H}^\pi_{\mathrm{reg}}$. The creation and annihilation operators associated to the representation π are defined as

$$a^\pi(h):=\frac{1}{\sqrt{2}}(\phi^\pi(h)+\mathrm{i}\phi^\pi(\mathrm{i}h)),\quad a^{\pi*}(h):=\frac{1}{\sqrt{2}}(\phi^\pi(h)-\mathrm{i}\phi^\pi(\mathrm{i}h)).$$

For further reference let us note the identities

$$W^\pi(\mathrm{i}g)a^{\pi*}(h)W^\pi(-\mathrm{i}g)=a^{\pi*}(h)+\frac{1}{\sqrt{2}}(g|h),\quad W^\pi(\mathrm{i}g)a^\pi(h)W^\pi(-\mathrm{i}g)=a^\pi(h)+\frac{1}{\sqrt{2}}(h|g).$$

3.2 The Fock representation

Let $\mathfrak{h}$ be a Hilbert space. $\Gamma_{\mathrm{s}}(\mathfrak{h})$ will denote the symmetric Fock space over $\mathfrak{h}$. Ω will denote the corresponding vacuum vector and N the number operator.

If $h \in \mathfrak{h}$, then $a^*(h)$ denotes the corresponding creation operator, that is the operator defined on finite particle vectors Φ as

$$a^*(h)\Phi := h \otimes_{\mathrm{s}} \sqrt{N+1}\Phi.$$

The same symbol $a^*(h)$ denotes the closure of this operator. The annihilation operator is defined as $a(h) := a^*(h)^*$ and the field and Weyl operators are

$$\phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)), \qquad W(h) := \mathrm{e}^{\mathrm{i}\phi(h)}.$$

It is well known that

$$\mathfrak{h} \ni h \mapsto W(h) \in U(\Gamma_{\mathrm{s}}(\mathfrak{h})), \tag{3.2}$$

is a regular CCR representation. It is called the Fock representation. (See [BR], [BSZ]).

If $f \in \mathfrak{h}$, then $W(-\mathrm{i}f)\Omega$ is called the *coherent vector centered at* f. Note that it satisfies

$$\sqrt{2}\, a(h)W(-\mathrm{i}f)\Omega = (h|f)W(-\mathrm{i}f)\Omega.$$

This property characterizes coherent vectors, as is seen from Theorem 3.1.

In the remaining part of this section, $\mathfrak{g}$ will be a dense subspace of $\mathfrak{h}$ and f will be an antilinear functional on $\mathfrak{g}$. The action of f on $h \in \mathfrak{g}$ will be denoted by $(h|f)$, as in the scalar product.

The following theorem is well known, for the proof see eg. [De].

Theorem 3.1 *Let $\Psi \in \Gamma_{\mathrm{s}}(\mathfrak{h})$. Suppose that for any $h \in \mathfrak{g}$ we have*

$$\Psi \in \mathrm{Dom}(a(h)), \qquad \sqrt{2}a(h)\Psi = (h|f)\Psi.$$

Then the following is true:

(1) *If $f \in \mathfrak{h}$, then Ψ is proportional to $W(-\mathrm{i}f)\Omega$.*

(2) *If $f \notin \mathfrak{h}$, then $\Psi = 0$.*

3.3 Coherent representations

Note that

$$\mathfrak{g} \ni h \mapsto W^f(h) := W(h)\mathrm{e}^{\mathrm{i}\mathrm{Re}(f|h)} \in U(\Gamma_{\mathrm{s}}(\mathfrak{h})) \tag{3.3}$$

is a regular CCR representation in $\Gamma_{\mathrm{s}}(\mathfrak{h})$. We will call (3.3) the *f-coherent representation.*

The corresponding field, creation and annihilation operators will be denoted $\phi^f(h)$, $a^{f*}(h)$, $a^f(h)$. Clearly,

$$\begin{aligned} \phi^f(h) &= \phi(h) + \mathrm{Re}(h|f), \\ a^{f*}(h) &= a^*(h) + \tfrac{1}{\sqrt{2}}(f|h), \\ a^f(h) &= a(h) + \tfrac{1}{\sqrt{2}}(h|f). \end{aligned} \tag{3.4}$$

Note that the vacuum satisfies for $h \in \mathfrak{g}$:

$$\sqrt{2}\, a^f(h)\Omega = (h|f)\Omega.$$

Theorem 3.2 (1) *If $f \in \mathfrak{h}$, then $W^f(h) = W(\mathrm{i}f)W(h)W(-\mathrm{i}f)$, $h \in \mathfrak{g}$.*
(2) *If $f \notin \mathfrak{h}$, then there is no operator U such that*

$$W^f(h) = UW(h)U^*, \quad h \in \mathfrak{g}. \tag{3.5}$$

Proof. (1) is immediate. To prove (2), suppose that U satisfies (3.5). Then $a^f(h) = Ua(h)U^*$. Using $a(h)\Omega = 0$ and the last identity of (3.4) we see that

$$\sqrt{2}\, a(h)U^*\Omega = (h|f)U^*\Omega,$$

which means that $U^*\Omega$ satisfies the assumptions of Theorem 3.1. But $U^*\Omega \neq 0$. Hence $f \in \mathfrak{h}$. □

3.4 Coherent sectors

In this and the following subsection we consider an arbitrary CCR representation

$$\mathfrak{g} \ni h \mapsto W^\pi(h) \in U(\mathcal{H}). \tag{3.6}$$

We are going to describe how to extract f-coherent sub-representations of (3.6).

A vector $\Psi \in \mathcal{H}$ is called an *f-coherent vector* for (3.6) if for any $h \in \mathfrak{g}$ we have

$$\Psi \in \mathrm{Dom}(a^\pi(h)), \qquad \sqrt{2}a^\pi(h)\Psi = (h|f)\Psi.$$

Let $\mathcal{K}_f^\pi$ be the set of f-coherent vectors for (3.6). Elements of $\mathcal{K}_0^\pi$ will be called *vacua* for (3.6).

Theorem 3.3 (1) *$\mathcal{K}_f^\pi$ is a closed linear subspace.*
(2) *$\Psi \in \mathcal{K}_f^\pi$ iff*

$$(\Psi|W^\pi(h)\Psi) = \|\Psi\|^2 \mathrm{e}^{-\frac{1}{4}\|h\|^2 + \mathrm{i}\mathrm{Re}(f|h)}.$$

(3) *All vectors in $\mathcal{K}_f^\pi$ are analytic for $\phi^\pi(h)$, $h \in \mathfrak{g}$.*
(4) *If $\Psi_1, \Psi_2 \in \mathcal{K}_f^\pi$, then*

$$(\Psi_1|W^\pi(h)\Psi_2) = (\Psi_1|\Psi_2)\mathrm{e}^{-\frac{1}{4}\|h\|^2 + \mathrm{i}\mathrm{Re}(f|h)}.$$

Proof. (1) is obvious, since $a^\pi(h)$ are closed operators.

Let us prove (2) $\Leftarrow$. Let $\Psi \in \mathcal{H}$ and $\|\Psi\| = 1$. Taking the first two terms of the Taylor expansion of

$$t \mapsto (\Psi|W^\pi(th)\Psi) = \|\Psi\|^2 \mathrm{e}^{-\frac{1}{4}t^2\|h\|^2 + \mathrm{i}t\mathrm{Re}(f|h)},$$

we obtain

$$(\Psi|\phi^\pi(h)\Psi)=\mathrm{Re}(f|h),\qquad (\Psi|\phi^\pi(h)^2\Psi)=\frac{1}{2}\|h\|^2+(\mathrm{Re}(f|h))^2.$$

Similarly,

$$(\Psi|\phi^\pi(\mathrm{i}h)\Psi)=-\mathrm{Im}(f|h),\qquad (\Psi|\phi^\pi(\mathrm{i}h)^2\Psi)=\frac{1}{2}\|h\|^2+(\mathrm{Im}(f|h))^2.$$

Clearly,

$$[\phi^\pi(h),\phi^\pi(\mathrm{i}h)]=\mathrm{i}\|h\|^2.$$

Therefore,

$$\begin{aligned}\|(\sqrt{2}\,a^\pi(h)-(h|f))\Psi\|^2 = &\ \|(\phi^\pi(h)+\mathrm{i}\phi^\pi(\mathrm{i}h)-(h|f))\Psi\|^2\\ = &\ \Big(\Psi|\big(\phi^\pi(h)^2+\phi(\mathrm{i}h)^2+\mathrm{i}[\phi^\pi(h),\phi^\pi(\mathrm{i}h)]\\ &-2\phi^\pi(h)\mathrm{Re}(f|h)-2\phi^\pi(\mathrm{i}h)\mathrm{Im}(f|h)+|(f|h)|^2\big)\Psi\Big)=0.\end{aligned}$$

To prove (2) $\Rightarrow$, note that $\mathrm{Dom}(a^\pi(h))=\mathrm{Dom}(\phi^\pi(h))\cap\mathrm{Dom}(\phi^\pi(\mathrm{i}h))$. Hence if $\Psi\in\mathrm{Dom}(a^\pi(h))$, then the function

$$\mathbb{R}\ni t\mapsto F(t):=(\Psi|W^\pi(th)\Psi)$$

is C^1. Now

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}F(t) &= \tfrac{\mathrm{i}}{\sqrt{2}}(a^\pi(h)\Psi|W^\pi(th)\Psi)+\tfrac{\mathrm{i}}{\sqrt{2}}(\Psi|W^\pi(th)a^\pi(h)\Psi)-\tfrac{t}{2}\|h\|^2F(t)\\ &=(\mathrm{i}\mathrm{Re}(f|h)-\tfrac{t}{2}\|h\|^2)F(t).\end{aligned}$$

This implies that $F(t)=\|\Psi\|^2\mathrm{e}^{-\frac{1}{4}t^2\|h\|^2+\mathrm{i}t\mathrm{Re}(f|h)}$.

(3) follows immediately from (2).

(4) follows from (2) by polarization $\Box$.

Set

$$\mathcal{H}^\pi_{[f]}:=\mathrm{Span}^{\mathrm{cl}}\{W^\pi(h)\Psi\ :\ \Psi\in\mathcal{K}^\pi_f,\quad h\in\mathfrak{g}\},$$

where $\mathrm{Span}^{\mathrm{cl}}\mathcal{A}$ denotes the closure of the span of the set $\mathcal{A}\subset\mathcal{H}$. Let $P^\pi_{[f]}$ be the orthogonal projection onto $\mathcal{H}^\pi_{[f]}$. We will call $\mathcal{H}^\pi_{[f]}$ the *f-coherent sector* of (3.6). Set

$$\mathcal{H}^\pi_f:=\mathcal{K}^\pi_f\otimes\Gamma_{\mathrm{s}}(\mathfrak{h}).$$

$\mathcal{H}^\pi_{[0]}$ will be called the *Fock sector* of π. If $\mathcal{H}^\pi_{[f]}=\mathcal{H}$ (resp. $\mathcal{H}^\pi_{[0]}=\mathcal{H}$) we will say that the representation W^π is of *$f-$coherent type* (resp. of *Fock type*).

Theorem 3.4 (1) *$\mathcal{H}^\pi_{[f]}$ is an invariant subspace of (3.6) contained in $\mathcal{H}^\pi_{\mathrm{reg}}$.*

(2) *There exists a unique operator $\Omega^\pi_f:\mathcal{H}^\pi_f\to\mathcal{H}^\pi_{[f]}$ satisfying*

$$\Omega^\pi_f\ \Psi\otimes W(h)\Omega=\mathrm{e}^{-\mathrm{i}\mathrm{Re}(h|f)}W^\pi(h)\Psi,\quad \Psi\in\mathcal{K}^\pi_f,\quad h\in\mathfrak{g}.$$

The operator Ω^π_f is unitary.

(3)
$$\Omega_f^\pi\ 1\!\!1\otimes W(g) = \mathrm{e}^{-\mathrm{i}\mathrm{Re}(g|f)} W^\pi(g)\Omega_f^\pi, \quad g\in\mathfrak{g}. \tag{3.7}$$

Proof. (1) is obvious.

Let us prove (2). Let $\Psi_1,\Psi_2\in\mathcal{K}_f^\pi$, $h_1,h_2\in\mathfrak{g}$. Then, by Theorem 3.3 (4),

$$\begin{aligned}(\mathrm{e}^{-\mathrm{i}\mathrm{Re}(h_1|f)}W^\pi(h_1)\Psi_1|\mathrm{e}^{-\mathrm{i}\mathrm{Re}(h_2|f)}W^\pi(h_2)\Psi_2) &= (\Psi_1|\Psi_2)\mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(h_1|h_2)-\frac14\|h_1-h_2\|^2}\\ &= (\Psi_1|\Psi_2)(W(h_1)\Omega|W(h_2)\Omega)\end{aligned}$$

Hence for $\alpha_j\in\mathbb{C}$, $\Psi_j\in\mathcal{K}_f^\pi$, $h_j\in\mathfrak{g}$.

$$\Big\|\sum_j\alpha_j\mathrm{e}^{-\mathrm{i}\mathrm{Re}(h_j|f)}W^\pi(h_j)\Psi_j\Big\|^2 = \Big\|\sum_j\alpha_j\Psi_j\otimes W(h_j)\Omega\Big\|^2.$$

Therefore, Ω_f^π is well defined and isometric. It is obvious that its range equals $\mathcal{H}_{[f]}^\pi$.

To show (3), we note:

$$\begin{aligned}&\Omega_f^\pi\ 1\!\!1\otimes W(g)\ \Psi\otimes W(h)\Omega\\ =\ &\mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(g|h)}\Omega_f^\pi\ \Psi\otimes W(g+h)\Omega\\ =\ &\mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(g|h)}\mathrm{e}^{-\mathrm{i}\mathrm{Re}(g+h|f)}W^\pi(g+h)\Psi\\ =\ &\mathrm{e}^{-\mathrm{i}\mathrm{Re}(g+h|f)}W^\pi(g)W^\pi(h)\Psi\\ =\ &\mathrm{e}^{-\mathrm{i}\mathrm{Re}(g|f)}W^\pi(g)\ \Omega_f^\pi\ \Psi\otimes W(h)\Omega.\ \Box\end{aligned}$$

3.5 Comparison of coherent sectors

For $h\in\mathfrak{h}$ we set
$$W_f^\pi(h) := \Omega_f^\pi\, 1\!\!1\otimes W(h)\,\Omega_f^{\pi*}.$$

Theorem 3.5 (1) *The map*
$$\mathfrak{h}\ni h\mapsto W_f^\pi(h)\in U(\mathcal{H}_{[f]}^\pi)$$
is a regular CCR representation

(2)
$$\Omega_f^\pi\ \Psi\otimes W(h)\Omega = W_f^\pi(h)\Psi, \quad \Psi\in\mathcal{K}_f^\pi, \quad h\in\mathfrak{h}.$$

(3) *For $h\in\mathfrak{g}$ we have*
$$\begin{aligned}W_f^\pi(h) &= \mathrm{e}^{-\mathrm{i}\mathrm{Re}(f|h)}P_{[f]}^\pi W^\pi(h),\\ \phi_f^\pi(h) &= P_{[f]}^\pi(\phi^\pi(h)-\mathrm{Re}(f|h)),\\ a_f^{\pi*}(h) &= P_{[f]}^\pi\big(a^{\pi*}(h)-\tfrac{1}{\sqrt2}(h|f)\big),\\ a_f^\pi(h) &= P_{[f]}^\pi\big(a^\pi(h)-\tfrac{1}{\sqrt2}(f|h)\big).\end{aligned}$$

(1) and (2) are immediate.

If we multiply (3.7) from the right by $\Omega_f^{\pi *}$, use $P_{[f]}^\pi = \Omega_f^\pi \Omega_f^{\pi *}$ and the fact that $P_{[f]}^\pi$ commutes with $W^\pi(h)$, we obtain the first identity of (3). The other follow immediately. □

Remark 3.6 *Let us make a comment on the purpose of introducing the operators $W_f^\pi(h)$. As we see from Theorem 3.5 (3), for various applications, as long as $h \in \mathfrak{g}$ we could use $W^\pi(h)$ instead of $W_f^\pi(h)$. The advantage of the operators $W_f^\pi(h)$, however, lies in the fact that they are defined for any $h \in \mathfrak{h}$.*

Note also that $W_f^\pi(h)$ is a different object from the f-coherent representation $W^f(h)$ introduced earlier.

Theorem 3.7 *Let f, g be antilinear functionals on $\mathfrak{g}$.*

(1) *Assume that $g \in \mathfrak{h}$. Then*

  (i) $\mathcal{K}_{g+f}^\pi = W_f^\pi(-\mathrm{i}g)\mathcal{K}_f^\pi$.

  (ii) $\mathcal{H}_{[f]}^\pi = \mathcal{H}_{[f+g]}^\pi$ *and* $P_{[f]}^\pi = P_{[f+g]}^\pi$. *Consequently the $f-$coherent sector $\mathcal{H}_{[f]}^\pi$ depends only on the class $[f]$ of f in $\mathfrak{g}^*/\mathfrak{h}$.*

  (iii) *Set* $W_{\mathrm{coh},f}^\pi(-\mathrm{i}g) := W_f^\pi(-\mathrm{i}g)\Big|_{\mathcal{K}_f^\pi}$. *Then* $W_{\mathrm{coh},f}^\pi(-\mathrm{i}g)$ *is a unitary map from* $\mathcal{K}_f^\pi$ *to* $\mathcal{K}_{f+g}^\pi$.

  (iv) *We have* $W_{\mathrm{coh},f}^\pi(-\mathrm{i}g) = W_{f+g}^\pi(-\mathrm{i}g)\Big|_{\mathcal{K}_f^\pi}$ *and* $W_{\mathrm{coh},f+g}^\pi(\mathrm{i}g) = W_{\mathrm{coh},f}^\pi(-\mathrm{i}g)^*$.

  (v) $\Omega_f^\pi \ = \Omega_{f+g}^\pi \ W_{\mathrm{coh},f}^\pi(-\mathrm{i}g)\otimes W(\mathrm{i}g)$.

(2) *If $g \notin \mathfrak{h}$, then $\mathcal{H}_{[f]}^\pi \perp \mathcal{H}_{[f+g]}^\pi$.*

Proof. Let us first prove (1.i). W_f^π is a CCR representation, hence for $h \in \mathfrak{g}$

$$a_f^\pi(h)W_f^\pi(-\mathrm{i}g) = W_f^\pi(-\mathrm{i}g)(a_f^\pi(h) + \tfrac{1}{\sqrt{2}}(h|g)).$$

Therefore,

$$a^\pi(h)W_f^\pi(-\mathrm{i}g) = W_f^\pi(-\mathrm{i}g)(a^\pi(h) + \tfrac{1}{\sqrt{2}}(h|g)).$$

This implies $W_f^\pi(-\mathrm{i}g)\mathcal{K}_f^\pi \subset \mathcal{K}_{f+g}^\pi$. An analogous reasoning shows the converse inclusion.

(1.ii) and (1.iii) follow immediately from (1.i).

To prove (1.iv) note that $W_f^\pi(-\mathrm{i}g) = W_{f+g}^\pi(-\mathrm{i}g)$, which follows from $\mathrm{Re}(g|\mathrm{i}g) = 0$.

Let us prove (1.v). Let $\Psi \in \mathcal{K}_f^\pi$ and $h \in \mathfrak{g}$.

$$\begin{aligned}
& \Omega_{f+g}^\pi \ W_{\mathrm{coh},f}^\pi(-\mathrm{i}g)\otimes W(\mathrm{i}g) \ \Psi\otimes W(h)\Omega \\
= \ & W_{f+g}^\pi(\mathrm{i}g)\Omega_{f+g}^\pi \ W_f^\pi(-\mathrm{i}g)\Psi\otimes W(h)\Omega \\
= \ & W_{f+g}^\pi(\mathrm{i}g)W_{f+g}^\pi(h)W_f^\pi(-\mathrm{i}g)\Psi \\
= \ & W_{f+g}^\pi(\mathrm{i}g)W_{f+g}^\pi(h)W_{f+g}^\pi(-\mathrm{i}g)\Psi \\
= \ & W_f^\pi(h)\Psi \\
= \ & \Omega_f^\pi \ \Psi \otimes W(h)\Omega.
\end{aligned}$$

Let us prove (2). Let us first show that

$$0 \neq \Phi \in \mathcal{K}^\pi_{f+g} \cap \mathcal{H}^\pi_{[f]} \ \Rightarrow\ g \in \mathfrak{h}. \tag{3.8}$$

In fact, for $h \in \mathfrak{g}$ we have

$$1\!\!1 \otimes a(h)\, \Omega_f^{\pi *}\Phi = \Omega_f^{\pi *}\big(a^\pi(h) - \frac{1}{\sqrt{2}}(h|f)\big)\Phi = \frac{1}{\sqrt{2}}(h|g)\Omega_f^{\pi *}\Phi.$$

But $\mathrm{Ran}\Omega_f^\pi = \mathcal{H}^\pi_{[f]}$, hence $\Omega_f^{\pi *}\Phi \neq 0$. By Theorem 3.1, this implies $g \in \mathfrak{h}$.

Now suppose that $\mathcal{H}^\pi_{[f]}$ is not perpendicular to $\mathcal{H}^\pi_{[f+g]}$. Then there exist vectors $\Psi_1 \in \mathcal{K}^\pi_f$, $\Psi_2 \in \mathcal{K}^\pi_{f+g}$, $h_1, h_2 \in \mathfrak{g}$ such that

$$(W^\pi(h_1)\Psi_1 | W^\pi(h_2)\Psi_2) \neq 0. \tag{3.9}$$

Set $\Phi := P^\pi_{[f]}\Psi_2$. Clearly, $\Phi \in \mathcal{H}^\pi_{[f]}$. Note that $P^\pi_{[f]}$ commutes with $a^\pi(h)$. Hence $\Phi \in \mathcal{K}^\pi_{f+g}$. Clearly, $(W^\pi(-h_2)W^\pi(h_1)\Psi_1|\Phi)$ equals the left hand side of (3.9), hence is nonzero. Therefore, $\Phi \neq 0$. By (3.8), this implies $g \in \mathfrak{h}$. □

4 Covariant CCR representations

4.1 Definition of a covariant CCR representation

In this section we describe properties of a CCR representation equipped with a dynamics.

Let $\mathfrak{h}$ and $\mathcal{H}$ be Hilbert spaces. Let $\mathfrak{g}$ be a dense subspace of $\mathfrak{h}$. Let

$$\mathfrak{g} \ni h \mapsto W^\pi(h) \in U(\mathcal{H}) \tag{4.1}$$

be a CCR representation. Let ω be a self-adjoint operator on $\mathfrak{h}$ and H a self-adjoint operator on $\mathcal{H}$. We say that the triple (W^π, ω, H) is a covariant CCR representation iff $\mathfrak{g}$ is invariant w.r.t. $\mathrm{e}^{\mathrm{i}t\omega}$ and

$$\mathrm{e}^{\mathrm{i}tH}W^\pi(h)\mathrm{e}^{-\mathrm{i}tH} = W^\pi(\mathrm{e}^{\mathrm{i}t\omega}h),\ t \in \mathbb{R},\ h \in \mathfrak{g}.$$

4.2 Operators $\mathrm{d}\Gamma(\cdot)$

Let $\mathrm{d}\Gamma(\omega)$ be defined in the usual way as a self-adjoint operator on $\Gamma_\mathrm{s}(\mathfrak{h})$. Recall that $W(h)$ denote the Weyl operators on $\Gamma_\mathrm{s}(\mathfrak{h})$. It is well known that

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\omega)}W(h)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\omega)} = W(\mathrm{e}^{\mathrm{i}t\omega}h).$$

Therefore, the triple $(W, \omega, \mathrm{d}\Gamma(\omega))$ is a covariant CCR representation (by W we mean the Fock representation over $\mathfrak{h}$ recalled in Subsection 3.2).

For further reference let us note the following identities, where we set $z = \frac{1}{\sqrt{2}}\omega g$:

$$W(\mathrm{i}g)\mathrm{d}\Gamma(\omega)W(-\mathrm{i}g) = \mathrm{d}\Gamma(\omega) + a^*(z) + a(z) + (z|\omega^{-1}z),$$

$$[W(g), \mathrm{d}\Gamma(\omega)] = -\mathrm{i}a^*(z)W(g) + \mathrm{i}W(g)a(z).$$

4.3 Van Hove Hamiltonians

Let $\mathfrak{h}_n$ for $n \in \mathbb{N}$ be the scale of Hilbert spaces associated with the operator ω^{-1}. This means that for $n \geq 0$, $\mathfrak{h}_n = \mathrm{Dom}(\omega^{-n/2})$, $\mathfrak{h}_{-n}$ is the space of continuous antilinear functionals on $\mathfrak{h}_n$. (An alternative notation for $\mathfrak{h}_{-n}$ is $(|\omega|^{-n/2} + 1)\mathfrak{h})$.

Let $f \in \mathfrak{h}_{-1}$. Set

$$z := \frac{1}{\sqrt{2}}\omega f \in (\omega^{1/2} + \omega)\mathfrak{h}.$$

It is easy to see that

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(f|\mathrm{e}^{\mathrm{i}t\omega}f)} W\left(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})f\right)\Gamma(\mathrm{e}^{\mathrm{i}t\omega}) \in U(\Gamma_{\mathrm{s}}(\mathfrak{h})). \tag{4.2}$$

is a strongly continuous unitary group. Therefore there exists a unique self-adjoint operator $\mathrm{d}\Gamma_f(\omega)$, that we will call the *Van Hove Hamiltonian*, such that (4.2) equals $\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_f(\omega)}$. Formally, the van Hove Hamiltonian is given by the following expression:

$$\mathrm{d}\Gamma_f(\omega) := \mathrm{d}\Gamma(\omega) + a^*(z) + a(z) + (z|\omega^{-1}z).$$

(In [De] it is called a van Hove Hamiltonian of the second kind).

Note that the infimum of the spectrum of $\mathrm{d}\Gamma_f(\omega)$ equals 0 and

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_f(\omega)} W(h)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma_f(\omega)} = \exp\left(\mathrm{i}\mathrm{Re}(f|(\mathrm{e}^{\mathrm{i}t\omega}-1)h)\right) W(\mathrm{e}^{\mathrm{i}t\omega}h).$$

If $f \in \mathfrak{h}$, then $\mathrm{d}\Gamma_f(\omega) = W(\mathrm{i}f)\mathrm{d}\Gamma(\omega)W(-\mathrm{i}f)$.

Theorem 4.1 (1) *If $f, g \in \mathfrak{h}_{-1}$, then the following identities holds:*

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_{f+g}(\omega)} = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{\mathrm{i}t\omega}g) + \mathrm{i}\mathrm{Im}(f|(\mathrm{e}^{\mathrm{i}t\omega}-1)g)} W(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g)\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_f(\omega)}.$$

(2) *If moreover $g \in \mathfrak{h}$, then:*

$$\mathrm{d}\Gamma_{f+g}(\omega) = W(\mathrm{i}g)\mathrm{d}\Gamma_f(\omega)W(-\mathrm{i}g)$$

Note that if we consider the f-coherent representation

$$\mathfrak{h}_1 \ni h \mapsto W^f(h) := W(h)\mathrm{e}^{\mathrm{i}\mathrm{Re}(f|h)}, \tag{4.3}$$

then it satisfies

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_f(\omega)} W^f(h)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma_f(\omega)} = W^f(\mathrm{e}^{\mathrm{i}t\omega}h).$$

Thus, the triple $\left(W^f, \omega, \mathrm{d}\Gamma_f(\omega)\right)$ is a covariant CCR representation.

4.4 Hamiltonian in the Fock sector

In the remaining part of this section we consider a covariant representation $\big(W^\pi, \omega, H\big)$, as at the beginning of this section. The following facts are immediate [DG3]:

Theorem 4.2 (1) *The space of vacua* $\mathcal{K}_0^\pi$ *is* $\mathrm{e}^{\mathrm{i}tH}$ *invariant.*

(2) *The Fock sector* $\mathcal{H}_{[0]}^\pi$ *is* $\mathrm{e}^{\mathrm{i}tH}$ *invariant.*

On $\mathcal{H}_0^\pi = \mathcal{K}_0^\pi \otimes \Gamma_{\mathrm{s}}(\mathfrak{h})$ we define the operator

$$H_0^\pi := \Omega_0^{\pi *} H \Omega_0^\pi.$$

Theorem 4.3 *We have*

$$H_0^\pi := K_0^\pi \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega),$$

where $K_0^\pi := H\Big|_{\mathcal{K}_0^\pi}$. *Moreover,*

$$H\Omega_0^\pi = \Omega_0^\pi H^\pi.$$

4.5 Hamiltonian in a coherent sector

One can generalize the constructions described in the previous subsection to the case of coherent sectors.

Theorem 4.4 (1) *Let* $\mathfrak{g}$ *be a dense subspace of* $\mathfrak{h}$ *and let* f *be an antilinear functional on* $\mathfrak{g}$. *Then* $\mathrm{e}^{\mathrm{i}tH}\mathcal{K}_f^\pi = \mathcal{K}_{\mathrm{e}^{\mathrm{i}t\omega} f}^\pi$.

(2) *If in addition* $f \in \mathfrak{h}_{-2}$, *then* $\mathcal{H}_{[f]}^\pi$ *is* $\mathrm{e}^{\mathrm{i}tH}$*-invariant.*

Proof. (1) Let $\Psi \in \mathcal{K}_f^\pi$. Then

$$\begin{aligned}
(\mathrm{e}^{\mathrm{i}tH}\Psi | W(h)\mathrm{e}^{\mathrm{i}tH}\Psi) &= (\Psi | W(\mathrm{e}^{-\mathrm{i}t\omega}h)\Psi) \\
&= \|\Psi\|^2 \mathrm{e}^{-\frac{1}{4}\|h\|^2 + \mathrm{i}\mathrm{Re}(f|\mathrm{e}^{-\mathrm{i}t\omega}h)} \\
&= \|\Psi\|^2 \mathrm{e}^{-\frac{1}{4}\|h\|^2 + \mathrm{i}\mathrm{Re}(\mathrm{e}^{\mathrm{i}t\omega}f|h)}.
\end{aligned}$$

(2) Since $f \in \mathfrak{h}_{-2}$, we have $(\mathrm{e}^{\mathrm{i}t\omega} - 1)f \in \mathfrak{h}$. Hence by Theorem 3.7, $\mathcal{H}_{[f]}^\pi = \mathcal{H}_{[\mathrm{e}^{\mathrm{i}t\omega}f]}^\pi$. Thus it suffices to apply (1). $\Box$

Set

$$H_f^\pi := \Omega_f^{\pi *} H \Omega_f^\pi.$$

Theorem 4.5 *Suppose that* $\mathfrak{g} = \mathfrak{h}_1$ *and* $f \in \mathfrak{h}_{-1}$.

(1) *There exists a unique operator* K_f^π *on* $\mathcal{K}_f^\pi$ *such that*

$$H_f^\pi := K_f^\pi \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma_f(\omega),$$

(2) $\Omega_f^\pi H_f^\pi = H\Omega_f^\pi$.

Proof. Let $h \in \mathfrak{h}_1$. We first check that

$$\begin{aligned} & \mathrm{e}^{\mathrm{i}tH_f^\pi}\ 1\!\!1 \otimes W(h)\,\mathrm{e}^{-\mathrm{i}tH_f^\pi} \\ = \ & \Omega_f^{\pi*}\mathrm{e}^{\mathrm{i}tH}\,W^\pi(h)\mathrm{e}^{-\mathrm{i}\mathrm{Re}(f|h)}\,\mathrm{e}^{-\mathrm{i}tH}\Omega_f^\pi \\ = \ & \Omega_f^{\pi*}\,W^\pi(\mathrm{e}^{\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}\mathrm{Re}(f|h)}\Omega_f^\pi \\ = \ & 1\!\!1 \otimes W(\mathrm{e}^{\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}\mathrm{Re}(f|h)+\mathrm{i}\mathrm{Re}(f|\mathrm{e}^{\mathrm{i}t\omega}h)} \\ = \ & 1\!\!1 \otimes \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_f(\omega)}W(h)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma_f(\omega)}. \end{aligned}$$

Since linear combinations of $W(h)$, $h \in \mathfrak{h}_1$, are weakly dense in $B(\Gamma_\mathrm{s}(\mathfrak{h}))$, for $B \in B(\Gamma_\mathrm{s}(\mathfrak{h}))$ we have

$$\mathrm{e}^{\mathrm{i}tH_f^\pi}\ 1\!\!1\otimes B\ \mathrm{e}^{-\mathrm{i}tH_f^\pi} = \mathrm{e}^{\mathrm{i}t1\!\!1\otimes\mathrm{d}\Gamma_f(\omega)}\ 1\!\!1\otimes B\ \mathrm{e}^{-\mathrm{i}t1\!\!1\otimes\mathrm{d}\Gamma_f(\omega)}.$$

By Lemma A.2, this implies that $H_f^\pi - 1\!\!1\otimes\mathrm{d}\Gamma_f(\omega)$ is of the form $K_f^\pi\otimes 1\!\!1$ for some self-adjoint operator K_f^π on $\mathcal{K}_f^\pi$. □

4.6 Comparison of coherent sectors of a covariant representation

In this subsection we assume that $\mathfrak{g} = \mathfrak{h}_1$ and $f \in \mathfrak{h}_{-1}$.

Theorem 4.6 *Let $g \in \mathfrak{h}$. Then*

$$H_{g+f}^\pi = W_{\mathrm{coh},f}^\pi(-\mathrm{i}g)\otimes W(\mathrm{i}g)\,H_f^\pi\,W_{\mathrm{coh},f}^{\pi*}(-\mathrm{i}g)\otimes W^*(\mathrm{i}g).$$

$$K_{g+f}^\pi = W_{\mathrm{coh},f}^\pi(-\mathrm{i}g)K_f^\pi W_{\mathrm{coh},f}^{\pi*}(-\mathrm{i}g).$$

Proof. This follows immediately from Theorem 3.7. □

Other natural objects that can be introduced in the context of coherent sectors are the following self-adjoint operators:

$$\begin{aligned} K_{[f]}^\pi &:= \Omega_f^\pi\ K_f^\pi\otimes 1\!\!1\ \Omega_f^{\pi*}, \\ \mathrm{d}\Gamma_{[f]}^\pi(\omega) &:= \Omega_f^\pi\ 1\!\!1\otimes\mathrm{d}\Gamma_f(\omega)\ \Omega_f^{\pi*}. \end{aligned}$$

Clearly, they give a natural decomposition of the operator H on the sector $\mathcal{H}_{[f]}^\pi$:

$$HP_{[f]}^\pi = K_{[f]}^\pi + \mathrm{d}\Gamma_{[f]}^\pi(\omega). \tag{4.4}$$

The decomposition (4.4) depends only on the class $[f]$ of f in $\mathfrak{g}^*/\mathfrak{h}$.

Theorem 4.7 *If $g \in \mathfrak{h}$, then $K_{[f+g]}^\pi = K_{[f]}^\pi$ and $\mathrm{d}\Gamma_{[f+g]}^\pi(\omega) = \mathrm{d}\Gamma_{[f]}^\pi(\omega)$.*

Theorem 4.7 follows from Theorem 3.7 (iii) and Theorem 4.1 (2).

5 Asymptotic CCR representations

5.1 Construction of asymptotic CCR representations

Suppose that ω is a self-adjoint operator with an absolutely continuous spectrum on a Hilbert space $\mathfrak{h}$. Let $\mathcal{K}$ be an additional Hilbert space and H a self-adjoint operator on $\mathcal{H} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathfrak{h})$. Let $\mathfrak{g}$ be a subspace of $\mathfrak{h}$ invariant w.r.t. $\mathrm{e}^{it\omega}$.

Throughout this section we make the following assumption:

Assumption 5.A *For any $h \in \mathfrak{g}$, there exists*

$$\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} 1\!\!1 \otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH} =: W^{\pm}(h).$$

It is easy to see that the above assumption implies the following theorem:

Theorem 5.1 (1) *We have*

$$W^{\pm}(h_1)W^{\pm}(h_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h_1|h_2)}W^{\pm}(h_1+h_2), \quad h_1, h_2 \in \mathfrak{g}.$$

In other words,

$$\mathfrak{g} \ni h \mapsto W^{\pm}(h) \in U(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathfrak{h})), \tag{5.1}$$

are CCR representations.

(2)

$$\mathrm{e}^{\mathrm{i}tH} W^{\pm}(h)\mathrm{e}^{-\mathrm{i}tH} = W^{\pm}(\mathrm{e}^{\mathrm{i}t\omega}h), \quad h \in \mathfrak{g}.$$

In other words, $(W^{\pm}, \omega, H)$ are covariant CCR representations .

We will call (5.1) the *asymptotic CCR representations*. Let $\phi^{\pm}(h)$, $a^{\pm}(h)$, $a^{\pm*}(h)$, etc, denote the field, annihilation, creation operators, etc. associated with the representations (5.1). All these objects will be called "asymptotic" (or, if there will be a need for a greater precision, "outgoing/incoming").

5.2 Wave and scattering operators

For any antilinear functional f on $\mathfrak{g}$ we can define the space of asymptotic f-coherent vectors $\mathcal{K}_f^{\pm}$, the asymptotic spaces $\mathcal{H}_f^{\pm}$, the asymptotic Hamiltonian in the f-coherent sector $H_f^{\pm}$, etc. The intertwining operators $\Omega_f^{\pm}$ will be called the *f-coherent wave operators*.

In the physical interpretation of these concepts an important role is played by the so-called *scattering operators*:

$$S_{g,f} := \Omega_g^{+*}\Omega_f^{-}.$$

Note that they satisfy

$$S_{g,f}H_f^{-} = H_g^{+}S_{g,f}.$$

Suppose that we prepare a state in the f-coherent sector. It is natural to describe it by a density matrix ρ, which is a positive trace 1 operator on $\mathcal{H}_f^{-}$.

Suppose that we measure an observable within the sector g. We can describe it by a self-adjoint operator $A \in B(\mathcal{H}_g^+)$. Then according to the standard rules of quantum mechanics, the expectation value of the measurement is given by

$$\mathrm{Tr} S_{g,f} \rho S_{g,f}^* A.$$

5.3 Fock sector of asymptotic representations

Theorem 5.2 *Eigenvectors of H are contained in the Fock sector $\mathcal{K}_0^\pm$.*

Proof. We will show first the following property of Weyl operators on the Fock space:

$$\mathrm{w-}\lim_{t\to\infty} W(\mathrm{e}^{\mathrm{i}t\omega}h) = \exp(-\tfrac{1}{4}\|h\|^2). \tag{5.2}$$

Let Ψ_1, Ψ_2 be vectors with a finite number of particles. Then, by the absolute continuity of ω, $a(\mathrm{e}^{\mathrm{i}t\omega}h)^n\Psi_i \to 0$ when $t \to \infty$. Hence

$$(\Psi_1|W(\mathrm{e}^{\mathrm{i}t\omega}h)\Psi_2) = \exp(-\tfrac{1}{4}\|h\|^2)(\mathrm{e}^{-\frac{\mathrm{i}}{\sqrt{2}}a(\mathrm{e}^{\mathrm{i}t\omega}h)}\Psi_1|\mathrm{e}^{\frac{\mathrm{i}}{\sqrt{2}}a(\mathrm{e}^{\mathrm{i}t\omega}h)}\Psi_2) \to \exp(-\tfrac{1}{4}\|h\|^2)(\Psi_1|\Psi_2).$$

Since $W(\mathrm{e}^{\mathrm{i}t\omega}h)$ is uniformly bounded, this proves (5.2).

Assume that $H\Psi = \lambda\Psi$. Then

$$\begin{aligned}(\Psi|W^\pm(h)\Psi) &= \lim_{t\to\pm\infty}(\Psi|\mathrm{e}^{\mathrm{i}tH}\mathbb{1}\otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}\Psi)\\ &= \lim_{t\to\pm\infty}(\Psi|\mathbb{1}\otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h)\Psi) = \|\Psi\|^2\exp(-\tfrac{1}{4}\|h\|^2). \ \Box\end{aligned}$$

6 Dressing operators

6.1 Dressing operator for a pair of CCR representations

Suppose that $\mathfrak{h}$, $\mathcal{H}$ are Hilbert spaces and $\mathfrak{g}$ is a dense subspace of $\mathfrak{h}$. Consider two CCR representations

$$\mathfrak{g} \ni h \mapsto W^\pi(h) \in U(\mathcal{H}), \tag{6.1}$$

$$\mathfrak{g} \ni h \mapsto W^\pi_{\mathrm{ren}}(h) \in U(\mathcal{H}). \tag{6.2}$$

For the representation (6.1) we use the notation described in the previous three sections. All the objects constructed from (6.2) will have an additional subscript ren (for "renormalized"). For instance, ϕ^π_{ren}, a^π_{ren} and $a^{\pi*}_{\mathrm{ren}}$ will denote the field, annihilation and creation operators for (6.2).

Let g be an antilinear functional on $\mathfrak{g}$. We say that $U^\pi \in U(\mathcal{H})$ is a *g-dressing operator* between (6.2) and (6.1) if for $h \in \mathfrak{g}$, we have

$$W^\pi(h)U^\pi = U^\pi W^\pi_{\mathrm{ren}}(h)\mathrm{e}^{\mathrm{i}\mathrm{Re}(h|g)}.$$

Theorem 6.1 (1) *If $h \in \mathfrak{g}$, then*

$$\phi^\pi(h)U^\pi = U^\pi(\phi^\pi_{\mathrm{ren}}(h) + \mathrm{Re}(g|h)),$$

$$a^{\pi*}(h)U^\pi = U^\pi(a^{\pi*}_{\mathrm{ren}}(h) + \tfrac{1}{\sqrt{2}}(g|h)),$$

$$a^\pi(h)U^\pi = U^\pi(a^\pi_{\mathrm{ren}}(h) + \tfrac{1}{\sqrt{2}}(h|g)).$$

(2) *Let f be an antilinear functional on $\mathfrak{g}$. Then $\mathcal{K}^\pi_{g+f} = U^\pi \mathcal{K}^\pi_{\mathrm{ren},f}$.*

(3) *Set $U^\pi_{\mathrm{coh},f} := U^\pi\Big|_{\mathcal{K}^\pi_{\mathrm{ren},f}}$. Then $U^\pi_{\mathrm{coh},f}$ is a unitary operator from $\mathcal{K}^\pi_{\mathrm{ren},f}$ to $\mathcal{K}^\pi_{f+g}$.*

(4) *$\mathcal{H}^\pi_{[g+f]} = U^\pi \mathcal{H}^\pi_{\mathrm{ren},[f]}$.*

(5) *$\Omega^\pi_{g+f} = U^\pi \Omega^\pi_{\mathrm{ren},f}\ U^{\pi *}_{\mathrm{coh},f} \otimes 1\!\!1$*

Proof. (1) is immediate.

Consider $\Psi \in \mathcal{K}^\pi_{\mathrm{ren},f}$. Then

$$(U^\pi \Psi | W^\pi(h) U^\pi \Psi) = \mathrm{e}^{\mathrm{i}\mathrm{Re}(h|g)} (\Psi | W^\pi_{\mathrm{ren}}(h)\Psi) = \mathrm{e}^{-\frac{1}{4}\|h\|^2 + \mathrm{i}\mathrm{Re}(h|f+g)} \|\Psi\|^2.$$

This proves (2), which implies (4) and (3).

To show (5), we compute for $h \in \mathfrak{g}$, $\Psi \in \mathcal{K}_{\mathrm{ren},f}$,

$$\begin{aligned} & U^\pi \Omega^\pi_{\mathrm{ren},f}\ U^{\pi *}_{\mathrm{coh},f} \otimes 1\!\!1\ \Psi \otimes W(h)\Omega = U^\pi \Omega^\pi_{\mathrm{ren},f}\ U^{\pi *}\Psi \otimes W(h)\Omega \\ = \ & \mathrm{e}^{-\mathrm{i}\mathrm{Re}(h|f)} U^\pi W^\pi_{\mathrm{ren}}(h) U^{\pi *}\Psi = \mathrm{e}^{-\mathrm{i}\mathrm{Re}(h|f+g)} W^\pi(h)\Psi = \Omega^\pi_{f+g}\ \Psi \otimes W^\pi(h)\Omega. \ \Box \end{aligned}$$

6.2 Dressing operators for a pair of covariant representations

Suppose that H and H^{ren} are self-adjoint operators on $\mathcal{H}$ and ω is a self-adjoint operator on $\mathfrak{h}$. We assume that $\mathfrak{g} = \mathfrak{h}_1$. Consider two covariant CCR representations (W^π, ω, H) and $(W^\pi_{\mathrm{ren}}, \omega, H_{\mathrm{ren}})$. Recall that this means that the representations of CCR (6.1) and (6.2) satisfy

$$\mathrm{e}^{\mathrm{i}tH} W^\pi(h) \mathrm{e}^{-\mathrm{i}tH} = W^\pi(\mathrm{e}^{\mathrm{i}t\omega}h). \tag{6.3}$$

$$\mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}} W^\pi_{\mathrm{ren}}(h) \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} = W^\pi_{\mathrm{ren}}(\mathrm{e}^{\mathrm{i}t\omega}h). \tag{6.4}$$

Let $g \in \mathfrak{h}_{-2}$ and let $U^\pi \in U(\mathcal{H})$ be a g-dressing operator between (6.1) and (6.2). We say that it is a *covariant g-dressing operator* between the covariant representations (6.3) and (6.4) if

$$\begin{aligned} \mathrm{e}^{\mathrm{i}tH} U^\pi \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} &= U^\pi W^\pi_{\mathrm{ren}}(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g) \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)} \\ &= W^\pi(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g) U^\pi \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}, \qquad t \in \mathbb{R}. \end{aligned}$$

Theorem 6.2 *Suppose that $\mathfrak{g} = \mathfrak{h}_1$ and $f, g \in \mathfrak{h}_{-1}$. Then*

$$K^\pi_{g+f} = U^\pi_{\mathrm{coh},f} K^\pi_{\mathrm{ren},f} U^{\pi *}_{\mathrm{coh},f}.$$

Proof. Recall that

$$\begin{aligned} \Omega^{\pi *}_{\mathrm{ren},f} H_{\mathrm{ren}} \Omega^\pi_{\mathrm{ren},f} &= H^\pi_{\mathrm{ren},f} &= K^\pi_{\mathrm{ren},f} \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma_f(\omega), \\ \Omega^{\pi *}_{g+f} H \Omega^\pi_{g+f} &= H^\pi_{g+f} &= K^\pi_{g+f} \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma_{g+f}(\omega). \end{aligned}$$

Hence

$$
\begin{aligned}
& \mathrm{e}^{\mathrm{i}tU^{\pi*}_{\mathrm{coh},f}K^{\pi}_{g+f}U^{\pi}_{\mathrm{coh},f}} \otimes \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_{g+f}(\omega)} \\
= \ & U^{\pi*}_{\mathrm{coh},f}\otimes 1\!\!1 \ \mathrm{e}^{\mathrm{i}tH^{\pi}_{g+f}} \ U^{\pi}_{\mathrm{coh},f}\otimes 1\!\!1 \\
= \ & U^{\pi*}_{\mathrm{coh},f}\otimes 1\!\!1 \ \Omega^{\pi*}_{g+f}\mathrm{e}^{\mathrm{i}tH}\Omega^{\pi}_{g+f} \ U^{\pi}_{\mathrm{coh},f}\otimes 1\!\!1 \\
= \ & \Omega^{\pi*}_{\mathrm{ren},f}U^{\pi*}\mathrm{e}^{\mathrm{i}tH}U^{\pi}\Omega^{\pi}_{\mathrm{ren},f} \\
= \ & \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}\Omega^{\pi*}_{\mathrm{ren},f}W^{\pi}_{\mathrm{ren}}(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g)\mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}\Omega^{\pi}_{\mathrm{ren},f} \\
= \ & \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)+\mathrm{i}\mathrm{Re}(f|\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g)} \ 1\!\!1\otimes W(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g) \ \mathrm{e}^{\mathrm{i}tH^{\pi}_{\mathrm{ren},f}} \\
= \ & \mathrm{e}^{\mathrm{i}tK^{\pi}_{\mathrm{ren},f}} \otimes \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{\mathrm{i}t\omega}g)+\mathrm{i}\mathrm{Im}(f|(\mathrm{e}^{\mathrm{i}t\omega}-1)g)}W(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g)\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_f(\omega)} \\
= \ & \mathrm{e}^{\mathrm{i}tK^{\pi}_{\mathrm{ren},f}} \otimes \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_{g+f}(\omega)}. \ \square
\end{aligned}
$$

6.3 Coherent asymptotic renormalization

Let $g \in \mathfrak{h}_{-1}$. Suppose that H_{ren} is a self-adjoint operator on $\mathcal{H}$. Set

$$
\begin{aligned}
U(t) \ &= \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}\mathrm{e}^{\mathrm{i}tH}W(\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}t\omega})g)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} \\
&= \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}t1\!\!1\otimes\mathrm{d}\Gamma_g(\omega)}\mathrm{e}^{\mathrm{i}t1\!\!1\otimes\mathrm{d}\Gamma(\omega)}\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}.
\end{aligned} \tag{6.5}
$$

Clearly, $\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)$ is well defined and $(1-\mathrm{e}^{-\mathrm{i}t\omega})g \in \mathfrak{h}$, therefore $U(t)$ is well defined. Moreover, in (6.5) we used the identity from Theorem 4.1.

Suppose the following assumption holds:

Assumption 6.A $\mathrm{s-}\lim_{t\to\pm\infty} U(t)$ *and* $\mathrm{s-}\lim_{t\to\pm\infty} U^*(t)$ *exist.*

Under Assumption 6.A we set $U^{\pm} := \mathrm{s-}\lim_{t\to\pm\infty} U(t)$. Clearly, $\mathrm{s-}\lim_{t\to\pm\infty} U^*(t) = U^{\pm*}$.

Theorem 6.3 *Suppose Assumption 5.A holds for the Hamiltonian H and the space $\mathfrak{g} = \mathfrak{h}_1$. Suppose also that Assumption 6.A is satisfied. Then the following is true:*

(1) *Assumption 5.A holds for the operator H_{ren} with $\mathfrak{g} = \mathfrak{h}_1$, that means, for any $h \in \mathfrak{h}_1$, there exists*

$$
\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}} 1\!\!1\otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} =: W^{\pm}_{\mathrm{ren}}(h).
$$

(2)

$$
\begin{aligned}
W^{\pm}_{\mathrm{ren}}(h_1)W^{\pm}_{\mathrm{ren}}(h_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h_1|h_2)}W^{\pm}_{\mathrm{ren}}(h_1+h_2), \quad h_1, h_2 \in \mathfrak{h}_1, \\
\mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}W^{\pm}_{\mathrm{ren}}(h)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} = W^{\pm}_{\mathrm{ren}}(\mathrm{e}^{\mathrm{i}t\omega}h), \quad h \in \mathfrak{h}_1.
\end{aligned}
$$

In other words, the triples $(W^{\pm}_{\mathrm{ren}}, \omega, H_{\mathrm{ren}})$ are covariant CCR representations .

(3) *For $h \in \mathfrak{h}_1$, we have*

$$\begin{aligned} W^{\pm}(h)U^{\pm} &= U^{\pm}W^{\pm}_{\mathrm{ren}}(h)\mathrm{e}^{\mathrm{i}\mathrm{Re}(h,g)}, \\ \mathrm{e}^{\mathrm{i}tH}U^{\pm}\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} &= U^{\pm}W^{\pm}_{\mathrm{ren}}(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g)\mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)} \\ &= W^{\pm}(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g)U^{\pm}\mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}. \end{aligned}$$

Therefore, $U^{\pm}$ are covariant g-dressing operators between the covariant CCR representations $(W^{\pm}_{\mathrm{ren}}, \omega, H_{\mathrm{ren}})$ and $(W^{\pm}, \omega, H)$.

Proof. We have

$$\begin{aligned} & \mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}\,1\!\!1\otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} \\ = \ & \mathrm{e}^{\mathrm{i}\mathrm{Re}((1-\mathrm{e}^{-\mathrm{i}t\omega})g|\mathrm{e}^{-\mathrm{i}t\omega}h)}\mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}\ 1\!\!1\otimes W(-\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}t\omega})g)W(\mathrm{e}^{-\mathrm{i}t\omega}h)W(\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}t\omega})g)\ \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} \\ = \ & \mathrm{e}^{\mathrm{i}\mathrm{Re}((1-\mathrm{e}^{-\mathrm{i}t\omega})g|\mathrm{e}^{-\mathrm{i}t\omega}h)}U(t)^*\mathrm{e}^{\mathrm{i}tH}\,1\!\!1\otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}U(t) \\ \to \ & \mathrm{e}^{-\mathrm{i}\mathrm{Re}(g|h)}U^{\pm *}W^{\pm}(h)U^{\pm}, \end{aligned}$$

where we used $\lim_{t\to\infty}(g|\mathrm{e}^{-\mathrm{i}t\omega}h) = 0$, which follows from the Riemann-Lebesgue lemma. This proves (1), (2) and the first identity of (3).

Let us now prove the second identity of (3). We compute:

$$\begin{aligned} & \mathrm{e}^{\mathrm{i}tH}U(s)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} \\ = \ & \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}s\omega}g)}\mathrm{e}^{\mathrm{i}(t+s)H}W(\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}s\omega})g)\mathrm{e}^{-\mathrm{i}(s+t)H_{\mathrm{ren}}} \\ = \ & \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}s\omega}g)}\mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}((\mathrm{e}^{-\mathrm{i}(s+t)\omega}-\mathrm{e}^{-\mathrm{i}s\omega})g|(1-\mathrm{e}^{-\mathrm{i}(s+t)\omega})g)} \\ & \times\mathrm{e}^{\mathrm{i}(s+t)H}W(\mathrm{i}(\mathrm{e}^{-\mathrm{i}(s+t)\omega}-\mathrm{e}^{-\mathrm{i}s\omega})g)W(\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}(s+t)\omega})g)\mathrm{e}^{-\mathrm{i}(s+t)H_{\mathrm{ren}}} \\ = \ & \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}\mathrm{e}^{\mathrm{i}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}s\omega}(1-\mathrm{e}^{-\mathrm{i}t\omega})g)}\mathrm{e}^{\mathrm{i}(s+t)H}W(\mathrm{i}(\mathrm{e}^{-\mathrm{i}(s+t)\omega}-\mathrm{e}^{-\mathrm{i}s\omega})g)\mathrm{e}^{-\mathrm{i}(s+t)H} \\ & \times\mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}(s+t)\omega}g)}\mathrm{e}^{\mathrm{i}(s+t)H}W(\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}(s+t)\omega})g)\mathrm{e}^{-\mathrm{i}(s+t)H_{\mathrm{ren}}} \\ \to \ & \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}W^{\pm}(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})g)U^{\pm}, \end{aligned}$$

where we used the Riemann-Lebesgue lemma to show that $\lim\limits_{s\to\infty}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}s\omega}(1-\mathrm{e}^{-\mathrm{i}t\omega})g) = 0$. □

7 Pauli-Fierz Hamiltonians

In this section we apply the abstract formalism developed in Sections 3-5 to a class of Pauli-Fierz Hamiltonians. We will formulate a set of assumptions that will guarantee a satisfactory scattering theory and the existence of a dressing operator.

7.1 Coupling Fock space

Let $\mathfrak{h}$, $\mathcal{K}$ be Hilbert spaces. Let $\mathfrak{h}_1$ and $\mathcal{K}_1$ be dense subspaces of $\mathfrak{h}$ and $\mathcal{K}$. Let $\overset{\circ}{\otimes}$ denote the algebraic tensor product. Let

$$(\mathcal{K}_1\overset{\circ}{\otimes}\mathfrak{h}_1) \times \mathfrak{h}_1 \ni (\Psi_1, \Psi_2) \mapsto (\Psi_1|v\Psi_2) \in \mathbb{C}$$

be a sesquilinear form.

Let $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathfrak{h}_1)$ denote the algebraic Fock space over the vector space $\mathfrak{h}_1$. We define the annihilation form and creation forms $\mathrm{Wick}(v^*)$ and $\mathrm{Wick}(v)$ as the forms on $\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathfrak{h})$ with the domain $\mathcal{K}_1\overset{\circ}{\otimes}\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathfrak{h}_1) \subset \mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathfrak{h})$ as follows: if $h_1, h_2 \in \mathfrak{h}_1$ and $\Psi_1, \Psi_2 \in \mathcal{K}_1$, then

$$(\Psi_2\otimes h_2^{\otimes m}|\mathrm{Wick}(v)\Psi_1\otimes h_1^{\otimes n}) = \begin{cases} \sqrt{m}(\Psi_2\otimes h_2|vh_1)(h_2|h_1)^n, & m = n+1 \\ 0, & m \neq n+1; \end{cases}$$

$$(\Psi_2\otimes h_2^{\otimes m}|\mathrm{Wick}(v^*)\Psi_1\otimes h_1^{\otimes n}) = \begin{cases} \sqrt{n}(\Psi_2|v^*\Psi_1\otimes h_1)(h_2|h_1)^m & m = n-1, \\ 0 & m \neq n-1 \end{cases}$$

Note that if v is bounded, then $\mathrm{Wick}(v)$ and $\mathrm{Wick}(v^*)$ extend to closed operators adjoint to one another. We will write $\mathrm{Wick}(v_1 + v_2^*)$ for $\mathrm{Wick}(v_1) + \mathrm{Wick}(v_2^*)$.

For a vector $z \in \mathfrak{h}$ the operators $|z) \in B(\mathbb{C}, \mathfrak{h})$ and its adjoint $(z| \in B(\mathfrak{h}, \mathbb{C})$ are defined in the in the usual way:

$$\mathbb{C} \ni \lambda \mapsto |z)\lambda := \lambda z \in \mathfrak{h}, \qquad \mathfrak{h} \ni h \mapsto (z|h := (z|h) \in \mathbb{C}.$$

Note that the usual creation and annihilation operators correspond to the case $\mathcal{K} = \mathbb{C}$: if $z \in \mathfrak{h}$, then

$$\mathrm{Wick}(|z)) = a^*(z), \qquad \mathrm{Wick}((z|) = a(z).$$

For further reference let us note the identities

$$\begin{aligned} W(\mathrm{i}g)\mathrm{Wick}(v)W(-\mathrm{i}g) &= \mathrm{Wick}(v) + \tfrac{1}{\sqrt{2}}\mathbb{1}\otimes(g|v, \\ W(\mathrm{i}g)\mathrm{Wick}(v^*)W(-\mathrm{i}g) &= \mathrm{Wick}(v^*) + \tfrac{1}{\sqrt{2}}v^*\mathbb{1}\otimes|g). \end{aligned} \tag{7.1}$$

(In the above identities we dropped the factors $\otimes\mathbb{1}_{\Gamma_{\mathrm{s}}(\mathfrak{h})}$).

Let us note the following inequalities:

Lemma 7.1 *For $\Psi \in \Gamma_{\mathrm{s}}(\mathfrak{h})$, $R > 0$ and a positive operator ω on $\mathfrak{h}$ we have*

$$\|\mathrm{Wick}(v^*)\Psi\|^2 \leq (\Psi|\mathbb{1}\otimes\mathrm{d}\Gamma(\omega)\Psi)\|v^*\ \mathbb{1}\otimes\omega^{-1}\ v\|; \tag{7.2}$$

$$|(\Psi|\mathrm{Wick}(v^*)\Psi)| \leq \|\mathbb{1}\otimes\omega^{-\frac{1}{2}}v(K+R)^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})}\|(K+R)^{\frac{1}{2}}\otimes\mathbb{1}\Psi\|\|\mathbb{1}\otimes\mathrm{d}\Gamma(\omega)^{\frac{1}{2}}\Psi\|. \tag{7.3}$$

Proof. The proof of the first inequality can be found e.g in [DJ] and [GGM]. The second inequality is proved e.g in [GGM, Corollary 3.10]). For the reader's convenience we will show how the first inequality implies the second.

Set $\tilde{v} := v(K+R)^{-1/2}$. Now

$$\begin{aligned}
|(\Psi|\mathrm{Wick}(v^*)\Psi)| &= |\big((R+K)^{1/2}\otimes\mathbb{1}\ \Psi \mid \mathrm{Wick}(\tilde{v}^*)\Psi\big)| \\
&\leq \|(R+K)^{1/2}\otimes\mathbb{1}\ \Psi\|\|\mathrm{Wick}(\tilde{v}^*)\Psi\| \\
&\leq \|(R+K)^{1/2}\otimes\mathbb{1}\ \Psi\|\|1\otimes\omega^{-1/2}\ \tilde{v}\|\|\mathbb{1}\otimes \mathrm{d}\Gamma(\omega)^{1/2}\Psi\|.
\end{aligned}$$

□

7.2 Pauli-Fierz Hamiltonians

Consider a positive operator K on $\mathcal{K}$ and a positive operator ω on $\mathfrak{h}$. The operator

$$H_0 := K\otimes\mathbb{1} + \mathbb{1}\otimes \mathrm{d}\Gamma(\omega),\ \text{acting on}\ \mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathfrak{h})$$

will be called a free Pauli-Fierz Hamiltonian.

The following assumption is weaker than Assumption 2.B:

Assumption 7.A *v is a form on $\mathcal{K}_1\otimes\mathfrak{h}_1 \times \mathcal{K}_1$ such that* $\limsup\limits_{R\to\infty} \|\omega^{-1/2}v(K+R)^{-1/2}\| < 1/2.$

From the inequality (7.3) one deduces the following theorem:

Theorem 7.2 *Under Assumption 7.A, the quadratic form*

$$\mathrm{Wick}(v+v^*)$$

is form bounded wrt H_0 with the bound less than 1. Therefore, by the KLMN theorem, we can define the Pauli-Fierz Hamiltonian as the self-adjoint operator

$$H := H_0 + \mathrm{Wick}(v+v^*),$$

with the same form domain as H_0.

7.3 Asymptotic CCR representations for Pauli-Fierz Hamiltonians

As before, let $\mathfrak{h}_n$ be the scale of Hilbert spaces associated with ω^{-1}.

The following assumption can be called the *short range condition* and is the equivalent of Assumption 2.C:

Assumption 7.B *There exists a subspace $\mathfrak{g}\subset\mathfrak{h}_1\cap\mathrm{Dom}(\omega^{1/2})$ dense in $\mathfrak{h}_1$ in the topology of $\mathfrak{h}_1$ such that for $h\in\mathfrak{g}$ and almost all $t\in\mathbb{R}$, the operator*

$$B(t) := \big(\mathbb{1}_{\mathcal{K}}\otimes(\mathrm{e}^{-\mathrm{i}t\omega}h|\ v + \mathrm{hc}\big)(1+K)^{-1/2} \tag{7.4}$$

is bounded and

$$\int_0^\infty \|B(t)\|\mathrm{d}t < \infty.$$

Remark 7.3 *Note that in (7.4)* $1\!\!1_{\mathcal{K}}\otimes(\mathrm{e}^{-\mathrm{i}t\omega}h|\ v$ *denotes an operator in* $B(\mathcal{K})$ *and* hc *stands for its hermitian conjugate, that is the operator* $v^*\ 1\!\!1_{\mathcal{K}}\otimes|\mathrm{e}^{-\mathrm{i}t\omega}h)$.

Theorem 7.4 *Suppose Assumptions 7.A and 7.B hold. Then*

(1) *For all* $h\in\mathfrak{h}_1$ *there exist*

$$W^{\pm}(h) := \mathrm{s}-\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}1\!\!1\otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}. \tag{7.5}$$

(2) *The map*

$$\mathfrak{h}_1\ni h\mapsto W^{\pm}(h) \tag{7.6}$$

is strongly continuous.

Consequently $(W^{\pm},\omega,H)$ *are two regular covariant CCR representations.*

(3) *For all* $h\in\mathfrak{h}_1$

$$W^{\pm}(h)(\mathrm{i}+H)^{-1/2} = \lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}1\!\!1\otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h)(\mathrm{i}+H)^{-1/2}\mathrm{e}^{-\mathrm{i}tH}. \tag{7.7}$$

(4) *The map*

$$\mathfrak{h}_1\ni h\mapsto W^{\pm}(h)(\mathrm{i}+H)^{-1/2} \tag{7.8}$$

is norm continuous.

(5) *for all* $h\in\mathfrak{h}_1$, $\mathrm{Dom}(H+c)^{\frac{1}{2}}\subset\mathrm{Dom}(\phi^{\pm}(h))$ *and*

$$\phi^{\pm}(h)(H+c)^{-\frac{1}{2}} = \mathrm{s}-\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}\phi(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{\mathrm{i}tH}(H+c)^{-\frac{1}{2}}.$$

(6) *for any* $\epsilon>0$ *the CCR representations* $W^{\pm}$ *are of Fock type when restricted to* $1\!\!1_{[\epsilon,+\infty[}(\omega)\mathfrak{h}$.

Proof. For shortness, we drop $1\!\!1_{\mathcal{K}}\otimes$ in the formulas below. We have

$$W(\mathrm{e}^{-\mathrm{i}t\omega}h) = \mathrm{e}^{-\mathrm{i}tH_0}W(h)\mathrm{e}^{\mathrm{i}tH_0},$$

which implies that $t\mapsto(1+H_0)^{-1}W(\mathrm{e}^{-\mathrm{i}t\omega}h)(1+H_0)^{-1}$ is C^1 and

$$\begin{aligned}
&\partial_t(1+H_0)^{-1}W(\mathrm{e}^{-\mathrm{i}t\omega}h)(1+H_0)^{-1}\\
=\ &-(1+H_0)^{-1}[H_0,\mathrm{i}W(\mathrm{e}^{-\mathrm{i}t\omega}h)](1+H_0)^{-1}\\
=\ &-\tfrac{1}{\sqrt{2}}(1+H_0)^{-1}\left(a^*(\mathrm{e}^{-\mathrm{i}t\omega}h)W(\mathrm{e}^{-\mathrm{i}t\omega}h)-W(\mathrm{e}^{-\mathrm{i}t\omega}h)a(\mathrm{e}^{-\mathrm{i}t\omega}h)\right)(1+H_0)^{-1}.
\end{aligned} \tag{7.9}$$

Using the fact that $\mathrm{e}^{\mathrm{i}t\omega}h\in\mathrm{Dom}(\omega^{-1/2})$ we see that $(1+H_0)^{-1/2}a^*(\mathrm{e}^{-\mathrm{i}t\omega}h)$ is bounded. Therefore, from (7.9) and Lemma A.1 we can actually conclude that

$$t\mapsto(1+H_0)^{-1/2}W(\mathrm{e}^{-\mathrm{i}t\omega}h)(1+H_0)^{-1/2}$$

is C^1 and

$$\begin{array}{rl} & \partial_t(1+H_0)^{-1/2}W(\mathrm{e}^{-\mathrm{i}t\omega}h)(1+H_0)^{-1/2} \\ = & -\frac{1}{\sqrt{2}}(1+H_0)^{-1/2}\left(a^*(\mathrm{e}^{-\mathrm{i}t\omega}h)W(\mathrm{e}^{-\mathrm{i}t\omega}h)-W(\mathrm{e}^{-\mathrm{i}t\omega}h)a(\mathrm{e}^{-\mathrm{i}t\omega}h)\right)(1+H_0)^{-1/2}. \end{array} \tag{7.10}$$

But $(1+H_0)^{-1/2}(c+H)^{1/2}$ is bounded, so $t\mapsto(c+H)^{-1/2}W(\mathrm{e}^{-\mathrm{i}t\omega}h)(c+H)^{-1/2}$ is C^1 and we can replace $(1+H_0)^{-1/2}$ with $(c+H)^{-1/2}$ in (7.10).

Now, $t\mapsto(c+H)^{-1}\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}(c+H)^{-1}$ is C^1 and we have

$$\begin{array}{rl} & \partial_t(c+H)^{-1}\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}(c+H)^{-1} \\ = & (c+H)^{-1}\mathrm{e}^{\mathrm{i}tH}\mathrm{i}[H,W(\mathrm{e}^{-\mathrm{i}t\omega}h)]\mathrm{e}^{-\mathrm{i}tH}(c+H)^{-1} \\ & +\mathrm{e}^{\mathrm{i}tH}\left(\partial_t(c+H)^{-1}W(\mathrm{e}^{-\mathrm{i}t\omega}h)(c+H)^{-1}\right)\mathrm{e}^{-\mathrm{i}tH} \\ = & \mathrm{e}^{\mathrm{i}tH}(c+H)^{-1}\mathrm{i}[\mathrm{Wick}(v+v^*),W(\mathrm{e}^{-\mathrm{i}t\omega}h)](c+H)^{-1}\mathrm{e}^{-\mathrm{i}tH} \\ = & \frac{1}{\sqrt{2}}(c+H)^{-1}\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}t\omega}h)\left((\mathrm{e}^{-\mathrm{i}t\omega}h|v-v^*|\mathrm{e}^{-\mathrm{i}t\omega}h)\right)\mathrm{e}^{-\mathrm{i}tH}(c+H)^{-1}, \end{array}$$

where, in the last step we used the identities (7.1). Eventually, using again Lemma A.1, we can write

$$\begin{array}{rl} & \partial_t\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}(c+H)^{-1/2} \\ = & \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}t\omega}h)\left((\mathrm{e}^{-\mathrm{i}t\omega}h|v-v^*|\mathrm{e}^{-\mathrm{i}t\omega}h)\right)\mathrm{e}^{-\mathrm{i}tH}(c+H)^{-1/2}. \end{array} \tag{7.11}$$

The norm of (7.11) can be estimated by

$$c\left\|\left((\mathrm{e}^{-\mathrm{i}t\omega}h|v-v^*|\mathrm{e}^{-\mathrm{i}t\omega}h)\right)(1+K)^{-1/2}\right\|.$$

By Assumption 7.B, if $h\in\mathfrak{g}$, this is integrable. Therefore, by the Cook method there exists

$$\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}t\omega}h)(\mathrm{i}+H)^{-1/2}\mathrm{e}^{-\mathrm{i}tH}. \tag{7.12}$$

If $h\in\mathfrak{h}_1$, then we will find a sequence (h_n) in $\mathfrak{g}$ such that $h_n\to h$ in the norm of $\mathfrak{h}_1$. Clearly, $\|h_n\|_{\mathfrak{h}_1}$ is uniformly bounded. Now, using Lemma A.1 and estimate (7.3) we get

$$\begin{array}{rl} & \sup_t\|\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}(c+H)^{-1/2}-\mathrm{e}^{\mathrm{i}tH}W(\mathrm{e}^{-\mathrm{i}t\omega}h_n)\mathrm{e}^{-\mathrm{i}tH}(c+H)^{-1/2}\| \\ \leq & \sup_t c\|(W(\mathrm{e}^{-\mathrm{i}t\omega}h)-W(\mathrm{e}^{-\mathrm{i}t\omega}h_n))(1+\mathrm{d}\Gamma(\omega))^{-1/2}\| \\ \leq & c_1(\|h-h_n\|+\|\phi(h-h_n)(1+\mathrm{d}\Gamma(\omega))^{-1/2}\| \\ \leq & c_2\left(\|h-h_n\|+\|\omega^{-1/2}(h-h_n)\|\right). \end{array}$$

This proves the existence of the norm limit (7.12) for an arbitrary $h\in\mathfrak{h}_1$, and also shows

$$\lim_{n\to\infty}W^{\pm}(h_n)(c+H)^{-1/2}=W^{\pm}(h)(c+H)^{-1/2}.$$

This proves (3) and (4). Now (1) and (2) follow by a simple density argument. The proof of (5) can be done as in eg [Ge2, Thm. 8.2]. It remains to prove (6). We will use the notion of the *number quadratic form* associated to a regular CCR representation (see eg [DG3, Sect. 4.2]). Let us fix $\epsilon > 0$ and let $\mathfrak{f}$ be a finite dimensional subspace in $\mathfrak{h}_\epsilon := 1\!\!1_{[\epsilon,+\infty[}(\omega)\mathfrak{h}$. Let $n_{\mathfrak{f}}^{\pm}$ be the quadratic form equal to:

$$n_{\mathfrak{f}}^{\pm}(u,u) = \sum_{i=1}^{n} \|a^{\pm}(f_i)u\|^2, \text{ with domain } \bigcap_{i=1}^{n} \mathrm{Dom}(a^{\pm}(f_i)).$$

where $(f_1,\ldots,f_n)$ is an orthonormal basis of $\mathfrak{f}$. It is easy to see that $n_{\mathfrak{f}}^{\pm}$ does not depend on the choice of the o.n.b. of $\mathfrak{f}$. One can then define the *number quadratic forms* $n^{\pm}$ as:

$$n^{\pm} := \sup_{f\subset\mathfrak{h}_\epsilon,\, \dim\mathfrak{f}<\infty} n_{\mathfrak{f}}^{\pm}.$$

Then (see e.g. [DG3, Thm. 4.3]) the CCR representations $W^{\pm}$ are of Fock type iff $n^{\pm}$ are densely defined. We claim that there exist a constant C, independent of $\mathfrak{f}\subset\mathfrak{h}_\epsilon$ such that:

$$n_{\mathfrak{f}}^{\pm}(u,u) \leq C(u,(H+c)u),\ u\in\mathrm{Dom}((H+c)^{\frac{1}{2}}), \tag{7.13}$$

which implies that $\mathrm{Dom}((H+c)^{\frac{1}{2}}) \subset \mathrm{Dom}(n^{\pm})$ and hence completes the proof of (6). In fact using (5), we obtain for $u\in\mathrm{Dom}((H+c)^{\frac{1}{2}})$:

$$n_{\mathfrak{f}}^{\pm}(u,u) = \lim_{t\to\pm\infty}(\mathrm{e}^{-\mathrm{i}tH}u, \mathrm{d}\Gamma(\mathrm{e}^{\mathrm{i}t\omega}\pi_{\mathfrak{f}}\mathrm{e}^{-\mathrm{i}t\omega})\mathrm{e}^{-\mathrm{i}tH}u), \tag{7.14}$$

where $\pi_{\mathfrak{f}}$ is the orthogonal projection on $\mathfrak{f}$. Next we have $\mathrm{e}^{\mathrm{i}t\omega}\pi_{\mathfrak{f}}\mathrm{e}^{-\mathrm{i}t\omega} \leq 1\!\!1_{[\epsilon,+\infty[}(\omega) \leq \epsilon^{-1}\omega$, and hence $\mathrm{d}\Gamma(\mathrm{e}^{\mathrm{i}t\omega}\pi_{\mathfrak{f}}\mathrm{e}^{-\mathrm{i}t\omega}) \leq \epsilon^{-1}\mathrm{d}\Gamma(\omega) \leq C(H+c)$, uniformly w.r.t. $\mathfrak{f}$. By (7.14) this implies (7.13) and completes the proof of the theorem. □

7.4 Renormalized Pauli-Fierz Hamiltonian

The following assumption is weaker than Assumption 2.D:

Assumption 7.C *We assume that*

$$\begin{aligned}
&v = |z) \otimes 1\!\!1_{\mathcal{K}} + v_{\mathrm{ren}},\\
&z\in\mathfrak{h},\quad v_{\mathrm{ren}}\in B(\mathrm{Dom}(K^{\frac{1}{2}}), \mathcal{K}\otimes\mathfrak{h}),\\
&(z|(1+\omega^{-1})z) < \infty,\\
&\|\omega^{-1}v_{\mathrm{ren}}(1{+}K)^{-1/2}\| < \infty.
\end{aligned}$$

Set

$$g := \sqrt{2}\omega^{-1}z.$$

Note that $g\in\omega^{-1/2}\mathfrak{h}$.

The assumption below is the equivalent of Assumption 2.F:

Assumption 7.D *For almost all $t \in \mathbb{R}$, the operator*

$$C(t) := \big(1\!\!1_{\mathcal{K}} \otimes (\mathrm{e}^{-\mathrm{i}t\omega} g|\ v_{\mathrm{ren}} + \mathrm{hc}\big)(1{+}K)^{-1/2}$$

is bounded and

$$\int_0^\infty \|C(t)\| \mathrm{d}t < \infty.$$

Introduce the renormalized Hamiltonian

$$H_{\mathrm{ren}} := K_{\mathrm{ren}} \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega) + \mathrm{Wick}(v_{\mathrm{ren}} + v_{\mathrm{ren}}^*),$$

where

$$\begin{aligned} v_{\mathrm{ren}} &:= v - 1\!\!1_{\mathcal{K}} \otimes |z), \\ K_{\mathrm{ren}} &:= K + (z|\omega^{-1}z) - 1\!\!1 \otimes (\omega^{-1}z|\ v\ -\ v^*\ 1\!\!1 \otimes |\omega^{-1}z) \\ &= K - (z|\omega^{-1}z) - 1\!\!1 \otimes (\omega^{-1}z\ |v_{\mathrm{ren}}\ -\ v_{\mathrm{ren}}^* 1\!\!1 \otimes\ |\omega^{-1}z). \end{aligned}$$

Note that if $g \in \mathfrak{h}$, then

$$H = W(\mathrm{i}g) H_{\mathrm{ren}} W(-\mathrm{i}g).$$

As in (6.5), set

$$U(t) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)} \mathrm{e}^{\mathrm{i}tH} W(\mathrm{i}(1 - \mathrm{e}^{-\mathrm{i}t\omega})g) \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}.$$

Theorem 7.5 (1) *Suppose Assumptions 7.A, 7.C and 7.D hold. Then there exist*

$$U^{\pm} := \mathrm{s-}\lim_{t \to \pm\infty} U(t).$$

Moreover,

$$U^{\pm *} = \mathrm{s-}\lim_{t \to \pm\infty} U^*(t).$$

(2) *Suppose in addition Assumption 7.B. Then there exist the limits*

$$\mathrm{s-}\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}} 1\!\!1 \otimes W(\mathrm{e}^{-\mathrm{i}t\omega}h) \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} =: W_{\mathrm{ren}}^{\pm}(h).$$

Moreover, $U^{\pm}$ are covariant g-dressing operators between the representations $(W_{\mathrm{ren}}^{\pm}, \omega, H_{\mathrm{ren}})$ and $(W^{\pm}, \omega, H)$ and satisfy all the properties described in Section 6.

Proof. We have

$$\begin{aligned} & \tfrac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}t 1\!\!1 \otimes \mathrm{d}\Gamma_g(\omega)} \\ = \ & \mathrm{i}\mathrm{e}^{\mathrm{i}tH} \left(\mathrm{Wick}(v + v^*) + K \otimes 1\!\!1 - 1\!\!1 \otimes a^*(z) - 1\!\!1 \otimes a(z) - (z|\omega z)\right) \mathrm{e}^{-\mathrm{i}t 1\!\!1 \otimes \mathrm{d}\Gamma_g(\omega)} \\ = \ & \mathrm{i}\mathrm{e}^{\mathrm{i}tH} \left(\mathrm{Wick}(v_{\mathrm{ren}} + v_{\mathrm{ren}}^*) + K_{\mathrm{ren}} \otimes 1\!\!1 + (\omega^{-1}z|v_{\mathrm{ren}} + v_{\mathrm{ren}}^*|\omega^{-1}z)\right) \mathrm{e}^{-\mathrm{i}t 1\!\!1 \otimes \mathrm{d}\Gamma_g(\omega)} \end{aligned}$$

and

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\mathrm{i}t 1\!\!1 \otimes \mathrm{d}\Gamma(\omega)} \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} = -\mathrm{i}\mathrm{e}^{\mathrm{i}t 1\!\!1 \otimes \mathrm{d}\Gamma(\omega)} \left(K_{\mathrm{ren}} \otimes 1\!\!1 + \mathrm{Wick}(v_{\mathrm{ren}} + v_{\mathrm{ren}}^*)\right) \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}.$$

Hence,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}U(t) &= \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}t1\!\!1\otimes\mathrm{d}\Gamma_g(\omega)}\Big)\mathrm{e}^{\mathrm{i}t1\!\!1\otimes\mathrm{d}\Gamma(\omega)}\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} + \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}t1\!\!1\otimes\mathrm{d}\Gamma_g(\omega)}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}t1\!\!1\otimes\mathrm{d}\Gamma(\omega)}\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}\\
&= \mathrm{i}\mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}\mathrm{e}^{\mathrm{i}tH}\Big(\mathrm{Wick}(v_{\mathrm{ren}}+v_{\mathrm{ren}}^*)+(\omega^{-1}z|v_{\mathrm{ren}}+v_{\mathrm{ren}}^*|\omega^{-1}z)\\
&\quad -W(\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}t\omega})g)\mathrm{Wick}(v_{\mathrm{ren}}+v_{\mathrm{ren}}^*)W(-\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}t\omega})g)\Big)W(\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}t\omega})g)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}\\
&= \mathrm{i}\mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(g|\mathrm{e}^{-\mathrm{i}t\omega}g)}\mathrm{e}^{\mathrm{i}tH}\Big((\mathrm{e}^{-\mathrm{i}t\omega}g|v_{\mathrm{ren}}+v_{\mathrm{ren}}^*|\mathrm{e}^{-\mathrm{i}t\omega}g)\Big)W(\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}t\omega})g)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}.
\end{aligned}
$$

Therefore

$$\int_0^\infty \|\tfrac{\mathrm{d}}{\mathrm{d}t}U(t)(\mathrm{i}+H_{\mathrm{ren}})^{-1/2}\|\mathrm{d}t<\infty.$$

This means that Assumption 6.A holds and we can apply the results of Section 6. □

7.5 Covariance of renormalized objects

The renormalization depends on the splitting of v given in Assumption 7.C into a singular scalar part and the regular part. This splitting is to some extent arbitrary. In this subsection we study how various "renormalized" objects depend on this splitting.

We will replace g by $\tilde{g}=g+h$, for $h\in\mathfrak{h}$ and denote with tildes the new objects obtained with the function $\tilde{g}$.

Theorem 7.6 *Suppose Assumptions 7.A, 7.B, 7.C and 7.D hold. Let $f\in\mathfrak{h}_{-1}$. Then*

(1) $\tilde{H}_{\mathrm{ren}}=W(-\mathrm{i}h)H_{\mathrm{ren}}W^*(\mathrm{i}h)$.

(2) $\tilde{W}^{\pm}_{\mathrm{ren}}(h_1)=W(-\mathrm{i}h)W^{\pm}_{\mathrm{ren}}(h_1)W^*(\mathrm{i}h)$, $h_1\in\mathfrak{h}_1$.

(3) $\tilde{\mathcal{K}}^{\pm}_{\mathrm{ren},f}=W(-\mathrm{i}h)\mathcal{K}^{\pm}_{\mathrm{ren},f}$.

(4) $\tilde{\mathcal{H}}^{\pm}_{\mathrm{ren},[f]}=W(-\mathrm{i}h)\mathcal{H}^{\pm}_{\mathrm{ren},[f]}$.

(5) $\tilde{\mathcal{H}}^{\pm}_{\mathrm{ren},f}=W(-\mathrm{i}h)\mathcal{H}^{\pm}_{\mathrm{ren},f}$.

(6) $\tilde{K}^{\pm}_{\mathrm{ren},f}=W(-\mathrm{i}h)K^{\pm}_{\mathrm{ren},f}W(\mathrm{i}h)$.

(7) $\tilde{H}^{\pm}_{\mathrm{ren},f}=W(-\mathrm{i}h)\otimes 1\!\!1\, H^{\pm}_{\mathrm{ren},f}\, W(\mathrm{i}h)\otimes 1\!\!1$.

(8) *If in addition $h\in\mathfrak{h}_1$, then there exists $\tilde{U}^{\pm}$ and $\tilde{U}^{\pm}=W^{\pm}(-\mathrm{i}h)U^{\pm}\ 1\!\!1\otimes W(\mathrm{i}h)$.*

Proof. Direct computation proves (1). To prove (2) we compute for $h_1\in\mathfrak{h}_1$

$$
\begin{aligned}
&\tilde{W}^{\pm}_{\mathrm{ren}}(h_1)\\
=\ & \mathrm{s-}\lim_{t\to\pm\infty} W(-\mathrm{i}h)\mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}W(\mathrm{i}h)W(\mathrm{e}^{-\mathrm{i}\omega t}h_1)W(-\mathrm{i}h)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}W(\mathrm{i}h)\\
=\ & \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}\mathrm{Re}(h|\mathrm{e}^{-\mathrm{i}t\omega}h_1)}W(-\mathrm{i}h)\mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}W(\mathrm{e}^{-\mathrm{i}\omega t}h_1)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}}W(\mathrm{i}h)\\
=\ & W(-\mathrm{i}h)W^{\pm}_{\mathrm{ren}}(h_1)W(\mathrm{i}h),
\end{aligned}
$$

since $(h|\mathrm{e}^{-\mathrm{i}t\omega}h_1) \to 0$ when $t \to \pm\infty$ by the Riemann-Lebesgue lemma.

(1) and (2) directly imply all the statements but (8), which we prove below:

$$\begin{array}{rl} & U^*(t)\tilde{U}(t)W(-\mathrm{i}h) \\ = & \mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\omega)}\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_g(\omega)}\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma_{\tilde{g}}(\omega)}\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\omega)}W(-\mathrm{i}h)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} \\ = & \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h|\mathrm{e}^{\mathrm{i}t\omega}h)-\mathrm{i}\mathrm{Im}(g|(\mathrm{e}^{\mathrm{i}t\omega}-1)h)}\mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\omega)}W(-\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t\omega})h)\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\omega)}W(-\mathrm{i}h)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} \\ = & \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h|\mathrm{e}^{\mathrm{i}t\omega}h)-\mathrm{i}\mathrm{Im}(g|(\mathrm{e}^{\mathrm{i}t\omega}-1)h)}\mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}W(\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}t\omega})h)W(-\mathrm{i}h)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} \\ = & \mathrm{e}^{-\mathrm{i}\mathrm{Im}(\tilde{g}|\mathrm{e}^{\mathrm{i}t\omega}h)+\mathrm{i}\mathrm{Im}(g|h)}\mathrm{e}^{\mathrm{i}tH_{\mathrm{ren}}}W(-\mathrm{i}\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ren}}} \\ \underset{\text{strongly}}{\to} & \mathrm{e}^{\mathrm{i}\mathrm{Im}(g|h)}W_{\mathrm{ren}}^{\pm}(-\mathrm{i}h), \end{array}$$

where we used the Riemann-Lebesgue lemma, and the fact that $h \in \mathfrak{h}_1$, $\tilde{g} \in \mathfrak{h}_{-1}$ to show that $\lim_{t\to\infty}(\tilde{g}|\mathrm{e}^{\mathrm{i}t\omega}h) = 0$. Therefore

$$\tilde{U}^{\pm} = \mathrm{e}^{\mathrm{i}\mathrm{Im}(g|h)}U^{\pm}W_{\mathrm{ren}}^{\pm}(-\mathrm{i}h)W(\mathrm{i}h) = W^{\pm}(-\mathrm{i}h)U^{\pm}W(\mathrm{i}h). \ \Box$$

A Appendix

In the appendix we prove a number of technical lemmas needed in Section 7.

A.1 Differentiability of operator valued functions

Lemma A.1 *Consider a function*

$$]-\epsilon,\epsilon[\ni t \mapsto C(t) \in B(\mathcal{H}). \tag{A.1}$$

Suppose that for some dense subspaces $\mathcal{B}$, $\mathcal{D}$ and $\Phi \in \mathcal{B}$, $\Psi \in \mathcal{D}$ the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi|C(t)\Psi) \tag{A.2}$$

exists. Suppose that $]-\epsilon,\epsilon[\ni t \mapsto C'(t) \in B(\mathcal{H})$ is a continuous function and (A.2) equals $(\Phi|C'(t)\Psi)$. Then (A.1) is norm differentiable and its derivative equals $C'(t)$, that means

$$\lim_{s\to 0}\frac{C(t+s)-C(t)}{s} = C'(t). \tag{A.3}$$

Proof. It suffices to prove (A.3) for $t = 0$. For $\Phi \in \mathcal{B}$ and $\Psi \in \mathcal{D}$,

$$\Big(\Phi|\big(s^{-1}(C(s)-C(0)-C'(0)\big)\Psi\Big) \quad = s^{-1}\int_0^s(\Phi|\big(C'(s_1)-C'(0)\big)\Psi\Big)\mathrm{d}s_1.$$

Hence

$$\Big(\Phi|\big(s^{-1}(C(s)-C(0)-C'(0)\big)\Psi\Big) \quad \leq \sup\{\|C'(s_1)-C'(0)\| \ : \ |s_1|<|s|\}\|\Phi\|\|\Psi\|.$$

Thus

$$\|s^{-1}\big(C(s)-C(0)\big)-C'(0)\| \leq \sup\{\|C'(s_1)-C'(0)\| \ : \ |s_1|<s\} \to 0. \ \Box$$

A.2 1-parameter groups of $*$-automorphisms

Let

$$\mathbb{R} \ni t \mapsto \alpha^t \tag{A.4}$$

be a group $*$-automorphisms of the $*$-algebra $B(\mathcal{H})$. We say that it is pointwise weakly continuous, if

$$t \mapsto (\Phi|\alpha^t(A)\Psi), \quad A \in B(\mathcal{H}), \quad \Phi, \Psi \in \mathcal{H},$$

is continuous. It is well known that if (A.4) is a pointwise weakly continuous group of $*$-automorphisms of the $*$-algebra $B(\mathcal{H})$, then there exists a self-adjoint operator H, unique up to an additive constant, such that $\alpha^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}$, see [BR, vol. I, Ex. 3.2.14 and 3.2.35].

Lemma A.2 *Suppose that α^t and H are as above. Assume that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and there exists a self-adjoint operator H_2 on $\mathcal{H}_2$ such that for any $A_2 \in B(\mathcal{H}_2)$ we have*

$$\alpha^t(1\!\!1 \otimes A_2) = 1\!\!1 \otimes \mathrm{e}^{\mathrm{i}tH_2} A_2 \mathrm{e}^{-\mathrm{i}tH_2}.$$

Then there exists a unique self-adjoint operator H_1 on $\mathcal{H}_1$ such that

$$H = H_1 \otimes 1\!\!1 + 1\!\!1 \otimes H_2.$$

Proof. For $A_1 \in B(\mathcal{H}_1)$, $\alpha^t(A_1 \otimes 1\!\!1)$ commutes with the operators of the form $1\!\!1 \otimes A_2$, $A_2 \in B(\mathcal{H}_2)$. Hence $\alpha^t(A_1 \otimes 1\!\!1)$ is of the form $B_1 \otimes 1\!\!1$ with $B_1 \in B(\mathcal{H}_1)$. Therefore,

$$\alpha_1^t(A_1) \otimes 1\!\!1 := \alpha^t(A_1 \otimes 1\!\!1)$$

defines a pointwise weakly continuous group of $*$-automorphisms of the $*$-algebra $B(\mathcal{H}_1)$. Therefore, there exists a self-adjoint operator $\tilde{H}_1$ on $\mathcal{H}_1$ such that

$$\alpha_1^t(A_1) = \mathrm{e}^{\mathrm{i}t\tilde{H}_1} A_1 \mathrm{e}^{-\mathrm{i}t\tilde{H}_1}.$$

Set

$$\tilde{H} = \tilde{H}_1 \otimes 1\!\!1 + 1\!\!1 \otimes H_2.$$

Clearly,

$$\mathrm{e}^{\mathrm{i}tH} A_1 \otimes A_2 \mathrm{e}^{-\mathrm{i}tH} = \mathrm{e}^{\mathrm{i}t\tilde{H}} A_1 \otimes A_2 \mathrm{e}^{-\mathrm{i}t\tilde{H}}.$$

By the weak density,

$$\mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH} = \mathrm{e}^{\mathrm{i}t\tilde{H}} A \mathrm{e}^{-\mathrm{i}t\tilde{H}},$$

for all $A \in B(\mathcal{H})$. Hence $c := \tilde{H} - H$ is a constant. We set $H_1 := \tilde{H}_1 - c$. □

A.3 Continuity of Weyl operators

Proposition A.3 *For $h_1, h_2 \in \mathfrak{h}$,*

$$\|(W(h_1) - W(h_2))\Psi\| \leq 2\sin\big(\|h_1 - h_2\|(\|h_1\| + \|h_2\|)\big)\|\Psi\| + 2\|\sin\tfrac{\phi(h_1-h_2)}{2}\Psi\|.$$

Proof. We have

$$\begin{aligned} W(h_1) - W(h_2) &= W(h_1)(\mathbb{1} - \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h_1|h_2)}) \\ &+ \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h_1|h_2)} W(h_1)(\mathbb{1} - W(h_2 - h_1)). \end{aligned}$$

We note also that

$$\mathrm{Im}(h_1|h_2) = \frac{1}{2}\mathrm{Im}\left((h_1|h_2 - h_1) + (h_1 - h_2|h_2)\right).$$

Hence

$$|\mathrm{Im}(h_1|h_2)| \leq \frac{1}{2}(\|h_1\| + \|h_2\|)\|h_1 - h_2\|.$$

Moreover

$$|\mathrm{e}^{\mathrm{i}s} - 1| = 2\left|\sin\frac{s}{2}\right|.$$

□

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