

UNIVERSITÉ PARIS-SUD XI



FACULTÉ
DES SCIENCES
D'ORSAY

**APPLICATIONS
OF
NONCROSSING PARTITIONS
TO
QUANTUM GROUPS**

AMAURY FRESLON



Département de Mathématiques d'Orsay

Copenhagen – May 2019

UNIVERSITY OF
COPENHAGEN



MASTERCLASS

SUBFACTORS AND QUANTUM GROUPS

A combinatorial approach to some problems in
Operator Algebras



APRIL 29 - MAY 3

DEPARTMENT OF MATHEMATICAL SCIENCES

Lecturers:

- Michael Brannan (Texas A&M Univ.)
- Amaury Freslon (Univ. Paris-Sud 11)
- Stefaan Vaes (KU Leuven)

Speakers:

- Laura Mančinska (QMATH)
- Adam Skalski (IMPAN)
- Roland Vergnioux (Univ. de Caen Normandie)

Organizers:

Rubén Martos & Ryszard Nest

*Registration before
20 March 2019

FOREWORD

These are extended notes from a lecture series given at the University of Copenhagen from May 29 to June 3, 2019 on the occasion of the Masterclass *Subfactors and Quantum Groups*. The aim of these lectures was to give an introduction to the theory of compact quantum groups and to some of their combinatorial aspects linked to noncrossing partitions. This is a recent and rapidly growing field of research, enriched by its many connections to other parts of mathematics like free probability, representation theory or quantum information theory. We therefore decided, in an attempt to make these lectures stimulating, to focus on applications.

All participants of the masterclass were assumed to have a background in operator algebras, roughly equivalent to a first graduate course on the subject. This is the reason why basic definitions and results concerning C^* -algebras and von Neumann algebras are usually used in the text without references. However, all the applications we consider involve a specific class of compact quantum groups which can be treated entirely in an algebraic way. This enabled us to give an account of the theory which only involves undergraduate level algebra. In that way, assuming a small amount of operator algebraic statements in the last two lectures, these notes should be readable to any graduate students with some algebra background. For convenience, we also included a small appendix with the proofs of two results in representation theory which are needed at some point.

The present document was first intended as a support for the lectures, not as an official publication. As a consequence, we have not done justice to all the authors involved in the development of the theory. Because we do not want to lengthen these notes by turning them into a book-style text, we will rather refer the reader to the two existing books on the theory of compact quantum groups, namely [Tim08] and [NT13], for historical and bibliographical comments. Moreover, we have only picked a few results that we considered striking or that we were asked to present by the organizers. Giving an account of all the works extending these or which were developed in the same vein would make a long survey paper, doomed to quickly become obsolete. As a consequence, we simply mentioned a few related works inside the text, in the form of comments or of remarks.

It is a pleasure to thank the organizers of the Masterclass, Rubèn Martos and Ryszard Nest, for their invitation which gave me the occasion to write these notes, as well as my co-lecturers Michael Brannan and Stefaan Vaes for providing a rich and stimulating environment. I am also grateful to Moritz Weber for reading a preliminary version and sharing his remarks and comments on this text.

CONTENTS

| | |
|--|-----|
| FOREWORD | v |
| CONTENTS | vii |
| LECTURE 1 INTRODUCING QUANTUM GROUPS | 1 |
| 1.1 The graph isomorphism game | 1 |
| 1.2 The quantum permutation algebra | 2 |
| 1.3 Compact matrix quantum groups | 5 |
| 1.3.1 A first definition | 6 |
| 1.3.2 The quantum orthogonal group | 8 |
| 1.3.3 The unitary case | 9 |
| 1.4 Representation theory | 10 |
| 1.4.1 Finite-dimensional representations | 10 |
| 1.4.2 Intertwiners | 11 |
| 1.4.3 Structure of the representation theory | 13 |
| LECTURE 2 PARTITIONS ENTER THE PICTURE | 17 |
| 2.1 Invariants of the (quantum) orthogonal group | 17 |
| 2.1.1 The orthogonal group : pair partitions | 17 |
| 2.1.2 The quantum orthogonal group : noncrossing partitions | 18 |
| 2.2 Partition quantum groups | 20 |
| 2.2.1 Partition maps | 20 |
| 2.2.2 Operations on partitions | 22 |
| 2.2.3 Tannaka-Krein reconstruction | 23 |
| 2.2.4 Examples of partition quantum groups | 27 |
| LECTURE 3 ALGEBRAIC APPLICATION : REPRESENTATION THEORY | 31 |
| 3.1 Building projections | 31 |
| 3.1.1 Linear independence of noncrossing partition operators | 31 |
| 3.1.2 Projective partitions | 33 |
| 3.2 From partitions to representations | 35 |
| 3.2.1 Irreducibility | 35 |
| 3.2.2 Fusion rules | 36 |
| 3.3 Examples | 38 |
| 3.3.1 Quantum orthogonal group | 38 |
| 3.3.2 Quantum permutation group | 38 |
| 3.3.3 Quantum hyperoctahedral group | 39 |

| | | |
|-------------------|--|-----------|
| LECTURE 4 | PROBABILISTIC APPLICATION : THE HAAR STATE | 41 |
| 4.1 | From algebra to analysis | 41 |
| 4.1.1 | A discrete detour | 41 |
| 4.1.2 | Positivity of the Haar integral | 42 |
| 4.2 | Weingarten calculus | 44 |
| 4.3 | Applications | 46 |
| 4.3.1 | Spectral measures | 46 |
| 4.3.2 | Truncated characters | 47 |
| 4.3.3 | Single coefficients and freeness | 49 |
| LECTURE 5 | ANALYTIC APPLICATION : THE CONNES EMBEDDING PROBLEM | 51 |
| 5.1 | Approximating operator algebras by matrices | 51 |
| 5.2 | The proof | 52 |
| 5.2.1 | Topological generation | 52 |
| 5.2.2 | At least six points | 53 |
| 5.2.3 | At most five points | 54 |
| 5.3 | Further examples | 55 |
| APPENDIX | TWO THEOREMS ON COMPLEX MATRIX ALGEBRAS | 57 |
| REFERENCES | | 59 |

LECTURE 1

INTRODUCING QUANTUM GROUPS

The purpose of this lecture series is to show how the combinatorics of partitions can be used to study compact quantum groups. This of course first requires to introduce the notion of compact quantum group. Since we will later on focus on applications to concrete examples, we will not need the theory of topological quantum groups in full generality, hence we will only introduce a subclass named *compact matrix quantum groups*.

We believe that there is no better way to introduce a new concept than by giving examples. We will therefore spend some time introducing one of the most important examples of compact matrix quantum groups, due to S. Wang in [Wan98] and called the *quantum permutation groups*.

1.1 THE GRAPH ISOMORPHISM GAME

There are several ways of motivating the definition of quantum permutation groups, because these objects are related to several important notions like quantum isometry groups in the sense of noncommutative geometry (see for instance [Bic03] or [Ban05]) or quantum exchangeability in the sense of free probability (see for instance [KS09]). In this lecture, we will start from a very recent connection, discovered in [LMR17], between quantum permutation groups and quantum information theory. This connection appears through a *game*, called a *graph isomorphism game* and introduced in [AMR⁺18], which we now describe.

As always in quantum information theory, the game is played by two players named *Alice* (denoted by A) and *Bob* (denoted by B). In this so-called *graph isomorphism game*, they cooperate to win against the *Referee* (denoted by R) leading the game. The rules are given by two finite graphs X and Y with vertex sets having the same cardinality, which are known to A and B . At each round of the game, R sends a vertex $v_A \in X$ to A and a vertex $v_B \in X$ to B . Each of them answers with a vertex $w_A \in Y$, $w_B \in Y$ of the other graph and they win the round if the following condition is matched :

The relation¹ between v_A and v_B is the same as the one between w_A and w_B ².

Now, the crucial point is that once the game starts, A and B **cannot communicate in any way**. The question one asks is then : under which condition on the graphs X and Y can the players devise a strategy which wins whatever the given vertices are? It is not very difficult to see that the answer is the following (see [AMR⁺18, Sec 3.1] for a proof) :

Proposition 1.1.1. *There exists a perfect strategy if and only if X and Y are isomorphic. Moreover, perfect strategies are then in one-to-one correspondence with isomorphisms between X and Y .*

This settles the problem in classical information theory, but in the quantum world, A and B can refine their strategy without communicating through the use of *entanglement*. This means

1. Here by relation we mean either being equal, being adjacent, or not being adjacent.

2. This is not the most general version of the graph isomorphism game. We refer the reader to [AMR⁺18] for a more comprehensive exposition.

that they can set up a quantum mechanical system and then split it into two parts, such that manipulating one part instantly modifies the other one. We will not go into the details, but it turns out that this gives more strategies, which are said to be *quantum*³. Using this quantum strategies, the previous proposition can be improved in the following way, where A_X denotes the adjacency matrix of the graph X (see [LMR17, Thm 4.4] for a proof) :

THEOREM 1.1.2 (Lupini-Mančinska-Roberson) There exists a perfect *quantum* strategy if and only if there exists a matrix $P = (p_{ij})_{1 \leq i, j \leq N}$ with coefficients in $\mathcal{B}(H)$ for some Hilbert space H , such that :

- p_{ij} is an orthogonal projection for all $1 \leq i, j \leq N$,
- $\sum_{k=1}^N p_{kj} = \text{Id}_H = \sum_{k=1}^N p_{ik}$ for all $1 \leq i, j \leq N$,
- $A_X P = P A_Y$.

Remark 1.1.3. From the perspective of quantum physics, this definition is very natural. Indeed, a family of orthogonal projections summing up to one is somehow the same as a *measurement system*, i.e. a complete system of observables. In this context, the projections come from the eigenspaces of the observables. We are therefore considering a collection of measurement systems with compatibility conditions coming from the graphs.

Remark 1.1.4. It is not straightforward to produce a pair of graphs for which there is a perfect quantum strategy but no classical one. The first example, given in [LMR17, Fig 1 and 2], has 24 vertices and may be the smallest possible one.

An intriguing point of Theorem 1.1.2 is the operator-valued matrices which appear in the statement. To understand them, let us consider the case $H = \mathbf{C}$. Then, the coefficients are scalars and since they are projections, they all equal either 0 or 1. Moreover, the sum on any row is 1, hence there is exactly one non-zero coefficient on each row. The same being true for the columns, we have a permutation matrix! We should therefore think of the operator-valued matrices as quantum version of permutations and this leads to the following definition :

DEFINITION 1.1.5. Let H be a Hilbert space. A *quantum permutation matrix* in H is a matrix $P = (p_{ij})_{1 \leq i, j \leq N}$ with coefficients in $\mathcal{B}(H)$ such that :

- p_{ij} is an orthogonal projection for all $1 \leq i, j \leq N$,
- $\sum_{k=1}^N p_{kj} = \text{Id}_H = \sum_{k=1}^N p_{ik}$ for all $1 \leq i, j \leq N$.

Moreover, with this point of view the last point of Theorem 1.1.2 has a nice interpretation. Indeed, if σ is a permutation and the corresponding matrix P_σ satisfies

$$A_X P_\sigma = P_\sigma A_Y,$$

then this means that the bijection induced by σ between the vertices of X and those of Y respects the edges. In other words, it is a graph isomorphism. Therefore, if the conditions of Theorem 1.1.2 are matched, one says that the graphs are *quantum isomorphic*.

1.2 THE QUANTUM PERMUTATION ALGEBRA

The brief discussion of Section 1.1 suggests that quantum permutation matrices are interesting objects which require further study. However, their definition lacks several important features of classical permutation matrices. In particular, there is no obvious way to "compose" quantum permutation matrices, especially if they do not act on the same Hilbert space, so that one

3. The concept of quantum strategy turns out to be quite subtle, depending on the type of operators allowed. We here use the term in a purposely vague sense.

could recover an analogue of the group structure of permutations. To overcome this problem, it is quite natural from an (operator) algebraic point of view to introduce a universal object associated to quantum permutation matrices.

DEFINITION 1.2.1. Let $\mathcal{A}_s(N)$ be the universal $*$ -algebra generated by N^2 elements $(p_{ij})_{1 \leq i, j \leq N}$ such that

1. $p_{ij}^2 = p_{ij} = p_{ij}^*$,
2. For all $1 \leq i, j \leq N$, $\sum_{k=1}^N p_{ik} = 1 = \sum_{k=1}^N p_{kj}$,
3. For all $1 \leq i, j, k, \ell \leq N$, $p_{ij}p_{ik} = \delta_{jk}p_{ij}$ and $p_{ij}p_{\ell j} = \delta_{i\ell}p_{ij}$.

This will be called the *quantum permutation algebra* on N points.

Remark 1.2.2. The third condition in the definition may seem strange since it is automatically satisfied for projections in a Hilbert space. However, a $*$ -algebra may not have a faithful representation on a Hilbert space, hence Condition (3) does not directly follow from the two other ones.

We now have a nice object to study, but the link to the classical permutation group is somewhat blurred. To clear it, let us consider the function $c_{ij} : S_N \rightarrow \mathbf{C}$ defined by

$$c_{ij}(\sigma) = \delta_{j\sigma(i)}.$$

This is nothing but the function sending the permutation matrix of σ to its (i, j) -th coefficient. In particular, c_{ij} always takes the value 0 or 1, hence $c_{ij}^* = c_{ij} = c_{ij}^2$. Similarly, it is straightforward to check that Conditions (2) and (3) are satisfied. Hence, by universality, there is a $*$ -homomorphism

$$\pi_{\text{ab}} : \begin{cases} \mathcal{A}_s(N) & \rightarrow \mathcal{F}(S_N) \\ p_{ij} & \mapsto c_{ij} \end{cases}$$

where $\mathcal{F}(S_N)$ is the algebra of all functions from S_N to \mathbf{C} . Moreover, since the functions c_{ij} obviously generate the whole algebra $\mathcal{F}(S_N)$, π_{ab} is onto.

We will now use this link to investigate a possible "group-like" structure on $\mathcal{A}_s(N)$. At the level of the coefficient functions, the group product satisfies the following equation :

$$c_{ij}(\sigma_1\sigma_2) = \sum_{k=1}^N c_{ik}(\sigma_1)c_{kj}(\sigma_2) = \left(\sum_{k=1}^N c_{ik} \otimes c_{kj} \right) (\sigma_1, \sigma_2).$$

Here, we have used the fact that the injective map

$$\iota : \begin{cases} \mathcal{F}(S_N) \otimes \mathcal{F}(S_N) & \rightarrow \mathcal{F}(S_N \times S_N) \\ f \otimes g & \mapsto ((\sigma_1, \sigma_2) \mapsto f(\sigma_1)g(\sigma_2)) \end{cases}$$

is onto for dimension reasons. Considering the elements p_{ij} as "coefficient functions", this suggests to encode a kind of "group law" through the map

$$\Delta : p_{ij} \rightarrow \sum_{k=1}^N p_{ik} \otimes p_{kj}. \tag{1.1}$$

Proposition 1.2.3. *There exists a unique $*$ -homomorphism $\Delta : \mathcal{A}_s(N) \rightarrow \mathcal{A}_s(N) \otimes \mathcal{A}_s(N)$ satisfying the formula (1.1).*

Proof. Let us set, for $1 \leq i, j \leq N$,

$$q_{ij} = \sum_{k=1}^N p_{ik} \otimes p_{kj}.$$

We claim that the q_{ij} 's satisfy Conditions (1) to (3) of Definition 1.2.1. The existence of Δ then follows from the universal property. ■

Exercise 1. Prove the claim in the preceding proof.

Solution. It is clear that $q_{ij}^* = q_{ij}$. Let us now compute the square :

$$\begin{aligned} q_{ij}^2 &= \sum_{k,\ell=1}^N p_{ik} p_{i\ell} \otimes p_{kj} p_{\ell j} \\ &= \sum_{k,\ell=1}^N \delta_{k\ell} p_{ik} \otimes p_{kj} \\ &= q_{ij}. \end{aligned}$$

We have therefore checked Condition (1). Moreover,

$$\begin{aligned} \sum_{i=1}^N q_{ij} &= \sum_{k,i=1}^N p_{ik} \otimes p_{kj} \\ &= \sum_{k=1}^N \left(\sum_{i=1}^N p_{ik} \right) \otimes p_{kj} \\ &= \sum_{k=1}^N 1 \otimes p_{kj} \\ &= 1 \otimes 1 \end{aligned}$$

hence Condition (2) is also satisfied. Eventually, for $j \neq j'$,

$$q_{ij} q_{ij'} = \sum_{k,\ell=1}^N p_{ik} p_{i\ell} \otimes p_{kj} p_{\ell j'}$$

The first tensor in the sum vanishes unless $k = \ell$, but in that case the second one vanishes and Condition (3) follows. ■

The map Δ is called the *coproduct* and is a reasonable substitute for matrix multiplication (i.e. the group law of a matrix group). In particular, it satisfies an analogue of the associativity property of the group law, called *coassociativity* :

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta.$$

Exercise 2. Prove that the coproduct on $\mathcal{A}_s(N)$ is indeed coassociative. Check also that the corresponding equation on the coefficient functions in S_N is equivalent to associativity of the composition of permutations.

Solution. Because Δ is a $*$ -algebra homomorphism, it is enough to check coassociativity on the generators, and this is straightforward :

$$\begin{aligned} (\Delta \otimes \text{id}) \circ \Delta(u_{ij}) &= \sum_{k=1}^N \Delta(u_{ik}) \otimes u_{kj} \\ &= \sum_{k,\ell=1}^N u_{i\ell} \otimes u_{\ell k} \otimes u_{kj} \\ &= \sum_{\ell=1}^N u_{i\ell} \otimes \Delta(u_{\ell j}) \\ &= (\text{id} \otimes \Delta) \circ \Delta(u_{ij}). \end{aligned}$$

As for the second assertions, we have already seen that $\Delta(c_{ij})(g, h) = c_{ij}(g, h)$. Thus,

$$(\Delta \otimes \text{id}) \circ \Delta(c_{ij})(\sigma_1, \sigma_2, \sigma_3) = \Delta(c_{ij})(\sigma_1 \sigma_2, \sigma_3) = c_{ij}((\sigma_1 \sigma_2) \sigma_3)$$

while

$$(\text{id} \otimes \Delta) \circ \Delta(c_{ij})(\sigma_1, \sigma_2, \sigma_3) = \Delta(c_{ij})(\sigma_1, \sigma_2 \sigma_3) = c_{ij}(\sigma_1(\sigma_2 \sigma_3))$$

so that coassociativity is equivalent to $f((\sigma_1 \sigma_2) \sigma_3) = f(\sigma_1(\sigma_2 \sigma_3))$ for all $f \in \mathcal{F}(S_N)$ and $\sigma_1, \sigma_2, \sigma_3 \in S_N$, which is in turn equivalent to the associativity of the group law. ■

The coproduct certainly indicates that we are on the right track to product a "group-like" structure on the quantum permutation algebra. However, we still need a neutral element and an inverse but instead of trying to translate each of them, we will take advantage of the fact that we are considering a matrix group. Indeed, for any permutation, the corresponding matrix is orthogonal, so that for any permutation σ ,

$$\sum_{k=1}^N c_{ik}(\sigma) c_{jk}(\sigma) = \delta_{ij} = \sum_{k=1}^N c_{ki}(\sigma) c_{kj}(\sigma). \quad (1.2)$$

Since this holds for any σ , it can be written as an equality of functions in $\mathcal{F}(S_N)$ and it turns out that the same equality holds in $\mathcal{A}_s(N)$:

Proposition 1.2.4. *For any $1 \leq i, j \leq N$,*

$$\sum_{k=1}^N p_{ik} p_{jk} = \delta_{ij} = \sum_{k=1}^N p_{ki} p_{kj}. \quad (1.3)$$

Proof. This is a direct consequence of Conditions (1) to (3). ■

This means that the quantum permutation algebra is somehow "made of orthogonal quantum matrices" and this property should contain all information about the unit and the inverse. As a conclusion, the algebra $\mathcal{A}_s(N)$ with its generators $(u_{ij})_{1 \leq i, j \leq N}$ seems to have all the properties one can expect for a "group-like" object. It therefore deserves the name of *quantum group* that we will define in the next section.

Remark 1.2.5. The fact that Condition (1.3) yields a full "group-like" structure can be encoded in the two following maps :

- The *antipode* $S : \mathcal{A}_s(N) \rightarrow \mathcal{A}_s(N)$, which is a $*$ -antihomomorphism induced by

$$p_{ij} \mapsto p_{ji}.$$

Since the transpose of P is its inverse, this plays the rôle of the inverse map.

- The *counit* $\varepsilon : \mathcal{A}_s(N) \rightarrow \mathbf{C}$, which is a $*$ -homomorphism induced by

$$p_{ij} \mapsto \delta_{ij}.$$

Since the matrix $(\delta_{ij})_{1 \leq i, j \leq N}$ is the identity, this play the rôle of the neutral element.

Equation (1.3) then becomes

$$m \circ (\text{id} \otimes S) = \varepsilon = m \circ (S \otimes \text{id}),$$

where $m : \mathcal{A}_s(N) \otimes \mathcal{A}_s(N) \rightarrow \mathcal{A}_s(N)$ is the multiplication map. Our focus in these lectures is on the matricial aspect of quantum groups, and we will therefore never use these maps. Note however that $(\mathcal{A}_s(N), \Delta, \varepsilon, S)$ is a *Hopf algebra*.

1.3 COMPACT MATRIX QUANTUM GROUPS

Our study of the quantum permutation algebras has given us enough motivation to introduce a notion of compact quantum group. There is a nice and complete theory of these objects, which was developed by S.L. Woronowicz in [Wor98]. There are to our knowledge two books explaining this theory in detail, [Tim08] and [NT13] to which the reader may refer for alternative and more comprehensive expositions.

1.3.1 A FIRST DEFINITION

The purpose of these lecture is to give some examples of the interaction between the combinatorics of partitions and the theory of compact quantum groups. The most striking examples involve compact quantum groups which belong to a specific class which is, in a sense, simpler to define and handle. It was introduced by S.L. Woronowicz in [Wor87] as a generalization of compact groups of matrices and as a first attempt to a general definition of compact quantum groups. We will therefore focus on this class for the moment, even though our definition differs from [Wor87, Def 1.1] and is closer to [Wan95, Def 2.1'].

DEFINITION 1.3.1. An *orthogonal compact matrix quantum group* of size N is given by a $*$ -algebra \mathcal{A} generated by N^2 self-adjoint elements $(u_{ij})_{1 \leq i, j \leq N}$ such that

1. There exist a $*$ -homomorphism $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ such that for all $1 \leq i, j \leq N$,

$$\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj},$$

2. For all $1 \leq i, j \leq N$,

$$\sum_{k=1}^N u_{ik} u_{jk} = \delta_{ij} = \sum_{k=1}^N u_{ki} u_{kj}.$$

Denoting by $u \in M_N(\mathcal{A})$ the matrix with coefficients $(u_{ij})_{1 \leq i, j \leq N}$, we will denote the orthogonal compact matrix quantum group by (\mathcal{A}, u) .

Remark 1.3.2. Let us briefly explain how this relates to [Wor87, Def 1.1]. First note that since the generators have norm less than one because of Condition (2), \mathcal{A} has a universal enveloping C^* -algebra $i : \mathcal{A} \rightarrow A$. By universality, the map

$$(i \otimes i) \circ \Delta : \mathcal{A} \rightarrow i(\mathcal{A}) \otimes i(\mathcal{A}) \subset A \otimes A$$

extends to a map $\tilde{\Delta} : A \rightarrow A \otimes A$ ⁴. The coassociativity is easily checked on the elements $v_{ij} = i(u_{ij})$ and therefore extends to the closure of the algebra that they generate, which is A . Only the properties linked to the antipode remain to be proven, but they follow from the fact that u is orthogonal, hence its transpose is invertible. Indeed, setting $S(v_{ij}) = v_{ji}$ and extending this map to $i(\mathcal{A})$ by antimultiplicativity, all the properties of [Wor87, Def 1.1] are satisfied.

As the notation "i" suggests, it turns out that \mathcal{A} always embeds in its universal enveloping C^* -algebra. This is however a quite non-trivial fact which will not be needed in these lectures and will be proven in Lecture 4, as a Corollary of the existence of the Haar state (see Corollary 4.1.6).

By analogy with our reasoning on S_N , \mathcal{A} will be thought of as the algebra of functions on a non-existent "quantum space". However, if we consider general "compact quantum spaces", we cannot use all the functions like for S_N . Since we work in an algebraic setting, the correct analogy is the algebra of *regular functions*, that is to say functions which are polynomial in the matrix coefficients. The usual notation for this is $\mathcal{O}(G)$, whence the notation $\mathcal{A} = \mathcal{O}(G)$ if G denotes the orthogonal compact matrix quantum group. We can now formalize the properties of the quantum permutation algebras established in Section 1.2 :

DEFINITION 1.3.3. For any integer N , the pair $(\mathcal{A}_s(N), P)$ is an orthogonal compact matrix quantum group, where $P = (p_{ij})_{1 \leq i, j \leq N}$. It is called the *quantum permutation group* on N points and is usually referred to using the notation S_N^+ .

Consequently, we may from now on write $\mathcal{O}(S_N^+)$ instead of $\mathcal{A}_s(N)$. This quantum group was first defined by S. Wang in [Wan98]. It is natural (and crucial for applications to quantum information theory) to wonder whether this is really different from S_N .

Exercise 3. Prove that for $N = 1, 2, 3$, $S_N^+ = S_N$. Prove that for any $N \geq 4$, $\mathcal{O}(S_N^+)$ is noncommutative, hence not isomorphic to S_N .

4. Note that we can take any C^* -algebra tensor product here.

Solution. For $N = 1$, $\mathcal{A}_s(1)$ is generated by one self-adjoint projection, hence is isomorphic to $\mathbf{C} = \mathcal{O}(S_1)$. For $N = 2$, observe that the relations force

$$P = \begin{pmatrix} p_{11} & 1-p_{11} \\ 1-p_{11} & p_{11} \end{pmatrix}$$

making $\mathcal{A}_s(2)$ abelian, hence equal to $\mathcal{O}(S_2)$.

For $N = 3$, we give here a simple argument from [LMR17]. It is enough to prove that p_{11} commutes with p_{22} since any independent permutation of the rows and columns of P yields an automorphism of $\mathcal{A}_s(N)$ by the universal property. We start by observing that

$$u_{11}u_{22} = u_{11}u_{22}(u_{11} + u_{12} + u_{13}) = u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13}.$$

But

$$u_{11}u_{22}u_{13} = u_{11}(1 - u_{21} - u_{23})u_{13} = u_{11}u_{13} - u_{11}u_{21}u_{13} - u_{11}u_{23}u_{13} = 0,$$

hence

$$u_{11}u_{22} = u_{11}u_{22}u_{11} = (u_{11}u_{22}u_{11})^* = u_{22}u_{11}.$$

For $N \geq 4$, consider a Hilbert space H and two orthogonal projections $q_1, q_2 \in \mathcal{B}(H)$ which do not commute. Then, consider the matrix

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

and complete it to an $N \times N$ matrix by putting it in the upper left corner, setting the other diagonal coefficients to 1 and all the other coefficients to 0. This yields a quantum permutation matrix, hence a $*$ -homomorphism $\pi : \mathcal{O}(S_N^+) \rightarrow \mathcal{B}(H)$. Because $\pi(u_{11}) = p$ and $\pi(u_{33}) = q$ do not commute, we infer that $\mathcal{O}(S_N^+)$ is not commutative. ■

To get a better understanding of Definition 1.3.1, it is worth working out the link with the classical case.

Exercise 4. Let (\mathcal{A}, u) is an orthogonal compact matrix quantum group such that \mathcal{A} is commutative. Prove that there exists a compact subgroup G of O_N and an isomorphism

$$\mathcal{A} \simeq \mathcal{O}(G)$$

sending u_{ij} to c_{ij} .

Hint : $\mathcal{O}(O_N)$ is the universal $*$ -algebra generated by elements $(c_{ij})_{1 \leq i, j \leq N}$ which pairwise commute and form an orthogonal matrix⁵.

Solution. By universality, there is a surjective $*$ -homomorphism

$$\pi : \mathcal{O}(O_N) \rightarrow \mathcal{A}$$

sending c_{ij} to u_{ij} . Set now $I = \ker(\pi)$, which is a polynomial ideal, and let

$$G = \{M \in O(N) \mid p(M) = 0 \text{ for all } p \in I\}.$$

This is a closed, hence compact, subset of O_N and $\mathcal{A} \simeq \mathcal{O}(G)$, so that it only remains to prove that G is a subgroup. If $p \in I$, let us write

$$\Delta(p) = \sum_i p_i \otimes q_i \in \mathcal{O}(O_N) \otimes \mathcal{O}(O_N)$$

5. This is straightforward once one notices that the algebra of regular functions on the algebraic manifold $M_N(\mathbf{C})$ is just $\mathbf{C}[X_{ij} \mid 1 \leq i, j \leq N]$.

with linearly independent tensors. Because

$$(\pi \otimes \pi) \circ \Delta(p) = \Delta \circ \pi(p) = 0,$$

for all i , either p_i or q_i belongs to I . Thus, for any $M_1, M_2 \in G$,

$$p(M_1 M_2) = \Delta(p)(M_1, M_2) = \sum_i p_i(M_1) q_i(M_2) = 0$$

and $M_1 M_2 \in G$. One can conclude with a sledgehammer argument : G is a group because it is a compact bisimplifiable semigroup. It is also possible to exploit the idea used for the coproduct a little more. Indeed, if S is the map induced by $c_{ij} \rightarrow c_{ji}$, then

$$\pi \circ S(p) = S \circ \pi(p) = 0$$

so that $p(M^{-1}) = p(M^t) = S(p)(M) = 0$ for any $M \in G$, hence $M^{-1} \in G$. Eventually, if ε is the map induced by $c_{ij} \mapsto \delta_{ij}$, one has $\varepsilon \circ \pi = \varepsilon$ so that $I \subset \ker(\varepsilon)$. As a consequence,

$$p(\text{Id}_N) = \varepsilon(p) = 0$$

for all $p \in I$, yielding $\text{Id}_N \in G$. ■

1.3.2 THE QUANTUM ORTHOGONAL GROUP

Before delving into the general theory of compact quantum groups, let us give another fundamental example which is also due to S. Wang but earlier in [Wan95]. After a look at Definition 1.3.1, it is natural to wonder about the "largest" possible orthogonal compact matrix quantum group. Its definition relies on the following simple fact :

Exercise 5. Let N be an integer and let $\mathcal{A}_o(N)$ be the universal $*$ -algebra generated by N^2 self-adjoint elements $(U_{ij})_{1 \leq i, j \leq N}$ such that

$$\sum_{k=1}^N U_{ik} U_{jk} = \delta_{ij} = U_{ki} U_{kj}.$$

Then, there exists a $*$ -homomorphism $\Delta : \mathcal{A}_o(N) \rightarrow \mathcal{A}_o(N) \otimes \mathcal{A}_o(N)$ such that for all $1 \leq i, j \leq N$,

$$\Delta(U_{ij}) = \sum_{k=1}^N U_{ik} \otimes U_{kj}.$$

Proof. The proof is similar to that of Proposition 1.2.3. We set

$$V_{ij} = \sum_{k=1}^N U_{ik} \otimes U_{kj}$$

and have to check that the corresponding matrix V is orthogonal. Indeed,

$$\begin{aligned} \sum_{k=1}^N V_{ik} V_{jk} &= \sum_{k, \ell, m=1}^N U_{i\ell} U_{jm} \otimes U_{\ell k} U_{mk} \\ &= \sum_{\ell, m=1}^N U_{i\ell} U_{jm} \otimes \left(\sum_{k=1}^N U_{\ell k} U_{mk} \right) \\ &= \sum_{\ell, m=1}^N U_{i\ell} U_{jm} \otimes \delta_{\ell m} \\ &= \sum_{\ell=1}^N U_{i\ell} U_{j\ell} \otimes 1 \\ &= \delta_{ij} 1 \otimes 1. \end{aligned}$$

The other equality is proved similarly and it then follows from universality that there exists a $*$ -homomorphism sending U_{ij} to V_{ij} . ■

This motivates the following definition :

DEFINITION 1.3.4. The pair $(\mathcal{A}_o(N), U)$ is an orthogonal compact matrix quantum group called the *quantum orthogonal group*. It is usually referred to using the notation O_N^+ .

As the name suggest, O_N^+ is linked to orthogonal groups. Indeed, if $c_{ij} : O_N \rightarrow \mathbf{C}$ are the matrix coefficient functions, then there is a surjective $*$ -homomorphism

$$\pi_{\text{ab}} : \mathcal{O}(O_N^+) \rightarrow \mathcal{O}(O_N)$$

sending U_{ij} to c_{ij} . Thus, O_N^+ is a "quantum version" of O_N just as S_N^+ is a "quantum version" of S_N .

Note that it follows from this discussion that there is a surjective $*$ -homomorphism $\mathcal{O}(O_N^+) \rightarrow \mathcal{O}(S_N^+)$, so that $\mathcal{O}(O_N^+)$ is not commutative as soon as $N \geq 2$. However, more is true in that case.

Proposition 1.3.5. *The $*$ -algebra $\mathcal{O}(O_N^+)$ is non-commutative as soon as $N \geq 2$.*

Proof. Let \mathbf{Z}_2 be the cyclic group of order 2, and consider the free product \mathbf{Z}_2^{*N} with canonical generators a_1, \dots, a_N . Seeing these elements as unitaries in $C^*(\mathbf{Z}_2^{*N})$, the diagonal matrix with coefficients a_1, \dots, a_N satisfies the relations of Definition 1.3.1. Hence, there is a surjective $*$ -homomorphism $\mathcal{O}(O_N^+) \rightarrow C^*(\mathbf{Z}_2^{*N})$ sending U_{ii} to a_i and U_{ij} to 0 for $i \neq j$. This proves in particular that U_{11} and U_{22} do not commute. ■

1.3.3 THE UNITARY CASE

Exercise 4 illustrates the fact that orthogonal compact matrix quantum groups generalize subgroups of O_N . This can be made rigorous in the following way : by universality, for any orthogonal compact matrix quantum group $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$, there is a surjective $*$ -homomorphism

$$\pi : \mathcal{O}(O_N^+) \rightarrow \mathcal{O}(\mathbb{G})$$

sending U_{ij} to u_{ij} and therefore satisfying

$$\Delta \circ \pi(x) = (\pi \otimes \pi) \circ \Delta(x)$$

for all $x \in \mathcal{O}(\mathbb{G})$. Thus, orthogonal compact matrix quantum groups are "quantum subgroups" of O_N^+ .

One may wonder whether it is possible to consider analogues of closed subgroups of the unitary group U_N instead of the orthogonal one. This is possible, but we will not need it in this lecture series. Moreover, this more general setting involves subtleties which make some arguments tricky. This can already be seen on the following definition :

DEFINITION 1.3.6. A *unitary compact matrix quantum group* of size N is given by a $*$ -algebra \mathcal{A} generated by N^2 elements $(u_{ij})_{1 \leq i, j \leq N}$ such that

1. There exist a $*$ -homomorphism $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ such that for all $1 \leq i, j \leq N$,

$$\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj},$$

2. For all $1 \leq i, j \leq N$,

$$\sum_{k=1}^N u_{ik} u_{jk}^* = \delta_{ij} = \sum_{k=1}^N u_{ki}^* u_{kj}$$

and

$$\sum_{k=1}^N u_{ki} u_{kj}^* = \delta_{ij} = \sum_{k=1}^N u_{ik}^* u_{jk}$$

hold.

Remark 1.3.7. The relations in the previous definition mean that both the matrix u and its conjugate \bar{u} (the matrix where each coefficient is replaced with its adjoint) are unitary. The second one does not follow from the first one in general (see [Wan95, Sec 4.1] for a counter-example), so that both need to be included in the definition.

Even though we will focus on orthogonal compact matrix quantum groups in these lectures, and therefore may use the orthogonality assumption whenever it simplifies things, some statements are proved likewise for general unitary compact matrix quantum groups. In that case, we will give the general statement and proof.

1.4 REPRESENTATION THEORY

Now that we have a definition of a quantum group, it is time to investigate their general structure. It turns out that compact groups are mainly tractable because they have a very nice representation theory. It is therefore natural to start by looking for a suitable notion of representation for compact quantum groups.

1.4.1 FINITE-DIMENSIONAL REPRESENTATIONS

Following our now usual strategy, we will restate the notion of representation in terms of functions. Recall that for a group G , a continuous representation on a vector space V is a continuous group homomorphism

$$\rho : G \rightarrow \mathcal{L}(V).$$

Assume for instance that V is finite-dimensional so that we can identify $\mathcal{L}(V)$ with $M_n(\mathbf{C})$ for some n . Composing ρ with the coefficient functions produces new functions $(\rho_{ij})_{1 \leq i, j \leq n} \in \mathcal{O}(G)$ ⁶. The fact that ρ is a representation translates into two properties :

- The matrix $[\rho_{ij}(g)]$ is invertible for all $g \in G$,
- For any $1 \leq i, j \leq n$, $\rho_{ij}(gh) = \sum_{k=1}^n \rho_{ik}(g)\rho_{kj}(h)$.

The second point is reminiscent of the discussion around the definition of the coproduct and we therefore know how to translate it. As for the first one, it means that ρ is an invertible element in the algebra of polynomial functions from G to $M_n(\mathbf{C})$, which is isomorphic to the algebra of $n \times n$ matrices with coefficients being functions on G . As a conclusion, we may give the following definition :

DEFINITION 1.4.1. Let $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$ be an orthogonal compact matrix quantum group and let n be an integer. A continuous n -dimensional representation of \mathbb{G} is an element $v \in M_n(\mathcal{O}(\mathbb{G}))$ such that

- v is invertible,
- $\Delta(v_{ij}) = \sum_{k=1}^n v_{ik} \otimes v_{kj}$.

If moreover v is unitary, then it is said to be a *unitary representation*.

Example 1.4.2. The first, extremely important, example is u , which is a unitary representation. Since it defines the quantum group, it ought to determine all the representations. This idea will become clearer in Lecture 2, but because of this peculiar rôle, u is called the *fundamental representation* of \mathbb{G} .

Example 1.4.3. The second important example is the element

$$\varepsilon = 1 \in M_1(\mathcal{O}(\mathbb{G})) = \mathcal{O}(\mathbb{G}),$$

which is also a representation, called the *trivial representation*.

6. We are using here a result from the classical theory of representations of compact groups : if ρ is a continuous finite-dimensional representation, then its coefficients belong to $\mathcal{O}(G)$.

The standard operations on representations generalize to this setting and will be crucial for the sequel. For instance, if v and w are two finite-dimensional representations of dimension n and m respectively, then

$$v \otimes w = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in M_{n+m}(\mathcal{O}(\mathbb{G}))$$

is their *direct sum*, while

$$v \otimes w = (v_{ij}w_{k\ell})_{1 \leq i,j \leq n, 1 \leq k,\ell \leq m} \in M_n(\mathcal{O}(\mathbb{G})) \otimes M_m(\mathcal{O}(\mathbb{G})) \simeq M_{nm}(\mathcal{O}(\mathbb{G}))$$

is their *tensor product*.

1.4.2 INTERTWINERS

The heart of representation theory is understanding the link between various representations, which given by intertwiners. In the following definitions, we see scalar matrices as matrices with coefficients in $\mathcal{O}(\mathbb{G})$ through the embedding $\mathbf{C}.1 \subset \mathcal{O}(\mathbb{G})$.

- An *intertwiner* between v and w is a linear map $T : M_n(\mathbf{C}) \rightarrow M_k(\mathbf{C})$ such that

$$Tv = wT,$$

- The representations v and w are said to be *equivalent* if there exists an invertible intertwiner between them. If this intertwiner is moreover unitary, then they are said to be *unitarily equivalent*.
- The representation w is said to be a *subrepresentation* of v if there exists an isometric intertwiner between v and w .
- A representation is said to be *irreducible* if it has no subrepresentation except for 0 and itself.

To get more understanding of these notions, we can give an alternate picture. Let v be a representation of dimension n and let $(e_i)_{1 \leq i \leq n}$ be the canonical orthonormal basis of \mathbf{C}^n . Then, we can define a linear map $\rho_v : \mathbf{C}^n \rightarrow \mathcal{O}(\mathbb{G}) \otimes \mathbf{C}^n$ through the formula

$$\rho_v(e_i) = \sum_{k=1}^n v_{ik} \otimes e_k.$$

As one may expect, the previous definition can be easily translated :

Exercise 6. Prove that T intertwines v and w if and only if

$$(\text{id} \otimes T) \circ \rho_v = \rho_w \circ (\text{id} \otimes T).$$

Solution. The fact that T is an intertwiner is equivalent to the fact that for any $1 \leq i \leq \dim(w)$ and $1 \leq j \leq \dim(v)$,

$$\sum_{k=1}^n T_{ik} v_{kj} = \sum_{\ell=1}^m w_{i\ell} T_{\ell j}.$$

Tensoring with e_i and summing yields

$$(\text{id} \otimes T) \circ \rho_v(e_j) = \rho_w \circ (\text{id} \otimes T)(e_j),$$

hence the result. ■

In this picture, the direct sum translates into

$$\rho_{v \oplus w}(e_i) = \sum_{k=1}^n v_{ik} \otimes e_k \quad \text{and} \quad \rho_{v \oplus w}(f_j) = \sum_{\ell=1}^n w_{j\ell} \otimes e_\ell$$

while the tensor product becomes

$$\rho_{v \otimes w}(e_i \otimes f_j) = \sum_{k=1}^n \sum_{\ell=1}^m u_{ik} u_{j\ell} \otimes e_k \otimes f_\ell.$$

More importantly, a subspace $W \subset V$ is said to be *invariant* if

$$\rho_v(W) \subset \mathcal{O}(\mathbb{G}) \otimes W$$

and taking an orthonormal basis of W we can then build a subrepresentation w of v . Let us illustrate this with some simple examples involving the two compact quantum groups defined above.

Example 1.4.4. Consider the fundamental representation P of S_N^+ , let $(e_i)_{1 \leq i \leq N}$ be the canonical orthonormal basis of \mathbf{C}^N and set

$$\xi = \sum_{i=1}^N e_i.$$

It is a straightforward consequence of Condition (2) that $\rho_P(\xi) = 1 \otimes \xi$. In other words, ρ_P has a fixed vector, which is equivalent to P having a trivial subrepresentation.

This phenomenon is analogous to a well-known fact for permutation groups : the permutation representation ρ of S_N on \mathbf{C}^N decomposes as the direct sum of the trivial representation and an irreducible representation⁷. Let us show that the same holds for S_N^+ by exploiting the idea that S_N is a "subgroup" of S_N^+ .

Example 1.4.5. We set $V = \xi^\perp \subset \mathbf{C}^N$, which is invariant under ρ_P by unitarity, and consider a subspace $W \subset V$ which is invariant under ρ_P . Letting

$$\tilde{\rho} : V \rightarrow \mathcal{F}(S_N) \otimes V \simeq \mathcal{F}(S_N, V)$$

be the map sending x to $g \mapsto \rho(g)(x)$. We have the equality

$$(\pi_{\text{ab}} \otimes \text{id}_V) \circ \rho_P = \tilde{\rho},$$

from which it follows that W is invariant under $\tilde{\rho}$. Hence, $W = \{0\}$ or $W = V$ and V is irreducible for P .

We can use the same strategy for the fundamental representation of O_N^+ :

Exercise 7. Let $N \geq 2$ be an integer and consider the fundamental representation U of O_N^+ .

1. Show that U is irreducible.
2. Let $(e_i)_{1 \leq i \leq N}$ be the canonical basis of \mathbf{C}^N . Show that the vector

$$\xi = \sum_{i=1}^N e_i \otimes e_i$$

is fixed for $\rho_{U \otimes U}$.

Solution. The strategy is the same as for S_N^+ , so let us denote by ρ the defining representation of O_N on $V = \mathbf{C}^N$ and by $\tilde{\rho}$ the map $x \mapsto (g \mapsto \rho(g)(x))$.

1. Because $(\pi_{\text{ab}} \otimes \text{id}) \circ \rho_U = \tilde{\rho}$ and the right-hand side is irreducible, we infer that ρ_U , hence U is irreducible.

⁷ Simply notice that ξ^\perp contains the vector $e_1 - e_2$ and that letting S_N act on it one can obtain $e_i - e_j$ for any $i \neq j$, which generate ξ^\perp .

2. We compute

$$\begin{aligned}
\rho_{U \otimes U}(\xi) &= \sum_{i=1}^N \sum_{k,\ell=1}^N U_{ik} U_{i\ell} \otimes e_k \otimes e_\ell \\
&= \sum_{k,\ell=1}^N \left(\sum_{i=1}^N U_{ik} U_{i\ell} \right) \otimes e_k \otimes e_\ell \\
&= \sum_{k,\ell=1}^N \delta_{k\ell} \otimes e_k \otimes e_\ell \\
&= \mathbf{1} \otimes \xi.
\end{aligned}$$

■

As one sees from these examples, pushing further the study by considering higher tensor powers of u or P , one will have to deal with the full representation theory of S_N and O_N . We will see in Lecture 2 that there is another way of investigating the representation theory of these quantum groups which completely avoids the use of classical groups. This is fortunate because the representation theory of O_N^+ , for instance, turns out to be much simpler than that of O_N .

1.4.3 STRUCTURE OF THE REPRESENTATION THEORY

Observe that $\mathcal{O}(\mathbb{G})$ is spanned by products of coefficients of u , which are nothing but the coefficients of tensor powers of u . In other words, the whole orthogonal compact matrix quantum group can be recovered from its finite-dimensional representations. The main theorem of this section, which is fundamental, turns this observation into a tractable tool for the study of orthogonal compact matrix quantum groups. Before getting to this, let us warm up by observing that the celebrated Schur Lemma still holds in our setting.

Lemma 1.4.6 (Schur's Lemma). *Let \mathbb{G} be a unitary compact matrix quantum group and let v and w be irreducible representations of dimension n and m respectively. Then, if T is an intertwiner between v and w , then either $T = 0$ or T is invertible (hence v is equivalent to w). Moreover, in the latter case T is a scalar multiple of the identity.*

Proof. Consider the subspace $Z = \ker(T) \subset \mathbf{C}^n$. Then, for any $x \in Z$,

$$(\text{id} \otimes T) \circ \rho_v(x) = \rho_w \circ T(x) = 0$$

so that $\rho_v(Z) \subset \ker(\text{id} \otimes T) = \mathcal{O}(\mathbb{G}) \otimes Z$ ⁸. Hence, Z is stable, meaning that the restriction of v to it is a subrepresentation. By irreducibility, we conclude that either $Z = \mathbf{C}^n$, in which case $T = 0$, or $Z = \{0\}$. In the second case, applying the same reasoning to T^* shows that $\ker(T^*) = \{0\}$, implying that T is also surjective. Eventually, if T is invertible, let λ be one of its eigenvalues. Then, $T - \lambda \cdot \text{id}$ is an intertwiner between v and w and is not invertible, thus $T = \lambda \cdot \text{id}$ by the first part of the statement. ■

We can now turn to the statement and proof of the main theorem on the structure theory of unitary compact matrix quantum groups. It was first proven (in a slightly different version) by S.L. Woronowicz in [Wor87, Prop 4.6 and Lem 4.8] and is the cornerstone of the study of compact quantum groups, hence of the sequel of these lectures.

THEOREM 1.4.7 (Woronowicz) Let $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$ be a unitary compact matrix quantum group. Then,

1. Any finite-dimensional unitary representation splits as a direct sum of irreducible unitary representations,
2. Coefficients of inequivalent irreducible finite-dimensional representations are linearly

⁸ This can be proven using the fact that the tensor product of two algebraic bases is an algebraic basis of the tensor product.

independent,

3. Any finite-dimensional representation is equivalent to a unitary one.

Proof. 1. Let v be a finite-dimensional unitary representation of dimension n and assume that it is not irreducible. It therefore has an invariant subspace $W \subset \mathbf{C}^n$. Picking an orthonormal basis of W and completing it into an orthonormal basis of V , we get a unitary matrix $B \in M_n(\mathbf{C})$ such that $w = B^* v B$ is block upper triangular, i.e. has the form

$$w = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

Since it is unitary, $A_{12} = 0$ so that w is a direct sum of unitary sub-representations. The result therefore follows by induction.

2. Let $\{v^{(1)}, \dots, v^{(n)}\}$ be pairwise inequivalent irreducible representations with $v^{(i)}$ acting on a finite-dimensional vector spaces $V^{(i)}$. For a linear form $f \in \mathcal{O}(\mathbb{G})^*$, we denote by $\widehat{f}(v)$ the matrix with coefficients $(f(v_{ij}))_{1 \leq i, j \leq \dim(v)}$ and we set

$$B = \left\{ \sum_{i=1}^n \widehat{f}(v^{(i)}) \mid f \in \mathcal{O}(\mathbb{G})^* \right\} \subset \bigoplus_{i=1}^n \mathcal{L}(V^{(i)}) = \mathcal{B}.$$

We claim that this inclusion is an equality. The proof goes through several steps :

- (a) By definition B is a vector space. Moreover, setting $f * g = (f \otimes g) \circ \Delta \in \mathcal{O}(\mathbb{G})$, we have

$$(\widehat{f}(v)\widehat{g}(v))_{ij} = \sum_{k=1}^{\dim(v)} f(v_{ik})g(v_{kj}) = (\widehat{(f * g)}(v))_{ij}$$

so that B is an algebra.

- (b) Let p_i be the minimal central projection in \mathcal{B} corresponding to the i -th summand and consider

$$\pi_i : x \in B \mapsto p_i x p_i \in \mathcal{L}(V^{(i)}).$$

This is an algebra representation which is irreducible by irreducibility of $v^{(i)}$. Thus, by Burnside's Theorem (see Theorem A in the Appendix),

$$\pi_i(B) = \mathcal{L}(V^{(i)}).$$

- (c) We now claim that there is no intertwiner between π_i and π_j for $i \neq j$. Indeed, let $T : V^{(i)} \rightarrow V^{(j)}$ be such that $T \circ \pi_i = \pi_j \circ T$. This means that for any $f \in \mathcal{O}(\mathbb{G})^*$,

$$T \widehat{f}(v^{(i)}) = \widehat{f}(v^{(j)}) T.$$

By linearity, this is equivalent to

$$\widehat{f}(T v^{(i)} - v^{(j)} T) = 0$$

and since linear forms separate points, this is also equivalent to $T v^{(i)} = v^{(j)} T$. Because $v^{(i)}$ and $v^{(j)}$ are assumed to be irreducible and inequivalent, Lemma 1.4.6 forces $T = 0$, concluding the proof.

(d) Noticing that $\oplus_i \pi_i = \text{id}_B$, we conclude that⁹

$$\begin{aligned}
B &= \text{End}_B(B) \\
&= \text{End}_B\left(\bigoplus_{i=1}^n \pi_i(B)\right) \\
&= \bigoplus_{i=1}^n \text{End}_B(\pi_i(B)) \\
&= \bigoplus_{i=1}^n \text{End}_{\mathcal{L}(V^{(i)})}(\mathcal{L}(V^{(i)})) \\
&= \bigoplus_{i=1}^n \mathcal{L}(V^{(i)}) \\
&= \mathcal{B}.
\end{aligned}$$

Now let, for each $1 \leq i \leq n$, $(\lambda_{k,\ell}^{(i)})_{1 \leq k,\ell \leq \dim(V^{(i)})}$ be complex numbers and set

$$x = \sum_{i=1}^n \sum_{k,\ell=1}^{\dim(V^{(i)})} \lambda_{k,\ell} v_{k,\ell}^{(i)}.$$

If we denote by Λ_i the matrix with coefficients $\overline{\lambda_{k,\ell}^{(i)}}$, then

$$\Lambda = (\Lambda_1, \dots, \Lambda_n) \in \mathcal{B} = B,$$

thus there exists $f \in \mathcal{O}(\mathbb{G})$ such that for all i , $\widehat{f}(v^{(i)}) = \Lambda_i$. As a consequence,

$$f(x) = \sum_{i=1}^n \sum_{k,\ell=1}^{\dim(V^{(i)})} |\lambda_{k,\ell}|^2$$

and x therefore vanishes if and only if all the coefficients vanish, proving linear independence.

3. Observe that $\mathcal{O}(\mathbb{G})$ is spanned by the products of coefficients of u , that is to say coefficients of tensor powers of u which, by point 1, are linear combinations of coefficients of irreducible unitary representations. If now v is a finite-dimensional representation, its coefficients are in the linear span of coefficients of unitary representations, hence by point 2 it is equivalent to a unitary representation. ■

We have proved along the way a useful result which we restate here for later reference :

Lemma 1.4.8. *Let v and w be representations of a unitary compact matrix quantum group \mathbb{G} of dimension n and m respectively. Then, $T \in \mathcal{L}(V, W)$ is an intertwiner if and only if, for all $f \in \mathcal{O}(\mathbb{G})^*$,*

$$T \widehat{f}(v) = \widehat{f}(w) T.$$

Proof. The fact that T is an intertwiner reads, for any $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$\sum_{k=1}^n T_{ik} v_{kj} = \sum_{k=1}^m T_{kj} w_{ik}.$$

Applying f to both sides yields the only if condition. The converse follows because linear maps separate the points. ■

Let us outline the following crucial consequence of Theorem 1.4.7 :

9. For the first and last lines, observe that for any unital algebra B , the map $b \mapsto (x \mapsto x.b)$ gives an isomorphism $B \mapsto \text{End}_B(B)$ with inverse $T \mapsto T(1)$.

Corollary 1.4.9. *Let $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$ be an orthogonal compact matrix quantum group. Then, any irreducible representation is equivalent to a subrepresentation of $u^{\otimes k}$.*

Proof. Because $\mathcal{O}(\mathbb{G})$ is generated by the coefficients of u , it is the linear span of coefficients of irreducible subrepresentations of tensor powers of u . Thus, any finite-dimensional representation is equivalent to one of these by point 2 of Theorem 1.4.7. ■

LECTURE 2

PARTITIONS ENTER THE PICTURE

Now that the quantum group stage is set, it is time for the second main objects of these lectures to enter the picture, namely partitions. The idea to use partitions of finite sets to study the representation theory of compact groups dates, at least, to the work of R. Brauer [Bra37]. However, it stayed unnoticed to the quantum group community for some time, probably because the seminal works of T. Banica [Ban96] and [Ban99b] relied on the alternative picture of Temperley-Lieb algebras. Only with the breakthrough article of T. Banica and R. Speicher [BS09] did the systematic formalization and study of the relationship between partitions and compact quantum groups started to spread as a research subject of its own.

The introduction of partitions in the work of T. Banica and R. Speicher partly had to do with the combinatorial approach to free probability (which is beautifully explained in the book [NS06]) and its emerging connections with quantum group theory. We will describe some of these connections in Lecture 4, but for the present time we will rather follow the path of R. Brauer.

2.1 INVARIANTS OF THE (QUANTUM) ORTHOGONAL GROUP

2.1.1 THE ORTHOGONAL GROUP : PAIR PARTITIONS

In order to introduce the main ideas of the theory, we will start by revisiting the work of R. Brauer who, in [Bra37], studied the orthogonal group O_N using partitions of finite sets. Let us therefore start with the group O_N . The defining representation of O_N as matrices acting on $V = \mathbf{C}^N$ should by definition "contain everything". The precise meaning of this last expression is that, by Corollary 1.4.9, given any finite-dimensional representation π of O_N , there exists an integer k such that π is unitarily equivalent to a subrepresentation of $\rho^{\otimes k}$.

This means that we can focus on subrepresentations of tensor powers of ρ . Going back to the definition, we see that subrepresentations can be read off from the intertwiner spaces. Indeed, if T is an isometric intertwiner between π and $\rho^{\otimes k}$, then TT^* is an orthogonal projection intertwining $\rho^{\otimes k}$ with itself. In other words, it should in principle be possible to recover the whole representation theory of O_N from the algebra structure of the spaces

$$\text{Mor}_{O_N}(\rho^{\otimes k}, \rho^{\otimes k}) = \left\{ T : V^{\otimes k} \rightarrow V^{\otimes k} \mid T\rho^{\otimes k}(g) = \rho^{\otimes k}(g)T \text{ for all } g \in O_N \right\} = \rho^{\otimes k}(O_N).$$

The study of these algebras is usually known under the name of *Schur-Weyl duality*. In our setting, orthogonality allows us to further reduce the problem thanks to the following elementary result :

Proposition 2.1.1. *For any integer k , there exists a canonical linear isomorphism*

$$\Phi_k : \text{Mor}(\rho^{\otimes k}, \rho^{\otimes k}) \simeq \text{Mor}(\rho^{\otimes(2k)}, \varepsilon),$$

where ε denotes the trivial representation of O_N .

Proof. Since the elements of O_N are unitary, they leave the inner product on V invariant. However, this inner product does not yield a linear form on $V \otimes V$ since it is not bilinear but only sesquilinear. It can however be made linear using the *duality map* $D : V \otimes V \rightarrow \mathbf{C}$ defined by

$$D(x \otimes y) = \langle x, \bar{y} \rangle,$$

where \bar{y} is the image of y in the *conjugate Hilbert space* \bar{V} ¹. The key fact is that because the coefficients of matrices in O_N are real-valued, the map D is invariant under the representation $\rho^{\otimes 2}$. Now, any map

$$T : V^{\otimes k} \rightarrow V^{\otimes k}$$

can be turned into a map

$$\tilde{T} : V^{\otimes 2k} \simeq V^{\otimes k} \otimes V^{\otimes k} \rightarrow \mathbf{C}$$

via the formula

$$\tilde{T} : x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_k \rightarrow D(T(x_1 \otimes \cdots \otimes x_k), y_1 \otimes \cdots \otimes y_k).$$

One easily checks that T is an intertwiner if and only if \tilde{T} is, hence the result. \blacksquare

As a consequence, we are now looking for the *invariants* of the orthogonal group, that is to say the polynomial maps to \mathbf{C} which are invariant under a given representation. Let us practice a little by computing $\text{Mor}_{O_N}(\rho^{\otimes 2}, \varepsilon)$. We know that D yields a non-trivial element of this space. Moreover, because ρ is irreducible, it follows from Schur's Lemma that

$$\dim(\text{Mor}_{O_N}(\rho^{\otimes 2}, \varepsilon)) = \dim(\text{Mor}_{O_N}(\rho, \rho)) = 1,$$

so that $\text{Mor}_{O_N}(\rho^{\otimes 2}, \varepsilon) = \mathbf{C}D$.

We can extend this idea to build non-trivial elements of $\text{Mor}(\rho^{\otimes 2k}, \varepsilon)$ for any $k \geq 1$ by pairing tensors using the map D . To do this, we just need a partition p of $\{1, \dots, 2k\}$ into subsets of size 2. Such a partition is called a *pair partition* and the set of pair partition of $\{1, \dots, 2k\}$ is denoted by $\mathcal{P}_2(2k)$. Given such a pair partition p , we set

$$f_p : x_1 \otimes \cdots \otimes x_{2k} = \prod_{\{a,b\} \in p} D(x_a, x_b).$$

We can then produce many intertwiners by taking linear combinations and one of the main results of R. Brauer's work [Bra37] is that we indeed get everything :

THEOREM 2.1.2 (Brauer) For any integer k , we have

$$\begin{aligned} \text{Mor}(\rho^{\otimes 2k+1}, \varepsilon) &= \{0\} \\ \text{Mor}(\rho^{\otimes 2k}, \varepsilon) &= \text{Vect}\{f_p \mid p \in \mathcal{P}_2(2k)\}. \end{aligned}$$

2.1.2 THE QUANTUM ORTHOGONAL GROUP : NONCROSSING PARTITIONS

Building on the previous discussion, we now want to use the same strategy to investigate the spaces $\text{Mor}_{O_N^+}(U^{\otimes 2k}, \varepsilon)$. One easily checks that the map D is still an invariant of $U^{\otimes 2}$:

$$\begin{aligned} (\text{id} \otimes D) \circ \rho_{U^{\otimes 2}}(e_i \otimes e_j) &= (\text{id} \otimes D) \left(\sum_{k,\ell} U_{ik} U_{j\ell} \otimes e_k \otimes e_\ell \right) \\ &= \sum_{k=1}^N U_{ik} U_{jk} \\ &= \delta_{ij} \\ &= \rho_\varepsilon(D(e_i \otimes e_j)). \end{aligned}$$

1. Recall that this is just the same abelian group as V but with scalar multiplication given by $\lambda \cdot \bar{x} = \overline{\lambda \cdot x}$.

So what will be the difference between O_N and O_N^+ ? Let us look at $U^{\otimes 4}$ and the O_N -intertwiner of $\rho^{\otimes 4}$ given by the partition

$$p_{\text{cross}} = \{\{1, 3\}, \{2, 4\}\}.$$

Does this yield an interwiner for O_N^+ ? Let us compute :

Exercise 8. Prove that for any orthogonal compact matrix quantum group $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$,

$$(\text{id} \otimes f_{p_{\text{cross}}}) \circ \rho_{u^{\otimes 4}}(e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}) = \sum_{k, \ell=1}^N u_{i_1 k} u_{i_2 \ell} u_{i_3 k} u_{i_4 \ell} \quad (2.1)$$

and

$$\rho_{\varepsilon} \circ (\text{id} \otimes f_{p_{\text{cross}}}) = \delta_{i_1 i_3} \delta_{i_2 i_4}. \quad (2.2)$$

Solution. This is an elementary computation :

$$\begin{aligned} (\text{id} \otimes f_{p_{\text{cross}}}) \circ \rho_{u^{\otimes 4}}(e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}) &= \sum_{j_1, \dots, j_4=1}^N u_{i_1 j_1} \cdots u_{i_4 j_4} f_p(e_{j_1} \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4}) \\ &= \sum_{j_1, \dots, j_4=1}^N u_{i_1 j_1} \cdots u_{i_4 j_4} \delta_{j_1 j_3} \delta_{j_2 j_4} \\ &= \sum_{j_1, j_2=1}^N u_{i_1 j_1} u_{i_2 j_2} u_{i_3 j_1} u_{i_4 j_2} \end{aligned}$$

and Equation (2.1) follows from the changes of indices $k = j_1$, $\ell = j_2$. As for the second one, this is the definition of $f_{p_{\text{cross}}}$. \blacksquare

Understanding the meaning of this relation is the key to the world of partition quantum groups. We will therefore state it as a proposition :

Proposition 2.1.3. Let $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$ be an orthogonal compact matrix quantum group and let $p_{\text{cross}} = \{\{1, 3\}, \{2, 4\}\}$. Then, $f_{p_{\text{cross}}} \in \text{Mor}_{\mathbb{G}}(u^{\otimes 4}, \varepsilon)$ if and only if \mathbb{G} is a classical group.

Proof. We will play around with the equality (2.1) = (2.2) to show that the coefficients of u pairwise commute. More precisely, multiplying each side by $u_{i_4 j_2} u_{i_3 j_1}$, for two arbitrary indices $1 \leq j_1, j_2 \leq N$, and summing over i_4 and i_3 yields

$$\begin{aligned} \sum_{k, \ell, i_3, i_4=1}^N u_{i_1 k} u_{i_2 \ell} u_{i_3 k} u_{i_4 \ell} u_{i_4 j_2} u_{i_3 j_1} &= \sum_{i_3, i_4=1}^N \delta_{i_1 i_3} \delta_{i_2 i_4} u_{i_4 j_2} u_{i_3 j_1} \\ &= u_{i_2 j_2} u_{i_1 j_1}. \end{aligned}$$

The left-hand side above can be simplified using the fact that u is orthogonal. Indeed,

$$\sum_{i_4=1}^N u_{i_4 \ell} u_{i_4 j_2} = \delta_{\ell j_2}$$

and similarly for i_3 so that

$$\begin{aligned} \sum_{k, \ell, i_3, i_4=1}^N u_{i_1 k} u_{i_2 \ell} u_{i_3 k} u_{i_4 \ell} u_{i_4 j_2} u_{i_3 j_1} &= \sum_{k, \ell, i_3=1}^N u_{i_1 k} u_{i_2 \ell} u_{i_3 k} \delta_{\ell j_2} u_{i_3 j_1} \\ &= \sum_{k, \ell=1}^N u_{i_1 k} u_{i_2 \ell} \delta_{k j_1} \delta_{\ell j_2} \\ &= u_{i_1 j_1} u_{i_2 j_2}. \end{aligned}$$

Thus, we have proven that if f_p is an intertwiner, then the coefficients of u pairwise commute. Moreover, it is clear that the converse holds. It now follows that $\mathcal{O}(\mathbb{G})$ is commutative so that by Exercise 4, \mathbb{G} is in fact a classical group. \blacksquare

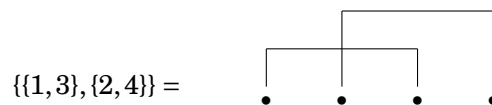
As a consequence of Proposition 1.3.5, f_p is not an intertwiner of O_N^+ and in the construction of R. Brauer we have used, without noticing it, the commutativity of $\mathcal{C}(O_N)$. So what is really the smallest space of intertwiners that one can build from D using pair partitions? The answer relies on the following definition :

DEFINITION 2.1.4. A partition is said to be *crossing* if there exists $k_1 < k_2 < k_3 < k_4$ such that

- k_1 and k_3 are in the same block,
- k_2 and k_4 are in the same block,
- the four elements are not in the same block.

Otherwise, it is said to be *noncrossing*.

To illustrate this notion we now introduce a very useful pictorial description of partitions : we draw k points in a row and then connect two points if and only if they belong to the same subset of the partition. It is then clear that for instance



cannot be drawn without letting the lines cross. It is therefore a crossing partition.

It is certainly not clear at the moment that non-crossing partitions have to do with the quantum orthogonal groups O_N^+ and there is indeed still some work needed to see the link. As a motivation, let us state the quantum analogue of R. Brauer's result, which was proven by T. Banica in [Ban96]. We will denote the set of non-crossing pair partitions on $2k$ points by $NC_2(2k)$.

THEOREM 2.1.5 (Banica) For any integer k , we have

$$\begin{aligned} \text{Mor}_{O_N^+}(\rho^{\otimes 2k+1}, \varepsilon) &= \{0\} \\ \text{Mor}_{O_N^+}(\rho^{\otimes 2k}, \varepsilon) &= \text{Vect}\{f_p \mid p \in NC_2(2k)\}. \end{aligned}$$

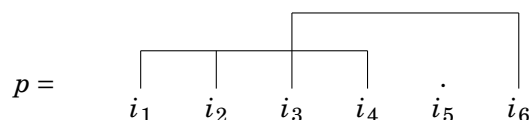
The proof will be a consequence of the general theory that we will develop in the remainder of this lecture.

2.2 PARTITION QUANTUM GROUPS

The previous section can be summarized in the following way : the representation theory of O_N is determined by the pair partitions P_2 while the representation theory of O_N^+ is determined by the noncrossing pair partitions NC_2 . This raises the question : what other (quantum) groups have their representation theory determined by partitions?

2.2.1 PARTITION MAPS

The answer requires to extend our setting. First, we will from now on consider arbitrary partitions of finite sets, not only those in pairs. To define a linear form associated to such a general partition p , we need to extend the definition of the maps f_p . For this purpose, let $p \in \mathcal{P}(k)$ and let $1 \leq i_1, \dots, i_k \leq N$. Place these indices on the points of p from left to right. Then, if whenever two indices are connected, they are equal, we set $\delta_p(i_1, \dots, i_k) = 1$. Otherwise, we set $\delta_p(i_1, \dots, i_k) = 0$. For instance, with the following partition



we get

$$\delta_p(i_1, i_2, i_3, i_4, i_5, i_6) = \delta_{i_1 i_2 i_4} \delta_{i_3 i_6}.$$

DEFINITION 2.2.1. Let $p \in \mathcal{P}(k)$ and let N be an integer. Then, we define a map $f_p : (\mathbf{C}^N)^{\otimes k} \rightarrow \mathbf{C}$ by the formula

$$f_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \delta_p(i_1, \dots, i_k).$$

Exercise 9. Check that for pair partitions, this coincides with the previous definition.

Solution. Simply observe that for a pair partition,

$$\delta_p(i_1, \dots, i_{2k}) = \prod_{\{a,b\} \subset p} \delta_{\sqcup}(i_a, i_b) = \prod_{\{a,b\} \subset p} D(e_{i_a} \otimes e_{i_b})$$

■

Second, if k and ℓ are integers, then we claimed earlier that the space $\text{Mor}_{\mathbb{G}}(V^{\otimes k}, V^{\otimes \ell})$ should be recoverable from the invariants of \mathbb{G} . Let us explain how this follows from extending the construction of the maps Φ_k introduced before. Indeed, using the duality map one can build a canonical isomorphism

$$\Psi_{k,\ell} : \mathcal{L}(V^{\otimes k}, V^{\otimes \ell}) \rightarrow \mathcal{L}(V^{\otimes(k+1)}, V^{\otimes(\ell-1)})$$

by the formula

$$\Phi_{k,1}(T)(x_1 \otimes \cdots \otimes x_k \otimes x_{k+1}) = \left(\text{id}_V^{\otimes(\ell-1)} \otimes D \right) (T(x_1 \otimes \cdots \otimes x_k) \otimes x_{k+1})$$

and iterating gives isomorphisms

$$\Phi_{k,\ell} = \Psi_{k+\ell-1,1} \circ \cdots \circ \Psi_{k,\ell} : \mathcal{L}(V^{\otimes k}, V^{\otimes \ell}) \rightarrow \mathcal{L}(V^{\otimes(k+\ell)}, \mathbf{C}).$$

In particular, if p is a partition of $\{1, \dots, k+\ell\}$, we can define an operator

$$T_p = \Phi_{k,\ell}^{-1}(f_p)$$

which is an intertwiner as soon as f_p is.

The drawback of our previous construction is that we have lost the pictorial description of the operator. To recover it, notice that we can also draw a partition $p \in \mathcal{P}(k+\ell)$ on two rows instead of one, drawing for instance k points corresponding to $\{1, \dots, k\}$ in the upper row and ℓ points corresponding to $\{k+1, \dots, k+\ell\}$ in the lower row. With this description, the operator T_p admits an explicit description which generalizes that of f_p . Let $(e_i)_{1 \leq i \leq N}$ be the canonical basis of $V = \mathbf{C}^N$. For a partition $p \in P(k+\ell)$ drawn with k upper points and ℓ lower points, we extend the definition of the function δ_p , taking now as arguments a k -tuple $\mathbf{i} = (i_1, \dots, i_k)$ and an ℓ -tuple $\mathbf{j} = (j_1, \dots, j_\ell)$, in the following way :

- We draw the indices of \mathbf{i} on the upper points of the partitions (from left to right) and the indices of \mathbf{j} on the lower points of p (from left to right),
- If whenever two points are connected by a string, their indices are equal, we set $\delta_p(\mathbf{i}, \mathbf{j}) = 1$,
- Otherwise, we set $\delta_p(\mathbf{i}, \mathbf{j}) = 0$.

With this in hand, we have :

Proposition 2.2.2. For any tuples \mathbf{i} and \mathbf{j} , we have

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1, \dots, j_\ell=1}^N \delta_p(\mathbf{i}, \mathbf{j}) e_{j_1} \otimes \cdots \otimes e_{j_\ell}.$$

Proof. Unwinding the definition of T_p , we have for any j_1, \dots, j_ℓ ,

$$\langle T_p(e_{i_1} \otimes \dots \otimes e_{i_k}), e_{j_1} \otimes \dots \otimes e_{j_\ell} \rangle = f_p(e_{i_1} \otimes \dots \otimes e_{i_k} \otimes e_{j_1} \otimes \dots \otimes e_{j_\ell}).$$

Moreover, it follows from our definition² that

$$\delta_p(\mathbf{i}, \mathbf{j}) = \delta_p(\mathbf{i} \otimes \mathbf{j}),$$

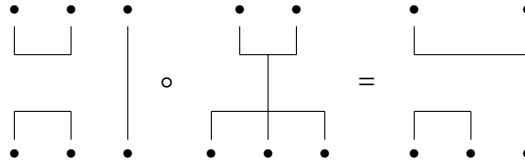
where \otimes denotes the concatenation of tuples, hence the result. \blacksquare

2.2.2 OPERATIONS ON PARTITIONS

Assume that we have a collection \mathcal{C} of partitions made of subsets $\mathcal{C}(k, \ell)$ for all integers k and ℓ and that we want to find an orthogonal compact matrix (quantum) group \mathbb{G} such that for all $k, \ell \in \mathbf{N}$,

$$\text{Mor}_{\mathbb{G}}(u^{\otimes k}, u^{\otimes \ell}) = \text{Vect}\{T_p \mid p \in \mathcal{C}(k, \ell)\}.$$

Obviously, the set \mathcal{C} must satisfy some stability conditions in order for the spaces above to be intertwiner spaces. For instance, when two intertwiners can be composed, their composition must again be an intertwiner so that we need to ensure that $T_q \circ T_p$ is a linear combination of maps T_r . This condition is linked to the following operation on partitions : given two partitions $p \in \mathcal{P}(k, \ell)$ et $q \in \mathcal{P}(\ell, m)$, we can perform their *vertical concatenation* $q \circ p$ by placing q below p and connecting the lower points of p to the corresponding ones in the upper row of q . This process may produce loops, which we erase and only remember their number, denoted by $\text{rl}(q, p)$. Here is an example with $\text{rl}(q, p) = 1$:



At the level of the operators T_p , this translates into the following formula :

$$T_q \circ T_p = \dim(V)^{\text{rl}(q,p)} T_{qp}. \quad (2.3)$$

Exercise 10. Prove Equation (2.3).

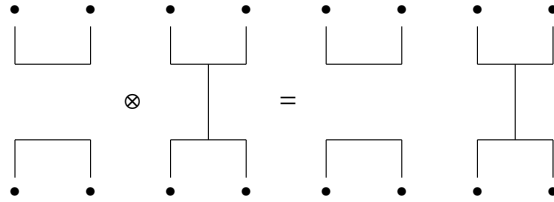
Solution. Assume that $p \in \mathcal{P}(k, \ell)$ and $q \in \mathcal{P}(\ell, m)$. For a tuple $\mathbf{i} = (i_1, \dots, i_k)$, we have

$$\begin{aligned} T_q \circ T_p(e_{i_1} \otimes \dots \otimes e_{i_k}) &= \sum_{j_1, \dots, j_\ell=1}^N \delta_p(\mathbf{i}, \mathbf{j}) T_q(e_{j_1} \otimes \dots \otimes e_{j_\ell}) \\ &= \sum_{j_1, \dots, j_\ell=1}^N \sum_{s_1, \dots, s_m=1}^N \delta_p(\mathbf{i}, \mathbf{j}) \delta_q(\mathbf{j}, \mathbf{s}) (e_{s_1} \otimes \dots \otimes e_{s_m}). \end{aligned}$$

It follows from the definition that if there is a \mathbf{j} such that $\delta_p(\mathbf{i}, \mathbf{j}) \neq 0$ and $\delta_q(\mathbf{j}, \mathbf{s}) \neq 0$, then $\delta_{qp}(\mathbf{i}, \mathbf{s}) \neq 0$. Conversely, for given tuples \mathbf{i} and \mathbf{s} , any tuple \mathbf{j} which coincides with \mathbf{i} on the lower row of p and with \mathbf{s} on the upper row of q will do. Therefore, the number of such tuples \mathbf{j} is the number of possible indexing of the loops which get removed in the composition qp . This gives the formula in the statement. \blacksquare

Stability under vertical concatenation is not sufficient to produce commutants of (quantum) group representations. For instance, if T_p and T_q are intertwiners, then $T_p \otimes T_q$ also is and must therefore come from partitions. This corresponds to the so-called *horizontal concatenation* : if p and q are two partitions of $\{1, \dots, 2k\}$ and $\{1, \dots, 2\ell\}$ respectively, then we build a partition $p \otimes q$ of $\{1, \dots, 2(k + \ell)\}$ by simply drawing q on the right of p . For instance

2. Note that we have two definitions of the symbol δ_p , which should not be confused : one has two arguments corresponding the choice of an upper and a lower row in p , while the original one has only one argument.



As for vertical concatenation, this operation has a nice interpretation at the level of operators, namely

$$T_p \otimes T_q = T_{p \otimes q} \quad (2.4)$$

Exercise 11. Prove Equation (2.4).

Solution. Assume that $p \in \mathcal{P}(k, \ell)$ and $q \in \mathcal{P}(m, s)$. For a tuple $\mathbf{i} = (i_1, \dots, i_{k+m})$, we have

$$\begin{aligned} (T_p \otimes T_q)(e_{i_1} \otimes \dots \otimes e_{i_{k+m}}) &= \left(\sum_{j_1, \dots, j_\ell=1}^N \delta_p((i_1, \dots, i_k), (j_1, \dots, j_\ell)) e_{j_1} \otimes \dots \otimes e_{j_\ell} \right) \\ &\otimes \left(\sum_{j_{\ell+1}, \dots, j_{\ell+s}=1}^N \delta_q((i_{k+1}, \dots, i_{k+m}), (j_{\ell+1}, \dots, j_{\ell+s})) e_{j_{\ell+1}} \otimes \dots \otimes e_{j_{\ell+s}} \right) \\ &= \sum_{j_1, \dots, j_{\ell+s}=1}^N \delta_{p \otimes q}(\mathbf{i}, \mathbf{j}) e_{j_1} \otimes \dots \otimes e_{j_{\ell+s}} \\ &= T_{p \otimes q}(e_{i_1} \otimes \dots \otimes e_{i_{k+m}}) \end{aligned}$$

■

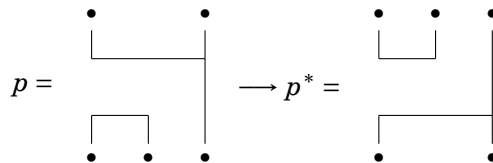
There is a third, very elementary operation which is needed. Because we are only considering unitary representations, if T is an intertwiner then so is its adjoint T^* . Noticing that

$$\begin{aligned} \delta_p(\mathbf{i}, \mathbf{t}) &= \left\langle \sum_{j_1, \dots, j_\ell=1}^N \delta_p(\mathbf{i}, \mathbf{j}) e_{j_1} \otimes \dots \otimes e_{j_\ell}, e_{t_1} \otimes \dots \otimes e_{t_\ell} \right\rangle \\ &= \left\langle e_{i_1} \otimes \dots \otimes e_{i_k}, \sum_{j_1, \dots, j_k=1}^N \delta_p(\mathbf{j}, \mathbf{t}) e_{j_1} \otimes \dots \otimes e_{j_k} \right\rangle, \end{aligned}$$

we see that

$$T_p^* = T_{p^*}, \quad (2.5)$$

where p^* is the partition obtained by reflecting p with respect to an horizontal axis between the two rows. Here is an example :



Do we now have everything needed to product an orthogonal compact matrix quantum group out of \mathcal{C} ? The answer is yes, but the proof of it requires a detour through a more abstract structure.

2.2.3 TANNAKA-KREIN RECONSTRUCTION

Reconstructing a group from its representations is the subject of *Tannaka-Krein duality*. The main point of this theory is that the representations of the group and their intertwiner can be assembled into one rich algebraic structure called a *tensor category*. The reader may for instance refer to the book [EGNO15] for a comprehensive introduction to tensor categories. For our purpose, we can restrict to a specific type of such categories for which we will use the following more "down-to-earth" definition.

DEFINITION 2.2.3. Let V be a finite-dimensional Hilbert space. A *concrete baby rigid C^* -tensor category* \mathcal{C} is a collection of spaces $\text{Mor}_{\mathcal{C}}(k, \ell) \subset \mathcal{L}(V^{\otimes k}, V^{\otimes \ell})$ for all k and ℓ such that

1. If $T \in \text{Mor}_{\mathcal{C}}(k, \ell)$ and $T' \in \text{Mor}_{\mathcal{C}}(k', \ell')$, then $T \otimes T' \in \text{Mor}_{\mathcal{C}}(k + k', \ell + \ell')$,
2. If $T \in \text{Mor}_{\mathcal{C}}(k, \ell)$ and $T' \in \text{Mor}_{\mathcal{C}}(\ell, r)$, then $T' \circ T \in \text{Mor}_{\mathcal{C}}(k, r)$,
3. If $T \in \text{Mor}_{\mathcal{C}}(k, \ell)$, then $T^* \in \text{Mor}_{\mathcal{C}}(\ell, k)$,
4. $\text{id} : x \mapsto x \in \text{Mor}_{\mathcal{C}}(1, 1)$,
5. $D : x \otimes y \mapsto \langle x, \bar{y} \rangle \in \text{Mor}_{\mathcal{C}}(2, 0)$.

If moreover $\sigma : x \otimes y \mapsto y \otimes x \in \text{Mor}_{\mathcal{C}}(2, 2)$, then \mathcal{C} is said to be *symmetric*.

The fundamental example is of course given by (quantum) groups :

Example 2.2.4. Let \mathbb{G} be an orthogonal compact matrix quantum group with a fundamental representation u of size N . Set $V = \mathbf{C}^N$ and

$$\text{Mor}_{\mathcal{C}}(k, \ell) = \text{Mor}_{\mathbb{G}}(u^{\otimes k}, u^{\otimes \ell}).$$

This defines a concrete baby rigid C^* -tensor category called the *representation category* of \mathbb{G} and denoted by $\mathcal{R}(\mathbb{G})$.

That this example is in fact the general case is the content of the quantum Tannaka-Krein theorem proved by S.L. Woronowicz in [Wor88] :

THEOREM 2.2.5 (Woronowicz) Let \mathcal{C} be a concrete baby rigid C^* -tensor category associated to a Hilbert space V . Then, there exists an orthogonal compact matrix quantum group \mathbb{G} with a fundamental representation u of dimension $\dim(V)$ such that for all k and l ,

$$\text{Mor}_{\mathbb{G}}(u^{\otimes k}, u^{\otimes \ell}) = \text{Mor}_{\mathcal{C}}(k, \ell).$$

Moreover, the quantum group \mathbb{G} is unique up to isomorphism and is a classical group if and only if \mathcal{C} is symmetric.

There are several proofs of this result using a different amount of categorical machinery, see for instance [NT13] or [Wor88]. However, due to our restricted definition of concrete rigid C^* -tensor category, we can give a very elementary proof which is close to that of [Mal18].

Proof. The idea is to build $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$ as a quotient by all the relations making the maps T intertwiners of tensor powers of its fundamental representation. So let N be the dimension of V and consider the universal complex algebra \mathcal{A} generated by N^2 elements X_{ij} with the involution given by $X_{ij}^* = X_{ji}$. Fixing an orthonormal basis $(e_i)_{1 \leq i \leq N}$ of V once and for all, we introduce some useful shorthand notations : if $\mathbf{i} = (i_1, \dots, i_k)$, $\mathbf{j} = (j_1, \dots, j_\ell)$ and $T \in \mathcal{L}(V^{\otimes k}, V^{\otimes \ell})$, then we set

$$\begin{aligned} e_{\mathbf{i}} &= e_{i_1} \otimes \dots \otimes e_{i_k} \\ T_{\mathbf{i}\mathbf{j}} &= \langle T(e_{\mathbf{i}}), e_{\mathbf{j}} \rangle \\ X_{\mathbf{i}\mathbf{j}} &= X_{i_1 j_1} \dots X_{i_k j_k} \text{ (if } k = \ell) \end{aligned}$$

Consider now the following non-commutative polynomial

$$P_{T, \mathbf{i}\mathbf{j}} = \sum_{\mathbf{s}} X_{\mathbf{i}\mathbf{s}} T_{\mathbf{s}\mathbf{j}} - T_{\mathbf{i}\mathbf{s}} X_{\mathbf{s}\mathbf{j}}.$$

Then, for a compact quantum group $(\mathcal{O}(\mathbb{G}), u)$, T intertwines $u^{\otimes k}$ with $u^{\otimes \ell}$ if and only if

$$P_{T, \mathbf{i}\mathbf{j}}(u) := P_{T, \mathbf{i}\mathbf{j}}((u_{ij})_{1 \leq i, j \leq N}) = 0$$

for all tuples \mathbf{i} and \mathbf{j} . Let us therefore consider the sets

$$\begin{aligned}\mathcal{I}_{k,\ell} &= \text{Vect} \{P_{T,\mathbf{i};\mathbf{j}}(X) \mid T \in \text{Mor}_{\mathcal{C}}(k,\ell), \mathbf{i} = (i_1, \dots, i_k), \mathbf{j} = (j_1, \dots, j_\ell)\} \\ \mathcal{I}_k &= \bigoplus_{i,j=0}^k \mathcal{I}_{i,j} \\ \mathcal{I} &= \bigoplus_{k \in \mathbf{N}} \mathcal{I}_k.\end{aligned}$$

According to our basic idea, let us set $\widetilde{\mathcal{A}} = \mathcal{A}/\mathcal{I}$ (this makes sense at least as a quotient of vector spaces) and let u_{ij} be the image of X_{ij} in this quotient. If $(\widetilde{\mathcal{A}}, u)$ is an orthogonal compact matrix quantum group, then it certainly is the one we are looking for. But for this to hold, \mathcal{I} must satisfy some extra conditions, namely

1. \mathcal{I} is an ideal, so that $\widetilde{\mathcal{A}}$ is an algebra,
2. $\mathcal{I} = \mathcal{I}^*$, so that $\widetilde{\mathcal{A}}$ is a $*$ -algebra,
3. $\Delta(\mathcal{I}) \subset \mathcal{I} \otimes \widetilde{\mathcal{A}} + \widetilde{\mathcal{A}} \otimes \mathcal{I}$, so that the coproduct is well-defined on $\widetilde{\mathcal{A}}$. Indeed, if $\pi_{\mathcal{I}}$ denotes the quotient map, then for any $x \in \mathcal{O}(O_N^+)$ and $y \in \mathcal{I}$,

$$(\pi_{\mathcal{I}} \otimes \pi_{\mathcal{I}}) \circ \Delta(x + y) = (\pi_{\mathcal{I}} \otimes \pi_{\mathcal{I}}) \circ \Delta(x)$$

so that the coproduct factors through the quotient.

As one may expect, these properties follow from the axioms of concrete baby rigid C^* -tensor categories, and here is how :

1. Observe that if \mathbf{i}', \mathbf{j}' are m -tuples, then

$$X_{\mathbf{i}';\mathbf{j}'} P_{T,\mathbf{i};\mathbf{j}}(X) = P_{\text{id}^{\otimes m} \otimes T, \mathbf{i}' \otimes \mathbf{i}; \mathbf{j}' \otimes \mathbf{j}}(X),$$

where \otimes denotes the concatenation of tuples. Thus, \mathcal{I} absorbs monomials, hence also polynomials, on the left. The same property on the right follows from a similar argument. As a consequence, \mathcal{I} is an ideal.

2. Denoting by $\mathbf{i}^{-1} = (i_k, \dots, i_1)$ the reversed tuple, we have

$$\begin{aligned}P_{T,\mathbf{i};\mathbf{j}}(X)^* &= \sum_{\mathbf{s}} X_{\mathbf{i}^{-1}\mathbf{s}^{-1}} \overline{T_{\mathbf{s}\mathbf{j}}} - X_{\mathbf{s}^{-1}\mathbf{j}^{-1}} \overline{T_{\mathbf{i}\mathbf{s}}} \\ &= \sum_{\mathbf{s}} X_{\mathbf{i}^{-1}\mathbf{s}^{-1}} T_{\mathbf{j}\mathbf{s}}^* - X_{\mathbf{s}^{-1}\mathbf{j}^{-1}} T_{\mathbf{s}\mathbf{i}}^* \\ &= \sum_{\mathbf{s}} X_{\mathbf{i}^{-1}\mathbf{s}^{-1}} S(T^*)_{\mathbf{s}^{-1}\mathbf{j}^{-1}} - X_{\mathbf{s}^{-1}\mathbf{j}^{-1}} S(T^*)_{\mathbf{i}^{-1}\mathbf{s}^{-1}} \\ &= P_{S(T^*), \mathbf{i}^{-1}; \mathbf{j}^{-1}}\end{aligned}$$

where $S(T)$ is defined to be the operator with coefficients

$$S(T)_{\mathbf{ij}} = T_{\mathbf{j}^{-1}\mathbf{i}^{-1}}.$$

To see that $S(T)$ is in $\text{Mor}_{\mathcal{C}}(\ell, k)$ if $T \in \text{Mor}_{\mathcal{C}}(k, \ell)$, let us define inductively operators

$$D_k : V^{\otimes 2k} \rightarrow \mathbf{C}$$

by

$$D_{k+1} = D_k \circ (\text{id}_{V^{\otimes k}} \otimes D \otimes \text{id}_{V^{\otimes k}}).$$

Then, a straightforward computation shows that

$$(\text{id}_{V^{\otimes k}} \otimes D_\ell) (\text{id}_{V^{\otimes k}} \otimes T \otimes \text{id}_{V^{\otimes \ell}}) (D_k^* \otimes \text{id}_{V^{\otimes \ell}}) = S(T).$$

3. This follows from a straightforward calculation :

$$\begin{aligned}
\Delta(P_{T,i,j}(X)) &= \sum_{t,s} X_{is} \otimes X_{st} T_{tj} - X_{ts} \otimes X_{sj} T_{it} \\
&= \sum_s X_{is} \otimes \left(\sum_t X_{st} T_{tj} - X_{tj} T_{st} \right) \\
&\quad + \left(\sum_t X_{tj} T_{st} - X_{ts} T_{it} \right) \otimes X_{sj} \\
&= \sum_s X_{is} \otimes P_{T,s,j}(X) + P_{T,i,s}(X) \otimes X_{sj}.
\end{aligned}$$

As a consequence, $\mathbb{G} = (\widetilde{\mathcal{A}}, u)$ is an orthogonal compact matrix quantum group (observe that the relations $P_{D,i,j}(u) = 0$ are precisely the orthogonality condition in Definition 1.3.1). Moreover, by construction,

$$\text{Mor}_{\mathbb{G}}(k, \ell) \subset \text{Mor}_{\mathbb{G}}(u^{\otimes k}, u^{\otimes \ell})$$

for all $k, \ell \in \mathbf{N}$. This implies that setting $u_k = \bigoplus_{i=0}^k u^{\otimes i}$ acting on $H_k = \bigoplus_{i=0}^k H^{\otimes i}$,

$$\mathcal{B}_k = \bigoplus_{i,j=0}^k \text{Mor}_{\mathbb{G}}(i, j) \subset \text{Mor}_{\mathbb{G}}(u_k, u_k).$$

To show the converse inclusion, let us first note that by universality, all the monomials in \mathcal{A} are linearly independent. Hence denoting by X_k the matrix with coefficients X_{ij} , there exists for any operator $T \in \mathcal{L}(H_k)$ a linear map $f \in \mathcal{A}^*$ such that $\widehat{f}(X_k) = T$. Setting $\mathcal{D}_k = \{\widehat{f}(X_k) \mid f \in \mathcal{I}_k\}$, we have

$$\begin{aligned}
\widehat{f}(X_k) \in \mathcal{B}'_k &\Rightarrow \omega([T, \widehat{f}(X_k)]) = 0, \forall T \in \mathcal{B}_k, \forall \omega \in \mathcal{L}(H_k)^* \\
&\Rightarrow (\omega \otimes f)([T, X_k]) = 0, \forall T \in \mathcal{B}_k, \forall \omega \in \mathcal{L}(H_k)^* \\
&\Rightarrow f((\omega \otimes \text{id})([T, X_k])) = 0, \forall T \in \mathcal{B}_k, \forall \omega \in \mathcal{L}(H_k)^* \\
&\Rightarrow f \in \mathcal{I}_k^\perp.
\end{aligned}$$

As a consequence, $\mathcal{B}'_k \subset \mathcal{D}_k$ so that by the Double Commutant Theorem (see Theorem B in the Appendix), $\mathcal{D}'_k \subset \mathcal{B}_k$. By Lemma 1.4.8, any $T \in \text{Mor}_{\mathbb{G}}(u^{\otimes k}, u^{\otimes \ell})$ commutes with $\widehat{f}(u_k)$ for all $f \in \mathcal{O}(\mathbb{G})^*$. Equivalently, T commutes with $\widehat{f}(X_k)$ for all $f \in \mathcal{I}_k^\perp$, that is to say with \mathcal{D}_k . In other words, $T \in \mathcal{D}'_k \subset \mathcal{B}_k$ and the equality is proved.

Let us turn to uniqueness. Let us consider an orthogonal compact matrix quantum group \mathbb{H} such that

$$\text{Mor}_{\mathbb{G}}(k, \ell) = \text{Mor}_{\mathbb{H}}(u^{\otimes k}, u^{\otimes \ell})$$

for all $k, \ell \in \mathbf{N}$. By definition there exists an ideal $\mathcal{J} \in \mathcal{A}$ such that $\mathcal{O}(\mathbb{H}) = \mathcal{A}/\mathcal{J}$ and we can, up to isomorphism, identify the copies of \mathcal{A} used for \mathbb{G} and for \mathbb{H} . Moreover, with the previous notations, $\mathcal{I} \subset \mathcal{J}$. If now \mathcal{J}_k denotes the intersection of \mathcal{J} with the span of coefficients of X_k , then the computations above show that

$$\{\widehat{f}(X_k) \mid f \in \mathcal{I}_k\}' = \text{Mor}_{\mathbb{H}}(u_k, u_k) = \mathcal{D}'_k = \mathcal{B}_k = \{\widehat{f}(X_k) \mid f \in \mathcal{I}_k\}'$$

so that $\mathcal{J}_k = \mathcal{I}_k$. Thus, $\mathcal{J} = \bigcup \mathcal{J}_k = \bigcup \mathcal{I}_k = \mathcal{I}$ and $\mathbb{H} \simeq \mathbb{G}$.

The last point concerns symmetry. But we have already seen that $\sigma \in \text{Mor}_{\mathbb{G}}(2, 2)$ is equivalent to having $f_{p_{\text{cross}}} \in \text{Mor}_{\mathbb{G}}(4, 0)$, and we proved in Proposition 2.1.3 that this is equivalent to \mathbb{G} being classical. \blacksquare

Recall that we want to apply Tannaka-Krein reconstruction to build a compact quantum group out of a category of partitions. Considering the axioms in Definition 2.2.3, we see that Axioms (1) to (3) correspond to the three operations defined in Section 2.2.2. As for the last two axioms, they follow from the following elementary computations :

- $T_{|} = \text{id}$,
- $T_{\sqcup} = D$.

To simplify later statement, let us give a name to this

DEFINITION 2.2.6. A *category of partitions* \mathcal{C} is a collection of sets of partitions $\mathcal{C}(k, \ell)$ for all integers k and ℓ such that

1. If $p \in \mathcal{C}(k, \ell)$ and $q \in \mathcal{C}(k', \ell')$, then $p \otimes q \in \mathcal{C}(k + k', \ell + \ell')$,
2. If $p \in \mathcal{C}(k, \ell)$ and $q \in \mathcal{C}(\ell, r)$, then $q \circ p \in \mathcal{C}(k, r)$.
3. If $p \in \mathcal{C}(k, \ell)$, then $p^* \in \mathcal{C}(\ell, k)$.
4. $| \in \mathcal{C}(1, 1)$.
5. $\sqcup \in \mathcal{C}(2, 0)$.

If moreover $\{\{1, 3\}, \{2, 4\}\} \in \mathcal{C}(2, 2)$, then \mathcal{C} is said to be *symmetric*.

We conclude this section by a theorem which rather a summary and rewriting of the whole construction that we have done :

THEOREM 2.2.7 (Banica-Speicher) Let N be an integer and let \mathcal{C} be a category of partitions. Then, there exists an orthogonal compact matrix quantum group $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$, where u has dimension N , such that for any $k, \ell \in \mathbf{N}$,

$$\text{Mor}_{\mathbb{G}}(k, \ell) = \text{Vect} \{T_p \mid p \in \mathcal{C}(k, \ell)\}.$$

Moreover, \mathbb{G} is a classical group if and only if \mathcal{C} is symmetric. The compact quantum group \mathbb{G} will be denoted by $\mathbb{G}_N(\mathcal{C})$ and called the *partition quantum group* associated to N and \mathcal{C} .

Remark 2.2.8. The quantum groups associated to a category of partitions as we defined them here were first introduced by T. Banica and R. Speicher in [BS09] under the name of *easy quantum groups*. It was already known to the authors that this setting could be extended to the unitary setting by adding black and white colours to the points of the partition. That formal description of this extended formalism was made in [TW17]. However, it had by then become clear that one can even allow arbitrary colourings of the partitions and that idea led to the general setting of *partition quantum groups* introduced in [Fre17].

Remark 2.2.9. One of the first questions raised by the definition of partition quantum groups is their classification, based on the categories of partitions. For easy quantum groups, there is a complete classification, as a result of many works including [BS09], [Web13], and [RW16]. The classification of the unitary versions of easy quantum groups is more subtle, but many results are now known, starting with [TW17] and followed by several authors like [Gro19] or [MW19]. As for the general setting of partition quantum groups, the only known results are restricted to noncrossing partitions with two colours, see [Fre19].

2.2.4 EXAMPLES OF PARTITION QUANTUM GROUPS

We conclude this lecture by some examples. To determine $\mathbb{G}_N(\mathcal{C})$, one only has to consider generators of \mathcal{C} , that is, a subset F such that \mathcal{C} is the smallest category of partitions containing F . Then, $\mathcal{O}(\mathbb{G}_N(\mathcal{C}))$ is the quotient of $\mathcal{O}(O_N^+)$ by the relations given by the fact that T_p is an intertwiner for $p \in F$.

Example 2.2.10. The smallest category of partition is the category NC_2 of *noncrossing pair partitions*. It is generated by \sqcup , whose associated intertwiner is the duality map D . Thus, the corresponding compact quantum group is O_N^+ . Moreover, adding the crossing partition $\{\{1, 3\}, \{2, 4\}\}$ yields the abelianization of O_N^+ , which is O_N . We have therefore recovered R. Brauer's Theorem 2.1.2, as well as T. Banica's Theorem 2.1.5.

The quantum permutation group of course also falls into this class, but this requires a bit of work.

Exercise 12. Let $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$ be an orthogonal compact matrix quantum group. Prove that $\mathcal{O}(\mathbb{G})$ satisfies the defining relations of $\mathcal{A}_s(N)$ if and only if it has the following intertwiners :

- Conditions (1) and (3) : $T_p \in \text{Mor}(u^{\otimes 2}, u)$ where p is the partition with one block in $\mathcal{P}(2, 1)$,
- Condition (2) : $T_q \in \text{Mor}(P, \varepsilon)$ where q is the singleton partition in $\mathcal{P}(1, 0)$.

Solution. We compute on the one hand

$$\begin{aligned} (\text{id} \otimes T_p) \circ \rho_{u^{\otimes 2}}(e_{i_1} \otimes e_{i_2}) &= \sum_{j_1, j_2=1}^N u_{i_1 j_1} u_{i_2 j_2} \otimes T_p(e_{j_1} \otimes e_{j_2}) \\ &= \sum_{j_1, j_2=1}^N u_{i_1 j_1} u_{i_2 j_2} \otimes \delta_{j_1 j_2} e_{j_1} \\ &= \sum_{j=1}^N u_{i_1 j} u_{i_2 j} \otimes e_j \end{aligned}$$

and on the other hand

$$\begin{aligned} \rho_u \circ T_p(e_{i_1} \otimes e_{i_2}) &= \rho_u(\delta_{i_1 i_2} e_{i_1}) \\ &= \delta_{i_1 i_2} \sum_{j=1}^N u_{i_1 j} \otimes e_j \end{aligned}$$

By linear independence, we conclude that for any i_1, i_2, j ,

$$u_{i_1 j} u_{i_2 j} = \delta_{i_1 i_2} u_{i_1 j}.$$

Moreover, T_p^* satisfies the corresponding relations with $u^* = u^t$, yielding Conditions (1) and (3)³. The converse straightforwardly follows from the same computation.

As for the second point,

$$\begin{aligned} (\text{id} \otimes T_q) \circ \rho_u(e_i) &= \sum_{j=1}^N u_{i j} \otimes T_q(e_j) \\ &= \sum_{j=1}^N u_{i j} \otimes 1 \end{aligned}$$

while $\varepsilon \circ T_q(e_j) = 1 \otimes 1$. Thus,

$$\sum_{j=1}^N u_{i j} = 1$$

and using u^* instead, we see that Condition (2) holds. Once again, the converse is straightforward. ■

Remark 2.2.11. We have included the map T_q in the statement of Exercise 12 because it is a nice and important example of partition map. However, it is redundant because of the orthogonality assumption. Indeed, if T_{\sqcup} and T_p are intertwiners, then so is

$$T_{\sqcup \otimes |} \circ T_p = T_{(\sqcup \otimes |) \circ p} = T_q.$$

Example 2.2.12. It follows from Exercise 12 that S_N^+ is a partition quantum group and that its category of partitions is generated by p and q . Let us show that this category of partitions is in fact NC . This can be done in two steps :

3. Recall that $u_{i,j}^* = u_{i,j}$ is part of the axioms of an orthogonal compact matrix quantum group.

1. Let $k \geq 1$ and consider the partition

$$\left(| \otimes | \otimes (\sqcup \otimes |)^{\otimes(k-1)} \otimes | \otimes | \right) \circ p^{\otimes k}. \quad (2.6)$$

This is a partition on $k+3$ points with only one block. In other words, any one-block partition is in the category of partitions $\langle p \rangle$ generated by p .

2. Let us prove by induction on n that any noncrossing partition on at most n points is in $\langle p \rangle$. This is clear for $n \leq 3$. If it is true for some n , let p be a partition on $n+1$ points. If p has only one block, then it is in $\langle p \rangle$ by the first point. Otherwise, because it is noncrossing there is an interval b in p , i.e. a block with all points consecutive (this results from a straightforward induction argument). Rotating p we can then write it as $b \otimes p'$ for some partition p' on at most n points. The result now follows from the induction hypothesis.

Abelianization yields that S_N corresponds to the category \mathcal{P} of all partitions, a result which was proved independently around the same time by P. Martin in [Mar94] and by V. Jones in [Jon93].

We conclude by another example, of a slightly different flavour :

Exercise 13. Let $\mathcal{O}(H_N^+)$ be the quotient of $\mathcal{O}(O_N^+)$ by the relations

$$u_{ij}u_{ik} = 0 = u_{ik}u_{ij} \quad (2.7)$$

for all $1 \leq i \leq N$ and $j \neq k$. Prove that this is the partition quantum group corresponding to the category NC_{even} of all even noncrossing partitions, meaning noncrossing partitions such that all blocks have even size. Why is this called the hyperoctahedral quantum group?

Solution. Let $p \in NC(2,2)$ be the partition with only one block. Then,

$$\begin{aligned} (\text{id} \otimes T_p) \circ \rho_{u^{\otimes 4}}(e_{i_1} \otimes e_{i_2}) &= \sum_{j_1, j_2=1}^N u_{i_1 j_1} u_{i_2 j_2} \otimes T_p(e_{j_1} \otimes e_{j_2}) \\ &= \sum_{j_1, j_2=1}^N u_{i_1 j_1} u_{i_2 j_2} \otimes \delta_{j_1 j_2} e_{j_1} \otimes e_{j_1} \\ &= \sum_{j=1}^N u_{i_1 j} u_{i_2 j} \otimes e_j \otimes e_j \end{aligned}$$

while

$$\begin{aligned} \rho_{u^{\otimes 2}} \circ T_p(e_{i_1} \otimes e_{i_2}) &= \delta_{i_1 i_2} \rho_{u^{\otimes 2}}(e_{i_1} \otimes e_{i_1}) \\ &= \delta_{i_1 i_2} \sum_{j=1}^N u_{i_1 j} u_{i_2 j} \otimes e_j. \end{aligned}$$

Replacing u by $u^* = u^t$, we get that T_p is an intertwiner if and only if the relations in the statement hold.

We have therefore shown that p generates the category of partitions of H_N^+ . Note that NC_{even} is indeed a category of partitions, since all the operations preserve even blocks. Thus, $\langle p \rangle \subset NC_{\text{even}}$ and the converse inclusion follows from the same argument as in Example 12 : the analogue of Equation (2.6) yields that any one-block partition on an even number of points is in $\langle p \rangle$, and from then on the proof goes by induction.

Consider the abelianization G of H_N^+ . Its coefficient functions satisfy Equation (2.7), so that on each row and each column of any element of G , there is at most one non-zero coefficient. Since G is moreover made of orthogonal matrices, it follows that there is exactly one non-zero coefficient on each row and column, and that this coefficient is -1 or 1 . In other words, G is the group of signed permutation matrices, i.e. permutation matrices where one allows -1 instead of 1 as a coefficient. This group is also known as the *hyperoctahedral group* H_N , hence the name of H_N^+ . ■

LECTURE 3

ALGEBRAIC APPLICATION : REPRESENTATION THEORY

Our goal in this lecture is to describe the representation theory of the quantum groups O_N^+ and S_N^+ . This was first done by T. Banica in [Ban96] and [Ban99b] respectively, using Temperley-Lieb categories and their variants. However, the setting of partition quantum groups allows us to take a more general approach. We will therefore give a description of the representation theory of any partition quantum group associated to *noncrossing* partitions, and then apply it to our favourite examples. This is the approach developed in [FW16].

3.1 BUILDING PROJECTIONS

Let us fix once and for all a category of partitions \mathcal{C} . Our first task is to find all the irreducible representations of $\mathbb{G}_N(\mathcal{C})$. By Corollary 1.4.9, we know that it is enough to find irreducible subrepresentations of $u^{\otimes k}$ for all $k \in \mathbf{N}$, which by definition are given by minimal projections in $\text{Mor}_{\mathbb{G}_N(\mathcal{C})}(u^{\otimes k}, u^{\otimes k})$.

3.1.1 LINEAR INDEPENDENCE OF NONCROSSING PARTITION OPERATORS

The good news is that we have a nice generating family of this space, namely the maps T_p for $p \in \mathcal{C}$. However, this generating family may not be linearly independent and this is a source of troubles. We will therefore, for the sake of simplicity, rule it out in this lecture thanks to the following result :

THEOREM 3.1.1 Let N be an integer,

1. If $N \geq 2$, then the linear maps $(T_p)_{p \in NC_2(k, \ell)}$ are linearly independent for all $k, \ell \in \mathbf{N}$,
2. If $N \geq 4$ and \mathcal{C} is any category of *noncrossing* partitions, then $(T_p)_{p \in \mathcal{C}(k, \ell)}$ are linearly independent for all $k, \ell \in \mathbf{N}$.

Proof. Our strategy will be to deduce the second point from the first one.

1. It is sufficient to prove that for any $k \in \mathbf{N}$, the vectors $\xi_p = f_p^*$ are linearly independent for all $p \in NC_2(k, 0)$. For this, we can try to show that the Gram determinant is non-zero. Given two partitions p and q ,

$$\begin{aligned} \langle \xi_p, \xi_q \rangle &= \sum_{i_1, \dots, i_k} \sum_{j_1, \dots, j_k} \langle \delta_p(\mathbf{i}) e_{i_1} \otimes \dots \otimes e_{i_k}, \delta_q(\mathbf{j}) e_{j_1} \otimes \dots \otimes e_{j_k} \rangle \\ &= \sum_{i_1, \dots, i_k} \delta_p(\mathbf{i}) \delta_q(\mathbf{i}). \end{aligned}$$

The last expression is the number of tuples which are compatible with both p and q . Let us denote by $p \vee q$ the partition obtained by gluing together blocks of p and q having a common point. Then,

$$\delta_p(\mathbf{i}) \delta_q(\mathbf{i}) = \delta_{p \vee q}(\mathbf{i})$$

so that

$$\langle \xi_p, \xi_q \rangle = N^{b(p \vee q)}.$$

The determinant of this matrix is known as the *meander determinant* and was computed by P. Di Francesco, O. Golinelli and E. Guitter in [DFGG97, Sec 5.2] (see also [BC10, Thm 6.1] for another proof), yielding the following result :

$$\det = \prod_{i=1}^k P_i(N)^{a_{k,i}},$$

where

$$a_{k,i} = \binom{2k}{k-i} - \binom{2k}{k-i-1} + \binom{2k}{k-i-2}$$

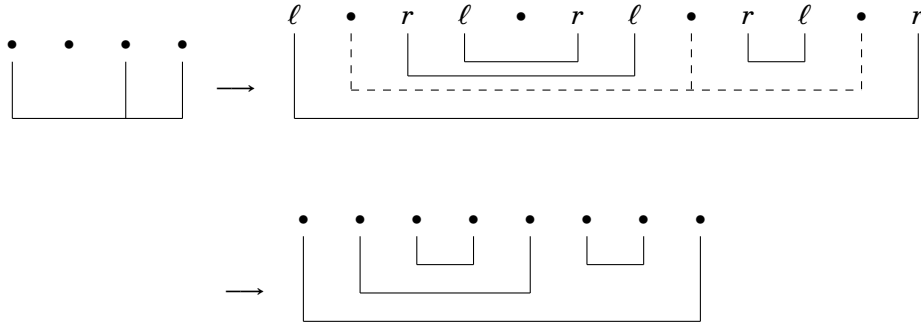
and P_i is the i -th *dilated Chebyshev polynomial of the second kind*, defined recursively by $P_0(X) = 1$, $P_1(X) = X$ and for any $i \geq 1$,

$$XP_i(X) = P_{i+1}(X) + P_{i-1}(X).$$

One easily checks that the roots of P_i are exactly $\{\cos(2j\pi/i) \mid 0 \leq j \leq i-1\}$ and the result follows.

2. The idea will be to reduce the problem to the previous case. For that, notice that given a partition $p \in NC_2(2k, 2\ell)$, one can produce another partition $\hat{p} \in NC(k, \ell)$ by gluing points two-by-two. Conversely, if $p \in NC(k, \ell)$, it can be "doubled" in the following way :

- (a) Assume for simplicity that p lies on one line. For each point a of p , draw a point a_ℓ on its left and a_r on its right.
- (b) Then, connect a_ℓ to b_r if a and b are connected and b is the closest (travelling from left to right cyclically) point of the block.
- (c) Here is an example :



Because of the linear independence proved in the first part, $\xi_p \mapsto \xi_{\hat{p}}$ yields a well-defined surjective linear map

$$\Phi : \text{Vect}\{\xi_p \mid p \in NC_2(2k, 0)\} \subset (\mathbf{C}^N)^{\otimes 2k} \rightarrow \text{Vect}\{\xi_p \mid p \in NC(k, 0)\} \subset (\mathbf{C}^{N^2})^{\otimes k}.$$

Now, it follows from [KS08, Prop 3.1], that for $p, q \in NC_2(2k)$,

$$\langle \xi_p, \xi_q \rangle = N^k \frac{\langle \xi_{\hat{p}}, \xi_{\hat{q}} \rangle}{\|\xi_{\hat{p}}\| \|\xi_{\hat{q}}\|}.$$

This means that the Gram matrices of the two families are conjugate by the diagonal matrix with coefficients $N^{k/2} \|\xi_{\hat{p}}\|$ which is invertible, hence the linear independence. ■

3.1.2 PROJECTIVE PARTITIONS

Assuming therefore $\mathcal{C} \in NC$ and $N \geq 4$, how can we build projections in $\text{Mor}_{\mathbb{G}_N(\mathcal{C})}(k, k)$? A natural thing to do is to look first for operators T_p which may be projections. The partition p should then satisfy $p = pp = p^*$, but one easily sees that this does not yield a projection in general for normalization reasons :

$$T_p T_p = N^{r(p,p)} T_p.$$

This is however easy to fix and leads to the following key definition :

DEFINITION 3.1.2. A partition p is said to be *projective* if $pp = p = p^*$. Then, there is a multiple S_p of T_p which is an orthogonal projection.

To questions immediately arise :

- Are all T_p 's which are proportional to a projection of this form ?
- Are there many projective partitions ?

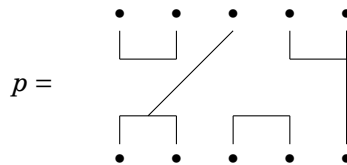
The answers to both questions rely on a fundamental fact that we will now explain. Given a partition p , we call *through-blocks* the blocks containing both upper and lower points and we denote their number by $t(p)$.

Proposition 3.1.3. Any noncrossing partition $p \in \mathcal{P}(k, \ell)$ can be written in a unique way in the form $p = p_d^* p_u$, where $p_u \in \mathcal{P}(k, t(p))$, $p_d \in \mathcal{P}(\ell, t(p))$ and both satisfy

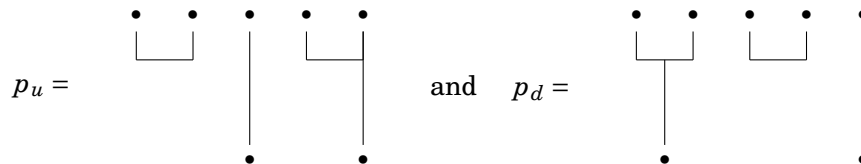
1. All lower point are in different blocks,
2. Each lower point is connected to at least one upper point,
3. If $i < j$ are lower points and $a(i), a(j)$ the leftmost upper point connected to i and j respectively, then $a(i) < a(j)$.

This is called the through-block decomposition of p .

Proof. This statement is obvious pictorially, for instance :



yields



This suggests the procedure to build the partitions. Let $b_1, \dots, b_{t(p)}$ be the through-bblocks of p ordered by their leftmost point in the upper row. Then, to build p_u we start with the upper row of p and connect b_i to the i -th (starting from the left) lower point. The construction of p_d is similar, using the lower parts of the through-blocks b'_1, \dots, b'_n . The crucial thing is that b_i and b'_i are the two parts of the same through-block. Indeed, if there exists $i < j$ and $k < \ell$ such that b_i is connected to b'_ℓ and b_j to b'_k , then this produces a crossing. As a consequence, $p_d^* p_u = p$.

To prove uniqueness, simply notice that if $p_d'^* p_u'$ is another through-block decomposition, then non-through-block parts must coincide since they are the non-through-blocks of p , and the through-blocks are completely determined by the properties in the statement. ■

Let us show how useful this is by answering the two previous questions in one shot :

Proposition 3.1.4. *A partition p is projective if and only if there exists a partition r such that $p = r^*r$. As a consequence,*

- T_p is a multiple of a partial isometry for all p ,
- T_p is a multiple of a projection if and only if p is projective.

Proof. If p is projective, then $p = p^*p$. Conversely, let r be any partition and let $r = r_d^*r_u$ be its through-block decomposition. The properties of Proposition 3.1.3 imply that

$$r_d r_d^* = |^{\otimes d} = r_u r_u^*.$$

Thus,

$$r^*r = r_u^*r_d r_d^*r_u = r_u^*r_u$$

and

$$(r_u^*r_u)^* (r_u^*r_u) = r_u^*r_u r_u^*r_u = r_u^*r_u. \quad \blacksquare$$

Using this we can now transfer the usual notions of equivalence and comparison of projection to projective partitions :

DEFINITION 3.1.5. Let p, q be two projective partitions. Then,

- We say that p is *dominated* by q , and write $p \preceq q$, if $pq = p = qp$,
- We say that p is *equivalent* to q , and write $p \sim q$, if there exists a partition r such that $r^*r = p$ and $rr^* = q$.

All this is nice and encouraging since it shows that partitions encode a structure comparable to that of matrix algebras. However, for a projective partition p , the projection S_p usually fails to be minimal. For instance, $T_{|^{\otimes k}} = \text{Id}$. To get smaller projections, we can use the comparison relation and subtract smaller projections. This yields

DEFINITION 3.1.6. Let \mathcal{C} be a category of noncrossing partitions and let $p \in \mathcal{C}$ be a projective partition. We set

$$R_p = \sup_{q \in \mathcal{C}, q \prec p} S_q$$

and

$$P_p = S_p - R_p$$

Note that it is not even clear that $P_p \neq 0$. A little linear algebra argument is needed to show that the supremum in the definition is a linear combination of maps T_r with $t(r) < t(p)$:

Proposition 3.1.7. *The projection R_p is a linear combination of maps T_r with $t(r) < t(p)$. As a consequence, $P_p \neq 0$.*

Proof. We first claim that if M is a direct sum of matrix algebras and $(P_i)_{i \in I}$ are orthogonal projections, then $R = \sup_{i \in I} P_i$ is a linear combination of projections $(Q_j)_{j \in J}$ such that for any $j \in J$, there exists $i \in I$ with $\text{Im}(Q_j) \subset \text{Im}(P_i)$. Indeed, there is a basis $(e_\ell)_{1 \leq \ell \leq s}$ of $\text{Im}(R)$ such that for every $1 \leq \ell \leq s$, there is an index $i \in I$ such that $e_\ell \in \text{Im}(P_i)$. Complete this basis with an orthonormal basis of the orthogonal complement of $\text{Im}(R)$ and let B be the change-of-basis matrix from this basis to the canonical basis of \mathbf{C}^n . This means that

$$R = B^{-1} \left(\sum_{1 \leq \ell \leq s} E_\ell \right) B,$$

where the (ℓ, ℓ) -th coefficient of E_ℓ is 1 and all the others are 0. Setting $Q_\ell = B^{-1}E_\ell B$ for $1 \leq \ell \leq s$, we get minimal projections summing up to R . Moreover, $\text{Im}(Q_\ell) = \mathbf{C}e_\ell \subset \text{Im}(P_i)$ for some i . Eventually, up to splitting the P_i 's in direct sums, we may assume that they belong to one of the blocks of M . Then, $Q_\ell \in M$ for all ℓ and the claim is proven.

Let now $R_p = \sum_\ell Q_\ell$ be the decomposition given by the previous claim. For each ℓ ,

$$Q_\ell = \sum_r \lambda_r T_r$$

but since its range is contained in the range of some T_q for $q < p$, we have

$$Q_\ell = T_q Q_\ell = \sum_r \lambda_r T_{qr}.$$

Because $t(qr) \leq t(q) < t(p)$, the proof of the first statement is complete. As for the second one, it follows from the linear independence of Theorem 3.1.1. \blacksquare

3.2 FROM PARTITIONS TO REPRESENTATIONS

We have constructed orthogonal projections in the space of self-intertwiners of $u^{\otimes k}$, we can therefore obtain subrepresentations from this :

DEFINITION 3.2.1. We set

$$u_p = P_p u^{\otimes k} P_p.$$

3.2.1 IRREDUCIBILITY

We will proceed to describe the whole representation theory of $\mathbb{G}_N(\mathcal{C})$ using the representations u_p . The first problem concerning u_p is irreducibility.

THEOREM 3.2.2 The representation u_p is irreducible for all projective partitions $p \in \mathcal{C}$. Moreover, for any irreducible representation v of $\mathbb{G}_N(\mathcal{C})$, there exists a projective partition $p \in \mathcal{C}$ such that $v \sim u_p$.

Proof. We have to prove that

$$P_p \text{Mor}_{\mathbb{G}_N(\mathcal{C})}(u^{\otimes k}, u^{\otimes k}) P_p = \mathbb{C} P_p$$

and it is of course enough to prove that $P_p T_r P_p \in \mathbb{C} P_p$ for all $r \in \mathcal{C}$. If $t(prp) < t(p)$, then r is an equivalence between two projective partitions strictly dominated by p , hence T_r is dominated by R_p . It follows that $P_p T_r P_p = 0$. If $t(prp) = t(p)$, consider the partition

$$q = p_u r p_u^*.$$

It has $t(p)$ lower points and $t(p)$ upper points. Moreover,

$$p r p = p_u^* q p_u$$

has $t(p)$ through-blocks so that in q , any lower point is connected to exactly one upper point. The only noncrossing partition having this property is $|\otimes^{t(p)}$, hence $p r p = p_u^* p_u = p$. It follows that

$$P_p T_r P_p = P_p T_p P_p \in \mathbb{C} P_p.$$

To prove the second part of the statement, it is enough to show that any irreducible subrepresentation of $u^{\otimes k}$ is equivalent to some u_p . By Theorem 1.4.7, this will follow from the fact that the supremum of the projections P_p is the identity. So let Q be this supremum. An easy induction on the number of through-blocks shows that Q dominates T_p for all p . In particular, it dominates $T_{|\otimes^k} = \text{Id}$, hence the result. \blacksquare

We now have found all the irreducible representations of $\mathbb{G}_N(\mathcal{C})$, but our list is certainly highly redundant. In other words, our second task is to decide whether u_p and u_q are equivalent or not. Once again, this matches perfectly with our previous definitions.

Proposition 3.2.3. *Let p and q be projective partitions in \mathcal{C} . Then, $u_p \sim u_q$ if and only if $p \sim q$.*

Proof. Assume first that $p \sim q$ and let $r \in \mathcal{C}$ be such that $r^*r = p$ and $rr^* = q$. It then follows from arguments similar to the proof of Theorem 3.2.2 that $V = P_q T_r P_p$ satisfies

$$V^*V = P_p \text{ and } VV^* = P_q,$$

i.e. V is an equivalence between u_p and u_q .

Conversely, assume that there exists a unitary intertwiner

$$V \in \text{Mor}_{\mathbb{G}_N(\mathcal{C})}(u_p, u_q) = P_q \text{Mor}_{\mathbb{G}_N(\mathcal{C})}(u^{\otimes k}, u^{\otimes \ell})P_p$$

and extend it to a partial isometry $W : \text{Mor}_{\mathbb{G}_N(\mathcal{C})}(u^{\otimes k}, u^{\otimes \ell})$. Then, there exists partitions $r_i \in \mathcal{C}$ such that

$$W = \sum_i \lambda_i T_{r_i}.$$

Let i be any index and note that because $WP_p = W$, $T_{r_i}P_p \neq 0$ hence $P_p T_{r_i^* r_i} P_p \neq 0$. But as we saw in the proof of Theorem 3.2.2, this implies that $r_i^* r_i = p$. Doing the same reasoning for $P_q W = W$ shows that $r_i r_i^* = q$, hence $p \sim q$. ■

3.2.2 FUSION RULES

The last step to describe the representation theory is to compute the so-called *fusion rules*. This means that for two irreducible representations v and w , we will find all the irreducible subrepresentations of $v \otimes w$. Note that the given two projective partitions $p \in \mathcal{C}(k, k)$ and $q \in \mathcal{C}(\ell, \ell)$, the intertwiner space of $u_p \otimes u_q$ is by definition

$$\text{Mor}_{\mathbb{G}_N(\mathcal{C})}(u_p, u_q) = (P_p \otimes P_q) \text{Mor}_{\mathbb{G}_N(\mathcal{C})}(u^{\otimes(k+\ell)}, u^{\otimes(k+\ell)}) (P_p \otimes P_q).$$

A good starting point is therefore to find the projective partitions r such that $(P_p \otimes P_q)P_r \neq 0$.

Proposition 3.2.4. *Let $X_{\mathcal{C}}(p, q)$ be the set of projective partitions $r \leq p \otimes q$ such that there is no projective partition $p' < p$ or $q' < q$ satisfying $r \leq p' \otimes q$ or $r \leq p \otimes q'$. Then, there exists a unitary equivalence*

$$u_p \otimes u_q \sim \sum_{r \in X_{\mathcal{C}}(p, q)} u_r.$$

Proof. We will proceed in two steps, analyzing the operators $(P_p \otimes P_q)P_r$.

1. Let us first prove that if $r \notin X_{\mathcal{C}}(p, q)$, then $(P_p \otimes P_q)P_r = 0$. Indeed,

$$\begin{aligned} P_p \otimes P_q &= (T_p - R_p) \otimes (T_q - R_q) \\ &= T_p \otimes T_q - (T_p \otimes R_q + R_p \otimes T_q - R_p \otimes R_q) \\ &= T_{p \otimes q} - (A + B - C). \end{aligned}$$

Noticing that $AB = BA = C$, we see that

$$R = A + B - C$$

is the supremum of the two commuting projections A and B . To conclude, let us note that a straightforward induction shows that for any noncrossing partition s , R_s is the supremum of the projections $P_{s'}$ for all $s' < s$. Thus, if $r \notin X_{\mathcal{C}}(p, q)$ then S_r is dominated either by $T_p \otimes R_q$ or by $R_p \otimes T_q$ and the same holds for $P_r < S_r$. It then follows that $(P_p \otimes P_q)P_r = 0$.

2. If now $r \in X_{\mathcal{C}}(p, q)$, P_r is a minimal projection by Theorem 3.2.2 hence there exists $\lambda \geq 0$ such that

$$P_r R P_r = \lambda P_r$$

and because P_r is not dominated by R , $\lambda < 1$. Thus, setting

$$V = (P_p \otimes P_q)P_r,$$

we have

$$V^*V = (1 - |\lambda|^2)P_r = \mu^{-2}P_r$$

for some $\mu > 0$. It follows that setting $W = \mu V$ we get an equivalence between P_r and a subprojection of $P_p \otimes P_q$. Therefore, the range of this subprojection yields a subrepresentation equivalent to u_r .

To conclude, simply notice that by the two points of this proof,

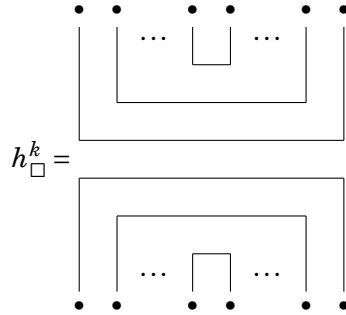
$$\sup_{r \in X_{\mathcal{C}}(p, q)} (P_p \otimes P_q)P_r(P_p \otimes P_q) = \sup_{r \leq p \otimes q} (P_p \otimes P_q)P_r(P_p \otimes P_q) = (P_p \otimes P_q).$$

■

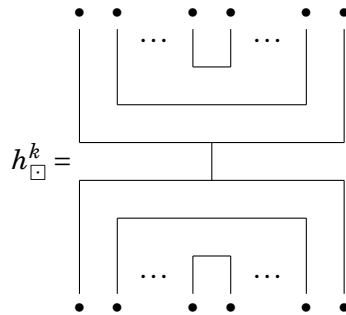
The previous result means that any irreducible subrepresentation of $u_p \otimes u_q$ is equivalent to u_r for some $r \in X_{\mathcal{C}}(p, q)$, and that any u_r appears at least once. But it does not say that different but equivalent r 's need both appear. To see this, we first need a better description of the set $X_{\mathcal{C}}(p, q)$. The intuitive idea is that a partition $r \in X_{\mathcal{C}}(p, q)$ is obtained by "mixing" some blocks of p with some blocks of q . Concretely, this mixing is done thanks to the following partitions :

DEFINITION 3.2.5. We denote by h_{\square}^k the projective partition in $NC(2k, 2k)$ where the i -th point in each row is connected to the $(2k - i + 1)$ -th point in the same row (i.e. an increasing inclusion of k blocks of size 2). If moreover we connect the points $1, k, 1'$ and k' , then we obtain another projective partition in $NC(2k, 2k)$ denoted by h_{\square}^k .

Here is a pictorial representation :



and



From this, we define binary operations on projective partitions, using $|$ to denote the identity partition :

$$\begin{aligned} p \square^k q &= (p_u^* \otimes q_u^*) \left(|^{\otimes t(p)-k} \otimes h_{\square}^k \otimes |^{\otimes t(q)-k} \right) (p_u \otimes q_u) \\ p \square^k q &= (p_u^* \otimes q_u^*) \left(|^{\otimes t(p)-k} \otimes h_{\square}^k \otimes |^{\otimes t(q)-k} \right) (p_u \otimes q_u) \end{aligned}$$

where $p = p_u^* p_u$ and $q = q_u^* q_u$ are the through-block decompositions. We are now ready to complete the description of the representation theory of partition quantum groups :

Proposition 3.2.6. *Let \mathcal{C} be a category of noncrossing partitions, let $N \geq 4$ be an integer and let $p, q \in \mathcal{C}$ be projective partitions. Then, $r \in X_{\mathcal{C}}(p, q) \setminus \{p \otimes q\}$ if and only if there exists $1 \leq k \leq \min(t(p), t(q))$ such that $r = p \square^k q$ or $r = p \square^k q$. Moreover,*

$$u_p \otimes u_q = u_{p \otimes q} \oplus \bigoplus_{k=1}^{\min(t(p), t(q))} v_{p \square^k q} \oplus v_{p \square^k q}$$

where $v_r = u_r$ if $r \in \mathcal{C}$ and $v_r = 0$ otherwise.

Proof. The first assertion is a deep result and its proof would take us too far for this lecture. We therefore simply hope to have given enough motivation for the reader to endeavour reading the proof of [FW16, Prop 2.28].

As for the second part of the statement, first notice that as a consequence of the first part, the partitions in $X_{\mathcal{C}}(p, q)$ all have a different number of through-blocks. But it follows from the definition of equivalence that two equivalent projective partitions have the same number of through-blocks. Thus, the irreducible subrepresentations of $u_p \otimes u_q$ are pairwise non-equivalent and this forces the direct sum decomposition. ■

3.3 EXAMPLES

We have now done the hard work and it is high time to be repaid of our efforts by easily deducing from the previous results the representation theory of our main examples.

3.3.1 QUANTUM ORTHOGONAL GROUP

As already mentioned, the representation theory of quantum orthogonal groups was first computed by T. Banica in [Ban96]. The proof relied on the identification of the category $\mathcal{R}(O_N^+)$ of representations of O_N^+ with the *Temperley-Lieb category* $\text{TL}(N)$. With our setting, the result follows from elementary calculations.

THEOREM 3.3.1 (Banica) For $N \geq 2$, the irreducible representations of O_N^+ can be labelled by the non-negative integers in such a way that $u^0 = \varepsilon$, $u^1 = U$ and for any $n \in \mathbf{N}$,

$$u^1 \otimes u^n = u^{n+1} \oplus u^{n-1}$$

Proof. Recall that if two projective partitions p and q are equivalent, then $t(p) = t(q)$. Assume now that p and q are noncrossing projective pair partitions with $t(p) = t(q)$. Denoting by $p = p_u^* p_u$ and $q = q_u^* q_u$ their through-block decompositions, observe that $p_u, q_u \in NC_2$, so that $r = q_u^* p_u \in NC(2)$. Since $r^* r = p$ and $r r^* = q$, we have proven that the equivalence class of p is given by its number of through blocks. Setting $u^n = u_{|\otimes n}$ therefore gives all irreducible representations.

Since the empty partition corresponds to the trivial representation, $u^0 = \varepsilon$. Since $NC_2(1, 1)$ only contains the identity partition, $P_1 = T_1 = \text{Id}$ so that $u^1 = U$. Eventually, $t(|\square|^n) = n - 1$ and $|\square|^n \notin NC_2$, hence the fusion rules. ■

3.3.2 QUANTUM PERMUTATION GROUP

The case of quantum permutation groups was studied by T. Banica in [Ban99b]. Once again, his strategy relies on the Temperley-Lieb category where the number of strands is doubled. Using noncrossing partition, the proof is almost the same as for O_N^+ .

THEOREM 3.3.2 (Banica) For $N \geq 4$, the irreducible representations of S_N^+ can be labelled by the non-negative integers in such a way that $u^0 = \varepsilon$, $P = \varepsilon \oplus u^1$ and for any $n \in \mathbf{N}$,

$$u^1 \otimes u^n = u^{n+1} \oplus u^n \oplus u^{n-1}$$

Proof. The same argument as for O_N^+ shows that equivalence classes of projective partitions in NC correspond to the number of through-blocks, so that we set $u^n = u_{| \circ n}$ and $u^0 = \varepsilon$. However, this time $NC(1, 1)$ has two elements, $|$ and the double-singleton. The second one gives a copy of the trivial representation since it has no through-block, yielding the decomposition. Eventually, $t(| \square |^n) = n - 1$ and $t(| \square |^n) = n$, hence the fusion rules. ■

3.3.3 QUANTUM HYPEROCTAEDRAL GROUP

The two preceding examples may have given the impression that the number of through-blocks is the only important data of a projective partition. To show that this is not the case, let us consider the quantum hyperoctahedral group H_N^+ . Recall that the corresponding category of partitions consists in all even partitions. Let $p \in NC_{\text{even}}(2, 2)$ be the partition with only one-block. Then, $t(p) = 1$ but $p_u^* | \in NC(1, 2)$ is a block of size three. Thus, u_p is not equivalent to $u_{|}$ and this turns out to be the only obstruction to equivalence :

Exercise 14. *Prove that the irreducible representations of H_N^+ can be indexed by words on $\{0, 1\}$, with $\varepsilon = u^\emptyset$, $u = u^1$ and $u_p = u^0$.*

Solution. Let $p \in NC(k, k)$ and $q \in NC(k', k')$ have only one block. Then, $p_u^* q_u \in NC(k', k)$ is even if and only if k and k' have the same parity. If now $p \in NC_{\text{even}}$, it follows from a straightforward induction that

$$p = p_1 \otimes \cdots \otimes p_{t(p)}$$

where for all $1 \leq i \leq t(p)$, p_i is a projective partitions with its endpoints connected and $t(p_i) = 1$. If \tilde{p}_i is the one-block partitions on the same number of points as p_i , then

$$p \sim \tilde{p}_1 \otimes \cdots \otimes \tilde{p}_{t(p)}$$

and we conclude that we can label the irreducible representations as in the statement. The last thing is to check that different labels do not yield equivalent partitions. Let $w \neq w'$ be different words and set $p^w = p_{w_1} \otimes \cdots \otimes p_{w_n}$ where $p_1 = |$ and $p_0 = p$. Let i be an integer such that $w_i \neq w'_i$. Then, $p_d^{w*} p_u^{w'}$ contains as a subpartition $p_{w_{id}}^* q_{w_{iu}}$ which is a block of size three. Thus, the partitions are not equivalent. ■

We still have to compute the fusion rules, but here again a surprise awaits us. Applying Proposition 3.2.6, we see that

$$\begin{aligned} u^1 \otimes u^0 &= u^{10} \oplus u^1 \oplus \varepsilon \\ u^0 \otimes u^1 &= u^{01} \oplus u^1 \oplus \varepsilon \end{aligned}$$

Because $u^{10} \neq u^{01}$, we see that the fusion rules are non-commutative! This is a purely quantum phenomenon. The complete description of the representation theory of H_N^+ was established by T. Banica and R. Vergnioux in [BV09]. To state it conveniently we need some notations. Let W be the set of words on $\{0, 1\} = \mathbf{Z}_2$ and endow it with the following operations :

- $\overline{w_1 \cdots w_n} = w_n^{-1} \cdots w_1^{-1}$,
- $w_1 \cdots w_n \bullet w'_1 \cdots w'_m = w_1 \cdots w_n w'_1 \cdots w'_m$,
- $w_1 \cdots w_n * w'_1 \cdots w'_m = w_1 \cdots w_{n-1} (w_n + w'_1) w'_2 \cdots w'_m$,

THEOREM 3.3.3 (Banica-Vergnioux) The irreducible representations of H_N^+ can be indexed by W in such a way that $\varepsilon = u^\emptyset$, $u = u^1$ and $u_p = u^0$. Moreover, given two words $w, w' \in W$, we have

$$u^w \otimes u^{w'} = \bigoplus_{w=v \bullet z, w'=\bar{z} \bullet v'} u^{v \bullet v'} \oplus u^{v * v'}$$

Proof. The first part of the statement was proven in Exercise 14. As for the fusion rules, let us consider two words $w = w_1 \cdots w_n$ and $w' = w'_1 \cdots w'_m$ and let $k \leq \max(m, n)$. Setting $p_0 = p$, $p_1 = |$ and

$$p_w = p_{w_0} \otimes \cdots \otimes p_{w_n},$$

we see that in $p_w \square^k p_{w'}$, $p_{w_{n-i+1}}$ is glued to $p_{w'_i}$. But if $w_{n-i+1} \neq w'_i$, then this yields a block of size three, which therefore does not belong to NC_{even} . Thus, $p_w \square^k p_{w'} \in NC_{\text{even}}$ if and only if the last k letter of w match the first k letters of w' , i.e. $w = v \bullet z$ and $w' = z \bullet v'$ with z of length k . Moreover, the through blocks of $p_w \square p_{w'}$ are just the first $n - i$ through-blocks of p_w followed by the last $m - i$ through-blocks of $p_{w'}$ and this corresponds to $p_{v \bullet v'}$. Considering now $p_w \square^2 p_{w'}$, the same decomposition is needed for it to be in NC_{even} , but this time the result contains an extra through-block obtained by gluing $p_{w_{n-k+1}}$ and $p_{w'_k}$. If they have the same parity, then the result has eight points hence is equivalent to p_0 while it is equivalent to p_1 if their parity differ. Hence, this is equivalent to $p_{v * v'}$. ■

LECTURE 4

PROBABILISTIC APPLICATION : THE HAAR STATE

We have taken the bias, throughout these lectures, to work exclusively in an algebraic setting. This may seem odd given that the name of compact groups suggest that they are *compact* objects. From an operator algebraic perspective, compact quantum groups should therefore be given by unital C*-algebras instead of *-algebras. There is indeed a bridge between our setting and the more usual definition of compact quantum groups using operator algebras. Since the goal of Lecture 5 will be to prove analytical results about S_N^+ using combinatorics, we will now describe the path from algebra to analysis.

4.1 FROM ALGEBRA TO ANALYSIS

To explore the analytical properties of a compact matrix quantum group \mathbb{G} , one may simply consider the universal enveloping C*-algebra of $\mathcal{O}(\mathbb{G})$, to which the coproduct extends by universality. To understand why this is unsatisfying, let us give examples of compact matrix quantum group of a different type than those encountered up to now.

4.1.1 A DISCRETE DETOUR

Let Γ be a finitely generated discrete group with a generating set $\{g_1, \dots, g_N\}$. Recall that its *group algebra* is the vector space $\mathbf{C}[\Gamma]$ of all finitely supported linear combinations of elements of Γ , endowed with the product coming from the group law. Let us consider the matrix

$$u = \text{diag}(g_1, \dots, g_N) \in M_N(\mathbf{C}[\Gamma]).$$

Then, $(\mathbf{C}[\Gamma], u)$ is a compact matrix quantum group in the sense of Definition 1.3.6. It is usually denoted by $\widehat{\Gamma}$ and called the *dual* of Γ .

Thus, finitely generated discrete groups are examples of compact matrix quantum groups and the enveloping C*-algebra $\mathbf{C}[\Gamma]$ is nothing but the universal group C*-algebra $C^*(\Gamma)$. It is well-known that this is not the correct operator algebra to look at when investigating geometric or probabilistic properties of Γ (like amenability, property (T) and so on). One should instead consider the reduced C*-algebra $C_r^*(\Gamma)$ and its associated von Neumann algebra $L(\Gamma)$. These reduced operator algebras are constructed by embedding $\mathbf{C}[\Gamma]$ into $\mathcal{B}(\ell^2(\Gamma))$ through the left regular representation. Alternately, this embedding is the GNS construction (we refer the reader to [Bla06, II.6.4]) of the state¹ δ_e on Γ .

To find an analogue of this, first note that the irreducible representations of $\widehat{\Gamma}$ are all one-dimensional and given (up to equivalence) by the elements of g . Since δ_e vanishes by definition on all elements $g \neq e$, the following definition is natural :

DEFINITION 4.1.1. Let \mathbb{G} be a compact matrix quantum group, let $\text{Irr}(\mathbb{G})$ be the set of equivalence classes of irreducible representations of \mathbb{G} and let u^α be a representative of $\alpha \in \text{Irr}(\mathbb{G})$. The *Haar*

1. A *state* on a *-algebra is a positive linear map φ such that $\varphi(1) = 1$, see for instance [Bla06, II.6.2.1].

integral of \mathbb{G} is the linear map $h : \mathcal{O}(\mathbb{G}) \rightarrow \mathbf{C}$ defined on the basis of coefficients of irreducible representations by

$$h\left(u_{ij}^\alpha\right) = \delta_\varepsilon(\alpha).$$

The name needs some explanations. To understand where it comes from, first note that for any $\alpha \in \text{Irr}(\mathbb{G})$ and any $1 \leq i, j \leq \dim(\alpha)$,

$$(h \otimes \text{id}) \circ \Delta\left(u_{ij}^\alpha\right) = h\left(u_{ij}^\alpha\right) \cdot 1 = (\text{id} \otimes h) \circ \Delta\left(u_{ij}^\alpha\right). \quad (4.1)$$

This is an invariance property which recalls that of the Haar measure on a classical compact group. In particular if h' is another linear form on $\mathcal{O}(\mathbb{G})$ satisfying Equation (4.1), then

$$h(x) = h(x)h'(1) = h'((h \otimes \text{id}) \circ \Delta(x)) = (h \otimes h') \circ \Delta(x) = h'(x)h(1) = h'(x).$$

With this we can give a better justification of the name "Haar integral" :

Exercise 15. Prove that if G is a compact matrix group, then the Haar integral on $\mathcal{O}(G)$ coincides with the integration with respect to the Haar measure.

Solution. Notice first that for any $f \in \mathcal{O}(G)$, $\Delta(f)(x, y) = f(xy)$. If now φ denotes the linear map given by integration with respect to the Haar measure μ , then

$$(\varphi \otimes \text{id}) \circ \Delta(f)(y) = \int_G f(xy) d\mu(x) = \int_G f(x) d\mu(x) = \varphi(f) \cdot 1$$

and the same works for $(\text{id} \otimes \varphi) \circ \Delta$. As explained above, this implies that $\varphi = h$. ■

4.1.2 POSITIVITY OF THE HAAR INTEGRAL

The only non-trivial fact about h is that it is positive. This will indeed be the first important result of this lecture. To prove it, let us first give a straightforward, though important, corollary of Theorem 1.4.7 :

Corollary 4.1.2. For any irreducible unitary representation v , $\bar{v} = v^{*t}$ is also an irreducible unitary representation.

Proof. By Corollary 1.4.9, v is unitarily equivalent to a subrepresentation of $u^{\otimes k}$ for some $k \in \mathbf{N}$. Thus, there exist a unitary matrix B and a unitary representation w satisfying

$$B^* u^{\otimes k} B = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}.$$

As a consequence, v^{*t} is a direct summand of $B^t u^{\otimes k} B^{*t}$, hence is a unitary representation. If V is the subspace of $(\mathbf{C}^N)^{\otimes k}$ on which v acts, then \bar{v} naturally acts on the conjugate Hilbert space \bar{V} . Moreover, T intertwines v with itself if and only if $\bar{T} = T^{*t}$ intertwines \bar{v} with itself. Thus,

$$\dim(\text{Mor}_{\mathbb{C}}(\bar{v}, \bar{v})) = \dim(\text{Mor}_{\mathbb{C}}(v, v)) = 1$$

and \bar{v} is irreducible. ■

DEFINITION 4.1.3. Let \mathbb{G} be a compact matrix quantum group and let $v = (v_{ij})_{1 \leq i, j \leq n}$ be an irreducible representation acting on $V = \mathbf{C}^n$. Then, $\bar{v} = (v_{ij}^*)_{1 \leq i, j \leq n}$ is called the *conjugate representation* of v .

Let us now proceed to prove that the Haar integral is a state. The Haar state for compact matrix quantum groups was first constructed by S.L. Woronowicz in [Wor87] in a completely different way :he directly proved the existence of a state satisfying Equation (4.1). Our more algebraic approach is due to [DK94].

THEOREM 4.1.4 (Woronowicz) Let (\mathbb{G}, u) be an orthogonal compact matrix quantum group. Then, the Haar integral h is a state on $\mathcal{O}(\mathbb{G})$. Moreover, it is faithful.

From now on, h will be called the *Haar state* of \mathbb{G} .

Proof. It turns out that positivity and faithfulness will be proved at the same time. For the sake of clarity, we proceed in several steps.

1. Let v be any finite-dimensional representation acting on V , and let W be the subspace of fixed vectors. The proof of the first point in Theorem 1.4.7 shows that v decomposes along $W \oplus W^\perp$. Since v_{W^\perp} does not contain the trivial representation, $\widehat{h}(v)$ vanishes on W^\perp , while it acts by the identity on W by definition. Thus it is the orthogonal projection onto the subspace of fixed vectors.
2. Let us show that for any irreducible representation v acting on V , the vector

$$\xi = \sum_{i=1}^{\dim(V)} e_i \otimes \bar{e}_i \in V \otimes \bar{V}$$

is fixed for $v \otimes \bar{v}$. Indeed,

$$\begin{aligned} \rho_{v \otimes \bar{v}}(\xi) &= \sum_{i=1}^{\dim(V)} \sum_{j,k=1}^{\dim(V)} v_{ij} v_{ik}^* e_j \otimes \bar{e}_k \\ &= \sum_{i=1}^{\dim(V)} \sum_{j,k=1}^{\dim(V)} \delta_{j,k} e_j \otimes \bar{e}_k \\ &= 1 \otimes \xi. \end{aligned}$$

3. We now claim that for any $\alpha, \beta \in \text{Irr}(\mathbb{G})$,

$$\text{Mor}_{\mathbb{G}}(\varepsilon, v^\alpha \otimes \overline{v^\beta}) \simeq \text{Mor}_{\mathbb{G}}(v^\beta, v^\alpha).$$

Indeed, if $T \in \mathcal{B}(V^\beta, V^\alpha)$, then

$$\xi_T = \sum_{i=1}^{\dim(v^\beta)} T(e_i) \otimes \bar{e}_i$$

is fixed for $v^\alpha \otimes \overline{v^\beta}$ if and only if T is an intertwiner, where $(e_i)_{1 \leq i \leq \dim(v^\beta)}$ is an orthonormal basis. This is a simple computation :

$$\begin{aligned} \left(\rho_{v^\alpha \otimes \overline{v^\beta}} \right) (\xi_T) &= \sum_{i=1}^{\dim(v^\beta)} \rho_{v^\alpha} \circ T(e_i) \otimes \rho_{\overline{v^\beta}} \bar{e}_i \\ &= \sum_{i=1}^{\dim(v^\beta)} (\text{id} \otimes T) \circ \rho_{v^\beta}(e_i) \otimes \rho_{\overline{v^\beta}} \bar{e}_i \\ &= (\text{id} \otimes T) \circ \rho_{v^\beta \otimes \overline{v^\beta}} \left(\sum_{i=1}^{\dim(v^\beta)} e_i \otimes \bar{e}_i \right) \\ &= (\text{id} \otimes T) \left(\sum_{i=1}^{\dim(v^\beta)} 1 \otimes e_i \otimes \bar{e}_i \right) \\ &= 1 \otimes \xi_T. \end{aligned}$$

Conversely, $\xi \in V^\alpha \otimes \overline{V^\beta}$ is a fixed vector for $v^\alpha \otimes \overline{v^\beta}$, if and only if

$$T_\xi : x \mapsto (\text{id} \otimes \bar{x}^*)(\xi)$$

is an intertwiner between v^β and v^α , because $\xi_{T_\xi} = \xi$ and $T_{\xi_T} = T$.

4. It follows from Schur's Lemma (Lemma 1.4.6) that $\text{Mor}_{\mathbb{G}}(v^{(\beta)}, v^{(\alpha)}) = \{0\}$ if $\alpha \neq \beta$. In particular,

$$\widehat{h}(v^{(\alpha)} \otimes \bar{v}^{(\beta)}) = 0$$

in that case.

5. For $\alpha = \beta$, the space $\text{Mor}_{\mathbb{G}}(\varepsilon, v^{(\alpha)} \otimes \bar{v}^{(\alpha)})$ is one-dimensional (again by Schur's Lemma) and is spanned by

$$\xi = \sum_{i=1}^{\dim(v^{(i)})} e_i \otimes \bar{e}_i.$$

Thus, if P_{ξ} denotes the orthogonal projection onto $\mathbf{C}\xi$, then

$$\begin{aligned} \widehat{h}(v_{ij}^{(\alpha)} \otimes v_{kl}^{(\alpha)*}) &= \langle e_j \otimes \bar{e}_l, P_{\xi}(e_i \otimes \bar{e}_k) \rangle \\ &= \delta_{ik} \delta_{jl} \frac{1}{\dim(v^{(i)})}. \end{aligned}$$

6. If now

$$x = \sum_{\alpha \in \text{Irr}(\mathbb{G})} \sum_{i,j=1}^{\dim(v^{(\alpha)})} \lambda_{i,j}^{\alpha} v_{ij}^{(\alpha)},$$

then

$$h(xx^*) = \sum_{\alpha \in \text{Irr}(\mathbb{G})} \sum_{i,j=1}^{\dim(v^{(\alpha)})} |\lambda_{i,j}^{\alpha}|^2 \geq 0$$

and this vanishes if and only if $x = 0$. ■

Remark 4.1.5. It follows from the proof above that h is *tracial* in the sense that $h(xy) = h(yx)$ for any $x, y \in \mathcal{O}(\mathbb{G})$. This is not true for arbitrary compact quantum groups, in fact it is equivalent to the property that for any unitary representation v, \bar{v} is also unitary (in general it is only equivalent to a unitary representation).

We can now build an embedding of $\mathcal{O}(\mathbb{G})$ into a Hilbert space. The sesquilinear form

$$\langle x, y \rangle = h(xy^*)$$

is an inner product on $\mathcal{O}(\mathbb{G})$ by Theorem 4.1.4. Taking the completion, we get a Hilbert space denoted by $L^2(\mathbb{G})$ together with a faithful representation

$$\lambda : \mathcal{O}(\mathbb{G}) \rightarrow \mathcal{B}(L^2(\mathbb{G})).$$

The weak closure of the range of λ is then a von Neumann algebra denoted by $L^{\infty}(\mathbb{G})$. We can now prove as a corollary a result which was mentioned in Remark 1.3.2 after the definition of orthogonal compact matrix quantum groups :

Corollary 4.1.6. *Let \mathbb{G} be a compact matrix quantum group. Then, $\mathcal{O}(\mathbb{G})$ embeds as a dense subalgebra of its universal enveloping C^* -algebra.*

4.2 WEINGARTEN CALCULUS

The definition of the Haar state given in the proof of Theorem 4.1.4 is in a sense explicit since it is given in the basis of coefficients of irreducible representations. However, when doing computations, one sometimes has to compute the Haar state for arbitrary polynomials in the coefficients of the fundamental representation u . There is no practical general formula for this, but if we restrict to partition quantum groups, then it is possible to express the image of these polynomials using the corresponding category of partitions.

When trying to compute $h(u_{i_1 j_1} \cdots u_{i_k j_k})$ for arbitrary indices, the basic idea is to recall that $\widehat{h}(u^{\otimes k})$ is the orthogonal projection onto the subspace of fixed points of $u^{\otimes k}$. Therefore, if we have enough data on this subspace, we may be able to derive information on its orthogonal projector. And it turns out that being a partition quantum group precisely means that we have a combinatorial description of the space $\text{Mor}_{\mathbb{G}}(\varepsilon, u^{\otimes k})$, which is the space of fixed points.

More precisely, if \mathcal{C} is a category of partitions and $\mathbb{G} = \mathbb{G}_N(\mathcal{C})$, then writing $\xi_p = f_p^*$ we have

$$\text{Mor}_{\mathbb{G}}(\varepsilon, u^{\otimes k}) = \text{Vect} \{ \xi_p \mid p \in \mathcal{C}(0, k) \}.$$

Now, to describe the orthogonal projection, a convenient tool is the Gram matrix of a basis. This one was already computed in the proof of Theorem 3.1.1 and we summarize this in a definition :

DEFINITION 4.2.1. The *Gram matrix* associated to \mathcal{C}, k and N is the $|\mathcal{C}(0, k)| \times |\mathcal{C}(0, k)|$ -matrix

$$\text{Gr}_N(\mathcal{C}, k)$$

with coefficients

$$\text{Gr}(\mathcal{C}, k)_{p, q} = N^{b(p \vee q)}$$

for $p, q \in \mathcal{C}(0, k)$. The corresponding *Weingarten matrix* is, if it exists,

$$W_N(\mathcal{C}, k) = \text{Gr}_N(\mathcal{C}, k)^{-1}.$$

As one may expect from this definition, the final formula for the Haar state will involve W_N rather than Gr_N , so that it will not be completely explicit. We will however show in the next section that it is possible to obtain asymptotic estimates on the coefficients of the Weingarten matrix which give free probabilistic information on $\mathcal{O}(\mathbb{G})$. The first proof of this result was done by T. Banica and B. Collins for O_N^+ and U_N^+ in [BC07] and then extended to arbitrary partition quantum groups with the same argument in [BS09, Thm 5.4].

THEOREM 4.2.2 (Banica-Collins) Let \mathcal{C} be a category of partitions and let \mathbf{N} be an integer. For any $k \in \mathbf{N}$, if $\{\xi_p \mid p \in \mathcal{C}(0, k)\}$ is linearly independent, then

$$h(u_{i_1 j_1} \cdots u_{i_k j_k}) = \sum_{p, q \in \mathcal{C}(0, k)} \delta_p(i_1, \dots, i_k) \delta_q(j_1, \dots, j_k) W_N(\mathcal{C}, k)_{pq}. \quad (4.2)$$

Proof. Let us denote by $W \subset (\mathbf{C}^N)^{\otimes k}$ the subspace of fixed vectors. As already mentioned, the left-hand side in Equation (4.2) is a coefficient of the orthogonal projection P_W onto W . Consider the map $\Phi : (\mathbf{C}^N)^{\otimes k} \rightarrow W$ given by

$$\Phi(x) = \sum_{p \in \mathcal{C}(0, k)} \langle x, \xi_p \rangle \xi_p.$$

This is a surjective map but it is not idempotent. Indeed,

$$\Phi(\xi_p) = \sum_{q \in \mathcal{C}(0, k)} \langle \xi_p, \xi_q \rangle \xi_q = \text{Gr}_N(\mathcal{C}, k) \xi_p.$$

In other words, $\Phi = \text{Gr}_N(\mathcal{C}, k) \circ P_W$, which readily yields $P_W = W_N(\mathcal{C}, k) \circ \Phi$. The proof now ends with an easy computation :

$$\begin{aligned} \langle \widehat{h}(u^{\otimes k}) e_{i_1} \otimes \cdots \otimes e_{i_k}, e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle &= \langle W_N(\mathcal{C}, k) \circ \Phi(e_{i_1} \otimes \cdots \otimes e_{i_k}), e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle \\ &= \sum_{p \in \mathcal{C}(0, k)} \langle e_{i_1} \otimes \cdots \otimes e_{i_k}, \xi_p \rangle \langle W_N(\mathcal{C}, k)(\xi_p), e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle \\ &= \sum_{p \in \mathcal{C}(0, k)} \delta_p(\mathbf{i}) \langle W_N(\mathcal{C}, k)(\xi_p), e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle \\ &= \sum_{p, q \in \mathcal{C}(0, k)} \delta_p(\mathbf{i}) W_N(\mathcal{C}, k)_{pq} \langle \xi_q, e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle \\ &= \sum_{p, q \in \mathcal{C}(0, k)} \delta_p(\mathbf{i}) \delta_q(\mathbf{j}) W_N(\mathcal{C}, k)_{pq}. \end{aligned}$$

■

4.3 APPLICATIONS

We will now give applications of Theorem 4.2.2 with a probabilistic flavour. This will only be a glimpse of a field of its own, which owes much to the combinatorial approach to free probability theory (see the book [NS06]). In particular, the use of *free cumulants* leads to spectacular applications of Weingarten calculus to *noncommutative de Finetti theorems* as in [BCS12] or to asymptotics of random matrices with noncommutative entries *à la Diaconis-Shahshahani* like in [BCS11].

4.3.1 SPECTRAL MEASURES

To be more precise, consider an element $x \in L^\infty(\mathbb{G})$ which is self-adjoint. Then, it generates a von Neumann subalgebra $\langle x \rangle \subset L^\infty(\mathbb{G})$ which is commutative, hence isomorphic to $L^\infty(\text{Sp}(x))$ by Borel functional calculus (see for instance [Bla06, I.4]). The restriction of h to this algebra is still a state, hence coincides with integration with respect to a Borel probability measure μ_x . We can therefore see x as a random variable and wonder about its *spectral measure* μ_x . We mainly have access to the moments of μ_x , which are given by

$$m_k(\mu_x) = \int_{\text{Sp}(x)} t^k d\mu_x(t) = h(x^k).$$

This means that we will compute moments and then try to reconstruct the probability measure. This is not always possible, but it will work on our cases. For later use, let us recall some facts about one of the most important probability distributions in free probability.

DEFINITION 4.3.1. The *semicircle distribution* (or *Wigner distribution*) μ_{sc} is the probability distribution on $[-2, 2]$ with density

$$\frac{1}{\pi} \sqrt{4 - x^2}$$

with respect to the Lebesgue measure.

The computation of the moments of μ_{sc} is a standard exercise in undergraduate integration.

Exercise 16. Prove that the moments of the semicircle distribution are given by $m_{2k+1}(\mu_{\text{sc}}) = 0$ and

$$m_{2k}(\mu_{\text{sc}}) = \frac{1}{k+1} \binom{2k}{k}.$$

Solution. Observe that because μ_{sc} has an even density, all its odd moments vanish. We can therefore focus on even moments and compute

$$\begin{aligned} m_{2k}(\mu_{\text{sc}}) &= \frac{1}{\pi} \int_{-2}^2 x^{2k} \sqrt{4 - x^2} dx \\ &= \frac{2^{2k+2}}{\pi} \int_0^2 \left(\frac{x}{2}\right)^{2k} \sqrt{1 - \left(\frac{x}{2}\right)^2} dx \\ &= \frac{2^{2k+2}}{\pi} \int_0^{\pi/2} (\sin(\theta))^{2k} \cos(\theta)^2 d\theta \\ &= \frac{2^{2k+2}}{\pi} \left(\frac{\pi}{2} \frac{(2k)!}{(2^k (k!))^2} - \frac{\pi}{2} \frac{(2k+2)!}{(2^{k+1} ((k+1)!))^2} \right) \\ &= \frac{1}{k+1} \binom{2k}{k}. \end{aligned}$$

■

4.3.2 TRUNCATED CHARACTERS

We will now investigate the character of the fundamental representation, that is to say the element

$$\chi = \sum_{i=1}^N u_{ii}.$$

Based on our probabilistic intuition, we will use from now on the following fancy but suggestive notation for the Haar state : $\int_{\mathbb{G}}$. The moments of χ are easy to compute and do not require Weingarten calculus :

Proposition 4.3.2. *Let \mathcal{C} be a category of noncrossing partitions and let $N \geq 4$. Then,*

$$m_k(\chi) = |\mathcal{C}(k, 0)|.$$

Proof. Denoting by $\text{Fix}(v)$ the subspace of fixed points of ρ_v , we have by definition

$$\begin{aligned} m_k(\chi) &= \int_{\mathbb{G}_N(\mathcal{C})} \chi^k \\ &= \text{Tr}(\widehat{h}(u^{\otimes k})) \\ &= \dim(\text{Fix}(u^{\otimes k})) \\ &= \dim(\text{Vect}\{\xi_p \mid p \in \mathcal{C}(k, 0)\}) \\ &= |\mathcal{C}(k, 0)| \end{aligned}$$

where the last lines comes from the linear independence of the partition vectors proven in Theorem 3.1.1. ■

Let us illustrate this result in the case of the quantum orthogonal groups :

Example 4.3.3. Assume that $\mathcal{C} = NC_2$ and first observe that for odd k , $NC_2(k, 0) = 0$. We now have to compute the number of noncrossing pairings on $2k$ points and this can be done by induction on k . Denoting by $(C_k)_{k \in \mathbb{N}}$ the numbers we are looking for, we have $C_1 = 1$. Moreover, consider a noncrossing pair partition p on $2k$ points and let ℓ be the point connected to 1. Then, p induces noncrossing pairings on $\{2, \dots, \ell - 1\}$ and $\{\ell + 1, \dots, 2k\}$. Conversely, given such pairings one can reconstruct p with the condition that 1 should be connected to ℓ . In other words, we have

$$C_k = \sum_{i=2}^{2k} C_{i-1} C_{2k-i+1}.$$

This uniquely defines the sequence $(C_k)_{k \in \mathbb{N}}$, and it turns out that the moments of the semicircle distribution satisfy this recursion relation. Thus, χ is a semicircular element.

For quantum permutation group, we can resort to the "doubling trick" explained in the proof of Theorem 3.1.1 :

Example 4.3.4. For S_N^+ , we have to compute the number of noncrossing partitions on k points. But the bijection $p \mapsto \widehat{p}$ used in the proof of Theorem 3.1.1 shows that

$$|NC(k, 0)| = |NC_2(2k, 0)|.$$

As a consequence, $\chi_{S_N^+}$ has the same distribution as $\chi_{O_N^+}^2$. This is known as the *free Poisson distribution* (or the *Marschenko-Pastur distribution*) with parameter 1. It is supported on $[0, 4]$ and has density

$$\frac{1}{2\pi} \sqrt{\frac{4}{x^2} - 1}$$

with respect to the Lebesgue measure.

The quantum hyperoctahedral group is more involved and the distribution of $\chi_{H_N^+}$ is the *free Bessel distribution* introduced in [BBCC11]. Instead of proving this we will, following the work of T. Banica and R. Speicher in [BS09], try to get more understanding on the previous results by considering *truncated characters* in the following sense :

DEFINITION 4.3.5. Let \mathcal{C} be a category of noncrossing partitions and let $N \geq 4$ be an integer. The truncated characters are the elements

$$\chi_t = \sum_{i=1}^{\lfloor tN \rfloor} u_{ii}.$$

for $t \in [0, 1]$.

The previous result can then be refined thanks to the Weingarten formula. This first requires an estimate on the Gram and Weingarten matrices. In the sequel, $O(N^{-1/2})$ means a matrix all of whose coefficients are dominated by $N^{-1/2}$.

Lemma 4.3.6. *Let \mathcal{C} be a category of partitions, let N, k be integers and let $\Gamma_N(\mathcal{C}, k)$ be the diagonal of $\text{Gr}_N(\mathcal{C}, k)$. Then,*

$$\begin{aligned} \text{Gr}_N(\mathcal{C}, k) &= \Gamma_N(\mathcal{C}, k) \left(1 + O\left(N^{-1/2}\right) \right) \\ \text{W}_N(\mathcal{C}, k) &= \Gamma_N(\mathcal{C}, k)^{-1} \left(1 + O\left(N^{-1/2}\right) \right) \end{aligned}$$

Proof. For clarity we will omit \mathcal{C} and k in the computations. The trick is to consider the coefficients of $\Gamma_N^{-1/2} \text{Gr}_N \Gamma_N^{-1/2}$:

$$\begin{aligned} \left(\Gamma_N^{-1/2} \text{Gr}_N \Gamma_N^{-1/2} \right)_{pq} &= \left(\Gamma_N^{-1/2} \right)_{pp} (\text{Gr}_N)_{pq} \left(\Gamma_N^{-1/2} \right)_{qq} \\ &= N^{b(p \vee q) - \frac{b(p)+b(q)}{2}}. \end{aligned}$$

If $p = q$, the result is 1. Otherwise, there is at least two blocks of p or of q which are merged in $p \vee q$, hence the result is less than $N^{-1/2}$. In other words, the matrix

$$B_N = \Gamma_N^{-1/2} \text{Gr}_N \Gamma_N^{-1/2} - \text{Id}$$

has all its coefficients dominated by $N^{-1/2}$, yielding the first part of the statement.

As for the second part, we have

$$\begin{aligned} \Gamma_N^{1/2} \text{W}_N^{-1} \Gamma_N^{1/2} &= (\text{Id} + B_N)^{-1} \\ &= \text{Id} + \sum_{n=1}^{+\infty} (-1)^n B_N^n \end{aligned}$$

and each term in the sum is dominated by $N^{-1/2}$, yielding the second part. ■

THEOREM 4.3.7 (Banica-Speicher) Let $N \geq 4$, let \mathcal{C} be a category of partitions and let u be the fundamental representation of $\mathbb{G}_N(\mathcal{C})$. Then,

$$\lim_{N \rightarrow +\infty} \int_{\mathbb{G}_N(\mathcal{C})} \chi_t^k = \sum_{p \in \mathcal{C}(k)} t^{b(p)}.$$

Proof. For clarity, we will omit k and \mathcal{C} in the notations since no confusion is possible. Let us first consider the sum of the first ℓ diagonal coefficients. We claim that

$$\int_{\mathbb{G}_N(\mathcal{C})} (u_{11} + \dots + u_{\ell\ell})^k = \text{Tr}(\text{W}_N \text{Gr}_\ell).$$

Indeed, by Theorem 4.2.2,

$$\begin{aligned} \int_{\mathbb{G}_N(\mathcal{C})} \left(\sum_{i=1}^{\ell} u_{ii} \right)^k &= \sum_{i_1, \dots, i_k=1}^{\ell} \int_{\mathbb{G}} u_{i_1 i_1} \cdots u_{i_k i_k} \\ &= \sum_{i_1, \dots, i_k=1}^{\ell} \sum_{p, q \in \mathcal{C}} \delta_p(\mathbf{i}) \delta_q(\mathbf{i}) W_N(p, q) \\ &= \sum_{p, q \in \mathcal{C}} \left(\sum_{i_1, \dots, i_k=1}^{\ell} \delta_p(\mathbf{i}) \delta_q(\mathbf{i}) \right) W_N(p, q). \end{aligned}$$

If both δ_p and δ_q do not vanish on \mathbf{i} , then this means that \mathbf{i} matches $p \vee q$. Because the indices run over a set of cardinality ℓ , there are therefore $\ell^{b(p \vee q)} = (\text{Gr}_{\ell})_{pq}$ tuples yielding a non-zero contribution, hence the result.

Using Lemma 4.3.6, we can now write

$$\begin{aligned} \int_{\mathbb{G}_N(\mathcal{C})} \chi_t^k &= \text{Tr}(W_N \text{Gr}_{\ell}) \\ &= \text{Tr}(\Gamma_N^{-1} \Gamma_{\ell}) + N^{-1} \text{Tr}(O(\Gamma_N^{-1} \Gamma_{\ell})) \\ &= \left(\sum_{p \in \mathcal{C}(k)} \ell^{b(p)} N^{-b(p)} \right) (1 + O(N^{-1})) \\ &= \left(\sum_{p \in \mathcal{C}(k)} \left(\frac{\lfloor tN \rfloor}{N} \right)^{b(p)} \right) (1 + O(N^{-1})) \end{aligned}$$

and this converges to the announced limit as N goes to infinity. ■

Example 4.3.8. For $\mathcal{C} = NC_2$, the limit of the odd moments vanish since there is no pair partition on an odd number of points, and we have

$$\lim_{N \rightarrow +\infty} \int_{O_N^+} \chi_t^{2k} = \sum_{p \in NC_2(2k)} t^k = t^k \frac{1}{k+1} \binom{2k}{k}.$$

This is the same as the semi-circle distribution, except that the radius of the circle has been changed to $2t$ instead of 2. We therefore see that the distribution of χ_t smoothly approximates the distribution of χ .

Example 4.3.9. For $\mathcal{C} = NC$, the computation requires the theory of *free cumulants* (see the book [NS06]) which we will not introduce here. Let us simply mention that using it, one can prove that χ_t is asymptotically free Poisson with parameter t , i.e. its distribution has density

$$\frac{1}{2\pi} \sqrt{\frac{4}{x^2} + \left(1 - \frac{1+t}{x}\right)^2}$$

with respect to the Lebesgue measure.

4.3.3 SINGLE COEFFICIENTS AND FREENESS

We will end with another result of the same type, but focusing on the joint behaviour of several elements. More precisely, let us consider all the coefficients u_{ij} of the fundamental representation of O_N^+ . Contrary to the character, the distribution of u_{ij} is very complicated and depends on N (see [BCZJ09] for an explicit computation). But the asymptotics can be easily obtained by the Weingarten formula :

Proposition 4.3.10. *For $N \geq 2$, set $x_{ij} = \sqrt{N}U_{ij}$, where U is the fundamental representation of O_N^+ . Then,*

$$\lim_{N \rightarrow +\infty} \int_{O_N^+} x_{ij}^k = \begin{cases} 0 & \text{if } k \text{ odd} \\ \frac{1}{\frac{k}{2} + 1} \binom{k}{k/2} & \text{if } k \text{ even} \end{cases}$$

In other words, the coefficients are asymptotically semicircular.

Proof. Because there is no pair partition on a odd number of points, Theorem 4.2.2 implies that the odd moments vanish. As for the even ones, using again Theorem 4.2.2 we have

$$\begin{aligned} \int_{O_N^+} x_{ij}^{2k} &= N^k \sum_{p,q \in NC_2(k)} \delta_p(i, \dots, i) \delta_q(j, \dots, j) (W_N)_{pq} \\ &= \sum_{p,q \in NC_2(k)} N^k (W_N)_{pq}. \end{aligned}$$

By Lemma 4.3.6, $N^k (W_N)_{pq} = N^k (\Gamma_N)_{pq}^{-1} (1 + O(N^{-1/2}))$. If $p = q$, since we are considering pair partitions on $2k$ points the number of blocks is k so that the right-hand side equals $1 + O(N^{-1/2})$. If $p \neq q$, the right-hand side equals

$$N^k (\Gamma_N)_{pp}^{-1} (1 + O(N^{-1/2}))_{pq} = O(N^{-1/2})_{pq}$$

which goes to 0 as N goes to infinity. Summing up,

$$\lim_{N \rightarrow +\infty} \int_{O_N^+} x_{ij}^k = |NC_2(k, 0)|$$

and the result follows. ■

Remark 4.3.11. For S_N^+ , the computation is trivial. Indeed, since p_{ij} is a projection, $p_{ij}^k = p_{ij}$ for all k so that all the moments are equal to N^{-1} .

We can even go further and compute the limit of *mixed moments*, that is to say arbitrary monomials involving the coefficients x_{ij} . In general in probability theory, it is difficult to compute such mixed moments because of the correlations between the variables. However, the situation greatly simplifies if these correlation vanish, i.e. if the variables are independent. In our setting, since the variables do not commute with one another, independence is not the correct notion. Nevertheless, there is a concept of *free independence* which translates the fact that there is "no correlation". We will not define this concept here because this would take us too far (but we once again highly recommend reading the book [NS06]) but simply give a formula for arbitrary joint moments. This requires some terminology.

DEFINITION 4.3.12. Given a monomial $X = x_{i_1 j_1} \cdots x_{i_k j_k}$ and a partition $p \in \mathcal{P}(k)$, we will say p *matches* X if for any points ℓ_1 and ℓ_2 connected by p , we have $i_{\ell_1} = i_{\ell_2}$ and $j_{\ell_1} = j_{\ell_2}$. The set of matching partitions will be denoted by $\mathcal{P}(X)$.

Proposition 4.3.13. *We have, for any monomial $X = x_{i_1 j_1} \cdots x_{i_k j_k}$,*

$$\lim_{N \rightarrow +\infty} \int_{O_N^+} X = \begin{cases} 0 & \text{if } k \text{ odd} \\ |\mathcal{P}(X) \cap NC_2(k)| & \text{if } k \text{ even} \end{cases}$$

Proof. The proof starts as for Proposition 4.3.10 using Theorem 4.2.2. The odd moments vanish so let us consider a moment of length $2k$:

$$\int_{O_N^+} X = \sum_{p,q \in NC(2k)} \delta_p(\mathbf{i}) \delta_q(\mathbf{j}) (W_N)_{p,q}.$$

The same computation as in the proof of Proposition 4.3.10 shows that as N goes to infinity, all the terms in the right-hand side vanish except for those with $p = q$. We therefore have

$$\lim_{N \rightarrow +\infty} \int_{O_N^+} X = \sum_{p \in NC_2(k)} \delta_p(\mathbf{i}) \delta_p(\mathbf{j}).$$

To conclude, simply notice that both δ -functions do not vanish, if and only if p matches X . ■

LECTURE 5

ANALYTIC APPLICATION : THE CONNES EMBEDDING PROBLEM

One important subject in the theory of operator algebras is be able to *approximate* the algebra using finite-dimensional data. The main example of such an approximation property is of course *amenability*, but there are several others like the *Haagerup property* and *weak amenability*. The book [BO08] gives a comprehensive overview of this subject for operator algebras and discrete groups. As for quantum groups, we refer the reader to the survey [Bra17].

5.1 APPROXIMATING OPERATOR ALGEBRAS BY MATRICES

But there are also other ways of approximating the structure of an operator algebra by finite-dimensional objects and one of the most famous ones comes from a question of A. Connes in [Con76] :

Problem. *Given a finite II_1 factor M , does there exist an ultraproduct¹ \mathcal{R}^ω of the hyperfinite II_1 factor \mathcal{R} in which M embeds ?*

We will refer to this as the *Connes embedding problem* and say that M has the *Connes embedding property* if the Connes embedding problem has a positive answer for M . We will not define the terms in this statement but rather give an equivalent "down-to-earth" definition.

Assume that we have a generating family a_1, \dots, a_n of M consisting in self-adjoint elements. Then, the Connes embedding problem has a positive answer for M if and only if for any $\varepsilon > 0$ and any integer $m > 0$, there exists an integer k and matrices $M_1, \dots, M_n \in M_k(\mathbf{C})$ such that for any $i_1, \dots, i_p \in \{1, \dots, n\}$ with $p \leq m$,

$$\left| h(a_{i_1} \cdots a_{i_p}) - \frac{1}{k} \text{Tr}(M_{i_1} \cdots M_{i_p}) \right| < \varepsilon.$$

In free probability, the preceding property corresponds to the fact that the noncommutative random variables (a_1, \dots, a_n) have enough microstates. This is important since it means that we can define their free entropy and therefore use free probability to investigate the structure of the von Neumann algebra M .

Remark 5.1.1. To this day, no example of II_1 factors without the Connes embedding property is known, and the existence of such a non-embeddable von Neumann algebra is related to an important conjecture in quantum information theory called *Tsireslon's problem* (see [JNP⁺11]).

In this lecture, we will prove the Connes embedding property for the von Neumann algebras $L^\infty(S_N^+)$ associated to the quantum permutation groups S_N^+ but we will not have to deal with von Neumann algebras at any moment. In fact, we will prove a stronger, purely algebraic property which also corresponds to embedding the algebra into matrices :

1. The reader may refer for instance to [] for the definition of this object.

DEFINITION 5.1.2. A $*$ -algebra A is said to be *residually finite-dimensional* (RFD for short) if there exists integers $(n_i)_{i \in I}$ such that there is an embedding of $*$ -algebras²

$$A \hookrightarrow \prod_{i \in I} M_{n_i}(\mathbf{C}).$$

A compact quantum group \mathbb{G} is said to be *residually finite* if $\mathcal{O}(\mathbb{G})$ is RFD.

In other words, finite-dimensional $*$ -representations separate the points, hence the name. If Γ is a discrete group and $A = \mathbf{C}[\Gamma]$, then this is the same as saying that Γ is a *residually finite group*. It turns out that this is stronger than the Connes embedding property, as proven in [BBCW19, Thm 2.1] :

THEOREM 5.1.3 (Bhattacharya-Brannan-Chirvasitu-Wang) Let \mathbb{G} be a residually finite orthogonal compact matrix quantum group. Then, $L^\infty(\mathbb{G})$ has the Connes embedding property.

Sketch of proof. Pick any faithful tracial state τ on the product of matrix algebras in which $\mathcal{O}(\mathbb{G})$ embeds and restrict it to a faithful tracial state $\tilde{\tau}$ on $\mathcal{O}(\mathbb{G})$ which is by construction *amenable* (i.e. a pointwise limit of traces which factor nicely through finite-dimensional algebras, see for instance [BO08, Def 6.2.1 and Thm 6.2.7]). It then follows from [Wor87, Prop 4.1] that any weak accumulation point of the sequence

$$\frac{1}{n} \sum_{i=1}^n \tilde{\tau}^{*i}$$

is the Haar state and amenability of traces is preserved under such limits, so that $\int_{\mathbb{G}}$ is amenable. Eventually E. Kirchberg proved in [Kir94, Prop 3.2] that the von Neumann algebra coming from the GNS construction of an amenable trace has the Connes embedding property. ■

We can therefore focus on residual finite-dimensionality. Here is the main result we want to discuss from now on :

THEOREM 5.1.4 (Brannan-Chirvasitu-F.) The quantum permutation groups S_N^+ are RFD for all N .

5.2 THE PROOF

5.2.1 TOPOLOGICAL GENERATION

The strategy for the proof of Theorem 5.1.4 for S_N^+ is induction on N , using the key notion of *topological generation*. This idea was first introduced by A. Chirvasitu in [Chi15] (though not under that name). To explain it, let us write $\mathbb{H} < \mathbb{G}$ if $\mathbb{G} = (\mathcal{O}(\mathbb{G}), u)$ and $\mathbb{H} = (\mathcal{O}(\mathbb{H}), v)$ are compact quantum groups with a surjective $*$ -homomorphism $\pi : C(\mathbb{G}) \rightarrow C(\mathbb{H})$ such that $\pi(u_{ij}) = v_{ij}$.

DEFINITION 5.2.1. Consider $\mathbb{G}_1, \mathbb{G}_2 < \mathbb{G}$ given by surjections π_1 and π_2 . We say that \mathbb{G} is *topologically generated* by \mathbb{G}_1 and \mathbb{G}_2 if the map

$$\pi := (\pi_1 \otimes \pi_2) \circ \Delta : \mathcal{O}(\mathbb{G}) \rightarrow \mathcal{O}(\mathbb{G}_1) \otimes \mathcal{O}(\mathbb{G}_2)$$

does not factor through any orthogonal compact matrix quantum group.

Remark 5.2.2. For the sake of simplicity, we will work exclusively with orthogonal compact matrix quantum groups in this section. However, all the general arguments carry to the setting of unitary compact matrix quantum groups (and even general compact quantum groups). We leave it to the interested reader to find the natural analogues of the statements and proofs.

The core result we need is the following :

Proposition 5.2.3 (Chirvasitu). *If \mathbb{G} is topologically generated by two RFD orthogonal compact matrix quantum subgroups \mathbb{G}_1 and \mathbb{G}_2 , then \mathbb{G} is RFD.*

2. No assumption is made concerning the cardinality of the set I .

Proof. Let \mathcal{I} be the intersection of the kernels of all finite-dimensional $*$ -representations of $\mathcal{O}(\mathbb{G})$, let $\mathcal{A} = \mathcal{O}(\mathbb{G})/\mathcal{I}$ and let v_{ij} be the image of u_{ij} in this quotient. By definition, because $\mathcal{O}(\mathbb{G}_1)$ and $\mathcal{O}(\mathbb{G}_2)$ are RFD, the map π factors through \mathcal{A} . To conclude we therefore have to show that (\mathcal{A}, v) is an orthogonal compact matrix quantum group. Note that \mathcal{I} is an intersection of $*$ -ideals, hence \mathcal{A} is a $*$ -algebra. Moreover, if $x \in \mathcal{I}$, and π is a finite-dimensional representation, then $(\pi \otimes \pi) \circ \Delta$ is also a finite-dimensional representation, hence $(\pi \otimes \pi) \circ \Delta(x) = 0$. Thus, if $q_{\mathcal{I}}$ is the quotient map, then there exists a $*$ -homomorphism $\tilde{\Delta} : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ uniquely determined by

$$\tilde{\Delta} \circ q_{\mathcal{I}} = (q_{\mathcal{I}} \otimes q_{\mathcal{I}}) \circ \Delta.$$

In particular,

$$\tilde{\Delta}(v_{ij}) = \sum_{k=1}^N v_{ik} \otimes v_{kj}$$

and (\mathcal{A}, v) is an orthogonal compact matrix quantum group. \blacksquare

In the sequel, we will need a criterion to prove topological generation, which is the following one :

Proposition 5.2.4. *Let \mathbb{G} be an orthogonal compact matrix quantum group and let $\mathbb{G}_1, \mathbb{G}_2 < \mathbb{G}$ be quantum subgroups. If for any $k \in \mathbf{N}$,*

$$\text{Mor}_{\mathbb{G}}(u^{\otimes k}, \varepsilon) = \text{Mor}_{\mathbb{G}_1}(u_1^{\otimes k}, \varepsilon) \cap \text{Mor}_{\mathbb{G}_2}(u_2^{\otimes k}, \varepsilon),$$

then \mathbb{G} is topologically generated by \mathbb{G}_1 and \mathbb{G}_2 .

Proof. First note that if f is invariant under $\rho_{u^{\otimes k}}$, then it is also invariant under

$$(\text{id} \otimes \pi_i) \circ \rho_{u^{\otimes k}} = \rho_{u_i^{\otimes k}}$$

so that the left-hand side is always contained in the right-hand side. Let us now consider the $*$ -ideal

$$\mathcal{I} = \bigcap_{k \in \mathbf{N}} \ker \left(\pi^{\otimes k} \circ \Delta^{(k)} \right).$$

Then, because

$$\left(\pi^{\otimes k} \circ \Delta^{(k)} \otimes \pi^{\otimes k} \circ \Delta^{(k)} \right) \circ \Delta = \pi^{2k} \circ \Delta^{(2k)},$$

we have $\Delta(\mathcal{I}) \subset \mathcal{O}(\mathbb{G}) \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{O}(\mathbb{G})$ by definition, hence the quotient $\mathcal{A} = \mathcal{O}(\mathbb{G})/\mathcal{I}$ has an orthogonal compact matrix quantum group structure coming from the images v_{ij} of u_{ij} . Moreover, by construction any orthogonal compact matrix quantum group through which π factors is a quotient of $\mathbb{H} = (\mathcal{A}, v)$.

Moreover, $\mathcal{I} \subset \ker(\pi_i)$ for $i = 1, 2$ so that $\pi_i \circ q_{\mathcal{I}} = \pi_i$, hence $\mathbb{G}_i < \mathbb{H}$. The comment at the beginning of the proof therefore yields

$$\text{Mor}_{\mathbb{H}}(v^{\otimes k}, \varepsilon) \subset \text{Mor}_{\mathbb{G}_1}(u_1^{\otimes k}, \varepsilon) \cap \text{Mor}_{\mathbb{G}_2}(u_2^{\otimes k}, \varepsilon).$$

Similarly, because $\mathbb{H} < \mathbb{G}$,

$$\text{Mor}_{\mathbb{G}}(u^{\otimes k}, \varepsilon) \subset \text{Mor}_{\mathbb{H}}(v^{\otimes k}, \varepsilon)$$

Using now our assumption, we deduce that $\text{Mor}_{\mathbb{G}}(u^{\otimes k}, \varepsilon) = \text{Mor}_{\mathbb{H}}(v^{\otimes k}, \varepsilon)$ for all $k \in \mathbf{N}$. The same equality then holds for any tensor powers of u and v , hence $\mathbb{G} = \mathbb{H}$ by Theorem 2.2.5. \blacksquare

5.2.2 AT LEAST SIX POINTS

The following result [BCF18, Thm 3.3 and Thm 3.12] is the key tool to prove residual finite-dimensionality. In this statement, we see the inclusion $S_{N-1}^+ < S_N^+$ through the surjection

$$\pi_1 : \mathcal{O}(S_N^+) \rightarrow \mathcal{O}(S_{N-1}^+)$$

sending u_{11} to 1.

THEOREM 5.2.5 (Brannan-Chirvasitu-F.) For any $N \geq 6$, the quantum permutation group S_N^+ is topologically generated by S_{N-1}^+ and S_N .

Proof of Theorem 5.2.5. Set $V = \mathbf{C}^N$ and let P_N, P_{N-1} and P_N^{class} denote the fundamental representations of S_N^+, S_{N-1}^+ and S_N respectively. We will say that a map is S_N^+ (resp. S_{N-1}^+ , resp. S_N) invariant if it commutes with appropriate tensor powers of P_N (resp. P_{N-1} , resp. P_N^{class}). By Proposition 5.2.4, it is enough to prove that if

$$f : V^{\otimes k} \rightarrow \mathbf{C}$$

is a linear map which is both S_{N-1}^+ -invariant and S_N -invariant, then f is S_N^+ -invariant.

Let us therefore consider such a map f and, for $1 \leq i \leq N$, let $V_i = e_i^\perp$. Because f is S_{N-1}^+ invariant, its restriction to $V_1^{\otimes k}$ is a linear combination of partitions maps : there exist complex numbers $(\lambda_p)_{p \in NC(k)}$ such that

$$f|_{V_1^{\otimes k}} = \sum_{p \in NC(k)} \lambda_p f_p.$$

Let us set

$$\tilde{f} = f - \sum_{p \in NC(k)} \lambda_p f_p.$$

This is still invariant under S_{N-1}^+ and S_N and vanishes on $V_1^{\otimes k}$. Our task is to show that it vanishes on the whole of $V^{\otimes k}$.

For this purpose, let us set $V_i' = \mathbf{C}e_i$, so that

$$V^{\otimes k} = \bigoplus_{\epsilon_1, \dots, \epsilon_k} V_1^{\epsilon_1} \otimes \dots \otimes V_1^{\epsilon_k}$$

where ϵ is either prime or nothing. Let us consider one of these summand where V_i appears ℓ times and denote it by W . Since S_{N-1}^+ acts trivially on V_1' , there exists a linear S_{N-1}^+ -equivariant isomorphism

$$\Phi : W \rightarrow V_1^{\otimes \ell}.$$

As a consequence, there exist complex numbers $(\mu_p)_{p \in NC(\ell)}$ such that

$$\tilde{f} \circ \Phi^{-1} = \sum_{p \in NC(\ell)} \mu_p f_p.$$

The idea now is to use the linear independence of the partition maps to conclude that $\mu_p = 0$ for all p , hence that $\tilde{f} \circ \Phi^{-1} = 0$.

To do this, set $V_{1,N} = V_1 \cap V_N$ and observe that

$$\Phi^{-1}(V_{1,N}^{\otimes \ell}) \subset V_N^{\otimes \ell}.$$

Now, by S_N -invariance, we can exchange e_1 and e_N without changing the value of \tilde{f} , hence it vanishes on $V_N^{\otimes \ell}$. Thus, $\tilde{f} \circ \Phi^{-1}$ vanishes on $V_{1,N}^{\otimes \ell}$. Since $N \geq 6$, $\dim(V_{1,N}) \geq 4$ so that noncrossing partition maps on $V_{1,N}^{\otimes \ell}$ are linearly independent by Theorem 3.1.1. This forces $\mu_p = 0$ for all p , hence $\tilde{f} = 0$. ■

5.2.3 AT MOST FIVE POINTS

So far, Theorem 5.2.5 is useless for an inductive proof since we do not know the base case : is $\mathcal{O}(S_5^+)$ RFD? Before addressing this question, let us comment on the case $N = 4$. In that case, we have a genuinely compact quantum group which can be shown to be isomorphic to the deformation $SO_{-1}(3)$. Moreover, it was shown by B. Collins and T. Banica in [BC08, Thm 4.1] that there is an embedding

$$\pi : C(S_4^+) \hookrightarrow C(SU(2), M_4(\mathbf{C})).$$

As a consequence, the $*$ -representations $\pi_g : x \mapsto \pi(x)(g)$ for all $g \in SU(2)$ separate the points, i.e. S_4^+ is RFD.

Only $N = 5$ remains, and this is indeed a non-trivial matter which can be solved using the classification of subfactors. It was showed by T. Banica in [Ban18, Thm 7.10] that S_5^+ enjoys a much stronger property : the canonical map

$$\pi_{\text{ab}} : \mathcal{O}(S_5^+) \rightarrow \mathcal{O}(S_5)$$

does not factor through any compact matrix quantum group. The idea of the proof is that by [Ban99a] and [TW18], any quantum subgroup of S_5^+ yields a subfactor planar algebra at index 5, with extra properties if it contains S_5 . Moreover, this correspondence is injective. Now one has to look at the complete list of subfactor planar algebras at index 5 satisfying the extra properties (see for instance the survey [JMS14]) and check that none of the corresponding quantum groups contains S_5 , which is not very difficult. We can now complete our proof :

Proof of Theorem 5.1.4 for S_N^+ . First, we can extend the statement of Theorem 5.2.5 to $N = 5$. Indeed, if we repeat the construction of the proof of Proposition 5.2.4, we get a quantum subgroup $\mathbb{H} < S_5^+$ which contains S_5 . Since it also contains S_4^+ , it must be equal to S_5^+ by the previous discussion. Thus, the map π does not factor through a compact matrix quantum group, i.e. S_4^+ and S_5 topologically generate S_5^+ . We can now conclude by induction starting from the fact that S_4^+ is RFD. ■

Remark 5.2.6. Let us mention the following interesting problem : is there a compact matrix quantum group through which the quotient map

$$\pi_{\text{ab}} : \mathcal{O}(S_N^+) \rightarrow \mathcal{O}(S_N)$$

factors? If not, then the inclusion $S_N < S_N^+$ is said to be *maximal*. For $N \leq 3$, the result trivially follows from the equality of the two quantum groups. For $N = 4$, it can be proven by checking the list of all quantum subgroups of S_4^+ given in [BB09] and we already mentioned the proof of T. Banica in [Ban18, Thm 7.10] for $N = 5$. This is all that is known to this day. Note that for $N > 5$, the subfactor approach will not work, because it is known that classification at index 6 and higher is much more involved.

5.3 FURTHER EXAMPLES

To conclude, we will prove that our other travelling companions O_N^+ and H_N^+ also have the Connes embedding property. We will start with H_N^+ , using a trick relying on the fact that H_N^+ decomposes as a *free wreath product* of the cyclic group \mathbf{Z}_2 by the quantum permutation group S_N^+ . We will not define this here (see [Bic04]) but simply use it as a black box.

THEOREM 5.3.1 The quantum group H_N^+ is RFD for all N , hence $L^\infty(H_N^+)$ has the Connes embedding property.

Proof. It was proven in [Bic04] that $H_N^+ = \mathbf{Z}_2 \wr_* S_N^+$ and this can be restated in the following way : consider the sequence of $*$ -algebras \mathcal{A}_k where $\mathcal{A}_0 = \mathcal{A}_s(N)$ and for any $k \geq 0$,

$$\mathcal{A}_{k+1} = \mathbf{C}[\mathbf{Z}_2] * \mathcal{A}_k / \langle [\mathbf{C}[\mathbf{Z}_2], u_{k+1j}] \mid 1 \leq j \leq N \rangle.$$

Then, $\mathcal{A}_N = \mathcal{O}(H_N^+)$. All that we need is therefore a stability result for residual finite-dimensionality. Let us first notice that $\langle u_{k+1} \mid 1 \leq j \leq N \rangle \simeq \mathbf{C}^N$ so that

$$\mathcal{A}_{k+1} \simeq \left(\mathbf{C}[\mathbf{Z}_2] \otimes \mathbf{C}^N \right) *_{\mathbf{C}^N} \mathcal{A}_k \hookrightarrow (\mathbf{C}[\mathbf{Z}_2] \otimes \mathcal{A}_k) *_{\mathbf{C}^N} (\mathbf{C}[\mathbf{Z}_2] \otimes \mathcal{A}_k).$$

It therefore suffices to show that $\mathcal{A} *_{\mathcal{B}} \mathcal{A}$ is RFD as soon as \mathcal{A} is and \mathcal{B} is finite-dimensional. We will not give the proof but simply explain the two steps :

1. Using a free product decomposition of a Hilbert space on which \mathcal{A} acts faithfully, it is easy to see that any element has non-zero image into a similar free product with \mathcal{A} replaced by a finite-dimensional quotient. It is therefore sufficient to do it for finite-dimensional \mathcal{A} .
2. The result is then a particular case of [ADEL04, Thm 4.2], where the strategy is to embed \mathcal{A} in a matrix algebra in a trace-preserving way. One can then use [BD04, Thm 2.3] to conclude. ■

Remark 5.3.2. The same strategy works to prove that if Γ is a finite group, then $\widehat{\Gamma} \wr S_N^+$ is RFD. Observing with the same argument as in [BCF18, Lem 2.13] that if Γ is residually finite, then $\widehat{\Gamma} \wr S_N^+$ is topologically generated by $\widehat{\Lambda} \wr S_N^+$ for all finite quotients Λ of Γ , we see that the free wreath product is residually finite as soon as Γ is.

For O_N^+ , there are two available proofs of residual finiteness. One uses topological generation in the spirit of Theorem 5.2.5 (this is [BCV17, Thm 4.1]) and the other one first proves the result for the free unitary quantum groups U_N^+ and then uses the fact that O_N^+ contains a "finite-index" subgroup which is also a subgroup of U_N^+ (this is [Chi15, Thm 3.1]). Here, we will indicate how the result for H_N^+ can be used to simplify the topological generation approach.

THEOREM 5.3.3 The quantum group O_N^+ is RFD for all N , hence $L^\infty(O_N^+)$ has the Connes embedding property.

Proof. We will use a topological generation result to prove the result by induction. First note that by definition, $H_N^+ < O_N^+$. Let now \mathcal{A} be the quotient of $\mathcal{O}(O_N^+)$ by the ideal generated by $U_{11} - 1$. We claim that $\mathcal{A} \simeq \mathcal{O}(O_{N-1}^+)$. Indeed, if we consider any matrix $(v_{ij})_{1 \leq i, j \leq (N-1)}$ of operators which is orthogonal, then the map sending U_{11} to 1, U_{1i}, U_{j1} to 0 and U_{ij} to $v_{(i-1)(j-1)}$ for $i, j > 1$ factors through a surjection from \mathcal{A} to the $*$ -algebra generated by the v_{ij} 's so that the claim is proven. We will write $B_N^+ = (\mathcal{A}, \nu)$, where ν is the image of U in the quotient.

Let us now show that \mathcal{A} in fact comes from a partition quantum group. Let $p \in NC(1, 0)$ be the singleton partition, whose corresponding linear map $T_p : \mathbf{C}^N \rightarrow \mathbf{C}$ sends all basis vectors to 1. Then,

$$(\text{id} \otimes T) \circ \rho_U(e_i) = \sum_{j=1}^N U_{ij} \otimes 1$$

while $\varepsilon \circ T(e_i) = 1$, so that if we add p to the category of partitions, we get the extra relation

$$\sum_{j=1}^N U_{ij} = 1$$

for all $1 \leq i, j \leq N$. In other words, B_N^+ is the partition quantum group associated to the category of partitions $NC_{1,2} = \langle p, NC_2 \rangle$.

A straightforward induction shows that $NC_{1,2}$ is the category of all partitions with blocks of size at most two. It therefore follows from Theorem 3.1.1 that for $N \geq 4$,

$$\begin{aligned} \text{Mor}_{B_N^+}(u^{\otimes k}, \varepsilon) \cap \text{Mor}_{H_N^+}(u^{\otimes k}, \varepsilon) &= \text{Vect}\{f_p \mid p \in NC_{2,1}(k)\} \cap \text{Vect}\{f_p \mid p \in NC_{\text{even}}(k)\} \\ &= \text{Vect}\{f_p \mid p \in NC_{2,1}(k) \cap NC_{\text{even}}(k)\} \\ &= \text{Vect}\{f_p \mid p \in NC_2(k)\} \\ &= \text{Mor}_{O_N^+}(U^{\otimes k}, \varepsilon), \end{aligned}$$

i.e. O_N^+ is topologically generated by H_N^+ and $B_N^+ \simeq O_{N-1}^+$.

To conclude we would need the result for O_3^+ . Unfortunately, this is the only case which is still open at the time these notes are written. It is nevertheless possible to "jump over it" thanks to [BCV17, Thm 4.2] : O_4^+ is topologically generated by the free product $O_2^+ \widehat{*} O_2^+$ and the permutation group S_4 . As a consequence, it is RFD, and we can now conclude. ■

APPENDIX : TWO THEOREMS ON COMPLEX MATRIX ALGEBRAS

For the sake of completeness, we give a detailed proof of two results from the representation theory of algebras which were used in these lectures. Since we only need them for algebras of matrices over the field of complex number, we will only give the proofs in that case, allowing us to simplify some of the arguments.

Let us recall a few basic facts for convenience : a subalgebra $A \subset M_n(\mathbf{C})$ is said to be *irreducible* if there is no vector in \mathbf{C}^n fixed by all the elements of A . The nature of irreducible matrix algebras is elucidated by the following celebrated result of Burnside :

THEOREM A (Burnside's Theorem) Let $A \subset M_n(\mathbf{C})$ be an irreducible subalgebra. Then, $A = M_n(\mathbf{C})$.

Proof. The proof proceeds in two steps. First, we will prove that A contains a rank 1 matrix. Then, we will deduce from this that it contains all rank 1 matrices, hence all matrices. We will do this following [HR80]. Before we start, note that by irreducibility, for any non-zero $x \in \mathbf{C}^n$,

$$A.x := \{T(x) \mid T \in A\} = \mathbf{C}^n.$$

For the first part, we will proceed by contradiction. Let $T \in A$ be a matrix with minimal rank and assume that $d = \text{rk}(T) \geq 2$. Then, there exists $x_1, x_2 \in \mathbf{C}^n$ such that the vectors $T(x_1)$ and $T(x_2)$ are linearly independent. Let us choose $S \in A$ such that $ST(x_1) = x_2$. Then, $T(x_1)$ and

$$TST(x_1) = T(x_2)$$

are linearly independent, so that the operator $TST - \lambda T$ is non-zero for all $\lambda \in \mathbf{C}$. However, notice that $TS - \mu \text{Id}$ is a linear operator on the range of T , hence it has an eigenvalue : there exists $\mu \in \mathbf{C}$ such that $TS - \mu \text{Id}$ is not invertible. Then,

$$TST - \mu T = (TS - \mu \text{Id})T$$

has rank strictly between 0 and d , contradicting minimality. As a conclusion, there is a rank-one matrix in A .

The rank-one matrix obtained in the previous paragraph can be written as the operator

$$T_{\phi, y} : x \mapsto \phi(x)y$$

for some linear form $\phi \in (\mathbf{C}^n)^*$ and a vector $y \in \mathbf{C}^n$. Because $Ay = \mathbf{C}^n$, we also have $T_{\phi, z} \in A$ for all $z \in \mathbf{C}^n$. Similarly, A acts on $(\mathbf{C}^n)^*$ by

$$(S, \psi) \mapsto \psi \circ S.$$

Assume that there is a fixed linear form η and pick vectors $y_1 \notin \ker(\eta)$ and $y_2 \in \ker(\eta)$. By irreducibility there exists $S \in A$ such that $S(y_1) = y_2$, but then

$$\eta \circ S(x_1) = 0 \neq \eta(x_1),$$

contradicting invariance. Therefore, A acts irreducibly on $(\mathbf{C}^n)^*$, hence $\{\phi \circ S \mid S \in A\} = (\mathbf{C}^n)^*$. Putting things together, for any $\psi \in (\mathbf{C}^n)^*$ and $z \in \mathbf{C}^n$, there exists $S_1, S_2 \in A$ such that

$$S_1 \circ T \circ S_2 : x \mapsto \psi(x)z.$$

As a consequence, A contains all rank-one matrices, hence equals $M_n(\mathbf{C})$. ■

Note that the argument only requires the existence of an eigenvalue for the matrix ST , hence works for any algebraically closed field.

Using this, we can prove our second result concerning the double commutant of a matrix $*$ -algebra. Recall that given a subalgebra $A \subset M_n(\mathbf{C})$, its *commutant* A' is by definition

$$A' = \{T \in M_n(\mathbf{C}) \mid AT = TA\}.$$

THEOREM B (Double Commutant Theorem) Let $A \subset M_n(\mathbf{C})$ be a subalgebra which is stable under taking adjoints. Then, $A'' = A$.

Proof. By definition $A \subset A''$. Moreover, if $V \subset \mathbf{C}^n$ is stable under the action of A , then by stability under taking adjoints, its orthogonal complement is also stable. As a consequence, there is an orthogonal decomposition

$$\mathbf{C}^n = V_1 \oplus \cdots \oplus V_m$$

into irreducible subspaces. By Burnside's Theorem **A**, we then have

$$A = \mathcal{L}(V_1) \oplus \cdots \oplus \mathcal{L}(V_m).$$

Consider now $T \in A''$ and let P_i be the orthogonal projection onto V_i . Since $P_i \in A'$, T commutes with it so that T decomposes as a block-diagonal operator :

$$\begin{aligned} T &= T \left(\sum_{i=1}^m P_i \right) \\ &= \sum_{i=1}^m TP_i \\ &= \sum_{i=1}^m P_i TP_i \end{aligned}$$

In other words,

$$T \in \mathcal{L}(V_1) \oplus \cdots \oplus \mathcal{L}(V_m) = 1$$

and the proof is complete. ■

REFERENCES

- [ADEL04] S. ARMSTRONG, K. DYKEMA, R. EXEL & H. LI – « On embeddings of full amalgamated free product C^* -algebras », *Proc. Amer. Math. Soc.* **132** (2004), no. 7, p. 2019–2030.
- [AMR⁺18] A. ATSERIAS, L. MANČINSKA, D. ROBERSON, R. ŠÁMAL, S. SEVERINI & A. VARVITSIOTIS – « Quantum and non-signalling graph isomorphisms », *Journal of Combinatorial Theory, Series B* (2018).
- [Ban96] T. BANICA – « Théorie des représentations du groupe quantique compact libre $O(n)$ », *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996), no. 3, p. 241–244.
- [Ban99a] — , « Representations of compact quantum groups and subfactors », *J. Reine Angew. Math.* **509** (1999), p. 167–198.
- [Ban99b] — , « Symmetries of a generic coaction », *Math. Ann.* **314** (1999), no. 4, p. 763–780.
- [Ban05] — , « Quantum automorphism groups of small metric spaces », *Pacific J. Math.* **219** (2005), no. 1, p. 27–51.
- [Ban18] — , « Homogeneous quantum groups and their easiness level », *Arxiv preprint* (2018).
- [BB09] T. BANICA & J. BICHON – « Quantum groups acting on 4 points », *J. Reine Angew. Math.* **626** (2009), p. 75–114.
- [BBCC11] T. BANICA, S. BELINSCHI, M. CAPITAINÉ & B. COLLINS – « Free Bessel laws », *Canadian J. Math.* **63** (2011), no. 1, p. 3–37.
- [BBCW19] A. BHATTACHARYA, M. BRANNAN, A. CHIRVASITU & S. WANG – « Kirchberg factorization and residual finiteness for discrete quantum groups », *J. Noncommut. Geom.* (2019).
- [BC07] T. BANICA & B. COLLINS – « Integration over compact quantum groups », *Publ. Res. Inst. Math. Sci.* **43** (2007), p. 277–302.
- [BC08] — , « Integration over the Pauli quantum group », *J. Geom. and Phys.* **58** (2008), no. 8, p. 942–961.
- [BC10] T. BANICA & S. CURRAN – « Decomposition results for Gram matrix determinants », *J. Math. Phys.* **51** (2010), no. 11, p. 113503.
- [BCF18] M. BRANNAN, A. CHIRVASITU & A. FRESLON – « Topological generation and matrix models for quantum reflection groups », *Arxiv preprint* (2018).
- [BCS11] T. BANICA, S. CURRAN & R. SPEICHER – « Stochastic aspects of easy quantum groups », *Prob. Theory Related Fields* **149** (2011), no. 3-4, p. 435–462.

- [BCS12] — , « De Finetti theorems for easy quantum groups », *Ann. Probab.* **40** (2012), no. 1, p. 401–435.
- [BCV17] M. BRANNAN, B. COLLINS & R. VERGNIUX – « The Connes embedding property for quantum group von Neumann algebras », *Trans. Amer. Math. Soc.* **369** (2017), p. 3799–3819.
- [BCZJ09] T. BANICA, B. COLLINS & P. ZINN-JUSTIN – « Spectral analysis of the free orthogonal matrix », *Int. Math. Res. Not.* **2009** (2009), no. 17, p. 3286–3309.
- [BD04] N. BROWN & K. DYKEMA – « Popa algebras in free group factors », *J. Reine Angew. Math.* **573** (2004), p. 157–180.
- [Bic03] J. BICHON – « Quantum automorphism groups of finite graphs », *Proc. Amer. Math. Soc.* **131** (2003), no. 3, p. 665–673.
- [Bic04] — , « Free wreath product by the quantum permutation group », *Algebr. Represent. Theory* **7** (2004), no. 4, p. 343–362.
- [Bla06] B. BLACKADAR – *Operator algebras*, Encyclopædia of Mathematical Sciences, vol. 122, Springer, 2006.
- [BO08] N. P. BROWN & N. OZAWA – *C*-algebras and finite-dimensional approximation*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, 2008.
- [Bra37] R. BRAUER – « On algebras which are connected with the semisimple continuous groups », *Ann. of Math.* **38** (1937), no. 4, p. 857–872.
- [Bra17] M. BRANNAN – « Approximation properties for locally compact quantum groups », *Banach Center Publications* **111** (2017), p. 185–232.
- [BS09] T. BANICA & R. SPEICHER – « Liberation of orthogonal Lie groups », *Adv. Math.* **222** (2009), no. 4, p. 1461–1501.
- [BV09] T. BANICA & R. VERGNIUX – « Fusion rules for quantum reflection groups », *J. Non-commut. Geom.* **3** (2009), no. 3, p. 327–359.
- [Chi15] A. CHIRVASITU – « Residually finite quantum group algebras », *J. Funct. Anal.* **268** (2015), no. 11, p. 3508–3533.
- [Con76] A. CONNES – « Classification of injective factors cases II, II_∞ , III_λ , $\lambda \neq 1$ », *Ann. Math.* **104** (1976), no. 1, p. 73–115.
- [DFGG97] P. DI FRANCESCO, O. GOLINELLI & E. GUITTER – « Meanders and the Temperley-Lieb algebra », *Comm. Math. Phys.* **186** (1997), no. 1, p. 1–59.
- [DK94] M. DIJKHUIZEN & T. KOORNWINDER – « CQG algebras : a direct algebraic approach to compact quantum groups », *Lett. Math. Phys.* **32** (1994), p. 315–330.
- [EGNO15] P. ETINGOF, S. GELAKI, D. NIKSHYCH & V. OSTRIK – *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, AMS, 2015.
- [Fre17] A. FRESLON – « On the partition approach to Schur-Weyl duality and free quantum groups – with an appendix by A. Chirvasitu », *Transform. Groups* **22** (2017), no. 3, p. 705–751.
- [Fre19] — , « On two-coloured noncrossing partition quantum groups », *Trans. Amer. Math. Soc.* (2019).
- [FW16] A. FRESLON & M. WEBER – « On the representation theory of partition (easy) quantum groups », *J. Reine Angew. Math.* **720** (2016), p. 155–197.

- [Gro19] D. GROMADA – « Classification of globally colorized categories of partitions », *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* (2019).
- [HR80] I. HALPERIN & P. ROSENTHAL – « Burnside’s theorem on algebras of matrices », *Amer. Math. Monthly* **87** (1980), p. 810.
- [JMS14] V. JONES, S. MORRISON & N. SNYDER – « The classification of subfactors of index at most 5 », *Bull. Amer. Math. Soc.* **51** (2014), no. 2, p. 277–327.
- [JNP⁺11] M. JUNGE, M. NAVASCUES, C. PALAZUELOS, D. PEREZ-GARCIA, V. SCHOLZ & R. WERNER – « Connes’ embedding problem and Tsirelson’s problem », *J. Math. Phys.* **52** (2011), no. 1, p. 012102.
- [Jon93] V. JONES – « The Potts model and the symmetric group », in *Subfactors : Proceedings of the Taniguchi Symposium on Operator Algebras (Kyuzeso, 1993)*, 1993, p. 259–267.
- [Kir94] E. KIRCHBERG – « Discrete groups with Kazhdan’s property T and factorization property are residually finite », *Math. Ann.* **299** (1994), no. 1, p. 551–563.
- [KS08] V. KODIYALAM & V. SUNDER – « Temperley-Lieb and non-crossing partition planar algebras », *Contemp. Math.* **456** (2008), p. 61–72.
- [KS09] C. KÖSTLER & R. SPEICHER – « A noncommutative de Finetti theorem : invariance under quantum permutations is equivalent to freeness with amalgamation », *Comm. Math. Phys.* **291** (2009), no. 2, p. 473–490.
- [LMR17] M. LUPINI, L. MANČINSKA & D. ROBERSON – « Nonlocal games and quantum permutation groups », *Arxiv preprint arXiv :1712.01820* (2017).
- [Mal18] S. MALACARNE – « Woronowicz Tannaka-Krein duality and free orthogonal quantum groups », *Math. Scand.* **122** (2018), no. 1, p. 151–160.
- [Mar94] P. MARTIN – « Temperley-Lieb algebras for non-planar statistical mechanics—the partition algebra construction », *J. Knot Theory Ramifications* **3** (1994), no. 1, p. 51–82.
- [MW19] A. MANG & M. WEBER – « Categories of two-colored pair partitions, part i : Categories indexed by cyclic groups », *The Ramanujan Journal* (2019).
- [NS06] A. NICA & R. SPEICHER – *Lectures on the combinatorics of free probability*, Lecture Note Series, vol. 335, London Mathematical Society, 2006.
- [NT13] S. NESHVEYEV & L. TUSET – *Compact quantum groups and their representation categories*, Cours Spécialisés, vol. 20, Société Mathématique de France, 2013.
- [RW16] S. RAUM & M. WEBER – « The full classification of orthogonal easy quantum groups », *Comm. Math. Phys.* **341** (2016), no. 3, p. 751–779.
- [Tim08] T. TIMMERMANN – *An invitation to quantum groups and duality. From Hopf algebras to multiplicative unitaries and beyond*, EMS Textbooks in Mathematics, European Mathematical Society, 2008.
- [TW17] P. TARRAGO & M. WEBER – « Unitary easy quantum groups : the free case and the group case », *Int. Math. Res. Not.* **18** (2017), p. 5710–5750.
- [TW18] P. TARRAGO & J. WAHL – « Free wreath product quantum groups and standard invariants of subfactors », *Adv. Math.* **331** (2018), p. 1–57.
- [Wan95] S. WANG – « Free products of compact quantum groups », *Comm. Math. Phys.* **167** (1995), no. 3, p. 671–692.
- [Wan98] S. WANG – « Quantum symmetry groups of finite spaces », *Comm. Math. Phys.* **195** (1998), no. 1, p. 195–211.

- [Web13] M. WEBER – « On the classification of easy quantum groups – The nonhyperoctahedral and the half-liberated case », *Adv. Math.* **245** (2013), no. 1, p. 500–533.
- [Wor87] S. WORONOWICZ – « Compact matrix pseudogroups », *Comm. Math. Phys.* **111** (1987), no. 4, p. 613–665.
- [Wor88] — , « Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups », *Invent. Math.* **93** (1988), no. 1, p. 35–76.
- [Wor98] — , « Compact quantum groups », *Symétries quantiques (Les Houches, 1995)* (1998), p. 845–884.