The Equivariant Tamagawa Number Conjecture for modular motives with coefficients in Hecke algebras

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Résumé

Sous des hypothèses faibles sur la représentation résiduelle, nous prouvons la Conjecture Équivariante sur les Nombres de Tamagawa pour les motifs modulaires à coefficients dans les anneaux de déformations universelles et les algèbres de Hecke en utilisant une combinaison nouvelle de la méthode des systèmes d'Euler et de celle des systèmes de Taylor-Wiles. Nous prouvons aussi la compatibilité de cette conjecture par spécialisation.

Abstract

Under mild hypotheses on the residual representation, we prove the Equivariant Tamagawa Number Conjecture for modular motives with coefficients in universal deformation rings and Hecke algebras using a novel combination of the methods of Euler systems and Taylor-Wiles systems. We also prove the compatibility of this conjecture with specialization.

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1 Introduction

1.1 Equivariant conjectures and Iwasawa theory of modular forms

The aim of this manuscript is to describe in terms of cohomological data the special values of L-functions of eigencuspforms in p-adic families parametrized by Hecke algebras.

Fix once and for all a prime $p \geq 3$. For Λ a p-adic ring (typically a compact p-ring), K.Kato stated in [41] an influential conjecture about smooth étale sheaves of Λ-modules on $\operatorname{Spec} \mathcal{O}_F[1/p]$ (here $\mathcal{O}_F$ is the ring of integers of a number field F) which he named there the generalized Iwasawa main conjecture for motives with coefficients in Λ although it is nowadays more commonly called the Equivariant Tamagawa Number Conjecture (or, more briefly, the ETNC) with coefficients in Λ. When applied to the étale sheaf of $\mathbb{Z}_p$-modules attached to the p-adic étale realization of a motive M over a number field F, the ETNC recovers the original Tamagawa Number Conjectures of [4] (see also [26]) on the p-adic valuation of the special values of the L-function of M. Compared to the original conjectures of Bloch and Kato, the novelty of the ETNC is that it is inherently a variational statement: it predicts not only the exact valuation of the algebraic part of the values at integers of the L-function of a motive M but also the p-adic variation of these special values as the étale realization of M ranges over the geometric points of a p-adic analytic family of G_F-representations parametrized by $\operatorname{Spec} \Lambda$.

The historically earliest non-trivial examples of such Λ-adic families were given by classical Iwasawa theory for Galois representations and motives, that is to say by the families which arise by deforming the determinant of a single motivic Galois representation, in which case Λ is equal to the classical Iwasawa algebra Λ_{Iw} of [71]. The ETNC for the family of cyclotomic twists of the Galois representation attached to an eigencuspform $f \in S_k(\Gamma_1(N))$, for instance, recovers the Iwasawa Main Conjecture for modular forms relating the analytic p-adic L-function of f to the direct limit on n of the Selmer groups $\mathrm{Sel}_{\mathbb{Q}(\zeta_{p^n})}(\rho_f)$. The ETNC ([41, Conjecture 3.2]) is a precise expression of the hope that there should exist an Iwasawa theory with coefficients in Λ describing the p-adic variation of special values in Λ-adic families just like classical Iwasawa theory describes the variation of special values in families of twists by characters.

In the last three decades, p-adic families of automorphic Galois representations parametrized by Hecke algebras have proven to be of considerable arithmetic interest, if only because they are believed to coincide with the p-adic families of Galois representations attached to universal deformations (in the sense of [56]) of residual automorphic Galois representations. This manuscript states and proves many cases of the ETNC with coefficients in Hecke algebras for the motives attached to rational eigencuspforms.

More precisely, let $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be an irreducible modular (equivalently, odd) representation. Denote by $N(\bar{\rho})$ its tame Artin conductor and choose $U^{(p)} \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(p\infty)})$ a compact open subgroup maximal outside $\Sigma \supset \{\ell | N(\bar{\rho})p\}$ a finite set of primes. As is recalled in section 3.1.1, to this data is attached a local, reduced, p-adic Hecke algebra $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ (the local factor corresponding to $\bar{\rho}$ in the inverse limit on the p-part of the level of the Hecke algebras generated by operators outside Σ; it depends in general on $U^{(p)}$ even though this dependence is suppressed in the notation) and an étale sheaf $T_{\Sigma,\mathrm{Iw}}$ of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-modules on $\operatorname{Spec} \mathbb{Z}[1/\Sigma]$ (the subscript Iw is here to remind the reader that $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is an algebra over Λ_{Iw}). It is conjectured (and often known) that $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is equidimensional of dimension 4. A prime x of $\mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$ or $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ (or more generally of a quotient thereof) is said to be classical if and only if the morphism $\mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p] \longrightarrow \mathbf{k}(x)$ with values in the residue field at x is the system of eigenvalues of a classical eigencuspform $f_x \in S_{k_x}(U^{(p)}U_p)$ of weight $k_x \geq 2$ twisted by a character χ_x of pro-p order. If x is classical and if $M(f_x)_{\mathrm{et},p}$ is the p-adic étale realization of the Grothendieck motive attached to f_x, then the fibre M_x of $T_{\Sigma,\mathrm{Iw}}$ at x is isomorphic as $\mathbf{k}(x)[G_{\mathbb{Q},\Sigma}]$-module to $M(f_x)_{\mathrm{et},p} \otimes \chi_x$.

An outline of our main results is as follows (we refer the reader to section 3.4 and to theorem 4.1.1 in the body of the text for details).

Theorem 1.1.1. *Assume that $\bar{\rho}$ satisfies the following two properties.*

1. (a) *Let p^* be $(-1)^{(p-1)/2}p$. Then $\bar{\rho}|_{G_{\mathbb{Q}(\sqrt{p^*})}}$ is irreducible.*
   (b) *If $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is an extension*
   $$0 \longrightarrow \chi \longrightarrow \bar{\rho}|_{G_{\mathbb{Q}_p}} \longrightarrow \psi \longrightarrow 0,$$
   then $\chi\psi^{-1} \neq 1$ and $\chi\psi^{-1} \neq \bar{\varepsilon}_{\mathrm{cyc}}$ (here $\varepsilon_{\mathrm{cyc}}$ is the p-adic cyclotomic character).

Assume moreover that $\bar{\rho}$ satisfies at least one of the following conditions.

2. (a) *There exists ℓ dividing exactly once $N(\bar{\rho})$, the representation $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is reducible and $\det \bar{\rho}$ is unramified outside p.*
   (b) *There exists ℓ dividing exactly once $N(\bar{\rho})$ and there exists a classical point $x \in \operatorname{Spec} \mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$ such that the eigenform f_x has rational coefficients and belongs to $S_2(\Gamma_0(N))$ with N square-free.*
   (c) *The order of the image of $\bar{\rho}$ is divisible by p and there exists an eigencuspform f attached to a classical point $x \in \operatorname{Spec} \mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$ for which the ETNC with coefficients in Λ_{Iw} holds.*

Finally, assume that the compact open subgroup $U^{(p)}$ satisfies the following property.

3. *If $\ell \nmid p$ belongs to $\Sigma \backslash \Sigma(\bar{\rho})$ then at least one of the following holds.*
   (a) *$\ell \not\equiv \pm 1 \bmod p$.*
   (b) *$\ell \equiv -1 \bmod p$ and for all modular specialization $x : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \bar{\mathbb{Q}}_p$, the restriction of ρ_x to $G_{\mathbb{Q}_\ell}$ is reducible.*
   (c) *$\ell \equiv 1 \bmod p$ and for all modular specialization $x : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \bar{\mathbb{Q}}_p$, the restriction of ρ_x to I_ℓ is scalar.*

Then the ETNC with coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is true for $T_{\Sigma,\mathrm{Iw}}$. Moreover, if x is a classical point of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$, then the ETNC with coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and the ETNC with coefficients in Λ_{Iw} are true for $M(f_x)_{\mathrm{et},p}$.

1.1.1 Consequences in Iwasawa theory

Before explaining in more details the meaning of theorem 1.1.1, we explain some of its Iwasawa-theoretic consequences for the convenience of readers more fluent in this language. As is explained in subsection 1.1.2 below, none of the corollaries stated in this subsection reflect the full range of consequences of theorem 1.1.1. In particular, the hypotheses under which they have been stated and their conclusions have been chosen with clarity in mind: the reader who wishes to derive optimal consequences is advised to consult directly section 4.1.

Classical Iwasawa theory The classical Iwasawa theory of the motive attached to an eigencuspform can refer to one of the following statements depending on the precise setting.

1. The main conjecture for modular motives in the sense of K.Kato ([43, Conjecture 12.10]). This conjecture, which is closely related to the ETNC for modular motives with coefficients in Λ_{Iw}, expresses the special values of the L-function of an eigencuspform twisted by p-power order characters in terms of an algebraic object (namely the so-called fundamental line with coefficients in Λ_{Iw}) and a zeta element with coefficients in Λ_{Iw}. Its precise statement is recalled in conjectures 2.3.3 and 2.3.4 below.

2. The main conjecture for nearly-ordinary eigencuspforms in the sense of R.Greenberg ([34, Conjecture 2.2]). This conjecture, which only applies to classical points of the nearly-ordinary quotient $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$ of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$, states an equality between the ideal generated in Λ_{Iw} by a p-adic L-function and the characteristic ideal of a Λ_{Iw}-adic Selmer group.

3. The main conjecture in the sense of B.Perrin-Riou ([62, 14]). This conjecture relates the p-adic L-function of [53, 81, 2] with an algebraic distribution constructed from Galois cohomology. At present, this conjecture applies only to eigencuspforms whose Galois representation restricted to the decomposition group at p is semi-stable.

4. The main conjecture for supersingular eigencuspforms (that is to say for eigencuspforms with $p|a_p(f)$) in the sense of R.Pollack, S.Kobayashi and I.Sprung [50, 64, 77]. These conjectures state an equality between a pair of ideals generated in Λ_{Iw} by p-adic L-functions on one hand and the characteristic ideals of a pair of Λ_{Iw}-adic Selmer groups on the other. At present, such conjectures are precisely formulated only for classical point of weight 2 or for which $a_p(f) = 0$.

As is apparent in this summary, the first formulation above-which is the one studied in this manuscript-is the only one which is currently known to apply unconditionally to all eigencuspforms. Theorem 1.1.1 implies many new cases of the main conjecture in the sense of Kato. As it is furthermore known that this conjecture implies all the other one when they apply, theorem 1.1.1 also implies many new cases of the Iwasawa main conjecture in the formulation of either Greenberg, Perrin-Riou or Pollack/Kobayashi/Sprung. For instance, the following corollary is an easy consequence of theorem 4.1.1 below and recent results of X.Wan and I.Sprung.

Corollary 1.1.2. *Let f be an eigencuspform attached to a classical point of* $\operatorname{Spec}\mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$ *such that the residual representation $\bar{\rho}_f$ satisfies the following hypotheses.*

1. *Let p^* be $(-1)^{(p-1)/2}p$. Then $\bar{\rho}_f|_{G_{\mathbb{Q}(\sqrt{p^*})}}$ is irreducible.*
2. *The local $G_{\mathbb{Q}_p}$-representation $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is irreducible.*
3. *There exists ℓ dividing exactly once $N(\bar{\rho}_f)$.*
4. *There exists an eigencuspform g of weight 2, with rational coefficients and such that $a_p(g) = 0$ attached to a classical point of of* $\operatorname{Spec}\mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$.

Then the classical Iwasawa main conjecture for f (either in the sense of Kato or, if these are known to make sense, in the sense of Perrin-Riou or Pollack/Kobayashi/Sprung) is true for f.

In the above corollary, the fourth condition can probably be removed thanks to ongoing work of I.Sprung.

Two-variable Iwasawa theory When $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is reducible, it is possible to state conjectures stating the equality of the ideal generated in $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$ by the two-variables p-adic L-function of [48, 23] and the characteristic series of a $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$-adic Selmer group of [61] (the so-called two-variable Iwasawa theory for nearly-ordinary eigencuspforms). See [34, 61, 32] for possible statements.

However, the precise statements to that effect known to this author only apply to rather restrictive settings (typically, they apply to a normal irreducible component of $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$, which has to be a Gorenstein ring). The ETNC with coefficients in $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$, of which conjecture 3.4.11 below is a precise statement, implies these conjectures when they apply. Combining theorem 4.1.1 below and [43, Theorem 12.4] and [76, Theorem 3.29] thus yields the following corollary, which proves many cases of [61, Conjecture 7.4] and [32, Conjecture 11.2.9].

Corollary 1.1.3. *Assume that $\bar{\rho}$ satisfies assumption 1 and that $U^{(p)}$ satisfies assumption 3 of theorem 1.1.1. Assume moreover the following.*

1. *The local representation $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is reducible.*
2. *There exists ℓ dividing exactly once $N(\bar{\rho})$ and $\det\bar{\rho}$ is unramified outside p.*

Then for each minimal prime ideal $\mathfrak{a}\in\operatorname{Spec}\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$ such that $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$ is normal, there is an equality

$$(L_p(\mathfrak{a}))\,\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}=\left(\operatorname{char}_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}}\operatorname{Sel}(\mathfrak{a})_{\mathrm{Iw}}\right)\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}.$$

between the ideal generated by the Mazur-Kitagawa p-adic L-function $L_p(\mathfrak{a})\in\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$ (which specializes to the cyclotomic p-adic L-function at each classical point of $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$) and the characteristic series of the $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$-adic Selmer module $\operatorname{Sel}(\mathfrak{a})_{\mathrm{Iw}}$ (which satisfies a perfect control theorem to the characteristic series of the cyclotomic extended Selmer group at each classical point of $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$).

We insist on the fact that the choice of a normal, irreducible component in corollary 1.1.3 is neither required for the statement of conjecture 3.4.11 below nor for our proof of it under the relevant hypotheses: it is solely made to be able to translate its statement in the language of p-adic L-functions (see proposition 3.4.18 below for details). Note also that the second assumption required for this translation (namely that $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$ be a Gorenstein ring) is automatically satisfied under the first two hypotheses of the corollary by [84, 80].

Iwasawa theory with coefficients in universal deformation rings Finally, assume that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible and denote by $R(\mathfrak{a})_{\mathrm{Iw}}$ the quotient of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ by a minimal prime ideal $\mathfrak{a}$. Under the strong assumption that $R(\mathfrak{a})_{\mathrm{Iw}}$ is a regular local ring, it is possible to translate the conclusion of theorem 1.1.1 into a perhaps more familiar equality between characteristic ideals of étale cohomology groups of $T(\mathfrak{a})_{\mathrm{Iw}}\overset{\mathrm{def}}{=}T_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}R(\mathfrak{a})_{\mathrm{Iw}}$.

Corollary 1.1.4. *Assume that $\bar{\rho}$ satisfies assumption 1 and that $U^{(p)}$ satisfies assumption 3 of theorem 1.1.1. Assume moreover the following.*

1. *The representation $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible.*
2. *The order of the image of $\bar{\rho}$ is divisible by p.*

Then the $R(\mathfrak{a})_{\mathrm{Iw}}$-modules $H^1(\mathbb{Z}[1/p],T(\mathfrak{a})_{\mathrm{Iw}})$ and $H^2_{\mathrm{et}}(\mathbb{Z}[1/p],T(\mathfrak{a})_{\mathrm{Iw}})$ are respectively of rank 1 and zero. Moreover, there exists a unique, non-zero class $\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}\in H^1_{\mathrm{et}}(\mathbb{Z}[1/p],T(\mathfrak{a})_{\mathrm{Iw}})$ satisfying the following two properties.

(i)

$$\operatorname{char}_{R(\mathfrak{a})_{\mathrm{Iw}}}H^2(\mathbb{Z}[1/p],T(\mathfrak{a})_{\mathrm{Iw}})\mid\operatorname{char}_{R(\mathfrak{a})_{\mathrm{Iw}}}\left(H^1(\mathbb{Z}[1/p],T(\mathfrak{a})_{\mathrm{Iw}})/\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}\right)\qquad(1.1.1.1)$$

(ii) Let $f\in S_k(\Gamma_1(N))$ be the eigencuspform attached to a classical point $\lambda:R(\mathfrak{a})_{\mathrm{Iw}}\longrightarrow\mathcal{O}$ and let $T(f)$ be a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice inside the p-adic étale realization of the motive $M(f)$ attached to f. Let $\mathbf{z}(f)_{\mathrm{Iw}}$ be the image of $\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}$ inside $H^1_{\mathrm{et}}(\mathbb{Z}[1/p],T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}})$. Then

$$\operatorname{char}_{\mathcal{O}_{\mathrm{Iw}}}H^2(\mathbb{Z}[1/p],T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}})\mid\operatorname{char}_{\mathcal{O}_{\mathrm{Iw}}}\left(H^1(\mathbb{Z}[1/p],T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}}\right)\quad(1.1.1.2)$$

Let ψ be a character of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ *with sufficiently large order and let* $1 \leq r \leq k-1$ *be an integer. Attached to* f, ψ *and* r *are a p-adic period map*

$$\mathrm{per}_p^{-1} : H^1(\mathbb{Z}[1/p], T(f) \otimes_{\mathcal{O}} \mathcal{O}_{\mathrm{Iw}}) \longrightarrow S_k(\Gamma_1(N)) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p$$

and, for all subfield F *of* $\mathbb{C}$*, a complex period map*

$$\mathrm{per}_{\mathbb{C}} : (S_k(\Gamma_1(N)) \otimes_{\mathbb{Q}} F) \otimes_F \mathbb{C} \longrightarrow \mathbb{C}.$$

Then there exists a subfield F *of* $\mathbb{C}$ *finite over* $\mathbb{Q}$ *such that the image* $\mathbf{z}(f)_{\mathrm{Iw}}$ *of* $\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}$ *inside* $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(f) \otimes \Lambda_{\mathrm{Iw}})$ *is sent through* per_p^{-1} *to* $S_k(\Gamma_1(N)) \otimes_{\mathbb{Q}} F$ *and such that*

$$\mathrm{per}_{\mathbb{C}}\left(\mathrm{per}_p^{-1}(\mathbf{z}(f)_{\mathrm{Iw}}) \otimes 1\right) = L_{\{p\}}(M(f)^*(1), \psi, r).$$

Assume moreover that the following condition holds.

3. *There exists a classical point* $x \in \mathrm{Spec}\, R(\mathfrak{a})_{\mathrm{Iw}}[1/p]$ *such that the Iwasawa main conjecture holds for* f*.*

Then both divisibilities (1.1.1.1) *and* (1.1.1.2) *are actually equalities.*

Under the hypotheses of corollary 1.1.4, the results of [84, 80, 6] show that the ring $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is the universal deformation ring of $\bar{\rho}$ parametrizing deformations which are unramified outside the set Σ, whence the title of this paragraph. Again, we point out that the choice of an irreducible component and the requirement that $R(\mathfrak{a})_{\mathrm{Iw}}$ be a regular ring is made in corollary 1.1.4 solely in order to formulate our results in the language of characteristic ideals (see proposition 3.4.17 for details).

1.1.2 Congruences and the ETNC

The main novelty of this manuscript, however, is in the proof of conjectures with coefficients in Hecke algebras and their compatibilities with specializations and thus lies beyond the world of classical Iwasawa theory.

Indeed, it should be first be noted that even for a single eigencuspform f, the ETNC with coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ for $M(f)_{\mathrm{et},p}$ is a strong refinement of the ETNC with coefficients in Λ_{Iw} for $M(f)_{\mathrm{et},p}$ (or equivalently of the Iwasawa main conjecture for f in any form): the former proves an equality of special values up to a unit in the Hecke algebra whereas the latter predicts such an equality only up to a unit in the normalization of the Hecke algebra, a potentially much larger ring (to illustrate with a close analogy, it is immediate to prove that the first étale cohomology group of the modular curves is free over the normalization of the Hecke algebra but much harder, and in fact not always true, that it is free over the Hecke algebra itself). For that reason, assuming that the ring of coefficients in our main results is a regular or even normal ring (as is needed to formulate them in terms of characteristic ideals) necessarily significantly weakens them.

Furthermore, the ETNC with coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ expresses the the special values of the L-function at integers in the critical strip of all eigencuspforms attached to classical primes of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ in terms of a zeta element $\mathbf{z}_{\Sigma,\mathrm{Iw}}$ and a fundamental line $\Delta_{\Sigma,\mathrm{Iw}}(T_{\Sigma,\mathrm{Iw}})$. Because $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is the universal deformation ring of $\bar{\rho}$ under the hypotheses of theorem 1.1.1, these are exactly the eigencuspforms with residual representation isomorphic to $\bar{\rho}$ so that theorem 1.1.1 predicts in particular congruences between special values of L-functions of congruent modular forms.[1] Thus, it settles in the affirmative the question asked at the end of the introduction of [55], whereas the classical Iwasawa main conjecture for a pair of congruent eigencuspforms or even the ETNC with coefficients in Hecke algebras for such a pair would not *a priori* have any bearing on this question. Reversing the perspective as in [44], or indeed as in the original works of Gauss and Dirichlet on class groups, theorem 1.1.1 also implies that the structure as $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-module of the Galois

[1]Congruent motives do not have congruent special values in general. See subsection 4.1.2 for examples of the congruences that do arise in this way.

cohomology of $M(f)_{\mathrm{et},p}$ is encoded in the special values of the L-functions of forms congruent to f. Because it provides a supplementary description of congruence between special values, the ETNC with coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is a much stronger statement than the collection of the ETNC for all individual classical primes of $\mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$.

To illustrate with concrete examples, section 4.1.2 contain several pairs of congruent eigencuspforms. As far as this author can see, the knowledge of the classical Iwasawa main conjecture (in any form) for one element of the pair or even for both would not in itself entail the non-triviality of the p-adic valuation of the special values of the L-functions of f_6, g_3 and g_4. This non-triviality is, however, an immediate consequence of the ETNC with coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$.

1.2 Outline of the proof

1.2.1 Weight-Monodromy, completed cohomology and fundamental lines

In analogy with the case of separated schemes of finite type over a finite field and in agreement with [40, 26], we understand the Tamagawa Number Conjecture to be a description of the values at integers of the L-function of a motive $M^*(1)$ (which we take for simplicity of exposition over $\mathbb{Q}$ and with coefficients in $\mathbb{Q}$ for the moment) in terms of a *fundamental line* and a *zeta element*; respectively a graded invertible $\mathbb{Z}_p$-module $\Delta_{\mathbb{Z}_p}(M)$ manufactured from the determinant of the étale cohomology of $\operatorname{Spec}\mathbb{Z}[1/p]$ with coefficients in the p-adic étale realization of M and a basis $\mathbf{z}(M) \in \Delta_{\mathbb{Z}_p}(M)$ whose image through p-adic and complex period maps computes $L(M^*(1), 0)$.[2] The equivariant refinement of [41, 31] (see also [40, 8] for the case of group algebra) is a generalization to motives with coefficients in Λ (equivalently, to p-adic families of motives parametrized by $\operatorname{Spec}\Lambda$) together with the compatibility of the conjecture with base change of ring of coefficients (see conjecture 2.3.1 below for a prototypical example in which $\Lambda = \Lambda_{\mathrm{Iw}}$)

Seen under this perspective, the first difficulty in the study of the ETNC for modular motives with coefficients in the Hecke algebra is in specifying what is the precise result that is to be proven. Indeed, the standard formulation of the ETNC, that of [41, Conjecture 3.2.1], assigns a central role to the determinant of the étale cohomology of $\operatorname{Spec}\mathbb{Z}[1/p]$ with coefficients in a bounded complex $\mathscr{F}$ of smooth étale sheaves of Λ-modules on $\operatorname{Spec}\mathbb{Z}[1/p]$ and consequently requires that $\mathrm{R}\Gamma_{\mathrm{et}}(\operatorname{Spec}\mathbb{Z}[1/p], \mathscr{F})$ be a perfect complex of Λ-modules. It is not known (and perhaps not expected) that the étale sheaf of Hecke-modules $\mathscr{F}$ coming from the first étale cohomology group of a modular variety X, or more generally from the étale cohomology in middle degree of a Shimura variety, is such that $\mathrm{R}\Gamma_{\mathrm{et}}(\operatorname{Spec}\mathbb{Z}[1/p], \mathscr{F})$ is a perfect complex of Hecke-modules (the problem being that the inertia invariants submodule of $H^1_{\mathrm{et}}(X \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_p)$ at a prime of bad reduction is not known to be a Hecke-module of finite projective dimension). Moreover, even when the formulation of the ETNC of [41] is known to apply (for instance if the coefficient ring is assumed to be local regular so that all bounded complexes are perfect), the set of characteristic zero specializations with which the ETNC as formulated in [41] is compatible is typically not known: for the conjecture to have some bearing on special values of L-functions of modular forms, this set should at least contain the specializations attached to classical points, though even this property was not known before this manuscript, whereas [41, Conjecture 3.2] seemingly states that this set contains all characteristic zero specializations of maximal generic ramification (but this stronger property does not actually hold; see [29, Section 4.2.2] for a numerical counterexample). Finally, several different p-adic Hecke algebras may act on a given modular motive depending on whether Hecke operators at places of bad reduction and p are considered and on whether they act on the full space of modular forms or just on newforms. These different choices lead to different conjectures which are *a priori* neither equivalent nor in fact obviously compatible with each other. For these reasons, even a precise unconditional

[2] When $\operatorname{Spec}\mathbb{Z}[1/p]$ is replaced by a finite separated scheme $X/\mathbb{F}_q$ and $M_{\mathrm{et},p}$ by a smooth étale sheaf on X, this formulation amounts to the completion, mostly by Grothendieck, of Weil's program of studies of L-functions as achieved in [35, Exposé III] and [15]. In personal discussions with the author, K.Kato recalled that his motivation in restating the Tamagawa Number Conjectures of [4] in this framework was to show that they could take the same external form as the Main Conjecture of Iwasawa theory while J-M.Fontaine credited P.Deligne for the insight that such a formulation could be desirable.

formulation of the ETNC with coefficients in Hecke algebras for modular motives seems to have been heretofore missing from the literature. We note that each of these problems is a manifestation of the fact that Tamagawa numbers (in the usual sense) are not well behaved in p-adic families.

The first main idea of this manuscript simultaneously solves these three problems thanks to the following crucial observation: the severe constraints conjecturally put on the action of the inertia group on the p-adic étale realization of a motive by the Weight-Monodromy Conjecture allow to refine the definition of the local complexes involved in the statement of the ETNC. This process yields refined fundamental lines which are not in general determinants of perfect complexes but rather canonical trivializations of invertible graded modules which themselves are the determinants of the sought for perfect complexes when these are known to exist. When the motive is of automorphic origin, the Local Langlands Correspondence further constrains the inertia action and our constructions are in this way shown to be compatible with the action of the Hecke algebra. Indeed, the very definition of the refined fundamental line for an automorphic motive singles out a specific local factor of the Hecke algebra which coincides with the universal deformation ring subject to natural conditions.

A conceptually satisfying property of the refined fundamental lines is that they are almost by construction shown to be compatible with change of rings of coefficients at motivic points; a property which generalizes the control theorem of [54] (and much subsequent works) in a probably optimal way. In order for them to be compatible with change of levels in the automorphic sense and with specializations at non-motivic points, it is necessary to alter further the definition suggested in [41]. The reason is as follows. The complex period map intervening in the Tamagawa Number Conjecture compares de Rham and Betti cohomology. For a single motive, this makes sense, but what is the Betti realization of a Λ-adic family of motives? A natural answer-the one implicit in [41] for instance-would be that it is its Λ-adic étale cohomology. It turns out however that in the Shimura variety context, it is the choice of the completed cohomology of [22] which yields the correct equivariance properties (this is closely related to the fact that local-global compatibility in p-adic families involves the normalization of the Langlands correspondance of [7] and not the classical normalization). Consequently, the fundamental lines of this manuscript are refined fundamental lines tensored with the determinant of completed cohomology. Independently of the hypotheses of theorem 1.1.1, we formulate precise conjectures relating this object to the variation of special values of L-functions and show that they are compatible with change of levels, passage to the new quotient and characteristic zero specializations.

1.2.2 Euler systems and Taylor-Wiles systems

The proof of theorem 1.1.1 is then by a novel amplification of the method of Euler/Kolyvagin systems, that is to say the combination of V.Kolyvagin's observation in [51] that Galois cohomology classes satisfying compatibility relations in towers of extensions reminiscent of the properties of partial Euler products yield systems of classes with coefficients in principal artinian rings whose local properties are sufficiently constrained to establish a crude bound on the order of some Galois cohomology groups or Selmer groups and the descent principle due to K.Rubin, which allows under suitable assumptions to translate a collection of crude bounds for many specializations with coefficients in artinian rings into a sharp bound in the limit, that is for objects with coefficients in Iwasawa algebras. When the ring of coefficients of the limit object is not known to be normal, as is the case with Hecke algebras, this descent principle meets quite formidable challenges, as it is of course entirely possible for an invertible module to be non-integral while all its specializations to discrete valuation rings are integral, in which case no contradiction can arise by naïve descent. For this reason, most accounts of the Euler/Kolyvagin systems method ([63, 69, 43, 57, 37, 38, 60, 27] for instance) assume that the ring of coefficients is normal, or even regular, and those which do not ([42, 28] for instance) typically prove weaker statement at the locus of non-normality of the coefficient ring. We point out that assuming the ring of coefficients in the statement of our main results to be normal obviates this difficulty, so that not only are such results weaker (as was pointed out in subsection 1.1.2), their proofs is also much easier and essentially amonts to the usual Euler system method applied to our refined fundamental lines.

Our second and most significant novel contribution allows us to bypass this difficulty by first resolving the singularities of the Hecke algebra using the method of Taylor-Wiles of [84, 80] systems as axiomatized and improved in [18, 30, 47] before applying the descent procedure. Under the two first hypotheses of theorem 1.1.1, there exists a Taylor-Wiles system $\{\Delta_Q\}_Q$ (indexed by finite set of well-chosen primes) of refined fundamental lines. This system yields a limit object Δ_∞ over a reduced, local ring R_∞ all of whose irreducible components are regular, local rings (hypotheses 1b and 3 of theorem 1.1.1 are necessary to establish the commutative algebra properties of the universal framed deformation ring of the local $G_{\mathbb{Q}_\ell}$-representation $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ for $\ell \in \Sigma$ required to establish this claim). If the limit object Δ_∞ is not integral, then it has non-integral specializations to discrete valuation rings. Even though Δ_∞ itself has no Galois interpretation, some of its specializations do, so that this non-integrality contradicts Kolyvagin's bound (or more accurately the sharper results of [42]). Hence Δ_∞ is integral. Then so are the Δ_Q and in particular the fundamental line $\Delta_{\Sigma,\mathrm{Iw}}$ we started with. We note that this argument is by nature extremely sensitive to the existence of a potential error term at any step and thus relies critically on the exact control property of the refined fundamental lines observed in section 1.2.1.

We record the following observation, which lies at the conceptual core of this manuscript: just as the conjectured compatibility of the Tamagawa Number Conjecture with the $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^s})/\mathbb{Q})$-action coming from the covering $\operatorname{Spec}\mathbb{Z}[\zeta_{Np^s}, 1/p] \longrightarrow \operatorname{Spec}\mathbb{Z}[1/p]$ implies that the collection of motivic zeta elements should form an Euler system, the conjectured compatibility of the Tamagawa Number Conjecture with the action of the Hecke algebra coming from the covering $X_{U'} \longrightarrow X_U$ of Shimura varieties with $U' \subset U$ implies that the collection of refined fundamental lines should form a Taylor-Wiles system. In both cases, the compatibilities we hope that conjectures on special values of L-functions satisfy therefore suggest powerful tools to actually establish the conjectures.

1.2.3 Organization of the manuscript

The second section of the manuscript is devoted to a precise statement of the ETNC with coefficients in Λ_{Iw} for modular motives and of our choice of normalizations therein. We mostly follow the practices of [41, 43] and when these two differ (for instance in the normalization of the p-adic period map), it is usually the first that we follow. We also review the known results on the Iwasawa Main Conjecture for modular forms that we need in subsection 2.3.3. The third section is devoted to a precise unconditional statement of the ETNC with coefficients in various Hecke algebras and to the proof that it is compatible with specializations. We treat small as well as large Hecke algebras and pay particular attention to the p-adic variation of Euler factors at places of bad reduction. The fourth and last section contains the strongest form of our main results, numerical examples of modular forms satisfying the Iwasawa Main Conjecture and of congruences between special values of L-functions that seem to be new and the proof of our main result. The manuscript ends with an appendix recording in particular our conventions relative to étale cohomology, the determinant functor and Selmer complexes.

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Notations

All rings are assumed to be commutative (and unital). The total quotient ring of a reduced ring R is denoted by $Q(R)$ and the fraction field of a domain A is denoted by $\mathrm{Frac}(A)$. If R is a domain with field of fraction K, if j denotes the morphism $j : \operatorname{Spec} K \longrightarrow \operatorname{Spec} R$ and if $\mathscr{F}$ is a sheaf on the étale site of $\operatorname{Spec} K$, we denote by $\mathrm{R}\,\Gamma_{\mathrm{et}}(R, \mathscr{F})$ the étale cohomology complex $\mathrm{R}\,\Gamma_{\mathrm{et}}(\operatorname{Spec} R, j_*\mathscr{F})$.

If F is a field, we denote by G_F the Galois group of a separable closure of F. If F is a number field and Σ is a finite set of places of F, we denote by F_Σ the maximal Galois extension of F unramified outside $\Sigma \cup \{v|\infty\}$ and by $G_{F,\Sigma}$ the Galois group $\operatorname{Gal}(F_\Sigma/F)$. The ring of integer of F is written $\mathcal{O}_F$. If v is a finite place of F, then $\mathcal{O}_{F,v}$ is the unit ball of F_v, ϖ_v is a fixed choice of uniformizing parameter and k_v is the residual field of $\mathcal{O}_{F,v}$. The reciprocity law of local class field theory is normalized so that ϖ_v is sent to (a choice of lift of) the geometric Frobenius morphism $\operatorname{Fr}(v)$. For all rational primes ℓ, we fix an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$, an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_\ell$ and an identification $\iota_{\infty,\ell} : \mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$ extending $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$.

The non-trivial element of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ is denoted by τ. If R is a ring in which 2 is a unit and M is an $R[\operatorname{Gal}(\mathbb{C}/\mathbb{R})]$-module (resp. m is an element of M), then $M^\pm$ (resp. $m^\pm$) denotes the eigenspace on which τ acts as ± 1 (resp. the projection of m to the $\pm$-eigenspace).

The p-adic cyclotomic character of $G_\mathbb{Q}$ is denoted by $\varepsilon_{\mathrm{cyc}}$. The field $\mathbb{Q}_\infty/\mathbb{Q}$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, that is to say the only Galois extension of $\mathbb{Q}$ with Galois group Γ isomorphic to $\mathbb{Z}_p$. For $n \in \mathbb{N}$, the number field $\mathbb{Q}_n$ is the sub-extension of $\mathbb{Q}_\infty$ with Galois group $\mathbb{Z}/p^n\mathbb{Z}$. If S is a reduced $\mathbb{Z}_p$-algebra, we write S_{Iw} for the completed group-algebra $S[[\Gamma]]$. To stick to usual notations, the 2-dimensional regular local ring $\mathbb{Z}_{p,\mathrm{Iw}} = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[X]]$ is denoted by Λ_{Iw}.

For G a group, a G-representation (T, ρ, R) is an R-module T free of finite rank together with a morphism

$$\rho : G \longrightarrow \operatorname{Aut}_R(T)$$

which is assumed to be continuous if G is a topological group. If F is a number field, Σ is a finite set of finite primes of $\mathcal{O}_F$ containing $\{v|p\}$, R is complete local noetherian ring of residual characteristic p which is a reduced $\mathbb{Z}_p$-algebra and (T, ρ, R) is a $G_{F,\Sigma}$-representation, then the $G_{F,\Sigma}$-representation $(T_{\mathrm{Iw}}, \rho_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ is the R_{Iw}-module $T \otimes_R R_{\mathrm{Iw}}$ with $G_{F,\Sigma}$-action on T through ρ and on R_{Iw} through the composition $\chi_\Gamma : G_{F,\Sigma} \twoheadrightarrow \operatorname{Gal}(F\mathbb{Q}_\infty/F) \hookrightarrow R_{\mathrm{Iw}}^\times$. We do not distinguish between the $G_{F,\Sigma}$-representation (T, ρ, R) and the corresponding étale sheaf T on $\operatorname{Spec} \mathcal{O}_F[1/\Sigma]$.

We refer to appendices A.1 and A.3 for notations and conventions regarding the determinant functor and complexes of cohomology with local conditions.

2 The ETNC for modular motives with coefficients in Λ_{Iw}

2.1 Modular curves, modular forms

2.1.1 Modular curves

Let $\mathbf{G}$ be the reductive group GL_2 over $\mathbb{Q}$ and let $\operatorname{Sh}(\mathbf{G}, \mathbb{C} - \mathbb{R})$ be the tower of Shimura curves attached to the Shimura datum $(\mathbf{G}, \mathbb{C} - \mathbb{R})$. For $U \subset \mathbf{G}(\mathbb{A}_\mathbb{Q}^{(\infty)})$ a compact open subgroup, the curve $Y(U) = \operatorname{Sh}_U(\mathbf{G}, \mathbb{C} - \mathbb{R})$ and its compactification along cusps $j : Y(U) \hookrightarrow X(U)$ are regular schemes over $\mathbb{Z}$ which are smooth over $\mathbb{Z}_\ell$ if U_ℓ is maximal and U is sufficiently small; *e.g* $U = U(N)$ and $N \geq 3$ (see [46, p. 305]). The set of complex points of $Y(U)$ is given by the double quotient

$$Y(U)(\mathbb{C}) \simeq \mathbf{G}(\mathbb{Q}) \backslash \left(\mathbb{C} - \mathbb{R} \times \mathbf{G}(\mathbb{A}_\mathbb{Q}^{(\infty)})/U\right)$$

and is an algebraic variety if U is sufficiently small. We consider the following compact open subgroups of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$.

$$U_0(N)=\prod_\ell U_0(N)_\ell=\prod_\ell\left\{g\in \mathrm{GL}_2(\mathbb{Z}_\ell)|g\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\bmod \ell^{v_\ell(N)}\right\}$$
$$U_1(N)=\prod_\ell U_1(N)_\ell=\prod_\ell\left\{g\in \mathrm{GL}_2(\mathbb{Z}_\ell)|g\equiv\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}\bmod \ell^{v_\ell(N)}\right\}$$
$$U(M,N)=\prod_\ell U(M,N)_\ell=\prod_\ell\left\{g\in U_1(N)_\ell|g\equiv\begin{pmatrix}1 & 0\\ * & *\end{pmatrix}\bmod \ell^{v_\ell(M)}\right\}$$
$$U(N)=U(N,N)=\prod_\ell\left\{g\in \mathrm{GL}_2(\mathbb{Z}_\ell)|g\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\bmod \ell^{v_\ell(N)}\right\}$$

For $U=U_?(*)$ with $?=\varnothing,0$ or 1 and $*=N$ or N,M, we write $Y_?(*)$ for $Y(U)$ and $X_?(*)$ for $X(U)$. For $U\subset\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$, we write $U=U_pU^{(p)}$ with $U_p=U\cap\mathbf{G}(\mathbb{Q}_p)$ and $U^{(p)}\subset\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(p\infty)})$ and we denote by $\Sigma(U)$ the finite set of finite places ℓ such that U_ℓ is not compact open maximal.

2.1.2 Hecke correspondences and twisted projections

Let g be an element of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$. Right multiplication by g induces a finite flat $\mathbb{Q}$-morphism

$$[\cdot g]:X(U\cap gUg^{-1})\longrightarrow X(g^{-1}Ug\cap U)$$

which defines the Hecke correspondence $T(g)=[UgU]$ on $X(U)$.

$$\begin{array}{ccc} X(U\cap gUg^{-1}) & \xrightarrow{[\cdot g]} & X(g^{-1}Ug\cap U)\\ \downarrow & & \downarrow\\ X(U) & \overset{[UgU]}{\dashrightarrow} & X(U)\end{array}\qquad(2.1.2.1)$$

For ℓ a prime number and $a\in\widehat{\mathbb{Q}}^\times$ a finite idèle, we denote by $T(\ell)$ the Hecke correspondence $[U\begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix}U]$ and by $\langle a\rangle$ the diamond correspondence $[U\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}U]$. The full classical Hecke algebra $\mathfrak{h}(U)$ of level U is the $\mathbb{Z}$-algebra generated by Hecke and diamond correspondences acting on $X(U)$.

Let U be a compact open subgroup and let ℓ be a finite prime which does not belong to $\Sigma(U)\cup\{p\}$, so that in particular U_ℓ is a maximal compact open subgroup of $\mathbf{G}(\mathbb{Q}_\ell)$. Let $g\in\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ be the element equal to the identity everywhere except at ℓ where it is equal to

$$g=\begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix}.$$

Let $U'\subsetneq U$ the compact open subgroup $U\cap U(\ell,1)$. We denote by

$$\pi_{U',U,g}:X(U')\longrightarrow X(U)$$

the composition of the right-multiplication $[\cdot g]$ with the natural projection from $X(g^{-1}U'g)$ to $X(U)$.

2.1.3 Cohomology

Betti and étale cohomology Let $\pi:E\longrightarrow Y(N)$ be the universal elliptic curve over $Y(N)$ and let $\bar\pi:\bar E\longrightarrow X(N)$ be the universal generalized elliptic curve over $X(N)$. For $k\geq 2$ an integer, let $\mathcal{H}_{k-2}$ be the local system $\mathrm{Sym}^{k-2}R^1\pi_*\mathbb{Z}$ on $Y(N)(\mathbb{C})$ and let $\mathcal{F}_{k-2}$ be $j_*\mathcal{H}_{k-2}$.

If $N \geq 3$, let $\mathrm{R}\Gamma_B(X(N)(\mathbb{C}), \mathcal{F}_{k-2})$ be the singular cohomology complex of the complex points of $X(N)$. If X is a quotient curve $G\backslash X(N)$ with $N \geq 3$ under the action of a finite group G and if A is a ring in which $|G|$ is invertible, we denote by $H^i(X(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} A)$ the cohomology group $H^i(X(N)(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} A)^G$ and note that it is also the cohomology of the complex $\mathrm{R}\Gamma_B(X(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} A)$ where X is seen as a Deligne-Mumford stack over A (in particular $H^i(X(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} A)$ is independent of the choice of N and G). We denote by $\mathrm{R}\Gamma_{\mathrm{et}}(X \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ the complex computing the cohomology groups

$$H^i_{\mathrm{et}}(X \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{Z}_p) = H^i(X(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

As usual, we denote by

$$\mathcal{M}_k(U(N)) = H^0(X(N), \pi_*(\Omega^1_{\bar{E}/X(N)})^{\otimes k})$$

the space of holomorphic modular forms of weight k and by

$$S_k(U(N)) = H^0(X(N), \pi_*(\Omega^1_{\bar{E}/X(N)})^{\otimes(k-2)} \otimes_{\mathcal{O}(X(N))} \Omega^1_{X(N)/\mathbb{Q}})$$

the space of holomorphic cusp forms.

For $U^{(p)} \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(p\infty)})$ a compact open subgroup and $\mathcal{O}$ a discrete valuation ring finite and flat over $\mathbb{Z}_p$, we denote by $\tilde{H}^1_c(U^{(p)}, \mathcal{O})$ the completed cohomology with compact support of [22], that is the direct limit

$$\tilde{H}^1_c(U^{(p)}, \mathcal{O}) = \varprojlim_{s} \varinjlim_{U_p} H^1_c(X(U^{(p)}U_p) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{O}/\varpi^s)$$

of the étale cohomology groups with compact support over all compact open subgroup U_p of $\mathbf{G}(\mathbb{Q}_p)$ followed by ϖ-completion.

Hecke action The Hecke algebra acts contravariantly on cohomological realizations of $X(U)$. As the Hodge decomposition realizes the $\mathbb{C}$-vector space of complex cusp forms $S_k(U)$ as a direct summand of $H^1(X(U)(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{C})$, the complex Hecke algebra $\mathfrak{h}(U) \otimes_{\mathbb{Z}} \mathbb{C}$ acts on $S_k(U)$. The $\mathbb{Z}$-submodule $S_k(U, \mathbb{Z}) \subset S_k(U)$ of cusp forms with integral q-expansion is stable under the action of $\mathfrak{h}(U)$ thereby induced. This defines an action of $\mathfrak{h}(U) \otimes_{\mathbb{Z}} A$ on $S_k(U, A) \overset{\text{def}}{=} S_k(U, \mathbb{Z}) \otimes_{\mathbb{Z}} A$ for all ring A. The complex $\mathrm{R}\Gamma_B(X(U)(\mathbb{C}), \mathcal{F}_{k-2})$ admits a representation as a bounded below (but not necessarily bounded above) complex of projective $\mathfrak{h}(U)$-modules.

An eigenform $f \in S_k(U)$ is an eigenvector under the action of all $T(\ell)$. To an eigenform f is attached an automorphic representation $\pi(f)$ of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ and the conductor of f is the the conductor $\mathfrak{c}(\pi(f))$as in [12, Theorem 1]. Two eigenforms are equivalent in the sense of Atkin-Lehner if they are eigenvectors for the same eigenvalues for all $T(\ell)$ except possibly finitely many. A newform $f \in S_k(U)$ is an eigenform such that for all $g \in S_k(U')$ equivalent to f in the sense of Atkin-Lehner, $\mathfrak{c}(\pi(f))$ divides $\mathfrak{c}(\pi(g))$.

We call $\mathfrak{h}(U) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ the classical p-adic Hecke algebra and denote it by $\mathbf{T}_{\mathrm{cl}}(U)$. It is a semi-local ring finite and free as $\mathbb{Z}_p$-module. To an eigenform f is attached a map λ_f from $\mathbf{T}_{\mathrm{cl}}(U)$ to $\bar{\mathbb{Q}}_p$ by $T(\ell)f = \lambda_f(T(\ell))f$ and, conversely, we say that a map λ from a quotient or sub-algebra of $\mathbf{T}_{\mathrm{cl}}(U)$ to a discrete valuation ring in $\bar{\mathbb{Q}}_p$ is modular if there exists an eigenform f such that $\lambda = \lambda_f$. The reduced Hecke algebra $\mathbf{T}^{\mathrm{red}}(U) \subset \mathbf{T}_{\mathrm{cl}}(U)$ is the sub $\mathbb{Z}_p$-algebra generated by the diamond operators $\langle \ell \rangle$ and the Hecke operators $T(\ell)$ for $\ell \notin \Sigma(U) \cup \{p\}$ (recall here that U_ℓ is a maximal compact open subgroup if $\ell \notin \Sigma(U)$). The new Hecke algebra $\mathbf{T}^{\mathrm{new}}(U)$ is the quotient of $\mathbf{T}_{\mathrm{cl}}(U)$ acting faithfully on the space of newforms of level U. Both $\mathbf{T}^{\mathrm{red}}(U)$ and $\mathbf{T}^{\mathrm{new}}(U)$ are finite, flat, reduced, semi-local $\mathbb{Z}_p$-algebras.

If $V_p \subset U_p \subset \mathbf{G}(\mathbb{Q}_p)$ and $U^{(p)} \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{p\infty})$ are compact open subgroups, there is a canonical surjection $\mathbf{T}^{\mathrm{red}}(V_pU^{(p)}) \twoheadrightarrow \mathbf{T}^{\mathrm{red}}(U_pU^{(p)})$. Denote by $\mathbf{T}^{\mathrm{red}}(U^{(p)})$ the inverse limit of the $\mathbf{T}^{\mathrm{red}}(U_pU^{(p)})$ over all compact open subgroup $U_p \subset \mathbf{G}(\mathbb{Q}_p)$. Then $\mathbf{T}^{\mathrm{red}}(U^{(p)})_{\mathcal{O}} = \mathbf{T}^{\mathrm{red}}(U^{(p)}) \otimes_{\mathbb{Z}_p} \mathcal{O}$ acts faithfully on $\tilde{H}^1_c(U^{(p)}, \mathcal{O})$ for all discrete valuation ring $\mathcal{O}$ finite flat over $\mathbb{Z}_p$.

2.2 Motives attached to modular forms

We review the properties of Grothendieck motives attached to eigencuspforms with coefficients in group used in the formulation of the Tamagawa Number Conjecture of [4] and its equivariant refinement in [40].

2.2.1 Modular motives

The category $CH(\mathbb{Q})$ of Chow motives is the pseudo-abelian envelope of the category of proper smooth schemes over $\mathbb{Q}$ with Tate twists inverted and with degree zero correspondences modulo rational equivalence as morphisms. A Chow motive is thus a triplet (X, e, r) where $X/\mathbb{Q}$ is proper and smooth, e is a projector of $CH^{\dim X}(X \times X)_{\mathbb{Q}}$ and r is an integer. For $k \geq 2$ and $r \in \mathbb{Z}$, a Chow motive of weight $k-1-2r$ (in the sense that the weight filtration of its Betti or étale realization is pure of weight $k-1-2r$) attached to the modular curve is constructed in [70].

Let $\bar{E}^{(k-2)}$ be the $(k-2)$-fold fiber product of $\bar{E}$ with itself over $X(N)$. Let KS_k be the canonical desingularization of $\bar{E}^{(k-2)}$ constructed in [16, n°5] (see also [70, Section 3]). The symmetric group $\mathfrak{S}_{k-2}$ acts on $\bar{E}^{(k-2)}$ by permutations, the $(k-2)$-th power of $(\mathbb{Z}/N\mathbb{Z})^2$ acts by translation and μ_2^{k-2} acts by inversion in the fibers. Let $\tilde{G}_{k-2}$ be the wreath product of $((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mu_2)^{k-2}$ with $\mathfrak{S}_{k-2}$. Then $\tilde{G}_{k-2}$ acts by automorphisms on $\bar{E}^{(k-2)}$ and thus on KS_k. Let ε be the character of $\tilde{G}_{k-2}$ which is trivial on $(\mathbb{Z}/N\mathbb{Z})^{2(k-2)}$, the product map on μ_2^{k-2} and signature on $\mathfrak{S}_{k-2}$. Let $\Pi_\varepsilon \in \mathbb{Z}[\frac{1}{2Nk!}][\tilde{G}_{k-2}]$ be the projector attached to ε. For $r \in \mathbb{Z}$, the triplet $(KS_k, \Pi_\varepsilon, r)$ is a Chow motive which we denote by $\mathcal{W}_N^{k-2}(r)$. Let ${}^B\mathcal{W}_N^{k-2}(r)$ (resp. ${}^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)$ resp. ${}^{\mathrm{dR}}\mathcal{W}_N^{k-2}(r)$) be its Betti (resp. étale p-adic resp. de Rham) realization.

For a number field L, a Grothendieck motive over $\mathbb{Q}$ with coefficients in L is an object in the category of motives over $\mathbb{Q}$ in which $\mathrm{Hom}(h(X), h(Y))$ is the group of algebraic cycles on $X \times Y$ of codimension $\dim Y$ tensored over $\mathbb{Q}$ with L modulo homological equivalence. If M is a Grothendieck motive, we denote by $M^*(1)$ its Cartier dual motive.

Fix a number field F containing all the eigenvalues of Hecke operators acting on eigenforms in $S_k(U(N))$. The image of $\mathcal{W}_N^{k-2}$ in the category of Grothendieck motive over $\mathbb{Q}$ with coefficients in F decomposes under the action of the Hecke correspondences. Let $f \in S_k(U_1(N))$ be a newform and denote by λ_f the corresponding modular map. Let $\mathcal{W}(f)(r)$ be the largest Grothendieck sub-motive of $\mathcal{W}_N^{k-2}(r)$ over $\mathbb{Q}$ with coefficients in F on which $\mathbf{T}^{\mathrm{red}}(U(N))$ acts through λ_f. For $\sigma : F \hookrightarrow \mathbb{C}$ an embedding of F into $\mathbb{C}$, we denote by $\mathcal{W}(f)_{B,\sigma}(r)$ (resp. $\mathcal{W}(f)_{\mathrm{dR}}(r)$, resp. $\mathcal{W}(f)_{\mathrm{et},p}(r)$) the Betti (resp. de Rham, resp. p-adic étale) realization of $\mathcal{W}(f)(r)$.

2.2.2 Realizations and comparison theorems

By [70, Theorem 1.2.1] and the comparison theorems in cohomology, $\mathcal{W}_N^{k-2}(r)$ and its realizations satisfy the following compatibilities.

Betti realization There is a canonical (and in particular Hecke-equivariant) isomorphism of $\mathbb{Q}[\mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-modules

$${}^B\mathcal{W}_N^{k-2}(r) = (2\pi i)^r H^{k-1}(KS_k(\mathbb{C}), \mathbb{Q})(\varepsilon) \overset{\mathrm{can}}{\simeq} (2\pi i)^r H^1(X(N)(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{Q})$$

between the Betti realization of $\mathcal{W}_N^{k-2}(r)$ and the Betti cohomology of the modular curve with coefficients in $\mathcal{F}_k \otimes_{\mathbb{Z}} \mathbb{Q}$. For $\sigma : F \hookrightarrow \mathbb{C}$, this isomorphism induces an isomorphism

$$\mathcal{W}(f)_{B,\sigma}(r) \overset{\mathrm{can}}{\simeq} (2\pi i)^r H^1(X(N)(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{Q})(f)$$

on the f-part of both sides. Here, the right-hand side is the largest quotient on which $\mathbf{T}^{\mathrm{red}}(U(N))$ acts through $\sigma \circ \lambda_f$.

de Rham realization There is a canonical isomorphism of $\mathbb{Q}$-vector spaces

$$^{\mathrm{dR}}\mathcal{W}_N^{k-2}(r) \overset{\mathrm{can}}{\simeq} H^{k-1}_{\mathrm{dR}}(KS_k/\mathbb{Q})(\varepsilon)(r).$$

Hence, the de Rham realization $^{\mathrm{dR}}\mathcal{W}_N^{k-2}(r)$ is equipped with a decreasing filtration $\mathrm{Fil}^i(^{\mathrm{dR}}\mathcal{W}_N^{k-2}(r))$ satisfying

$$\mathrm{Fil}^i\left(^{\mathrm{dR}}\mathcal{W}_N^{k-2}(r)\right) = \begin{cases} H^{k-1}_{\mathrm{dR}}(KS_k/\mathbb{Q})(\varepsilon)(r) & \text{if } i+r \leq 0, \\ S_k(U(N)) & \text{if } 1 \leq i+r \leq k-1, \\ 0 & \text{if } i+r > k-1. \end{cases}$$

There is a canonical comparison isomorphism of $\mathbb{C}$-vector spaces

$$^{\mathrm{dR}}\mathcal{W}_N^{k-2}(r) \otimes_{\mathbb{Q}} \mathbb{C} \overset{\mathrm{can}}{\simeq} {}^{B}\mathcal{W}_N^{k-2}(r) \otimes_{\mathbb{Q}} \mathbb{C} \tag{2.2.2.1}$$

compatible with the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}$ on the left-hand side and the diagonal action on the right-hand side. When $1 \leq r \leq k-1$, the isomorphism (2.2.2.1) induces on $\mathrm{Fil}^0(^{\mathrm{dR}}\mathcal{W}_N^{k-2}(r))$ a complex period map

$$S_k(U(N)) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow \left[(2\pi i)^r H^1(X(N)(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}\right]^+. \tag{2.2.2.2}$$

Composed with surjection on the quotient by $\left[(2\pi i)^r H^1(X(N)(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}\right]^+$, this map yields an isomorphism

$$\mathrm{per}_{\mathbb{C}} : S_k(U(N)) \otimes_{\mathbb{Q}} \mathbb{R} \simeq (2\pi i)^{r-1} H^1(X(N)(\mathbb{C}), \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{Q})^{(-1)^{r-1}} \otimes_{\mathbb{Q}} \mathbb{R}. \tag{2.2.2.3}$$

whose target is ${}^{B}\mathcal{W}_N^{k-2}(r-1)^+ \otimes_{\mathbb{Q}} \mathbb{R}$. For $\sigma : F \hookrightarrow \mathbb{C}$, the complex period map is compatible with projection on both sides onto the largest quotient on which $\mathbf{T}^{\mathrm{red}}(U(N))$ acts through $\sigma \circ \lambda_f$.

***p*-adic étale realization** There is a canonical isomorphism of $\mathbb{Q}_p[\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})]$-modules

$$^{\mathrm{et}}\mathcal{W}_N^{k-2}(r) = H^{k-1}_{\mathrm{et}}(KS_k \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_p)(\varepsilon)(r) \overset{\mathrm{can}}{\simeq} H^1_{\mathrm{et}}(X(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{F}_{k-2} \otimes_{\mathbb{Z}} \mathbb{Q}_p)(r).$$

There is a canonical comparison isomorphism

$${}^{B}\mathcal{W}_N^{k-2}(r) \otimes_{\mathbb{Q}} \mathbb{Q}_p \overset{\mathrm{can}}{\simeq} {}^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)$$

of $\mathbb{Q}_p[\mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-modules. The $G_{\mathbb{Q}_p}$-representation $^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)$ is a de Rham p-adic representation in the sense of [24, 1]. Its de Rham module

$$D_{\mathrm{dR}}(^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)) = H^0(G_{\mathbb{Q}_p}, {}^{\mathrm{et}}\mathcal{W}_N^{k-2}(r) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})$$

is equipped with a descending filtration

$$D^i_{\mathrm{dR}}(^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)) = H^0(G_{\mathbb{Q}_p}, {}^{\mathrm{et}}\mathcal{W}_N^{k-2}(r) \otimes_{\mathbb{Q}_p} B^i_{\mathrm{dR}})$$

indexed by $\mathbb{Z}$ such that

$$D^i_{\mathrm{dR}}(^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)) = \begin{cases} D_{\mathrm{dR}}(^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)) & \text{if } i+r \leq 0, \\ S_k(U(N)) \otimes_{\mathbb{Q}} \mathbb{Q}_p & \text{if } 1 \leq i+r \leq k-1, \\ 0 & \text{if } i+r > k-1. \end{cases} \tag{2.2.2.4}$$

The dual exponential map $\exp^*$ of [4] is a map

$$\exp^* : H^1(G_{\mathbb{Q}_p}, {}^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)) \longrightarrow D^0_{\mathrm{dR}}(^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)).$$

For all integers $1 \leq r \leq k-1$, the dual exponential map induces an inverse p-adic period map

$$\mathrm{per}_p^{-1} : H^1(G_{\mathbb{Q}_p}, {}^{\mathrm{et}}\mathcal{W}_N^{k-2}(r)) \longrightarrow S_k(U(N)) \otimes_{\mathbb{Q}} \mathbb{Q}_p. \tag{2.2.2.5}$$

The choice of the inverse normalization comes from the fact that the $\mathbb{Q}$-vector spaces of eigen-cuspforms should be thought as the absolute cohomology underlying all realizations and thus as the source of all comparison maps.

For $\mathfrak{p} \subset \mathcal{O}_F$ over p, there are variants of all the results above with coefficients in $F_{\mathfrak{p}}$ for the largest quotient on which $\mathbf{T}^{\mathrm{red}}(U(N))$ acts through λ_f.

2.2.3 Group-algebra coefficients

Let $K/\mathbb{Q}$ be a finite abelian extension with Galois group G (which may be trivial). The motive $h^0(\operatorname{Spec} K)$ is defined by the compatible system of realizations

$$\begin{aligned} h^0(\operatorname{Spec} K)_B &= \prod_{\sigma:K\hookrightarrow\mathbb{C}} \mathbb{Q} \simeq \mathbb{Q}[G], \\ h^0(\operatorname{Spec} K)_{\mathrm{et},\ell} &= \prod_{\sigma:K\hookrightarrow\bar{\mathbb{Q}}} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell[G], \\ h^0(\operatorname{Spec} K)_{\mathrm{dR}} &= K. \end{aligned}$$

The group G acts naturally on $h^0(\operatorname{Spec} K)_{\mathrm{dR}}$ and the de Rham filtration is given by $\operatorname{Fil}^0(K) = K$ and $\operatorname{Fil}^1(K) = 0$.

Let M be a Grothendieck motive over $\mathbb{Q}$ with coefficients in L and let $\iota : L \hookrightarrow \mathbb{C}$ be an embedding of L into $\mathbb{C}$. There exists a motive $M_K \overset{\mathrm{def}}{=} M \otimes_{\mathbb{Q}} h^0(\operatorname{Spec} K)$ over $\mathbb{Q}$ endowed with a natural action of the group-algebra $L[G]$. In a slight abuse of terminology, we say that M_K has coefficients in $L[G]$. Let S be a finite set of rational primes containing p and all the primes of ramification of K. For $\ell \notin S$, the Euler factor at ℓ of the p-adic étale realization of M_K is

$$\operatorname{Eul}_\ell(M_K, X) = \det(1 - \operatorname{Fr}(\ell)X|M_{K,\mathrm{et},p}^{I_\ell}) = \prod_{\chi\in\hat{G}} (1 - \chi(\operatorname{Fr}(\ell)) \operatorname{Fr}(\ell)X|M_{\mathrm{et},p}^{I_\ell}).$$

We assume that $\operatorname{Eul}_\ell(M_K, X)$ belongs to $L[G][X] \overset{\iota}{\hookrightarrow} \mathbb{C}[G][X]$ and does not depend on the choice of p. The S-partial L-function of M_K is

$$L_S(M_K, s) = \prod_{\ell\notin S} \frac{1}{\operatorname{Eul}_\ell(M_K, \ell^{-s})} = \left(\sum_{(n,S)=1} a_n \chi(n) n^{-s} \right)_{\chi\in\hat{G}} \in \mathbb{C}[G]^{\mathbb{C}}$$

with the natural action of G on $L_S(M_K, s)$. Here, we have assumed the usual conjecture that $L_S(M_K, \cdot)$ is well-defined for $\Re s$ large enough and admits a meromorphic continuation to $\mathbb{C}$ (the S-partial L-function depends on the choice of ι though we have suppressed this dependence from the notation). For $\sigma \in G$, denote by $\mathbb{N}_\sigma$ the set of natural numbers prime to S such that the Artin reciprocity map sends $n \in \mathbb{N}_\sigma$ to σ and write

$$L_S(M, \sigma, s) = \sum_{n\in\mathbb{N}_\sigma} a_n n^{-s}$$

for the σ-component of the L-function of M. Then $L_S(M_K, \cdot)$ is also equal to

$$\sum_{\sigma\in G} L_S(M, \sigma, \cdot)\sigma \in \mathbb{C}[G]^{\mathbb{C}}.$$

2.3 The ETNC with coefficients in $F[\operatorname{Gal}(\mathbb{Q}_n/\mathbb{Q})]$ and $\mathcal{O}_{\mathrm{Iw}}$

Henceforth, we constantly appeal to the determinant functor applied to Selmer complexes and étale cohomology complexes. We refer to the appendices for our choices of normalizations.

Let $f \in S_k(U(N))$ be a newform with eigenvalues in a number field F. Denote by M the motive $\mathcal{W}(f)$ over $\mathbb{Q}$ with coefficients in F and let M^* be the dual motive of M. We fix an embedding $\iota : F \hookrightarrow \mathbb{C}$ with respect to which we compute the Betti realization, the coefficients of L-functions and the comparison theorem(s).

Let $\mathfrak{p} \subset \mathcal{O}_F$ be a prime ideal of F above p and denote by $\mathcal{O}$ the ring $\mathcal{O}_{F,\mathfrak{p}}$. Recall that $\mathbb{Q}_n/\mathbb{Q}$ is the subfield of $\mathbb{Q}(\zeta_{p^{n+1}})$ with Galois group G_n over $\mathbb{Q}$ equal to $\mathbb{Z}/p^n\mathbb{Z}$. We denote by $V_{\mathbb{C},n}$

the Betti realization of $M \times_{\mathbb{Q}} \mathbb{Q}_n$. Write V for the étale p-adic realization of M. For $T \subset V$ a stable $\mathcal{O}[G_{\mathbb{Q}}]$-lattice, the $\mathcal{O}_{\mathrm{Iw}}$-module $T_{\mathrm{Iw}} = T \otimes_{\mathcal{O}} \mathcal{O}_{\mathrm{Iw}}$ is a $G_{\mathbb{Q}}$-stable module inside the $G_{\mathbb{Q}}$-representation $V_{\mathrm{Iw}} = M_{\mathrm{et},p} \otimes_{F_{\mathfrak{p}}} \Lambda_{\mathrm{Iw}}[1/p]$ and likewise $T \otimes_{\mathcal{O}} \mathcal{O}[G_n]$ is a Galois stable sub-module inside $V[G_n] = M_{\mathrm{et},p} \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}[G_n]$ for all $n \in \mathbb{N}$.

Fix integers $n \geq 1$ and $1 \leq r \leq k-1$.

2.3.1 Equivariant period maps

Denote $V(k-r-1) \otimes_{F_{\mathfrak{p}}} F_{\mathfrak{p}}[G_n]$ by $V_n(k-r-1)$. The inverse of the p-adic period map (2.2.2.5) composed with the determinant functor induces an isomorphism

$$\begin{array}{c}\mathrm{Det}_{F_{\mathfrak{p}}[G_n]}\, H^1(G_{\mathbb{Q}_p(\zeta_{p^n})}, V(k-r)) \\ \Big\downarrow \mathrm{per}_p^{-1} \\ \mathrm{Det}_{F_{\mathfrak{p}}[G_n]}\,(S(U(N))(f) \otimes_F F_{\mathfrak{p}}[G_n])\end{array}$$

Here $S(U(N))(f)$ is the largest quotient of $S(U(N))$ on which the Hecke algebra acts through the eigenvalues of f and $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p)$ acts on the cohomology through its quotient G_n. Taking the tensor product with the determinant of $V_n(k-r-1)^+$ yields an equivariant p-adic period map

$$\begin{array}{c}\mathrm{Det}_{F_{\mathfrak{p}}[G_n]}\, H^1(G_{\mathbb{Q}_p(\zeta_{p^n})}, V(k-r)) \otimes_{F_{\mathfrak{p}}[G_n]} \mathrm{Det}^{-1}_{F_{\mathfrak{p}}[G_n]}(V_n(k-r-1))^+ \\ \Big\downarrow \mathrm{per}_p^{-1} \\ \mathrm{Det}_{F_{\mathfrak{p}}[G_n]}\,(S(U(N))(f) \otimes_F F_{\mathfrak{p}}[G_n]) \otimes_{F_{\mathfrak{p}}[G_n]} \mathrm{Det}^{-1}_{F_{\mathfrak{p}}[G_n]}(V_n(k-r-1))^+.\end{array} \tag{2.3.1.1}$$

The complex period map (2.2.2.3) composed with the determinant functor induces an equivariant complex period map

$$\begin{array}{c}\left[\mathrm{Det}_{\mathbb{C}[G_n]}\,(S(U(N))(f) \otimes_F \mathbb{C}[G_n])\right] \otimes \left[\mathrm{Det}^{-1}_{\mathbb{C}[G_n]}(V_{\mathbb{C},n}(k-r-1)^+ \otimes_F \mathbb{C})\right] \\ \Big\downarrow \mathrm{per}_{\mathbb{C}} \\ \mathbb{C}[G_n]\end{array} \tag{2.3.1.2}$$

from the F-rational subspace

$$\mathrm{Det}^{-1}_{F[G_n]}\,(S(U(N))(f) \otimes_F F[G_n]) \otimes_{F[G_n]} \mathrm{Det}_{F[G_n]}(V_{\mathbb{C},n}(k-r-1))^+ \tag{2.3.1.3}$$

of the target of (2.3.1.1) tensored with $\mathbb{C}$ to the group-algebra $\mathbb{C}[G_n]$. For $\chi \in \hat{G}_n$, we denote by $\mathrm{per}_{\mathbb{C},\chi}$ the composition of $\mathrm{per}_{\mathbb{C}}$ with χ seen as group-algebra morphism with values in $\mathbb{C}$. Consequently, to any element $\mathbf{z}$ of the source of (2.3.1.1) whose image through per_p^{-1} lands in the F-rational submodule (2.3.1.3) is attached an element $\mathrm{per}_{\mathbb{C}}(\mathrm{per}_p^{-1}(\mathbf{z}) \otimes 1)$ of $\mathbb{C}[G_n]$ and a complex number $\mathrm{per}_{\mathbb{C},\chi}(\mathrm{per}_p^{-1}(\mathbf{z}) \otimes 1)$ for all $\chi \in \hat{G}_n$.

2.3.2 Statement of the ETNC

In this subsection, we review the statement of the ETNC of [41, Conjecture 3.2.1] for the p-adic family of motives $\{M_n = M_{\mathbb{Q}_n}\}_{n \geq 1}$ with coefficients in $F[G_n]$. To our fixed embedding of F into $\mathbb{C}$ is attached the p-partial G-equivariant complex L-function

$$L_{\{p\}}(M_n^*(1), s) = \sum_{\sigma \in G_n} L_{\{p\}}(M^*(1), \sigma, s)\sigma \in \mathbb{C}[G_n]^{\mathbb{C}}$$

of the twisted dual motive of M.

A specialization of $\mathcal{O}_{\mathrm{Iw}}$ is an $\mathcal{O}$-algebras morphism $\psi:\mathcal{O}_{\mathrm{Iw}}\longrightarrow S$ with values in a characteristic zero reduced ring S. We denote by T_ψ the $G_{\mathbb{Q}}$-representation $T_{\mathrm{Iw}}\otimes_{\mathcal{O}_{\mathrm{Iw}},\psi}S$ and say it is an S-specialization of T_{Iw}.

Conjecture 2.3.1. *There exist a* zeta morphism

$$Z(f)_{\mathrm{Iw}}:V_{\mathrm{Iw}}(-1)^+\longrightarrow \mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p],V_{\mathrm{Iw}})[1],$$

a fundamental line

$$\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})\overset{\mathrm{def}}{=}\mathrm{Det}^{-1}_{\mathcal{O}_{\mathrm{Iw}}}\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p],T_{\mathrm{Iw}})\otimes_{\mathcal{O}_{\mathrm{Iw}}}\mathrm{Det}^{-1}_{\mathcal{O}_{\mathrm{Iw}}}T_{\mathrm{Iw}}(-1)^+\subset\mathrm{Det}_{\mathcal{O}_{\mathrm{Iw}}[1/p]}\mathrm{Cone}(Z(f)_{\mathrm{Iw}})$$

and a unique basis $\mathbf{z}(f)_{\mathrm{Iw}}$ of $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})$ called the zeta element *of M with coefficients in $\mathcal{O}_{\mathrm{Iw}}$. For all S-specialization T_ψ, there exist a morphism*

$$Z(f)_\psi:V_\psi(-1)^+\longrightarrow \mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p],V_\psi)[1],$$

a fundamental line

$$\Delta_S(T_\psi)\overset{\mathrm{def}}{=}\mathrm{Det}^{-1}_S\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p],T_\psi)\otimes_S\mathrm{Det}^{-1}_S T_\psi(-1)^+\subset\mathrm{Det}_{S[1/p]}\mathrm{Cone}(Z(f)_\psi)$$

and a basis $\mathbf{z}(f)_\psi$ of $\Delta_S(T_\psi)$. The collection of pairs $(\mathbf{z}(f)_\psi,\Delta_S(T_\psi))$ satisfies the following properties.

1. *There exists a canonical isomorphism $\psi^\Delta:\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})\otimes_{\mathcal{O}_{\mathrm{Iw}}}S\overset{\mathrm{can}}{\simeq}\Delta_S(T_\psi)$ sending $\mathbf{z}(f)_{\mathrm{Iw}}\otimes 1$ to $\mathbf{z}(f)_\psi$.*

2. *The basis $\mathbf{z}(f)_\psi$ defines an isomorphism*

$$\mathrm{triv}_\psi:\Delta_S(T_\psi)\overset{\mathrm{can}}{\simeq}S$$

compatible with change of rings in the sense that the diagram

$$\begin{array}{ccc}\Delta_S(T_\psi) & \xrightarrow{\mathrm{triv}_\psi} & S\\ {\scriptstyle -\otimes_S S'}\downarrow & & \downarrow\\ \Delta_{S'}(T_\phi) & \xrightarrow{\mathrm{triv}_\phi} & S'\end{array}$$

is commutative whenever $\phi:\mathcal{O}_{\mathrm{Iw}}\longrightarrow S'$ factors through $\psi:\mathcal{O}_{\mathrm{Iw}}\longrightarrow S$. If S is a domain and if the image of $Z(f)_\psi$ is not torsion, then $\mathrm{Cone}(Z(f)_\psi\otimes_S\mathrm{Frac}(S)$ *is acyclic and* triv_ψ *coincides with the morphism*

$$\Delta_S(T_\psi)\subset\Delta_{\mathrm{Frac}(S)}(T_\psi\otimes\mathrm{Frac}(S))\overset{\mathrm{can}}{\simeq}\mathrm{Cone}(Z(f)_\psi\otimes_S\mathrm{Frac}(S))\overset{\mathrm{can}}{\simeq}\mathrm{Frac}(S)$$

where the last isomorphism is induced by the canonical isomorphism between the determinant of an acyclic complex and the coefficient ring.

3. *For $n\geq 1$ an integer, let $\chi\in\hat{G}_n$ be a character with values in $F_{\mathfrak{p}}$ (this can always be achieved by replacing F by a finite extension) and let $1\leq r\leq k-1$ be an integer. Let ψ be*

the $F_\mathfrak{p}$-specialization such that T_ψ is equal to $V(k-r)\otimes_{F_\mathfrak{p}} F_\mathfrak{p}[G_n]$. Then the map

$$\begin{array}{c}\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})\\ \big\downarrow \psi\\ \Delta_{F_\mathfrak{p}}(T_\psi)\\ \big\downarrow \mathrm{loc}_p\\ \mathrm{Det}_{F_\mathfrak{p}[G_n]} H^1(G_{\mathbb{Q}_p(\zeta_{p^n})}, V(k-r))\otimes_{F_\mathfrak{p}[G_n]} \mathrm{Det}^{-1}_{F_\mathfrak{p}[G_n]} T_\psi(-1)^+\\ \big\downarrow \mathrm{per}_p^{-1}\\ \mathrm{Det}_{F_\mathfrak{p}[G_n]}(S(U(N))(f)\otimes_F F_\mathfrak{p}[G_n])\otimes_{F_\mathfrak{p}[G_n]} \mathrm{Det}^{-1}_{F_\mathfrak{p}[G_n]} T_\psi(-1)^+\end{array}$$

obtained by localization at p of the étale cohomology composed with the equivariant p-adic period map sends the F-submodule generated by $\mathbf{z}(f)_{\mathrm{Iw}}$ into the F-rational subspace

$$\mathrm{Det}_{F[G_n]}(S(U(N))(f)\otimes_F F[G_n])\otimes \mathrm{Det}^{-1}_{F[G_n]}(V_{\mathbb{C},n}(k-r-1)\otimes_F F[G_n])^+.$$

Furthermore

$$\mathrm{per}_{\mathbb{C}}((\mathrm{per}_p^{-1}\circ \mathrm{loc}_p(\mathbf{z}(f)_\psi))\otimes 1) = L_{\{p\}}(M_n^*(1), r)\in \mathbb{C}[G_n] \tag{2.3.2.1}$$

and in particular

$$\mathrm{per}_{\mathbb{C},\chi}((\mathrm{per}_p^{-1}\circ \mathrm{loc}_p(\mathbf{z}(f)_\psi))\otimes 1) = L_{\{p\}}(M^*(1), \chi, r)\in \mathbb{C}. \tag{2.3.2.2}$$

Remarks: (i) Assertion 3, and especially equations (2.3.2.1) and (2.3.2.2) therein, expresses in which sense conjecture 2.3.1 predicts the special values of the L-function of motivic points and their variations alongside $\operatorname{Spec}\mathcal{O}_{\mathrm{Iw}}[1/p]$. Indeed, once the morphism $Z(f)_{\mathrm{Iw}}$ and the basis $\mathbf{z}(f)_{\mathrm{Iw}}$ of its cone are known, specializations and period maps compute the critical special values of $M^*(1)$. Conversely, because specializations extending $\chi^r_{\mathrm{cyc}}\chi$ for $1\leq r\leq k-1$ and $\chi\in\hat{G}_n$ for some $n\geq 1$ form a Zariski-dense subset of $\mathrm{Hom}(\mathcal{O}_{\mathrm{Iw}}, \bar{\mathbb{Q}}_p)$, there can be at most one element of $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})\otimes_{\mathcal{O}_{\mathrm{Iw}}}\mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})$ which satisfies assertion 3 independently of the truth of the other assertions of the conjecture. In particular, $\mathbf{z}(f)_{\mathrm{Iw}}$ does not depend on the choice of the lattice $T\subset V$. This is coherent with the assertion of the conjecture that $\mathbf{z}(f)_{\mathrm{Iw}}$ is a basis of $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})$ as this module also does not depend on the choice of T by Tate's formula.

(ii) Let us assume the truth of assertion 3 independently from the other. Then the cohomology $\mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})$ is concentrated in degree 1 and 2, the $\mathcal{O}_{\mathrm{Iw}}$-module $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})$ is of rank 1 and the $\mathcal{O}_{\mathrm{Iw}}$-module $H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})$ is torsion by [43, Theorem 12.4]. Equivalently, the Weak Leopoldt's Conjecture [62, Conjecture Section 1.3] is known for modular motives. By [39, 68], the complex numbers $L_{\{p\}}(M^*(1), \chi, r)$ do not vanish for all $1\leq r\leq k-1$ and all characters $\chi\in\hat{G}_n$. Hence $\mathbf{z}(f)_{\mathrm{Iw}}$ is non-zero and it then follows that

$$\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})\otimes_{\mathcal{O}_{\mathrm{Iw}}}\mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}}) = \mathrm{Det}_{\mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})}(0)\overset{\mathrm{can}}{\simeq}\mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}}).$$

There then exists an element of $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})$, which we also denote by $\mathbf{z}(f)_{\mathrm{Iw}}$ in a slight abuse of notation, such that the image of $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})$ inside $\mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})$ through the natural morphism

$$\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})\hookrightarrow \Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})\otimes_{\mathcal{O}_{\mathrm{Iw}}}\mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}}) = \mathrm{Det}_{\mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})}(0)\overset{\mathrm{can}}{\simeq}\mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})$$

is equal to

$$\mathrm{Det}^{-1}_{\mathcal{O}_{\mathrm{Iw}}} H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})\otimes_{\mathcal{O}_{\mathrm{Iw}}}\left(\mathrm{Det}_{\mathcal{O}_{\mathrm{Iw}}} H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}}\right)\subset \mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})$$

and hence to

$$\operatorname{char}_{\mathcal{O}_{\mathrm{Iw}}} H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}}) \otimes_{\mathcal{O}_{\mathrm{Iw}}} \operatorname{char}^{-1}_{\mathcal{O}_{\mathrm{Iw}}} H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}} \subset \operatorname{Frac}(\mathcal{O}_{\mathrm{Iw}})$$

by the structure theorem for finitely generated torsion modules over regular local rings. Assertion 2 for ψ equal to the identity is then seen to be equivalent to the equality

$$\operatorname{char}_{\mathcal{O}_{\mathrm{Iw}}} H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}}) \otimes_{\mathcal{O}_{\mathrm{Iw}}} \operatorname{char}^{-1}_{\mathcal{O}_{\mathrm{Iw}}} H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}} = \mathcal{O}_{\mathrm{Iw}} \subset \operatorname{Frac}(\mathcal{O}_{\mathrm{Iw}})$$

or equivalently

$$\operatorname{char}_{\mathcal{O}_{\mathrm{Iw}}} H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}}) = \operatorname{char}_{\mathcal{O}_{\mathrm{Iw}}} H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}}.$$

Conjecture 2.3.1 thus recovers [40, Conjecture (4.9)] for the modular motive M and [43, Conjecture 12.10]. By [43, Section 17.13] (resp. [45] and [14, Théorème 4.16]), it thus also recovers the Iwasawa Main Conjecture [34, Conjecture 2.2] for M when M has potentially crystalline ordinary reduction at p or equivalently when $\pi(f)_p$ is a principal series ordinary representation (resp. when M has potentially semi-stable but not potentially crystalline ordinary reduction at p or equivalently when $\pi(f)_p$ is a Steinberg representation). If f belongs to $S_2(\Gamma_0(N))$ and has rational coefficients, then conjecture 2.3.1 implies [50, Conjecture page 2] (when $a_p(f) = 0$) and [77, Main Conjecture 1.3] (in the general case).

(iii) If assertion 2 holds, then $\mathbf{z}(f)_{\mathrm{Iw}}$ (and so $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})$) determine $\mathbf{z}(f)_\psi$ (and so $\Delta_S(T_\psi)$) for all $\psi : \mathcal{O}_{\mathrm{Iw}} \longrightarrow S$. This assertion thus encodes the interpolation property of $\mathbf{z}(f)_{\mathrm{Iw}}$.

(iv) Let ψ be a specialization with values in a discrete valuation ring S such that ψ seen as having values in $\operatorname{Frac}(S)$ is as in assertion 3 with $L_{\{p\}}(M_n^*(1), r) \neq 0$. Then assertion 1 for $S = \mathcal{O}_{\mathrm{Iw}}$, assertion 2 for the identity specialization and ψ and assertion 3 together recover the Tamagawa Number Conjecture of [4] for the motive M twisted by ψ.

(v) Attentive readers will have remarked that our statement (2.3.2.2) uses the normalizations of [41], an article which however does not make completely explicit the link between zeta elements and special values in our case of interest, and not the specific treatment of modular motives in [43]. The reason for this choice actually lies deep. For a general compact p-ring, what the ETNC with coefficients in Λ predicts is the existence of a zeta element $\mathbf{z}_\Lambda$ with coefficients in Λ whose image through a motivic specialization and then through the canonical period maps attached to the specialized motive computes the values of the L-function at 0. A bolder conjecture would be to reverse the order of the operations and to ask in addition for the existence of a Λ-adic period map $\operatorname{per}_\Lambda$ such that $\operatorname{per}_\Lambda(\mathbf{z}_\Lambda)$ is a Λ-adic L-function which then interpolates special values after specialization at motivic points. The existence of such a universal normalization of the period maps is known for the family of motives M_n, and this is the choice made in [43]. However, even a precise formulation of the stronger conjecture is typically not known for the families of modular motives parametrized by Hecke algebras and deformations rings we consider in this manuscript so we stuck with the usual statement of the ETNC. For the convenience of the reader, we explain how to pass from the normalization of [41], which we follow, to that of [43], which deals with the same objects as we do. First note that [41, Section 3.2.6] predicts that the complex period map for the Betti cohomology $N_B(-1)^+$ of a strictly critical motive N is related to the value of the L-function of $N^*(1)$ at 0. As the Betti cohomology appearing in (2.3.1.2) is the Betti cohomology of $M(k-r)$, our formulation of the conjecture computes the special value of $M^*(r+1-k)$ at zero, and hence the special value of the motive attached to the dual eigencuspform f^* at r (the dual eigencuspform is the eigencuspform whose eigenvalues are the complex conjugates of those of f or, equivalently, the eigencuspform whose motive is the motive of f with the dual action of the Hecke algebra). Hence, equation (2.3.2.2) is equivalent to the statement that

$$\operatorname{per}_{\mathbb{C}}(\operatorname{per}_p^{-1} \circ \operatorname{loc}_p(\mathbf{z}(f)_{\mathrm{Iw}}) \otimes 1) = L_{\{p\}}(f^*, \chi, r) \in \mathbb{C}.$$

In the comparable equation in [43, Theorem 12.5], the period map is universally normalized to have V^+ as its source whereas our period map for the specialization $\chi^r_{\mathrm{cyc}}\chi$ is normalized to have $(V \otimes \chi)(k-r)(-1)^+$ as its source (and thus depends on r and χ). Consequently, we expect that, in the formula of [43, Theorem 12.5], the value of the L-function is multiplied by $(2\pi i)^{k-r-1}$ and the $+$-eigenspace is replaced by the $(-1)^{k-r-1}\chi(-1)$-eigenspace, as is indeed the case.

2.3.3 Review of known results on conjecture 2.3.1

Thanks to the awe-inspiring results of [43] and the remarkable progresses towards the Iwasawa Main Conjecture for modular forms in [50, 64, 23, 61, 76, 77, 82], much of conjecture 2.3.1 is known. We record here the following theorem.

Theorem 2.3.2 (Kato, Skinner-Urban, Kobayashi, Sprung, Wan)**.** *Let $f \in S_k(\Gamma_1(Np^s))$ be a classical eigencuspform. Then there exists a zeta element $\mathbf{z}(f)_{\mathrm{Iw}} \in \Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}}) \otimes_{\mathcal{O}_{\mathrm{Iw}}} \mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})$ satisfying assertion 3 of conjecture 2.3.1. Assume that f satisfies the following properties.*

1. *The residual representation $\bar{\rho}_f$ is irreducible.*
2. *The order of the image of $\bar{\rho}_f$ is divisible by p.*
3. *Either there exists a finite extension $K/\mathbb{Q}_p$ such that $\rho_f|G_K$ is crystalline (equivalently $\pi(f)_p$ is either principal series or supercuspidal) or $k > 2$.*

Then the trivialization $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}}) \otimes_{\mathcal{O}_{\mathrm{Iw}}} \mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}}) \simeq \mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})$ induced by $\mathbf{z}(f)_{\mathrm{Iw}}$ sends $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})^{-1}$ inside $\mathcal{O}_{\mathrm{Iw}}$. Assume in addition that f satisfies the following properties.

4. *Either f belongs to $S_k(\Gamma_0(N) \cap \Gamma_1(p^s))$, the semisimplification of $\bar{\rho}_f|G_{\mathbb{Q}_p}$ is isomorphic to $\chi \oplus \psi$ with $\chi \neq \psi$ and $a_p(f)$ is a p-adic unit or f belongs to $S_2(\Gamma_0(N))$ with N square-free, all its coefficients are rational and $a_p(f)$ is not a p-adic unit.*
5. *There exists $\ell \nmid p$ dividing exactly once the Artin conductor of $\bar{\rho}_f$.*

Then the trivialization $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}}) \otimes_{\mathcal{O}_{\mathrm{Iw}}} \mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}}) \simeq \mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})$ induced by $\mathbf{z}(f)_{\mathrm{Iw}}$ sends $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})$ to $\mathcal{O}_{\mathrm{Iw}}$.

Proof. The first assertion is [43, Theorem 12.5 (1)]. Under the supplementary assumptions 1, 2 and 3, the fact that the trivialization of $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}}) \otimes_{\mathcal{O}_{\mathrm{Iw}}} \mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}}) \simeq \mathrm{Frac}(\mathcal{O}_{\mathrm{Iw}})$ induced by $\mathbf{z}(f)_{\mathrm{Iw}}$ sends $\Delta_{\mathcal{O}_{\mathrm{Iw}}}(T_{\mathrm{Iw}})^{-1}$ inside $\mathcal{O}_{\mathrm{Iw}}$, which more concretely means that

$$\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}}) \mid \mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}}, \tag{2.3.3.1}$$

is [43, Theorem 12.5 (4)]. In fact, in *loc. cit.*, such a statement is proved under the marginally logically stronger hypothesis that the image of $\bar{\rho}_f$ contains $\mathrm{SL}_2(\mathbb{Z}_p)$ so we recall briefly how the argument goes under our hypotheses. Denote by $\mathbb{F}$ a finite field such that $\mathrm{GL}_2(\mathbb{F})$ contains the image G of $\bar{\rho}_f$.Then G acts absolutely irreducibly on $\mathbb{F}^2$ by hypothesis 1 and in particular all $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattices in $V(f)$ are isomorphic. Because G is of order divisible by p according to hypothesis 2, the classification of subgroups of $\mathrm{GL}_2(\mathbb{F})$ shows that G contains a non-trivial unipotent element. That the hypothesis that the image of ρ_f contains $\mathrm{SL}_2(\mathbb{Z}_p)$ is invoked in [43] to prove (2.3.3.1) only either because $\mathrm{SL}_2(\mathbb{F}_p)$ acts absolutely irreducibly on $\mathbb{F}_p^2$ or because it contains a non-trivial unipotent element.

That the reverse divisibility

$$\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}} \mid \mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\mathrm{Iw}}), \tag{2.3.3.2}$$

holds under the assumptions 4 and 5 remains to be shown. Assume first that $\rho_f|G_{\mathbb{Q}_p}$ is reducible. By [43, Section 17.13] (see especially the short exact sequence at the end of that section), the reverse divisibility (2.3.3.2) for the eigencuspform f is equivalent to the main conjecture in Iwasawa theory of modular forms of R.Greenberg and B.Mazur; see for instance [61, Conjecture 7.4] for a precise statement. Hence, it is true by [76, Theorem 3.29] once we check that the hypotheses of this theorem are verified. The hypotheses **(dist)** and **(irr)** of [76, Theorem 3.29] are true respectively by our assumption 4 and assumption 1. The third hypothesis of [76, Theorem 3.29] follows from assumption 5. The first, fourth and last hypotheses of [76, Theorem 3.29] are imposed there in order to establish the divisibility (2.3.3.1) but we have already checked it holds under our hypotheses.

Now we assume that $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible and that f satisfies the second set of assumptions of assumption 4 of the theorem. By [50, Theorem 7.4] and [77, Section 7], the reverse divisibility (2.3.3.2) for the eigencuspform f is then equivalent to [50] or to [77, Main Conjecture 7.21]. Hence, it is known by [82] if $a_p(f) = 0$ (resp. by [78] in the general case). $\square$

Despite these results, no non-tautological set of hypotheses is currently known to be sufficient to prove assertion 2 of conjecture 2.3.1 for specializations ψ such that the L-value of T_ψ at 0 vanishes at high order (indeed, such a result would imply in particular the Birch and Swinnerton-Dyer Conjecture for modular abelian variety over $\mathbb{Q}$). For this reason, we introduce the following weaker conjecture.

Conjecture 2.3.3. *Let $\mathbf{z}(f)_{\mathrm{Iw}} \in H^1_{\mathrm{et}}(\mathbb{Z}[1/p], V_{\mathrm{Iw}})$ be the unique class satisfying assertion 3 of conjecture 2.3.1. Then for all specializations $\psi : \mathcal{O}_{\mathrm{Iw}} \longrightarrow S$ with values in a characteristic zero reduced ring such that $\mathbf{z}(f)_\psi \overset{\mathrm{def}}{=} \psi(\mathbf{z}(f)_{\mathrm{Iw}})$ is non-zero, the trivialization $\Delta_S(T_\psi) \otimes_S Q(S) \simeq Q(S)$ induced by $\mathbf{z}(f)_\psi$ identifies $\Delta_S(T_\psi)^{-1}$ and S.*

A partial version of conjecture 2.3.3 is the following.

Conjecture 2.3.4. *With the same notations and hypotheses as in conjecture 2.3.3, the trivialization $\Delta_S(T_\psi) \otimes_S Q(S) \simeq Q(S)$ induced by $\mathbf{z}(f)_\psi$ sends $\Delta_S(T_\psi)^{-1}$ inside S.*

Under assumptions 1, 2 and 3 of 2.3.2, the results of [43] summed up there precisely assert that conjecture 2.3.4 holds. In corollary 4.1.2 below, we replace assumption 3 of theorem 2.3.2 by a weaker assumption on the local residual representation $\bar{\rho}_f|G_{\mathbb{Q}_p}$.

3 The ETNC with coefficients in Hecke rings

3.1 Galois representations

3.1.1 Modular levels

Let $\bar{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}_p$ and let

$$\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{F}})$$

be an irreducible, modular, residual $G_{\mathbb{Q}}$-representation. As in the introduction, denote by $N(\bar{\rho})$ the tame Artin conductor of $\bar{\rho}$ (defined as in [72] and hence prime to p).

A compact open subgroup $U^{(p)}$ is said to be allowable (with respect to $\bar{\rho}$) if there exists a maximal ideal $\mathfrak{m}_{\bar{\rho}}$ of $\mathbf{T}^{\mathrm{red}}(U^{(p)})$ such that

$$\begin{cases} \operatorname{tr} \bar{\rho}(\mathrm{Fr}(\ell)) = T(\ell) \bmod \mathfrak{m}_{\bar{\rho}} \\ \det \bar{\rho}(\mathrm{Fr}(\ell)) = \ell\langle\ell\rangle \bmod \mathfrak{m}_{\bar{\rho}} \end{cases}$$

for all $\ell \notin \Sigma(U^{(p)}) \cup \{p\}$ (we recall that $\Sigma(U^{(p)})$ is the set of primes out of which $U^{(p)} \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(p\infty)})$ is maximal). A finite set of primes $\Sigma \supset \{\ell | N(\bar{\rho})p\}$ is said to be allowable (with respect to $\bar{\rho}$) if there exists an allowable compact open subgroup $U^{(p)} \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(p\infty)})$ such that $\Sigma \supset \Sigma(U^{(p)})$.

If $U^{(p)}$ is allowable and if $\Sigma = \Sigma(U^{(p)}) \cup \{p\}$, we denote by $\mathbf{T}_{\Sigma,\mathrm{Iw}}(U^{(p)})$ the localization of $\mathbf{T}^{\mathrm{red}}(U^{(p)})$ at $\mathfrak{m}_{\bar{\rho}}$. Though it depends on general on $U^{(p)}$, we often write $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ for $\mathbf{T}_{\Sigma,\mathrm{Iw}}(U^p)$ (by [10, 52], if $U^{(p)}$ is sufficiently small, then $\mathbf{T}_{\Sigma,\mathrm{Iw}}(U^{(p)})$ depends up to isomorphism only on Σ but we will not always assume that $U^{(p)}$ is sufficiently small in that sense at primes $\ell \equiv \pm 1 \bmod p$). If $U^{(p)} \subset V^{(p)}$ is an inclusion of allowable subgroups, then $\Sigma' = \Sigma(U^{(p)}) \cup \{p\}$ contains $\Sigma = \Sigma(V^{(p)}) \cup \{p\}$ and there is a surjective morphism $\mathbf{T}_{\Sigma',\mathrm{Iw}}(U^{(p)}) \longrightarrow \mathbf{T}_{\Sigma,\mathrm{Iw}}(V^{(p)})$ which we simply denote by $\mathbf{T}_{\Sigma',\mathrm{Iw}} \longrightarrow \mathbf{T}_{\Sigma,\mathrm{Iw}}$. The $\mathbb{Z}_p$-algebra $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is flat and its relative dimension is at least 3 (see [33]). It is conjectured (and often known) that this lower bound is sharp.

For Σ an allowable finite set of primes, we denote by $\Sigma^{(p)}$ the set $\Sigma(U^{(p)})$ and by

$$N(\Sigma) = N(\bar{\rho}) \prod_{\ell \in \Sigma^{(p)}} \ell^{\dim_{\bar{\mathbb{F}}} \bar{\rho}_{I_\ell}}.$$

the maximal tame level attached to a modular specialization of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$. For such a Σ, there exists by definition a pseudocharacter

$$\operatorname{tr} \rho_{\mathfrak{m}_{\bar{\rho}}} : G_{\mathbb{Q},\Sigma} \longrightarrow \mathbf{T}_{\Sigma,\mathrm{Iw}}$$

of dimension 2 in the sense of [83, 79, 3]. If $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow S$ is a map of complete, local $\mathbb{Z}_p$-algebras, the map

$$\operatorname{tr} \rho_\psi = \psi \circ \operatorname{tr} \rho_{\mathfrak{m}_{\bar{\rho}}}$$

is a pseudocharacter coinciding with $\operatorname{tr} \rho_{\mathfrak{m}_{\bar{\rho}}}$ modulo the maximal ideal of S. By [11, 59], there exists a $G_{\mathbb{Q},\Sigma}$-representation (T_ψ, ρ_ψ, S) unique up to isomorphism whose trace is equal to $\operatorname{tr} \rho_\psi$. This holds in particular for $\psi = \mathrm{Id}$ in which case we write $(T_{\Sigma,\mathrm{Iw}}, \rho_\Sigma)$ for $(T_{\mathrm{Id}}, \rho_{\mathrm{Id}})$. When ψ has values in a domain, we denote by V_ψ the $G_{\mathbb{Q},\Sigma}$-representation $T_\psi \otimes_S \operatorname{Frac}(S)$.

If $\bar{\rho}$ is the residual representation attached to an eigencuspform f such that the local $G_{\mathbb{Q}_p}$-representation $\rho_f|_{G_{\mathbb{Q}_p}}$ is reducible, then $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ admits a quotient $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$ called the nearly-ordinary quotient (see [36]) which is isomorphic for all $k \geq 2$ to a local factor of the inverse limit on r of the sub-algebras of $\operatorname{End}_{\mathbb{Z}_p}(S_k(U_1(Np^r), \mathbb{Z}_p)^{\mathrm{ord}})$ generated by $T(\ell)$ for $\ell \notin \Sigma$ and by $T(p)$ where $S_k(U_1(Np^r), \mathbb{Z}_p)^{\mathrm{ord}}$ denotes the subspace of $S_k(U_1(Np^r), \mathbb{Z}_p)$ on which $T(p)$ acts as a unit. The $\mathbb{Z}_p$-algebra $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$ is flat and of relative dimension 2. If $\psi : \mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}} \longrightarrow S$ is a map of local $\mathbb{Z}_p$-algebras, the $G_{\mathbb{Q}_p}$-representation $\rho_\psi|_{G_{\mathbb{Q}_p}}$ is reducible.

3.1.2 Euler factors

Proposition-Definition 3.1.1. *If $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow S$ is a map of flat $\mathbb{Z}_p$-algebras, the algebraic Euler polynomial at $\ell \notin \Sigma$ a finite prime of T_ψ is*

$$\mathrm{Eul}_\ell(T_\psi, X) = \det(1 - \mathrm{Fr}(\ell) X | T_\psi) \in S[X]$$

and the algebraic Euler factor at ℓ of T_ψ is

$$\mathrm{Eul}_\ell(T_\psi) = \mathrm{Eul}_\ell(T_\psi, 1) \in S.$$

If the diagram

$$\begin{array}{ccc} \mathbf{T}_{\Sigma,\mathrm{Iw}} & \xrightarrow{\phi} & S \\ \downarrow \psi & \nearrow \kappa & \\ R & & \end{array} \qquad (3.1.2.1)$$

is commutative, then

$$\kappa(\mathrm{Eul}_\ell(T_\psi, X)) = \mathrm{Eul}_\ell(T_\phi, X). \qquad (3.1.2.2)$$

Proof. Everything is clear once observed that $T_\psi = T_\Sigma \otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}},\psi} S$ is an S-module free of rank 2 with a trivial action of I_ℓ. □

When S is not a domain and $\ell \in \Sigma$, the obvious generalization of proposition-definition 3.1.1 need not be true and it is thus not possible in general to extend the definition of the Euler factor at $\ell \in \Sigma$ to this case. For $\mathfrak{a}^{\mathrm{red}}$ a minimal prime ideal of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$, we denote by $(T(\mathfrak{a}^{\mathrm{red}}), \rho(\mathfrak{a}^{\mathrm{red}}), \mathbf{T}(\mathfrak{a}^{\mathrm{red}}))$ the specialization $(T_\psi, \rho_\psi, \mathbf{T}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}^{\mathrm{red}})$ attached to the natural projection $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \mathbf{T}(\mathfrak{a}^{\mathrm{red}})$ and write $V(\mathfrak{a}^{\mathrm{red}})$ for V_ψ.

Proposition-Definition 3.1.2. *If $\psi : \mathbf{T}(\mathfrak{a}^{\mathrm{red}}) \longrightarrow S$ is a map of flat $\mathbb{Z}_p$-algebras with values in a domain, the algebraic Euler polynomial of T_ψ at $\ell \nmid p$ a finite prime is*

$$\mathrm{Eul}_\ell(T_\psi, X) = \det\left(1 - \mathrm{Fr}(\ell) X | V_\psi^{I_\ell}\right) \in S[X]$$

and the algebraic Euler factor at ℓ of T_ψ is

$$\mathrm{Eul}_\ell(T_\psi) = \mathrm{Eul}_\ell(T_\psi, 1) \in S.$$

If λ_f is a modular map factoring through $T(\mathfrak{a}^{\mathrm{red}})$ and if the diagram

$$\begin{array}{ccc} \mathbf{T}(\mathfrak{a}^{\mathrm{red}}) & \xrightarrow{\lambda_f} & \bar{\mathbb{Q}}_p \\ \downarrow \psi & \searrow^{\phi} & \uparrow \\ R & \xrightarrow{\kappa} & S \end{array} \tag{3.1.2.3}$$

is commutative, then

$$\kappa(\mathrm{Eul}_\ell(T_\psi, X)) = \mathrm{Eul}_\ell(T_\phi, X). \tag{3.1.2.4}$$

Proof. We have to prove that Euler polynomials have coefficients in the ring of coefficients of the representation and that they obey the compatibility (3.1.2.4).

That $\mathrm{Eul}_\ell(T(\mathfrak{a}^{\mathrm{red}}), X)$, which *a priori* is a polynomial with coefficients in the normalization of $\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$, actually has coefficients in $\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$ itself is presumably well-known, but we include a brief proof. Denote by $\mathrm{Spec}^{\mathrm{cl}}\,\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$ the set of modular maps of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ factoring through $\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$. For $\psi \in \mathrm{Spec}^{\mathrm{cl}}\,\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$, let D_ψ be the determinant $\det \rho_\psi^{I_\ell}$ (in the sense of [13]) and consider the collection of determinants

$$\left\{ D_\psi : G_{\mathbb{Q}_\ell}/I_\ell \longrightarrow \bar{\mathbb{Q}}_p | \psi \in \mathrm{Spec}^{\mathrm{cl}}\,\mathbf{T}(\mathfrak{a}^{\mathrm{red}}) \right\}.$$

By the local-global compatibility in Langlands correspondance, D_ψ has values in $\psi(\mathbf{T}(\mathfrak{a}^{\mathrm{red}}))$ for all ψ in $\mathrm{Spec}^{\mathrm{cl}}\,\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$. By Zariski-density of $\mathrm{Spec}^{\mathrm{cl}}\,\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$ in $\mathrm{Spec}\,\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$ and [13, Exemple 2.32], there thus exists a unique determinant D with values in $\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$ such that $D(1-\mathrm{Fr}(\ell)X)$ is the characteristic polynomial of $\rho(\mathfrak{a}^{\mathrm{red}})^{I_\ell}$.

Next we show assertion (3.1.2.4) in the context of (3.1.2.3). It is enough by construction of T_ψ and T_ϕ to show that $T_\psi^{I_\ell}$ and $T_\phi^{I_\ell}$ have the same ranks over R and S respectively. In turn, this is implied by the statement that $T(\mathfrak{a}^{\mathrm{red}})^{I_\ell}$ and $T_{\lambda_f}^{I_\ell}$ have the same rank over $R(\mathfrak{a}^{\mathrm{red}})$ and $\bar{\mathbb{Q}}_p$ respectively. Non-zero elements of $\bar{\mathbb{Q}}_p$ are not in the kernel of λ_f so if $\sigma \in I_\ell$ acts on V non-trivially through a finite quotient, then its action is also non-trivial on T_{λ_f}. It is thus further enough to prove that $\mathrm{rank}_{R(\mathfrak{a}^{\mathrm{red}})}\,T(\mathfrak{a}^{\mathrm{red}})^U$ is larger than $\mathrm{rank}_{\bar{\mathbb{Q}}_p}\,T_{\lambda_f}^U$ for U a finite index subgroup of I_ℓ. By Grothendieck's monodromy theorem [73, Page 515], we can choose U such that $V(\mathfrak{a}^{\mathrm{red}})^U$ is quasi-unipotent, in which case $\mathrm{rank}_{R(\mathfrak{a}^{\mathrm{red}})}\,T(\mathfrak{a}^{\mathrm{red}})^U$ is at least 1 and is exactly 1 if the monodromy operator is of rank 1. If the action of monodromy on $T(\mathfrak{a}^{\mathrm{red}})$ is trivial, it is also trivial on T_{λ_f} and we are done. Now suppose monodromy acts non-trivially on $T(\mathfrak{a}^{\mathrm{red}})$. Let $\sigma \in G_{\mathbb{Q}_\ell}$ be a lift of $\mathrm{Fr}(\ell)$. Because the representation T_{λ_f} is a pure $G_{\mathbb{Q}_\ell}$-module by Ramanujan's conjecture (proved for modular forms in [9, Théorème A]), the eigenvalues of σ acting on T_{λ_f} are all non zero. Hence, the eigenvalues of σ acting on $V(\mathfrak{a}^{\mathrm{red}})$ are also non-zero and their quotient is equal to $\ell^{\pm 1}$ by the monodromy relation. Hence, the eigenvalues of σ on T_{λ_f} have different Weil weights so by Ramanujan's conjecture again, there is a non-trivial, hence necessarily rank 1, monodromy operator acting on T_{λ_f}. The rank of $T_{\lambda_f}^U$ is then at most 1, and so is less than $\mathrm{rank}_{R(\mathfrak{a}^{\mathrm{red}})}\,T(\mathfrak{a}^{\mathrm{red}})^U$.

It remains to show that $\mathrm{Eul}_\ell(T_\psi, X)$ has values in $S[X]$ for all specializations $\psi : \mathbf{T}(\mathfrak{a}^{\mathrm{red}}) \longrightarrow S$. Whenever there exists a normal subgroup U of I_ℓ with finite index such that $T(\mathfrak{a}^{\mathrm{red}})^U$ and T_ψ^U have the same ranks over $\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$ and S respectively, the first part of the proof shows that $\psi(\mathrm{Eul}_\ell(T(\mathfrak{a}^{\mathrm{red}}), X)) = \mathrm{Eul}_\ell(T_\psi, X)$ and so $\mathrm{Eul}_\ell(T_\psi, X)$ has coefficients in S. The existence of such a U does not obtain only if monodromy acts non-trivially on $T(\mathfrak{a}^{\mathrm{red}})$ and trivially on T_ψ. In that case, $T_\psi^{I_\ell}$ is a free S-module of rank 2 and so the algebraic Euler polynomial of T_ψ has coefficients in S. □

Definition 3.1.3. *Let $\psi : \mathbf{T}(\mathfrak{a}^{\mathrm{red}}) \longrightarrow S$ be a map of flat $\mathbb{Z}_p$-algebras with values in a domain and let $\ell \nmid p$ be a finite prime. If $\operatorname{rank}_S T_\psi^{I_v} = 1$, denote by α_v the unique eigenvalue of $\mathrm{Fr}(v)$ on $T_\psi^{I_v}$. The graded invertible module $\mathscr{X}_\ell(T_\psi)$ is defined as follows.*

$$\mathscr{X}_\ell(T_\psi) = \begin{cases} \operatorname{Det}_S \mathrm{R}\,\Gamma(G_{K_v}/I_v, T_\psi^{I_v}) & \text{if } \operatorname{rank}_S T_\psi^{I_v} \neq 1, \\ \operatorname{Det}_S [S \overset{1-\alpha_v}{\longrightarrow} S] & \text{if } \operatorname{rank}_S T_\psi^{I_v} = 1. \end{cases}$$

Here, the complex $[S \overset{1-\alpha_v}{\longrightarrow} S]$ is placed in degree $0, 1$.

The module $\mathscr{X}_\ell(T)$ recovers the determinant of the unramified cohomology of T when both are defined and is compatible with change of rings provided the rank of inertia invariants remains constant in the sense of the following lemma.

Lemma 3.1.4. *Let*

$$\begin{array}{ccc} \mathbf{T}(\mathfrak{a}^{\mathrm{red}}) & \overset{\phi}{\longrightarrow} & S' \\ {\scriptstyle\psi}\big\downarrow & \nearrow_{\kappa} & \\ S & & \end{array}$$

be a commutative diagram of integral flat $\mathbb{Z}_p$-algebras. If $T_\psi^{I_\ell}$ is a perfect complex of S-modules, then there is a canonical isomorphism

$$\mathscr{X}_\ell(T_\psi) \overset{\mathrm{can}}{\simeq} \operatorname{Det}_S \mathrm{R}\,\Gamma(G_{\mathbb{Q}_\ell}/I_\ell, T_\psi^{I_\ell}).$$

If $\operatorname{rank}_{S'} T_\phi^{I_\ell}$ is equal to $\operatorname{rank}_S T_\psi^{I_\ell}$, then $\mathscr{X}_\ell(T_\psi) \otimes_{S,\kappa} S'$ is canonically isomorphic to $\mathscr{X}_\ell(T_\phi)$.

Proof. If $T_\psi^{I_\ell}$ is a perfect complex of S-modules, then so is $\mathrm{R}\,\Gamma(G_{\mathbb{Q}_\ell}/I_\ell, T_\psi^{I_\ell})$ and the determinant $\operatorname{Det}_S \mathrm{R}\,\Gamma(G_{\mathbb{Q}_\ell}/I_\ell, T_\psi^{I_\ell})$ is well-defined. The first assertion of the lemma is non-tautological only if $\operatorname{rank}_S T_\psi^{I_\ell} = 1$. In that case, a finite projective resolution of $T_\psi^{I_\ell}$ yields a projective resolution of $(1 - \mathrm{Fr}(v))T_\psi^{I_\ell}$ and computing $\operatorname{Det}_S(\mathrm{Fr}(v) - 1)T_\psi^{I_\ell} \otimes_S \operatorname{Det}^{-1} T_\psi^{I_\ell}$ using these resolutions yields the desired result.

We now assume that $\operatorname{rank}_{S'} T_\phi^{I_\ell}$ and $\operatorname{rank}_S T_\psi^{I_\ell}$ are equal to r. If r is equal to zero, then both $\mathscr{X}_\ell(T_\psi) \otimes_S S'$ and $\mathscr{X}_\ell(T_\phi)$ are canonically isomorphic to $(S', 0)$. If $r = 1$, they are both canonically isomorphic to $\operatorname{Det}_{S'}[S' \overset{1-\mathrm{Fr}(\ell)}{\longrightarrow} S']$. If $r = 2$, then both T_ψ and T_ϕ are unramified so the canonical isomorphism

$$\mathrm{R}\,\Gamma(G_{\mathbb{Q}_\ell}/I_\ell, T_\psi^{I_\ell}) \overset{\mathrm{L}}{\otimes}_S S' \overset{\mathrm{can}}{\simeq} \mathrm{R}\,\Gamma(G_{\mathbb{Q}_\ell}/I_\ell, T_\psi \otimes_S S')$$

yields the result after taking determinant. The second assertion is thus true. □

3.1.3 Algebraic p-adic determinants

Definition 3.1.5. *Let $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow S$ be a map of flat $\mathbb{Z}_p$-algebras with values in a domain. The graded invertible S-module $\mathscr{X}(T_\psi)$ is defined to be*

$$\operatorname{Det}_S \mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma], T_\psi) \otimes_S \bigotimes_{\ell \in \Sigma^{(p)}} \mathscr{X}_\ell(T_\psi).$$

We recall from the appendix that the subscript c denotes étale cohomology compactly supported outside p. The properties of local Euler factors given in subsection 3.1.2 imply that the formation of $\mathscr{X}$ often commutes with base-change.

Proposition 3.1.6. *If λ_f is a modular map factoring through $\mathbf{T}(\mathfrak{a}^{\mathrm{red}})$ and if the diagram*

$$\begin{array}{ccc} \mathbf{T}(\mathfrak{a}^{\mathrm{red}}) & \xrightarrow{\lambda_f} & \bar{\mathbb{Q}}_p \\ \downarrow \psi & \searrow^{\phi} & \uparrow \\ R & \xrightarrow{\kappa} & S \end{array} \tag{3.1.3.1}$$

is commutative, or more generally if $T_\psi^{I_\ell}$ and $T_\phi^{I_\ell}$ have the same ranks over R and S respectively, then there is a canonical isomorphism $\mathscr{X}(T_\psi)\otimes_{R,\kappa}S\overset{\mathrm{can}}{\simeq}\mathscr{X}(T_\phi)$.

Proof. By the commutativity of $\mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma],-)$ with $-\overset{\mathrm{L}}{\otimes}_{R,\kappa}S$, we are reduced to showing that there exists a canonical isomorphism between $\mathscr{X}_\ell(T_\psi)\otimes_{R,\kappa}S$ and $\mathscr{X}_\ell(T_\phi)$ for all $\ell\in\Sigma^{(p)}$. By lemma 3.1.4, this amounts to showing that $T_\psi^{I_\ell}$ and $T_\phi^{I_\ell}$ have the same ranks over R and S respectively. This holds by assumption or follows from the proof of proposition-definition 3.1.2. $\square$

Though $\mathscr{X}(T_\psi)$ has *a priori* no special relevance for an arbitrary $S[G_{\mathbb{Q},\Sigma}]$-module T_ψ, we note that there are by construction canonical isomorphisms

$$\mathscr{X}(T_\psi)\overset{\mathrm{can}}{\simeq}\mathrm{Det}_S\,\mathrm{R}\Gamma_f(G_{\mathbb{Q},\Sigma},T_\psi)\overset{\mathrm{can}}{\simeq}\mathrm{Det}_S\,\mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p],T_\psi) \tag{3.1.3.2}$$

whenever all the objects appearing in (3.1.3.2) are well defined. This is for instance the case if S is a regular local ring or if T_ψ is a perfect complex of smooth étale sheaves on $\operatorname{Spec}\mathbb{Z}[1/p]$ (more concretely, if the S-module $T_\psi^{I_\ell}$ has finite projective dimension for all $\ell\in\Sigma^{(p)}$; this holds for instance if T_ψ is minimally ramified).

3.2 Cohomology of the tower of modular curves

3.2.1 Modular bases of the Betti cohomology of modular curves

Let U be a sufficiently small compact open subgroup of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ and let $k\geq 2$ and $1\leq r\leq k-1$ be positive integers. Denote by $\mathcal{H}_{k-2}^*$ the $\mathbb{Z}$-dual of the sheaf $\mathcal{H}_{k-2}$ on $Y(U)(\mathbb{C})$ of subsection 2.1.3. We consider the relative homology group $H_1(X(U)(\mathbb{C}),\{\text{cusps}\},\mathcal{H}_{k-2}^*)$ with respect to cusps and with coefficients in $\mathcal{H}_{k-2}^*$. Let ϕ be a continuous map from $]0,\infty[$ to $X(U)$ whose image is an arc from 0 to ∞. The sheaf $\phi^*\mathcal{H}^*$ is a rank 2 constant sheaf on $]0,\infty[$ whose stalk at $x\in]0,\infty[$ is isomorphic to $\mathbb{Z}xi+\mathbb{Z}$. As in [43, Section 4.7], denote by (e_1,e_2) the basis of $\phi^*\mathcal{H}^*$ such that, at $x\in]0,\infty[$, the stalk of e_1 is xi and the stalk of e_2 is 1 and let $\alpha\in\Gamma(]0,\infty[,\phi^*,\mathcal{H}_{k-2}^*)$ be the global section $e_1^{r-1}e_2^{k-r-1}$. The image of α in $H_1(X(U)(\mathbb{C}),\{\text{cusps}\},\mathcal{H}_{k-2}^*)$ through the map

$$H_1([0,\infty],\{0,\infty\},\phi^*\mathcal{H}_{k-2}^*)\longrightarrow H_1(X(U)(\mathbb{C}),\{\text{cusps}\},\mathcal{F}_{k-2}^*)$$

is denoted by $\delta_U^*(k,r)$. For f an eigencuspform of weight k and level U, the class $\delta_U^*(f,r)$ is the projection of $\delta_U^*(k,r)$ to $H_1(X(U)(\mathbb{C}),\{\text{cusps}\},\mathcal{F}_{k-2}^*)(f)$ where this last space is as in previous subsections the largest quotient on which the Hecke algebra acts through λ_f. Applying the pairing between relative homology and compactly supported cohomology then Poincaré duality yields classes $\delta_U(k,r)$ and $\delta_U(f,r)$ in $H^1_{\mathrm{et}}(Y(U),\mathcal{H}_{k-2})$ and $H^1_{\mathrm{et}}(X(U)(\mathbb{C}),\mathcal{F}_{k-2})(f)$ respectively. As in section 2.1.3, these definitions make sense for any U after tensor product with $\mathbb{Z}_p$ or $\mathbb{Q}$ by considering stack cohomology or taking invariants under U/U' for U' a sufficiently small normal compact open subgroup of U.

The classes $\delta_U^*(k,r),\delta_U(k,r),\delta_U^*(f,r)$ and $\delta_U(f,r)$ admit an alternate construction using completed cohomology solely in terms of the classes $\delta_U^*(2,1)$ (which do not require the introduction of the global section α and which are thus simply arcs in $Y(U)$) as we now recall. Write $U=U_pU^{(p)}$. For $s\geq 1$, define $\delta_{U,s}^*$ to be the image of the arc ϕ in $H_1(X(U)(\mathbb{C}),\{\text{cusps}\},\mathbb{Z}/p^s\mathbb{Z})$ and $\delta_{U,s}$

the class corresponding to $\delta^*_{U,s}$ through the pairing with compactly supported cohomology and Poincaré duality. If $U' \subset U$, the construction of $\delta_{U,s}(2,1)$ is compatible with the trace map

$$H^1(X(U')(\mathbb{C}), \mathbb{Z}/p^{s'}\mathbb{Z}) \longrightarrow H^1(X(U)(\mathbb{C}), \mathbb{Z}/p^s\mathbb{Z}).$$

Taking the inverse limit on s and on compact open subgroups $U_p \subset \mathbf{G}(\mathbb{Q}_p)$ thus defines an object $\tilde{\delta}_{U^{(p)}}$ in the Poincaré dual of the completed étale cohomology group $\tilde{H}^1_c(U^{(p)}, \mathbb{Z}_p)$. By [20, Proposition 4.3 and Corollary 4.5] (see also [22, Corollary 2.2.18, 4.3.2 and (4.3.4)]), for all newforms f of tame level N and weight k and all $1 \leq r \leq k-1$, the class $\tilde{\delta}_{U^{(p)}}$ yields by projection an element in $H^1(X_1(Np^s)(\mathbb{C}), \mathcal{F}_{k-2})$ which coincides with $\delta_{U_1(Np^s)}(f, r)$.

Let Σ be an allowable subset of finite primes. For $\mathcal{O}$ a discrete valuation ring finite and flat over $\mathbb{Z}_p$, we consider the completed cohomology

$$\tilde{\tilde{H}}^1_{\text{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} = \varprojlim_{s} \varprojlim_{U_p} H^1_{\text{et}}(X(U_1(N(\Sigma))^{(p)}U_p) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{O}/\varpi^s)_{\mathfrak{m}_{\bar{\rho}}}$$

where the inverse limit on $U_p \subset \mathbf{G}(\mathbb{Q}_p)$ is taken with respect to the trace map. The inclusion $\Gamma \simeq (1 + p\mathbb{Z}_p) \subset \mathbb{Q}_p^\times$ and the diagonal embedding of $\mathbb{Q}_p^\times$ in the diagonal torus of $\mathbf{G}(\mathbb{Q}_p)$ endows $\tilde{\tilde{H}}^1_{\text{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}$ with a structure of Λ_{Iw}-module and thus of $\mathbf{T}_{\Sigma,\text{Iw}}$-module. We put

$$\delta_{\Sigma,\text{Iw}} \overset{\text{def}}{=} (\tilde{\delta}_{U_1(N(\Sigma))} \otimes \chi_{\text{cyc}}^{-1})^+ \tag{3.2.1.1}$$

and

$$M_{\Sigma,\text{Iw}} \overset{\text{def}}{=} \mathbf{T}_{\Sigma,\text{Iw}} \cdot \delta_{\Sigma,\text{Iw}} \subset \tilde{\tilde{H}}^1_{\text{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}. \tag{3.2.1.2}$$

If $\psi : \mathbf{T}_{\Sigma,\text{Iw}} \longrightarrow S$ is a map of $\mathbb{Z}_p$-algebras, we denote by

$$\delta^\Sigma_{\psi,\text{Iw}} \in \tilde{\tilde{H}}^1_{\text{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \otimes_{\mathbf{T}_{\Sigma,\text{Iw}},\psi} S$$

the image of $\delta_{\Sigma,\text{Iw}}$ in $\tilde{\tilde{H}}^1_{\text{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \otimes_{\mathbf{T}_{\Sigma,\text{Iw}},\psi} S$ and by $M^\Sigma_{\psi,\text{Iw}}$ the S_{Iw}-module it generates; which is then $M_{\Sigma,\text{Iw}} \otimes_{\mathbf{T}_{\Sigma,\text{Iw}},\psi} S$.

Let $\mathfrak{a}^{\text{red}} \in \operatorname{Spec} \mathbf{T}_{\Sigma,\text{Iw}}$ be a minimal prime ideal. There then exist a unique compact open subgroup $U^{(p)} \subset \mathbf{G}(\mathbb{A}^{(p\infty)}_{\mathbb{Q}})$ containing $U_1(N(\Sigma))^{(p)}$ and a minimal prime ideal $\mathfrak{a} \in \operatorname{Spec} \mathbf{T}^{\text{new}}(U)$ such that $\mathbf{T}_{\Sigma,\text{Iw}}/\mathfrak{a}^{\text{red}}$ embeds in $R(\mathfrak{a})_{\text{Iw}} = \mathbf{T}^{\text{new}}(U)/\mathfrak{a}$. We denote by

$$\psi(\mathfrak{a}) : \mathbf{T}_{\Sigma,\text{Iw}} \longrightarrow R(\mathfrak{a})_{\text{Iw}}$$

the corresponding morphism. Let $(T(\mathfrak{a})_{\text{Iw}}, \rho(\mathfrak{a}), R(\mathfrak{a}))$ be the $G_{\mathbb{Q},\Sigma}$-representation attached to $\psi(\mathfrak{a})$. Denote by $N(\mathfrak{a})$ the tame Artin conductor of $T(\mathfrak{a})_{\text{Iw}} \otimes_{R(\mathfrak{a})_{\text{Iw}}} \operatorname{Frac}(R(\mathfrak{a})_{\text{Iw}})$. According to the equality (3.1.2.4) of proposition-definition 3.1.2, the tame conductor of $(V_\psi, \rho_\psi, \bar{\mathbb{Q}}_p)$ is also equal to $N(\mathfrak{a})$. Let $\delta(\mathfrak{a})_{\text{Iw}}$ be the class $\delta^\Sigma_{\psi(\mathfrak{a}),\text{Iw}}$ inside $\tilde{\tilde{H}}^1_{\text{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \otimes_{\mathbf{T}_{\Sigma,\text{Iw}}} R(\mathfrak{a})_{\text{Iw}}$ and let $M(\mathfrak{a})_{\text{Iw}}$ be the $R(\mathfrak{a})_{\text{Iw}}$-module it generates.

3.2.2 Change of level

Let $\Sigma \subset \Sigma'$ be a strict inclusion of allowable set of primes and denote for brevity $N(\Sigma)$ and $N(\Sigma')$ by N and N' respectively. Assume first that $\Sigma' = \Sigma \cup \{\ell\}$. For all $i \in \{-1, -2\}$, the twisted projections

$$\begin{aligned} \pi_{U_1(N'),U_1(N),\ell^i} : \quad & X_1(N') \longrightarrow X_1(N) \\ & [z, g]_{U_1(N')} \longmapsto [z, g \begin{pmatrix} \ell^i & 0 \\ 0 & 1 \end{pmatrix}]_{U_1(N)} \end{aligned}$$

induce covariant cohomological maps

$$\pi_{U_1(N'),U_1(N),\ell^i*} : H^1_{\text{et}}(X(U_1(N')U_p) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{O}/\varpi^s)_{\mathfrak{m}_{\bar{\rho}}} \longrightarrow H^1_{\text{et}}(X(U_1(N)U_p) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{O}/\varpi^s)_{\mathfrak{m}_{\bar{\rho}}}$$

for all $s \geq 1$ and $U_p \subset \mathbf{G}(\mathbb{Q}_p)$. These maps are compatible with $s' \geq s$ and $U'_p \subset U_p \subset \mathbf{G}(\mathbb{Q}_p)$ and are $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-equivariant after tensor product of the source with $\mathbf{T}_{\Sigma,\mathrm{Iw}}$.

In the following definition, recall that $\chi_\Gamma : G_{\mathbb{Q},\Sigma} \longrightarrow \Lambda_{\mathrm{Iw}}^\times$ is the composition $G_{\mathbb{Q},\Sigma} \twoheadrightarrow \Gamma \hookrightarrow \Lambda_{\mathrm{Iw}}^\times$.

Definition 3.2.1. *If $\Sigma \subsetneq \Sigma' = \Sigma \cup \{\ell\}$, define*

$$\pi_{\Sigma',\Sigma,\ell} : \tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\Sigma'))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \otimes_{\mathbf{T}_{\Sigma',\mathrm{Iw}}} \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}$$

to be the map induced by

$$1 - T(\ell)\chi_\Gamma(\mathrm{Fr}(\ell))\pi_{\Sigma',\Sigma,\ell^{-1}*} + \langle \ell \rangle \ell \chi_\Gamma(\mathrm{Fr}(\ell))^2 \pi_{\Sigma',\Sigma,\ell^{-2}*}.$$

For a general inclusion of allowable set of primes $\Sigma \subset \Sigma'$ such that $\Sigma' \backslash \Sigma = \{\ell_1, \cdots, \ell_m\}$, let

$$\pi_{\Sigma',\Sigma} : \tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\Sigma'))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \otimes_{\mathbf{T}_{\Sigma',\mathrm{Iw}}} \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \tag{3.2.2.1}$$

be the composition of the maps $\pi_{\Sigma \cup \{\ell_1, \cdots, \ell_j\}, \Sigma \cup \{\ell_1, \cdots, \ell_{j-1}\}, \ell_j}$ for j ranging from m to 1.

Consider as in subsection 3.2.1 a minimal prime $\mathfrak{a}^{\mathrm{red}}$ of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and the quotient $R(\mathfrak{a})_{\mathrm{Iw}}$ of $\mathbf{T}^{\mathrm{new}}$ in which $\mathbf{T}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$ embeds. Denote by $\Sigma(\mathfrak{a})$ the set of primes dividing $N(\mathfrak{a})$, that is to say the set of primes $\ell \nmid p$ at which $T(\mathfrak{a})_{\mathrm{Iw}}$ (and all its modular specializations) is ramified. For $\ell \in \Sigma$, denote by $e_\ell \in \{0, 1, 2\}$ the valuation at ℓ of $N(\Sigma)/N(\mathfrak{a})$.

Definition 3.2.2. *If $\Sigma \backslash \Sigma(\mathfrak{a}) = \{\ell\}$, define*

$$\pi_{\Sigma,\Sigma(\mathfrak{a}),\ell} : \tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}} R(\mathfrak{a}) \longrightarrow \tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\mathfrak{a}))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}$$

to be the map induced by

$$\begin{cases} 1 & \text{if } e_\ell = 0, \\ 1 - T(\ell)\chi_\Gamma(\mathrm{Fr}(\ell))\pi_{\Sigma,\Sigma(\mathfrak{a}),\ell^{-1}*} & \text{if } e_\ell = 1, \\ 1 - T(\ell)\chi_\Gamma(\mathrm{Fr}(\ell))\pi_{\Sigma,\Sigma(\mathfrak{a}),\ell^{-1}*} + \langle \ell \rangle \ell \chi_\Gamma(\mathrm{Fr}(\ell))^2 \pi_{\Sigma,\Sigma(\mathfrak{a}),\ell^{-2}*} & \text{else.} \end{cases}$$

For a general inclusion $\Sigma(\mathfrak{a}) \subset \Sigma$ sur that $\Sigma \backslash \Sigma(\mathfrak{a}) = \{\ell_1, \cdots, \ell_m\}$, let

$$\pi_{\Sigma,\Sigma(\mathfrak{a})} : \tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}} R(\mathfrak{a}) \longrightarrow \tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\mathfrak{a}))^{(p)}, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \tag{3.2.2.2}$$

be the composition of the maps $\pi_{\Sigma(\mathfrak{a}) \cup \{\ell_1, \cdots, \ell_j\}, \Sigma(\mathfrak{a}) \cup \{\ell_1, \cdots, \ell_{j-1}\}, \ell_j}$ for j ranging from m to 1.

The following proposition is a crucial ingredient in establishing the compatibility of the ETNC with change of levels and Hecke algebras.

Proposition 3.2.3. *Let $\Sigma \subset \Sigma'$ be two allowable set of primes. The map $\pi_{\Sigma',\Sigma}$ induces a map of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-modules*

$$\pi^{\mathrm{Iw}}_{\Sigma',\Sigma} : M_{\Sigma',\mathrm{Iw}} \otimes_{\mathbf{T}_{\Sigma',\mathrm{Iw}}} \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow M_{\Sigma,\mathrm{Iw}} \tag{3.2.2.3}$$

which sends $\delta_{\Sigma',\mathrm{Iw}}$ to

$$\delta_{\Sigma,\mathrm{Iw}} \prod_{\ell \in \Sigma' \backslash \Sigma} \mathrm{Eul}_\ell(T_{\Sigma,\mathrm{Iw}}). \tag{3.2.2.4}$$

If $\Sigma(\mathfrak{a}) \subset \Sigma$ is the smallest allowable set of primes such that $T(\mathfrak{a})$ is unramified outside $\Sigma(\mathfrak{a})$, then the map $\pi_{\Sigma,\Sigma(\mathfrak{a})}$ induces a map

$$\pi^{\mathrm{Iw}}_{\Sigma,\Sigma(\mathfrak{a})} : M_{\Sigma,\mathrm{Iw}} \otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}} R(\mathfrak{a})_{\mathrm{Iw}} \longrightarrow M(\mathfrak{a})_{\mathrm{Iw}} \tag{3.2.2.5}$$

which sends $\delta_{\Sigma,\mathrm{Iw}}$ to

$$\delta(\mathfrak{a})_{\mathrm{Iw}} \prod_{\ell \in \Sigma} \mathrm{Eul}_\ell(T(\mathfrak{a})_{\mathrm{Iw}}). \tag{3.2.2.6}$$

Proof. In order to prove that the maps (3.2.2.3) and (3.2.2.5) are well-defined and satisfy (3.2.2.4) and (3.2.2.6), it is enough to compute $\pi_{\Sigma',\Sigma}(\delta_{\Sigma',\mathrm{Iw}})$ and $\pi_{\Sigma,\Sigma(\mathfrak{a})}(\delta_{\Sigma,\mathrm{Iw}})$ respectively and to compare them with $\delta_{\Sigma,\mathrm{Iw}}$ and $\delta(\mathfrak{a})_{\mathrm{Iw}}$ respectively. The maps $\pi_{\Sigma',\Sigma,\ell^i*}$ are compatible with change of compact open subgroup U_p and s so it is enough to carry these comparisons for

$$H^1_{\mathrm{et}}(X_1(N(\Sigma')p^t)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O}/\varpi^s),\ H^1_{\mathrm{et}}(X_1(N(\Sigma)p^t)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O}/\varpi^s)$$

with t, s sufficiently large. Up to the isomorphism between

$$H^1(X(U)(\mathbb{C}),\mathcal{O}/\varpi^s)\simeq H_1(X(U)(\mathbb{C}),\{\mathrm{cusps}\},\mathcal{O}/\varpi^s)$$

induced by Poincaré duality, this is then the computation of [23, Page 558] (in order to check that the Euler factors appearing in [23] are indeed compatible with the statement of the proposition, notice that in the normalizations of this article the pseudocharacter attached to $\mathfrak{m}_{\bar\rho}$ sends the arithmetic, not the geometric, Frobenius morphism to $T(\ell)$, that the diamond operator $\langle\ell\rangle$ acts with weight k on modular forms of weight k, not weight $k-2$, and that the Euler factor is computed with respect to the specialization $\gamma\mapsto\chi_{\mathrm{cyc}}^{-1}(\gamma)$ of the cyclotomic variable or more concretely at the special value $r=1$, not for the full action of Γ). □

If $T(\mathfrak{a})_{\mathrm{Iw}}$ is unramified outside $\Sigma(\mathfrak{a})\subset\Sigma\subset\Sigma'$, then $\pi_{\Sigma',\Sigma(\mathfrak{a})}$ factors through $\pi_{\Sigma',\Sigma}$ and $\pi_{\Sigma,\Sigma(\mathfrak{a})}$. The compatibility (3.1.2.2) of unramified Euler factors with arbitrary change of ring of coefficients entails that the image of $\delta_{\Sigma',\mathrm{Iw}}$ through $\pi_{\Sigma',\Sigma(\mathfrak{a})}$ is equal to its image through $\pi_{\Sigma,\Sigma(\mathfrak{a})}\circ\pi_{\Sigma',\Sigma}$.

For a more general specialization $\psi:\mathbf{T}_{\Sigma,\mathrm{Iw}}\longrightarrow S$, different factorizations of ψ yield potentially distinct modular elements δ attached to ψ. First, one can consider $\delta^{\Sigma}_{\psi,\mathrm{Iw}}$ as in subsection 3.2.1. Second, one can consider choose a minimal prime ideal $\mathfrak{a}^{\mathrm{red}}$ of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ through which ψ factors and consider the image $\delta^{\mathfrak{a}}_{\psi,\mathrm{Iw}}$ of $\delta(\mathfrak{a})_{\mathrm{Iw}}$ through ψ. According to proposition 3.2.3, the class $\delta^{\mathfrak{a}}_{\psi,\mathrm{Iw}}$ satisfies

$$\delta^{\mathfrak{a}}_{\psi,\mathrm{Iw}}=\delta^{\Sigma}_{\psi,\mathrm{Iw}}\left(\prod_{\ell\in\Sigma^{(p)}}\psi(\mathrm{Eul}_\ell(T(\mathfrak{a})_{\mathrm{Iw}}))\right)^{-1}.$$

Note that when ψ factors through two distinct minimal ideals $\mathfrak{a}$ and $\mathfrak{b}$, the elements $\delta^{\mathfrak{a}}_{\psi,\mathrm{Iw}}$ and $\delta^{\mathfrak{b}}_{\psi,\mathrm{Iw}}$ may differ. Finally, one could consider

$$\delta_{\psi,\mathrm{Iw}}=\delta^{\Sigma}_{\psi,\mathrm{Iw}}\left(\prod_{\ell\in\Sigma^{(p)}}\mathrm{Eul}_\ell(T_\psi)\right)^{-1}.$$

Note that because of the action of $G_{\mathbb{Q},\Sigma}$ on Λ_{Iw}, the Euler factors $\mathrm{Eul}_\ell(T(\mathfrak{a})_{\mathrm{Iw}})$ and $\mathrm{Eul}_\ell(T_\psi)$ are indeed invertible for all $\ell\in\Sigma^{(p)}$. If ψ is a modular point, more generally if there exists a modular point factoring through ψ, even more generally if $\mathrm{rank}_{S_{\mathrm{Iw}}}T_\psi^{I_\ell}$ is equal to $\mathrm{rank}_{R(\mathfrak{a})_{\mathrm{Iw}}}T(\mathfrak{a})_{\mathrm{Iw}}^{I_\ell}$ for all $\ell\in\Sigma^{(p)}$ then $\delta^{\mathfrak{a}}_{\psi,\mathrm{Iw}}$ and $\delta_{\psi,\mathrm{Iw}}$ coincide. This holds in particular for ψ equal to $\psi(\mathfrak{a})$. In general, they may differ.

Definition 3.2.4. *Let $\psi:\mathbf{T}_{\Sigma,\mathrm{Iw}}\longrightarrow S$ be a specialization with values in a flat $\mathbb{Z}_p$-algebra and factoring through $R(\mathfrak{a})_{\mathrm{Iw}}$ for a minimal prime ideal $\mathfrak{a}\in\operatorname{Spec}\mathbf{T}^{\mathrm{new}}$. Denote by $M^{\Sigma}_{\psi,\mathrm{Iw}}$ (resp. $M^{\mathfrak{a}}_{\psi,\mathrm{Iw}}$, resp. $M_{\psi,\mathrm{Iw}}$) the S_{Iw}-module generated by $\delta^{\Sigma}_{\psi,\mathrm{Iw}}$ (resp. $\delta^{\mathfrak{a}}_{\psi,\mathrm{Iw}}$, resp. $\delta_{\psi,\mathrm{Iw}}$).*

When the context is unambiguous, we often omit the subscript Iw from the notation $M_{\psi,\mathrm{Iw}}$ and $\delta_{\psi,\mathrm{Iw}}$.

3.3 Zeta elements and zeta morphisms

In this subsection, we fix $\Sigma\supset\{\ell|N(\bar\rho)p\}$ an allowable set of primes and a classical modular specialization $\phi:\mathbf{T}_{\Sigma,\mathrm{Iw}}\longrightarrow\bar{\mathbb{Q}}_p$ of weight k. In other words, we fix an integer t such that ϕ is the system of eigenvalues attached to a classical normalized eigencuspform $f_\phi \in S_k(U(N(\Sigma)p^t))$ of weight $k \geq 2$.

The kernel x of the extension of ϕ to $\mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$ is then a classical point of $\operatorname{Spec}\mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$. By definition, the morphism ϕ then factors through $\psi_x : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ where $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ is the Iwasawa algebra attached to the local factor $\mathbf{T}^x_\Sigma$ of the classical Hecke algebra $\mathbf{T}^{\mathrm{cl}}(U(N(\Sigma)p^t))$ acting on $H^1_{\mathrm{et}}(X(N(\Sigma)p^t) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{F}_{k-2})_{\mathfrak{m}_{\bar\rho}}$. The eigencuspform f_ϕ is a newform of some tame level $N(\mathfrak{a}_x)|N(\Sigma)$ so there exists a unique minimal prime $\mathfrak{a}^{\mathrm{red}}_x$ of $\mathbf{T}^x_\Sigma$ and unique minimal prime $\mathfrak{a}_x$ of $\mathbf{T}^{\mathrm{new}}(U(N(\mathfrak{a}_x)p^t))$ acting on $H^1_{\mathrm{et}}(X(N(\mathfrak{a})p^t) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{F}_{k-2})_{\mathfrak{m}_{\bar\rho}}$ such that ϕ viewed as a morphism from $\mathbf{T}^x_\Sigma$ factors through

$$\mathbf{T}^x_\Sigma/\mathfrak{a}^{\mathrm{red}}_x \hookrightarrow R(\mathfrak{a}_x) = \mathbf{T}^{\mathrm{new}}(U(N(\mathfrak{a}_x)p^t))/\mathfrak{a}_x.$$

Denote this map by $\psi(\mathfrak{a}_x) : \mathbf{T}^x_\Sigma \longrightarrow R(\mathfrak{a}_x)$. Write $(T^x_\Sigma, \rho^x_\Sigma, \mathbf{T}^x_\Sigma)$ and $(T(\mathfrak{a}_x), \rho(\mathfrak{a}_x), R(\mathfrak{a}_x))$ for the $G_{\mathbb{Q},\Sigma}$-representations attached to ψ_x and $\psi(\mathfrak{a}_x)$ respectively. We also write $M^x_{\Sigma,\mathrm{Iw}}$ for the image of $M_{\Sigma,\mathrm{Iw}}$ through ψ_x and $M(\mathfrak{a}_x)_{\mathrm{Iw}}$ for its image through $\psi(\mathfrak{a}_x)$.

Our aim is to construct so called fundamental lines, zeta elements and zeta morphisms with coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$, $R(\mathfrak{a})_{\mathrm{Iw}}$ $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ and $R(\mathfrak{a}_x)_{\mathrm{Iw}}$; that is to say trivializations of free modules of rank 1 constructed from étale cohomology of deformations of $\bar\rho$ and completed cohomology. As the zeta element $\mathbf{z}(f)_{\mathrm{Iw}}$ computes the special values of the L-function of f (and its cyclotomic twists), one way to understand these objects is to consider them as p-adic interpolation of special values in families parametrized by Hecke algebra. We also prove an important part of the ETNC as formulated in [41]; namely that it is compatible with arbitrary specializations.

At first glance, the precise statements and proofs of this property in subsection 3.3.2 and 3.3.3 seem very similar. This is, however, partly deceptive: the fact that zeta morphisms with coefficients in reduced Hecke algebra are compatible with base change is mostly formal, whereas the similar property for zeta morphism with coefficients in integral domains is much subtler and requires the full strength of the Weight-Monodromy Conjecture as well as delicate properties of completed cohomology. On the other hand, the latter is also a much more precise result, as it takes into account Euler factors at places of bad reduction.

3.3.1 p-adic zeta elements

Let $n \geq 1$ be an integer. Write N for $N(\Sigma)$ for brevity and fix an integer $M \geq 1$.

To $(\alpha, \beta) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$ is attached in [43, Section 1.4] a Siegel unit

$$g_{\alpha,\beta} \in \mathcal{O}(Y(Np^n))^\times \otimes_{\mathbf{T}^{\mathrm{red}}(Np^n)} Q(\mathbf{T}^{\mathrm{red}}(Np^n))$$

satisfying remarkable distribution properties. As in [43, Section 2.2], the zeta element $\mathbf{z}_{Mp^n,Np^n}$ is defined to be

$$\mathbf{z}_{Mp^n,Np^n} = \left\{g_{1/Mp^n,0}, g_{0,1/Np^n}\right\} \in K_2(Y(Mp^n, Np^n)) \otimes_{\mathbf{T}^{\mathrm{red}}(Mp^n,Np^n)} Q(\mathbf{T}^{\mathrm{red}}(Mp^n, Np^n)).$$

If $m \geq n$, the morphism induced by the norm map

$$K_2(Y(Mp^m, Np^m)) \longrightarrow K_2(Y(Mp^n, Np^n))$$

sends $\mathbf{z}_{Mp^m,Np^m}$ to $\mathbf{z}_{Mp^n,Np^n}$. Hence, the system $\{\mathbf{z}_{Mp^n,Np^n}\}_{n\geq 1}$ is a compatible system in the inverse limit

$$\varprojlim_n K_2(Y(Mp^n, Np^n)) \otimes_{\mathbf{T}^{\mathrm{red}}(Mp^n,Np^n)} Q(\mathbf{T}^{\mathrm{red}}(Mp^n, Np^n)).$$

Following [43, Section 8.4 and 8.9], the composition of the Chern class map

$$K_2(Y(Mp^n, Np^n)) \longrightarrow H^2_{\mathrm{et}}(Y(Mp^n, Np^n), \mathbb{Z}/p^s\mathbb{Z})(2)$$

with the spectral sequence degeneracy map

$$H^2_{\mathrm{et}}(Y(Mp^n, Np^n), \mathbb{Z}/p^s\mathbb{Z}) \longrightarrow H^1(G_{\mathbb{Q}}, H^1_{\mathrm{et}}(Y(Mp^n, Np^n) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}/p^s\mathbb{Z}))$$

and the projection from $Y(Mp^n, Np^n)$ to $Y_1(Np^n) \otimes \mathbb{Q}(\zeta_{p^n})$ sends $\mathbf{z}_{Mp^n,Np^n}$ to an element

$$\mathbf{z}_{Np^n} \in H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma, \zeta_{p^n}], H^1_{\mathrm{et}}(Y_1(Np^n) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}/p^s\mathbb{Z})).$$

The norm compatibility of the $\mathbf{z}_{Mp^n,Np^n}$ implies that the system $\{\mathbf{z}_{Np^n}\}_{n\geq 1}$ is compatible with the trace map from $\mathbb{Q}(\zeta_{p^m})$ to $\mathbb{Q}(\zeta_{p^n})$ and with projection from $Y_1(Np^m)$ to $Y_1(Np^n)$ (see [43, Propositions 8.7 and 8.8]). Localizing at $\mathfrak{m}_{\bar{\rho}}$ and taking the inverse limit first on n then on s thus yields a zeta element $\mathbf{z}_{\Sigma,\mathrm{Iw}}$ in the space

$$\varprojlim_s \varprojlim_n H^1(G_{\mathbb{Q}(\zeta_{p^n}),\Sigma}, H^1_{\mathrm{et}}(Y_1(Np^n) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}/p^s\mathbb{Z})_{\mathfrak{m}_{\bar{\rho}}}).$$

The Shimura variety $Y_1(Np^n)$ being an affine curve, there are isomorphisms

$$\begin{aligned}\varprojlim_s \varinjlim_n H^1_c(Y_1(Np^n) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_p)/p^s &\simeq \varprojlim_s \varinjlim_n H^1_c(Y_1(Np^n) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}/p^s\mathbb{Z}) \\ &\simeq \tilde{H}^1_c(Y_1(Np^n) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_p).\end{aligned}$$

The observation that the first inverse system on s is Mittag-Leffler and Poincaré duality thus shows that $\mathbf{z}_{\Sigma,\mathrm{Iw}}$ belongs to $H^1(G_{\mathbb{Q},\Sigma}, \tilde{H}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)}, \mathbb{Z}_p)_{\mathfrak{m}_{\bar{\rho}}})$ (see appendix A.2 for the definition of this cohomology group). In fact, it is well-known that the construction of $\mathbf{z}_{\Sigma,\mathrm{Iw}}$ implies that it is unramified at $\ell \nmid p$ but we will not use this fact.

Definition 3.3.1. *Denote by $Z_{\Sigma,\mathrm{Iw}}$ the $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-module*

$$\mathbf{T}_{\Sigma,\mathrm{Iw}} \cdot \mathbf{z}_{\Sigma,\mathrm{Iw}} \subset H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], \tilde{H}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)}, \mathbb{Z}_p)_{\mathfrak{m}_{\bar{\rho}}})$$

generated by $\mathbf{z}_{\Sigma,\mathrm{Iw}}$.

We prove in subsection 3.3.2 below that $Z_{\Sigma,\mathrm{Iw}}$ is a free $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-module of rank 1.

3.3.2 Coefficients in $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$

The image of $\mathbf{z}_{\Sigma,\mathrm{Iw}}$ through ψ_x yields by [22, Corollary 4.3.2] an element

$$\mathbf{z}^x_{\Sigma,\mathrm{Iw}} \in H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T^x_\Sigma) \otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}} Q(\mathbf{T}^x_{\Sigma,\mathrm{Iw}}).$$

which actually lies in $H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T^x_\Sigma) \otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}} \mathbf{T}^x_{\Sigma,\mathrm{Iw}}[1/p]$ by [43, Section 13.12] (and in fact in $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T^x_\Sigma) \otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}} \mathbf{T}^x_{\Sigma,\mathrm{Iw}}[1/p]$ but we will not use this fact).

Lemma 3.3.2. *The complex $\mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}})$ is a perfect complex of $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$-modules acyclic outside degree 1 and 2. Its first cohomology group is torsion-free and its second cohomology group is torsion.*

Proof. The functor $\mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma], -)$ sends perfect complexes to perfect complexes and $\bar{\rho}$ is irreducible so the first assertion is standard.

Choose $(\mathbf{x}, \mathbf{y})$ a regular sequence in Λ_{Iw}. The isomorphisms

$$\mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}}) \overset{\mathrm{L}}{\otimes}_{\Lambda_{\mathrm{Iw}}} \Lambda_{\mathrm{Iw}}/\mathbf{x} \simeq \mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}}/\mathbf{x})$$
$$\mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}}/x) \overset{\mathrm{L}}{\otimes}_{\Lambda_{\mathrm{Iw}}/\mathbf{x}} \Lambda_{\mathrm{Iw}}/(\mathbf{x},\mathbf{y}) \simeq \mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}}/(\mathbf{x},\mathbf{y}))$$

and the fact that

$$H^0(G_{\mathbb{Q},\Sigma}, \bar{\rho}) = 0$$

show that $H^1_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}})[\mathbf{x}]$ and $H^1_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}}/\mathbf{x})[\mathbf{y}]$ are zero. Thus $H^1_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}})$ is a Λ_{Iw}-module of depth at least 2 and so is a free Λ_{Iw}-module of finite rank. As $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ is a Cohen-Macaulay local ring, it is free over Λ_{Iw} and so the above argument also shows that $H^1_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}})$ is of depth 2 as $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$-module and in particular torsion-free.

By definition, the modular map ϕ factors through $\mathbf{T}^x_\Sigma$. For all $\ell\nmid p$, there exists a cyclotomic twist of T_ϕ such that the eigenvalues of $\mathrm{Fr}(\ell)$ acting on $T_\phi^{I_\ell}$ are of non-zero weights. Hence $H^0(G_{\mathbb{Q}_\ell}, T_\phi)\otimes_{\Lambda_{\mathrm{Iw}}}\mathrm{Frac}(\Lambda_{\mathrm{Iw}})$ vanishes for all ℓ. By Poitou-Tate duality, this implies that the complexes

$$\mathrm{R}\,\Gamma(G_{\mathbb{Q}_\ell}, T_\phi)\overset{\mathrm{L}}{\otimes}_{\Lambda_{\mathrm{Iw}}}\mathrm{Frac}(\Lambda_{\mathrm{Iw}})$$

are acyclic and hence that $H^2_c(\mathbb{Z}[1/\Sigma], T_\phi)$ and $H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_\phi)$ become isomorphic after tensor product with $\mathrm{Frac}(\Lambda_{\mathrm{Iw}})$. As $H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_\phi)$ is Λ_{Iw}-torsion by [43, Theorem 12.4], so is $H^2_c(\mathbb{Z}[1/\Sigma], T_\phi)$. The isomorphism

$$\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}})\overset{\mathrm{L}}{\otimes}_{\mathbf{T}^x_\Sigma,\phi}\bar{\mathbb{Q}}_p\simeq\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma], T_\phi)$$

then shows that $H^2_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}})$ is $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$-torsion. □

Consider $\lambda:\mathbf{T}^x_{\Sigma,\mathrm{Iw}}\longrightarrow F_{\mathfrak{p}}[G_n]$ a modular specialization of $\mathbf{T}^x_\Sigma$ and M_λ the modular motive attached to λ. By [43, Theorem 5.6], $\lambda(\mathbf{z}^x_\Sigma)$ is equal to the product of $\mathbf{z}(f_\lambda)_{\mathrm{Iw}}$ with Euler factors at $\ell\in\Sigma^{(p)}$ (in fact, $\mathbf{z}(f_\lambda)_{\mathrm{Iw}}$ is defined in [43] as the quotient of $\lambda(\mathbf{z}^x_\Sigma)$ by these Euler factors). More precisely, the equality

$$\mathrm{per}_{\mathbb{C},\chi}(\mathrm{per}_p^{-1}\circ\mathrm{loc}_p\circ\lambda(\mathbf{z}^x_\Sigma)\otimes 1)=\left(\prod_{\ell\in\Sigma^{(p)}}\mathrm{Eul}_\ell(T_\lambda)\right)=L_\Sigma(M^*_\lambda(1),\chi,r)\qquad(3.3.2.1)$$

holds for all characters $\chi\in\hat{G}_n$. Applying this to ϕ and taking into account the fact that $\mathbf{z}(f_\phi)_{\mathrm{Iw}}$ is non-zero shows that $\mathbf{z}^x_{\Sigma,\mathrm{Iw}}$ generates a free $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$-module inside $H^1_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}})[1/p]$.

The morphism

$$M^x_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathbf{T}^x_{\Sigma,\mathrm{Iw}}[1/p]\longrightarrow H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T^x_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathbf{T}^x_{\Sigma,\mathrm{Iw}}[1/p]$$

which sends $\delta^x_{\Sigma,\mathrm{Iw}}$ to $\mathbf{z}^x_{\Sigma,\mathrm{Iw}}$ defines by lifting a morphism of complexes

$$Z^x_{\Sigma,\mathrm{Iw}}:M^x_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathbf{T}^x_{\Sigma,\mathrm{Iw}}[1/p]\longrightarrow\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], T^x_{\Sigma,\mathrm{Iw}})[1]\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathbf{T}^x_{\Sigma,\mathrm{Iw}}[1/p].\qquad(3.3.2.2)$$

In the above, we view $M^x_{\Sigma,\mathrm{Iw}}$ and $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T^x_{\Sigma,\mathrm{Iw}})$ as complexes concentrated in degree 0.

Definition 3.3.3. *Define $\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})$ to be the free $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$-module of rank 1*

$$\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})=\mathrm{Det}^{-1}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma], T^x_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathrm{Det}^{-1}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}M^x_{\Sigma,\mathrm{Iw}}.\qquad(3.3.2.3)$$

Let $\psi:\mathbf{T}^x_{\Sigma,\mathrm{Iw}}\longrightarrow S_{\mathrm{Iw}}$ be a specialization induced by a $\mathbb{Z}_p$-algebras morphism $\mathbf{T}^x_\Sigma\longrightarrow S$ with values in a reduced, flat $\mathbb{Z}_p$-algebra and such that $\delta_{\psi,\mathrm{Iw}}$ is non-zero. Specializing $\mathbf{z}^x_{\Sigma,\mathrm{Iw}}$ at ψ yields an element

$$\mathbf{z}_\psi\in H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\psi)\otimes_{S_{\mathrm{Iw}}}S_{\mathrm{Iw}}[1/p].$$

As for the case $\psi=\phi$ which is treated in the proof of lemma 3.3.2 above, there is an isomorphism of perfect complexes of $Q(\mathbf{T}^x_{\Sigma,\mathrm{Iw}})$-modules

$$\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma], T_\psi)\overset{\mathrm{L}}{\otimes}Q(\mathbf{T}^x_{\Sigma,\mathrm{Iw}})\simeq\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T_\psi)\overset{\mathrm{L}}{\otimes}Q(\mathbf{T}^x_{\Sigma,\mathrm{Iw}})$$

and $H^2_c(\mathbb{Z}[1/\Sigma], T_\psi)\overset{\mathrm{L}}{\otimes}Q(\mathbf{T}^x_{\Sigma,\mathrm{Iw}})$ vanishes. The morphism sending $\delta_{\psi,\mathrm{Iw}}$ to $\mathbf{z}_\psi$ consequently gives as above a morphism

$$Z_\psi:M_\psi\otimes_{S_{\mathrm{Iw}}}S_{\mathrm{Iw}}[1/p]\longrightarrow\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], T_\psi)[1]\otimes_{S_{\mathrm{Iw}}}S_{\mathrm{Iw}}[1/p].$$

Definition 3.3.4. *Define $\Delta_{S_{\mathrm{Iw}}}(T_\psi)$ to be the free S_{Iw}-module of rank 1*

$$\Delta_{S_{\mathrm{Iw}}}(T_\psi) = \mathrm{Det}^{-1}_{S_{\mathrm{Iw}}} \mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma], T_\psi) \otimes_{S_{\mathrm{Iw}}} \mathrm{Det}^{-1}_{S_{\mathrm{Iw}}} M_\psi. \tag{3.3.2.4}$$

Denote $\Delta_{S_{\mathrm{Iw}}}(T_\psi)$ by Δ_ψ for brevity. The various Δ_ψ are compatible with change of allowable levels and specializations.

Proposition 3.3.5. *The $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$-module $\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})$ is equipped with a canonical morphism*

$$\mathrm{triv}_\Sigma : \Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}}) \subset \Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}}) \otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}} Q(\mathbf{T}^x_{\Sigma,\mathrm{Iw}}) \overset{\mathrm{can}}{\simeq} Q(\mathbf{T}^x_{\Sigma,\mathrm{Iw}}).$$

More generally, if $\psi : \mathbf{T}^x_{\Sigma,\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$ is a specialization induced by a $\mathbb{Z}_p$-algebras morphism $\mathbf{T}^x_{\Sigma,\mathrm{Iw}} \longrightarrow S$ with values in a reduced, flat $\mathbb{Z}_p$-algebra such that $\mathbf{z}_\psi$ is a non zero-divisor, then the S_{Iw}-module $\Delta_{S_{\mathrm{Iw}}}(T_\psi)$ (which we also denote by Δ_ψ for brevity) is endowed with a canonical trivialization isomorphism

$$\mathrm{triv}_\psi : \Delta_\psi \subset \Delta_\psi \otimes_{S_{\mathrm{Iw}}} Q(S_{\mathrm{Iw}}) \overset{\mathrm{can}}{\simeq} Q(S_{\mathrm{Iw}}).$$

For all ψ as above, denote by $\mathrm{triv}_\psi(\Delta_\psi) \subset Q(S_{\mathrm{Iw}})$ the image of $\Delta_\psi \subset \Delta_\psi \otimes_{S_{\mathrm{Iw}}} Q(S_{\mathrm{Iw}})$ through triv_ψ. If the diagram

$$\begin{array}{ccc} \mathbf{T}^x_{\Sigma,\mathrm{Iw}} & \xrightarrow{\xi} & S'_{\mathrm{Iw}} \\ {\scriptstyle\psi}\downarrow & \nearrow_{\phi} & \\ S_{\mathrm{Iw}} & & \end{array}$$

of reduced flat $\mathbb{Z}_p$-algebra quotients of $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ is commutative, then there is a canonical isomorphism

$$\phi^\Delta : \Delta_\psi \otimes_{S,\phi} S' \overset{\mathrm{can}}{\simeq} \Delta_\xi$$

compatible with triv_ψ and triv_ξ in the sense that the rightmost downward arrow of the diagram

$$\begin{array}{ccc} \Delta_\psi & \longrightarrow & \mathrm{triv}_\psi(\Delta_\psi) \\ {\scriptstyle \phi^\Delta(-\otimes_{S,\phi}S')}\downarrow & & \downarrow{\scriptstyle\phi} \\ \Delta_\xi & \longrightarrow & \mathrm{triv}_\xi(\Delta_\xi) \end{array} \tag{3.3.2.5}$$

exists and makes the diagram commutative.

If $\Sigma \subset \Sigma'$ is an inclusion of allowable sets, then there is a canonical isomorphism

$$\Delta_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}(T^x_{\Sigma',x}) \otimes_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}} \mathbf{T}^x_{\Sigma,\mathrm{Iw}} \overset{\mathrm{can}}{\simeq} \Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})$$

compatible with triv *in the sense that the rightmost downward arrow of the diagram*

$$\begin{array}{ccc} \Delta_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}(T^x_{\Sigma',\mathrm{Iw}}) & \longrightarrow & \mathrm{triv}_{\Sigma'}(\Delta_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}(T^x_{\Sigma',\mathrm{Iw}})) \\ \downarrow & & \downarrow \\ \Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}}) & \longrightarrow & \mathrm{triv}_\Sigma(\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})) \end{array} \tag{3.3.2.6}$$

exists and makes the diagram commutative.

Proof. Fix ψ as in the proposition (possibly equal to the identity of $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$). Then the complex

$$\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma], T_\psi) \overset{\mathrm{L}}{\otimes}_{S_{\mathrm{Iw}}} Q(S_{\mathrm{Iw}})$$

is concentrated in degree 1. As in the proof of lemma 3.3.2, the Λ_{Iw}-module $H^1_c(\mathbb{Z}[1/\Sigma],T_\psi)$ is of depth 2 and hence free of finite rank. By Poitou-Tate duality, its rank is equal to the rank of T_ψ^- as Λ_{Iw}-module and so is equal to the rank of M_ψ as Λ_{Iw}-module. Consequently, the cone of the morphism

$$Z_\psi : M_\psi\otimes_{S_{\mathrm{Iw}}} Q(S_{\mathrm{Iw}})\longrightarrow \mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],T_\psi)[1]\otimes_{S_{\mathrm{Iw}}} Q(S_{\mathrm{Iw}})$$

is an acyclic complex and so $\mathrm{Det}_{Q(S_{\mathrm{Iw}})}\operatorname{Cone} Z_\psi=\mathrm{Det}_{Q(S_{\mathrm{Iw}})}0$ is canonically isomorphic to $Q(S_{\mathrm{Iw}})$. The inclusion of Δ_ψ in $\Delta_\psi\otimes_{S_{\mathrm{Iw}}}Q(S_{\mathrm{Iw}})$ composed with the canonical isomorphism $\mathrm{Det}_{Q(S_{\mathrm{Iw}})}\operatorname{Cone} Z_\psi\overset{\mathrm{can}}{\simeq}Q(S_{\mathrm{Iw}})$ then defines

$$\mathrm{triv}_\psi:\Delta_\psi\subset\Delta_\psi\otimes_{S_{\mathrm{Iw}}}Q(S_{\mathrm{Iw}})\overset{\mathrm{can}}{\simeq}Q(S_{\mathrm{Iw}}).$$

By definition, the classes $\mathbf{z}^x_{\Sigma,\mathrm{Iw}}$ and $\delta^x_{\Sigma,\mathrm{Iw}}$ are compatible with specializations. Hence, the compatibility of diagram (3.3.2.5) amounts to the compatibility of $\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],-)$ with arbitrary base-change of rings of coefficients and the functorial compatibility of Det with derived tensor product.

Fix $\Sigma\subset\Sigma'$ an inclusion of allowable sets. Then the isomorphism (3.2.2.3) of proposition 3.2.3 induces a canonical isomorphism

$$\mathrm{Det}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(M^x_{\Sigma',\mathrm{Iw}}\otimes_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}\mathbf{T}^x_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathrm{Det}^{-1}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}M^x_{\Sigma,\mathrm{Iw}}\overset{\mathrm{can}}{\simeq}\mathrm{Det}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}[S_{\mathrm{Iw}}\overset{d}{\longrightarrow}S_{\mathrm{Iw}}]$$

where the complex $[S_{\mathrm{Iw}}\overset{d}{\longrightarrow}S_{\mathrm{Iw}}]$ is placed in degree 0 and 1 and d is multiplication by

$$\prod_{\ell\in\Sigma'\backslash\Sigma}\mathrm{Eul}_\ell(T^x_{\Sigma,\mathrm{Iw}}).$$

Hence

$$\mathrm{Det}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(M^x_{\Sigma',\mathrm{Iw}}\otimes_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}\mathbf{T}^x_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathrm{Det}^{-1}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}M^x_{\Sigma,\mathrm{Iw}}\overset{\mathrm{can}}{\simeq}\bigotimes_{\ell\in\Sigma'\backslash\Sigma}\mathrm{Det}^{-1}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}S_{\mathrm{Iw}}/\mathrm{Eul}_\ell(T^x_{\Sigma,\mathrm{Iw}})$$

where each of the $S_{\mathrm{Iw}}/\mathrm{Eul}_\ell(T^x_{\Sigma,\mathrm{Iw}})$ is viewed as a complex in degree 0. On the other hand, the definition and base-change property of $\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma'],-)$ induces a canonical isomorphism between

$$\mathrm{Det}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\left(\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma'],T^x_{\Sigma',\mathrm{Iw}})\overset{\mathrm{L}}{\otimes}_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}\mathbf{T}^x_{\Sigma,\mathrm{Iw}}\right)\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathrm{Det}^{-1}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],T^x_{\Sigma,\mathrm{Iw}})$$

and

$$\bigotimes_{\ell\in\Sigma'\backslash\Sigma}\mathrm{Det}^{-1}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\mathrm{R}\,\Gamma(G_{\mathbb{Q}_\ell},T^x_{\Sigma,\mathrm{Iw}})\overset{\mathrm{can}}{\simeq}\bigotimes_{\ell\in\Sigma'\backslash\Sigma}\mathrm{Det}_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}S_{\mathrm{Iw}}/\mathrm{Eul}_\ell(T^x_{\Sigma,\mathrm{Iw}})$$

where each of the $S_{\mathrm{Iw}}/\mathrm{Eul}_\ell(T^x_{\Sigma,\mathrm{Iw}})$ is again seen as a complex in degree 0. Putting these isomorphism together yields a canonical isomorphism

$$\Delta_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}(T^x_{\Sigma',\mathrm{Iw}})\otimes_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}\mathbf{T}^x_{\Sigma,\mathrm{Iw}}\overset{\mathrm{can}}{\simeq}\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}}).\qquad(3.3.2.7)$$

The trivialization of $\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})$ induced by $\mathrm{triv}_{\Sigma'}$ is induced by the morphism sending $\delta_{\Sigma',\mathrm{Iw}}$ to $\mathbf{z}^x_{\Sigma',\mathrm{Iw}}$. After tensor product with $Q(\mathbf{T}^x_{\Sigma,\mathrm{Iw}})$, it thus sends $\delta_{\Sigma,\mathrm{Iw}}$ to $\mathbf{z}^x_{\Sigma,\mathrm{Iw}}$. Hence, the isomorphism (3.3.2.7) is compatible with the trivializations $\mathrm{triv}_{\Sigma'}$ and triv_Σ. □

We end this subsection by proving that, as announced at the end of subsection 3.3.1, $\mathbf{z}_{\Sigma,\mathrm{Iw}}$ generates a free $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-module inside $H^1_{\mathrm{et}}(G_{\mathbb{Q},\Sigma},\tilde{H}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)},\mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}})$.

Proposition 3.3.6. *The $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-module $Z_{\Sigma,\mathrm{Iw}}$ is free of rank 1.*

Proof. As $Z_{\Sigma,\mathrm{Iw}}$ is cyclic, it is enough to show that it is a faithful $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-module. Let $m \in \mathbf{T}_{\Sigma,\mathrm{Iw}}$ be an element annihilating $Z_{\Sigma,\mathrm{Iw}}$, let ψ be a modular specialization of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ corresponding to an eigencuspform f_ψ and let $\psi_y : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \mathbf{T}^y_{\Sigma,\mathrm{Iw}}$ be the specialization with values in the local factor of the classical reduced Hecke algebra of a certain level and weight through which ψ factors. Then the image of $\mathbf{z}_{\Sigma,\mathrm{Iw}}$ through ψ_y is equal to $\mathbf{z}^y_{\Sigma,\mathrm{Iw}}$ in the notations of this subsection so is a non zero-divisor by (3.3.2.1). Hence, m is in the kernel of ψ_y. By [21, Theorem 7.4.2], this means that m annihilates the subspace of locally $\mathrm{GL}_2(\mathbb{Q}_p)$-algebraic vectors of $\tilde{H}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)}, \mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}}$. As locally $\mathrm{GL}_2(\mathbb{Q}_p)$-algebraic vectors are dense in $\tilde{H}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)}, \mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}}$ and as the action of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ on this module is faithful, m is zero. □

3.3.3 Coefficients in $R(\mathfrak{a}_x)_{\mathrm{Iw}}$

Subsection 3.3.2 admits an important variant with $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ replaced with $R(\mathfrak{a}_x)_{\mathrm{Iw}}$.

Lemma 3.3.7. *The complex $\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})$ is acyclic outside degree 1 and 2. The $R(\mathfrak{a}_x)_{\mathrm{Iw}}$-module $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})$ is of depth 2 and of rank 1 whereas $H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})$ is $R(\mathfrak{a}_x)_{\mathrm{Iw}}$-torsion.*

Proof. That $\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})$ is acyclic outside degree 1 and 2 follows from the irreducibility of $\bar\rho$.

Consider ψ_x as having values in a discrete valuation ring $\mathcal{O}$, finite and flat over $\mathbb{Z}_p$ with uniformizing parameter ϖ. The short exact sequence

$$0 \longrightarrow T_{\psi_x} \xrightarrow{\varpi} T_{\psi_x} \longrightarrow T_{\psi_x}/\varpi \longrightarrow 0$$

induces an isomorphism between $H^1(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a}_x))[\varpi]$ and $H^0(G_{\mathbb{Q},\Sigma}, \bar\rho)$ which is zero under our ongoing assumption that $\bar\rho$ is absolutely irreducible. For γ a topological generator of Γ, the short sequence

$$0 \longrightarrow T(\mathfrak{a}_x)_{\mathrm{Iw}} \xrightarrow{\gamma-1} T(\mathfrak{a}_x)_{\mathrm{Iw}} \longrightarrow T_{\psi_x} \longrightarrow 0$$

of étale sheaves on $\operatorname{Spec}\mathbb{Z}[1/p]$ is exact so

$$H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})[\gamma - 1] = 0$$

and $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})/(\gamma - 1)$ embeds into $H^1(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a}_x))$ so is of depth 1. Hence the $R(\mathfrak{a}_x)_{\mathrm{Iw}}$-module $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})$ is of depth 2.

As $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}}) \otimes_{\Lambda_{\mathrm{Iw}}} \mathrm{Frac}(\Lambda_{\mathrm{Iw}})$ is of dimension 1 by [43, Theorem 12.4], the remaining assertions follow by Poitou-Tate duality. □

By [43, Section 13.9], there exists a non-zero element

$$\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}} \in H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}} R(\mathfrak{a}_x)_{\mathrm{Iw}}[1/p]$$

such that

$$\psi(\mathfrak{a}_x)(\mathbf{z}^x_{\Sigma,\mathrm{Iw}}) = \prod_{\ell \in \Sigma^{(p)}} \mathrm{Eul}_\ell(T(\mathfrak{a}_x)_{\mathrm{Iw}})\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}}. \tag{3.3.3.1}$$

This entails that

$$\lambda(\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}} \otimes 1) = \mathbf{z}(f_\lambda)_{\mathrm{Iw}}$$

for all modular specializations $\lambda : \mathbf{T}_\Sigma \longrightarrow \bar{\mathbb{Z}}_p$ factoring through $R(\mathfrak{a}_x)$. Concretely, if M_λ denote the motive attached to f_λ, then for all integers n, all modular specializations $\lambda : R(\mathfrak{a}_x)_{\mathrm{Iw}} \longrightarrow F_{\mathfrak{p}}[G_n]$ and all characters $\chi \in \hat{G}_n$, the element $\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}}$ satisfies

$$\mathrm{per}_{\mathbb{C}}(\mathrm{per}_p^{-1}(\mathrm{loc}_p(\lambda(\mathbf{z}(\mathfrak{a}_x)))) \otimes 1) = L_{\{p\}}(M^*_\lambda(1), \chi, r).$$

The $R(\mathfrak{a}_x)_{\mathrm{Iw}}$-module generated by $\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}}$ is included in $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})$ after localization at all height one prime, so if $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})$ has finite projective dimension as $R(\mathfrak{a}_x)_{\mathrm{Iw}}$-module (for instance if $R(\mathfrak{a}_x)_{\mathrm{Iw}}$ is a regular local ring), then $\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}}$ belongs to $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})$. It is not obvious to this author that this remains true otherwise.

As $\mathbf{z}(f_\lambda)_{\mathrm{Iw}}$ is non-zero, the $R(\mathfrak{a}_x)_{\mathrm{Iw}}$-module $\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}}$ generates inside $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})[1/p]$ is free of rank 1 by lemma 3.3.3. The morphism

$$M(\mathfrak{a}_x)_{\mathrm{Iw}} \otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}} R(\mathfrak{a}_x)_{\mathrm{Iw}}[1/p] \longrightarrow H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}} R(\mathfrak{a}_x)_{\mathrm{Iw}}[1/p]$$

which sends $\delta(\mathfrak{a}_x)_{\mathrm{Iw}}$ to $\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}}$ defines by lifting a morphism of complexes

$$Z(\mathfrak{a}_x)_{\mathrm{Iw}} : M(\mathfrak{a}_x)_{\mathrm{Iw}} \otimes R(\mathfrak{a}_x)_{\mathrm{Iw}}[1/p] \longrightarrow \mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})[1] \otimes R(\mathfrak{a}_x)_{\mathrm{Iw}}[1/p]. \tag{3.3.3.2}$$

In the above, we view $M(\mathfrak{a}_x)_{\mathrm{Iw}}$ and $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a}_x)_{\mathrm{Iw}})$ as complexes concentrated in degree 0 and all tensor products are over $R(\mathfrak{a}_x)_{\mathrm{Iw}}$.

Definition 3.3.8. *Define $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})$ to be the free $R(\mathfrak{a}_x)_{\mathrm{Iw}}$-module of rank 1*

$$\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}}) = \mathscr{X}(T(\mathfrak{a}_x)_{\mathrm{Iw}})^{-1} \otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}} \mathrm{Det}^{-1}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}} M(\mathfrak{a}_x)_{\mathrm{Iw}}. \tag{3.3.3.3}$$

Let $\psi : R(\mathfrak{a}_x)_{\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$ be a specialization induced by a $\mathbb{Z}_p$-algebras morphism $R(\mathfrak{a}_x) \longrightarrow S$ with values in a flat reduced $\mathbb{Z}_p$-algebra and such that

$$\mathbf{z}_\psi \overset{\mathrm{def}}{=} \psi(\mathbf{z}((\mathfrak{a}_x)_{\mathrm{Iw}})) \in H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_\psi) \otimes_{S_{\mathrm{Iw}}} S_{\mathrm{Iw}}[1/p]$$

is non-zero. The morphism sending $\delta_{\psi,\mathrm{Iw}}$ to $\mathbf{z}_\psi$ consequently gives as above a morphism

$$Z_\psi : M_\psi \otimes_{S_{\mathrm{Iw}}} S_{\mathrm{Iw}}[1/p] \longrightarrow \mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], T_\psi)[1] \otimes_{S_{\mathrm{Iw}}} S_{\mathrm{Iw}}[1/p].$$

Here again, it is not necessary to invert p when S happens to be a discrete valuation ring.

Definition 3.3.9. *Define $\Delta_{S_{\mathrm{Iw}}}(T_\psi)$ to be the free S_{Iw}-module of rank 1*

$$\Delta_{S_{\mathrm{Iw}}}(T_\psi) = \mathscr{X}(T_\psi)^{-1} \otimes_{S_{\mathrm{Iw}}} \mathrm{Det}^{-1}_{S_{\mathrm{Iw}}} M_\psi. \tag{3.3.3.4}$$

After inverting p, the modules $\Delta_{S_{\mathrm{Iw}}}(T_\psi)$ (including when ψ is the identity) are canonically isomorphic to the cone of a morphism of complexes. In general form, this is expressed by the following proposition.

Proposition 3.3.10. *The $R(\mathfrak{a}_x)_{\mathrm{Iw}}$-module $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})$ is equipped with a canonical morphism*

$$\mathrm{triv} : \Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}} T((\mathfrak{a}_x)_{\mathrm{Iw}}) \subset \Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}} \mathrm{Frac}(R(\mathfrak{a}_x)_{\mathrm{Iw}}) \overset{\mathrm{can}}{\simeq} \mathrm{Frac}(R(\mathfrak{a}_x)_{\mathrm{Iw}}).$$

More generally, if $\psi : R(\mathfrak{a}_x)_{\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$ is a specialization induced by a $\mathbb{Z}_p$-algebras morphism $R(\mathfrak{a}_x) \longrightarrow S$ with values in a flat reduced $\mathbb{Z}_p$-algebra and such that $\mathbf{z}_\psi$ is non-zero, then the S_{Iw}-module $\Delta_{S_{\mathrm{Iw}}}(T_\psi)$ (which we also denote by Δ_ψ for brevity) is endowed with a canonical trivialization isomorphism

$$\mathrm{triv}_\psi : \Delta_\psi \subset \Delta_\psi \otimes_{S_{\mathrm{Iw}}} Q(S_{\mathrm{Iw}}) \overset{\mathrm{can}}{\simeq} Q(S_{\mathrm{Iw}}).$$

For all ψ as above, denote by $\mathrm{triv}_\psi(\Delta_\psi) \subset Q(S_{\mathrm{Iw}})$ the image of $\Delta_\psi \subset \Delta_\psi \otimes_{S_{\mathrm{Iw}}} Q(S_{\mathrm{Iw}})$ through triv_ψ. If the diagram

$$\begin{array}{ccc} R(\mathfrak{a}_x)_{\mathrm{Iw}} & \xrightarrow{\ \xi\ } & S'_{\mathrm{Iw}} \\ {\scriptstyle\psi}\downarrow & \nearrow_{\phi} & \\ S_{\mathrm{Iw}} & & \end{array}$$

of reduced flat $\mathbb{Z}_p$-algebra quotients of $R(\mathfrak{a}_x)_{\mathrm{Iw}}$ is commutative, then there is a canonical isomorphism

$$\phi^{\Delta}:\Delta_{\psi}\otimes_{S,\phi}S'\overset{\text{can}}{\simeq}\Delta_{\xi}$$

compatible with triv_{ψ} *and* triv_{ξ} *in the sense that the rightmost downward arrow of the diagram*

$$\begin{array}{ccc}\Delta_{\psi} & \longrightarrow & \mathrm{triv}_{\psi}(\Delta_{\psi})\\ {\scriptstyle \phi^{\Delta}(-\otimes_{S,\phi}S')}\downarrow & & \downarrow{\scriptstyle\phi}\\ \Delta_{\xi} & \longrightarrow & \mathrm{triv}_{\xi}(\Delta_{\xi})\end{array}\tag{3.3.3.5}$$

exists and makes the diagram commutative.

Proof. By lemma 3.3.3, the complex $\mathrm{Cone}\, Z(\mathfrak{a}_x)_{\mathrm{Iw}}$ of $\mathrm{Frac}(R(\mathfrak{a}_x)_{\mathrm{Iw}})$-vector spaces is acyclic and there is thus a canonical isomorphism

$$\mathrm{Det}_{\mathrm{Frac}(R(\mathfrak{a}_x)_{\mathrm{Iw}})}\mathrm{Cone}\, Z(\mathfrak{a}_x)_{\mathrm{Iw}}\overset{\text{can}}{\simeq}\mathrm{Frac}(R(\mathfrak{a}_x)_{\mathrm{Iw}}).\tag{3.3.3.6}$$

As

$$\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})=\mathscr{X}(T(\mathfrak{a}_x)_{\mathrm{Iw}})^{-1}\otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}\mathrm{Det}^{-1}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}M(\mathfrak{a}_x)_{\mathrm{Iw}}$$

satisfies

$$\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})\otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}\mathrm{Frac}(R(\mathfrak{a}_x)_{\mathrm{Iw}})\overset{\text{can}}{\simeq}\mathrm{Det}_{\mathrm{Frac}(R(\mathfrak{a}_x)_{\mathrm{Iw}})}\mathrm{Cone}\, Z(\mathfrak{a}_x)_{\mathrm{Iw}}\tag{3.3.3.7}$$

by (3.1.3.2), the inclusion $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})$ inside $\mathrm{Det}_{\mathrm{Frac}(R(\mathfrak{a}_x)_{\mathrm{Iw}})}\mathrm{Cone}\, Z(\mathfrak{a}_x)_{\mathrm{Iw}}$ composed with the isomorphism (3.3.3.6) defines the morphism triv.

Now let ψ be a specialization as in the statement of the proposition. Then $H^2_{\mathrm{et}}(\mathbb{Z}[1/p],T_{\psi})$ is torsion as $\mathbb{Z}_p$-module so the complex $\mathrm{Cone}\, Z_{\psi}$ of $Q(S_{\mathrm{Iw}})$-modules is acyclic. Hence (3.1.3.2) and acyclicity define two canonical isomorphisms

$$\Delta_{S_{\mathrm{Iw}}}(T_{\psi})\otimes_{S_{\mathrm{Iw}}}Q(S_{\mathrm{Iw}})\overset{\text{can}}{\simeq}\mathrm{Det}_{Q(S_{\mathrm{Iw}})}\mathrm{Cone}\, Z_{\psi}\overset{\text{can}}{\simeq}Q(S_{\mathrm{Iw}}).$$

Composed with the inclusion $\Delta_{S_{\mathrm{Iw}}}(T_{\psi})\subset\Delta_{S_{\mathrm{Iw}}}(T_{\psi})\otimes_{S_{\mathrm{Iw}}}Q(S_{\mathrm{Iw}})$, this defines triv_{ψ}.

In order to prove the remaining statements, it is enough to prove that there exists a canonical isomorphism

$$\psi^{\Delta}:\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})\otimes_{R(\mathfrak{a}_x),\psi}S\overset{\text{can}}{\simeq}\Delta_{\psi}$$

compatible with triv and triv_{ψ} for all ψ as in the statement of the proposition. By construction, there exists a modular specialization λ factoring through ψ so $\mathrm{rank}_{S_{\mathrm{Iw}}}T_{\psi}^{I_{\ell}}$ is equal to $\mathrm{rank}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}T(\mathfrak{a}_x)_{\mathrm{Iw}}^{I_{\ell}}$ for all $\ell\in\Sigma^{(p)}$. Proposition 3.1.2 then yields a canonical isomorphism

$$\mathscr{X}(T(\mathfrak{a}_x)_{\mathrm{Iw}})\otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}S_{\mathrm{Iw}}\overset{\text{can}}{\simeq}\mathscr{X}(T_{\psi}).$$

By definition, $\mathbf{z}_{\psi}$ is the specialization of $\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}}$ and δ_{ψ} coincides with $\delta_{\psi}^{\mathfrak{a}_x}$ by proposition 3.2.3. Hence, there is a canonical isomorphism

$$\psi^{\Delta}:\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})\otimes_{R(\mathfrak{a}_x),\psi}S\overset{\text{can}}{\simeq}\Delta_{S_{\mathrm{Iw}}}(T_{\psi}).$$

□

A consequence of the results of subsection 3.1.3 and of proposition 3.2.3 is the compatibility of $\Delta_{\mathbf{T}^{x}_{\Sigma,\mathrm{Iw}}}(T^{x}_{\Sigma,\mathrm{Iw}})$ with $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})$.

Proposition 3.3.11. *The map $\psi(\mathfrak{a}_x)$ induces a canonical isomorphism*

$$\Delta_{\mathbf{T}^{x}_{\Sigma,\mathrm{Iw}}}\otimes_{\mathbf{T}^{x}_{\Sigma,\mathrm{Iw}}}R(\mathfrak{a}_x)_{\mathrm{Iw}}\overset{\text{can}}{\simeq}\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})$$

compatible with triv_{Σ} *and* triv.

Proof. The isomorphism (3.2.2.5) of proposition 3.2.3 induces a canonical isomorphism

$$\operatorname{Det}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(M^x_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}R(\mathfrak{a}_x)_{\mathrm{Iw}})\otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}\operatorname{Det}^{-1}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}M(\mathfrak{a}_x)_{\mathrm{Iw}}\overset{\mathrm{can}}{\simeq}\operatorname{Det}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}[R(\mathfrak{a}_x)_{\mathrm{Iw}}\overset{d}{\longrightarrow}R(\mathfrak{a}_x)_{\mathrm{Iw}}]$$

where the complex $[R(\mathfrak{a}_x)_{\mathrm{Iw}}\overset{d}{\longrightarrow}R(\mathfrak{a}_x)_{\mathrm{Iw}}]$ is placed in degree 0 and 1 and d is multiplication by

$$\prod_{\ell\in\Sigma}\operatorname{Eul}_\ell(T(\mathfrak{a}_x)_{\mathrm{Iw}}).$$

Hence

$$\operatorname{Det}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(M^x_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}R(\mathfrak{a}_x)_{\mathrm{Iw}})\otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}\operatorname{Det}^{-1}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}M(\mathfrak{a}_x)_{\mathrm{Iw}}\overset{\mathrm{can}}{\simeq}\bigotimes_{\ell\in\Sigma}\operatorname{Det}^{-1}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}R(\mathfrak{a}_x)_{\mathrm{Iw}}/\operatorname{Eul}_\ell(T(\mathfrak{a}_x)_{\mathrm{Iw}})$$

where each of the $R(\mathfrak{a}_x)_{\mathrm{Iw}}/\operatorname{Eul}_\ell(T(\mathfrak{a}_x)_{\mathrm{Iw}})$ is viewed as a complex in degree 0. On the other hand, the definition and base-change property of $\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],-)$ induces a canonical isomorphism between

$$\operatorname{Det}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}\left(\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],T^x_{\Sigma,\mathrm{Iw}})\overset{\mathrm{L}}{\otimes}_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}R(\mathfrak{a}_x)_{\mathrm{Iw}}\right)\otimes_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}\operatorname{Det}^{-1}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p],T(\mathfrak{a}_x)_{\mathrm{Iw}})$$

and

$$\bigotimes_{\ell\in\Sigma}\mathscr{X}_\ell(T(\mathfrak{a}_x)_{\mathrm{Iw}})^{-1}\overset{\mathrm{can}}{\simeq}\bigotimes_{\ell\in\Sigma}\operatorname{Det}_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}R(\mathfrak{a}_x)_{\mathrm{Iw}}/\operatorname{Eul}_\ell(T(\mathfrak{a}_x)_{\mathrm{Iw}})$$

where each of the $R(\mathfrak{a}_x)_{\mathrm{Iw}}/\operatorname{Eul}_\ell(T(\mathfrak{a}_x)_{\mathrm{Iw}})$ is again seen as a complex in degree 0. Putting these isomorphism together yields a canonical isomorphism

$$\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}\otimes_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}R(\mathfrak{a}_x)_{\mathrm{Iw}}\overset{\mathrm{can}}{\simeq}\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}}).\tag{3.3.3.8}$$

The trivialization of $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})$ induced by triv_Σ and this isomorphism comes from the morphism sending $\psi(\mathfrak{a}_x)(\delta^x_{\Sigma,\mathrm{Iw}})$ to $\psi(\mathfrak{a}_x)(\mathbf{z}^x_{\Sigma,\mathrm{Iw}})$ and hence from the morphism verifying

$$\delta(\mathfrak{a}_x)_{\mathrm{Iw}}\prod_{\ell\in\Sigma^{(p)}}\operatorname{Eul}_\ell(T(\mathfrak{a}_x)_{\mathrm{Iw}})\mapsto\mathbf{z}(\mathfrak{a}_x)_{\mathrm{Iw}}\prod_{\ell\in\Sigma^{(p)}}\operatorname{Eul}_\ell(T(\mathfrak{a}_x)_{\mathrm{Iw}}).$$

This is also the trivialization induced by $Z(\mathfrak{a}_x)_{\mathrm{Iw}}$ so the compatibility of (3.3.3.8) with triv and triv_Σ is proved. □

We conclude this subsection with a well-known computation relating $\mathrm{triv}_\psi(\Delta_\psi)$ with the invariants appearing in other formulation of the Iwasawa Main Conjecture when S_{Iw} is a normal Cohen-Macaulay local ring (keeping the notations of proposition 3.3.10).

Lemma 3.3.12. *Let $\psi:R(\mathfrak{a}_x)\longrightarrow S$ be a specialization with values in a normal Cohen-Macaulay ring and such that $\mathbf{z}_\psi$ is non-zero. Then*

$$\mathrm{triv}_\psi(\Delta_\psi)=\frac{\operatorname{char}_{S_{\mathrm{Iw}}}H^2_{\mathrm{et}}(\mathbb{Z}[1/p],T_\psi)}{\operatorname{char}_{S_{\mathrm{Iw}}}H^1_{\mathrm{et}}(\mathbb{Z}[1/p],T_\psi)/\mathbf{z}_\psi}S_{\mathrm{Iw}}.\tag{3.3.3.9}$$

Proof. As S is a Cohen-Macaulay ring, invertible ideals are uniquely characterized by their localizations at height 1 prime. Because S is furthermore normal, we may replace S_{Iw} by one of its localization at A a discrete valuation ring in the proof of (3.3.3.9). Then $\mathbf{z}_\psi$ belongs to $H^1_{\mathrm{et}}(\mathbb{Z}[1/p],T_\psi)$ and the complex

$$\operatorname{Cone}Z_\psi=\operatorname{Cone}(M_\psi\longrightarrow\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p],T_\psi)[1])$$

is a perfect complex of A-modules concentrated in degree 0 and 1 with torsion cohomology groups $H^1_{\mathrm{et}}(\mathbb{Z}[1/p],T_\psi)/\mathbf{z}_\psi$ and $H^2_{\mathrm{et}}(\mathbb{Z}[1/p],T_\psi)$. If M is a torsion A-module, the image of $\operatorname{Det}^{-1}_A M$ inside $\operatorname{Frac}(A)$ through the canonical isomorphism $\operatorname{Det}_{\operatorname{Frac}(A)}0\overset{\mathrm{can}}{\simeq}\operatorname{Frac}(A)$ is equal to $(\operatorname{char}_A M)A$ so the statement of the lemma follows. □

The typical outcome of the method of Euler systems, on which we rely, is that the numerator of the left-hand side of (3.3.3.9) divides its denominator or equivalently that $\operatorname{triv}_\psi(\Delta_\psi^{-1})$ is included in S_{Iw}.

3.3.4 Coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and $R(\mathfrak{a})_{\mathrm{Iw}}$

Lemma 3.3.13. *The complex $\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})$ is a perfect complex of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-modules with trivial cohomology outside degree 1, 2. After tensor product with $Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})$, $H^1_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})$ becomes free of rank 1 and $H^2_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})$ vanishes.*

The same assertions hold for $\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],T(\mathfrak{a})_{\mathrm{Iw}})$ after replacing $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ by $R(\mathfrak{a})_{\mathrm{Iw}}$ and $T_{\Sigma,\mathrm{Iw}}$ by $T(\mathfrak{a})_{\mathrm{Iw}}$.

Proof. By irreducibility of $\bar\rho$, $H^0_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})$ vanishes. The functor $\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],-)$ preserves perfect complexes and commutes with base change of ring of coefficients. Applying the latter property to $\psi_x:\mathbf{T}_{\Sigma,\mathrm{Iw}}\longrightarrow\mathbf{T}^x_\Sigma$ shows first that $H^3_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})$ vanishes, then that $H^2_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})$ is $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-torsion and finally that $H^1_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})$ is torsion-free. Pick a minimal prime $\mathfrak{p}$ of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$. After tensor product first with $\mathbf{T}_{\Sigma,\mathrm{Iw}}/\mathfrak{p}$ then with $\operatorname{Frac}(\mathbf{T}_{\Sigma,\mathrm{Iw}}/\mathfrak{p})$, the computation of the Euler-Poincaré characteristic shows that $H^1_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})$ is of rank 1. As $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is reduced, this establishes that $H^1_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})$ is free of rank 1.

The proofs for $\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],T(\mathfrak{a})_{\mathrm{Iw}})$ are similar but easier. □

We write $Z(\mathfrak{a})_{\mathrm{Iw}}$ for the $R(\mathfrak{a})_{\mathrm{Iw}}$-module

$$Z(\mathfrak{a})_{\mathrm{Iw}}=\left(Z_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}R(\mathfrak{a})_{\mathrm{Iw}}\right)\otimes_{R(\mathfrak{a})_{\mathrm{Iw}}}\bigotimes_{\ell\in\Sigma^{(p)}}\operatorname{Det}_{R(\mathfrak{a})_{\mathrm{Iw}}}\left(M(\mathfrak{a})_{\mathrm{Iw}}/\pi_{\Sigma,\Sigma(\mathfrak{a})}(M_{\Sigma,\mathrm{Iw}})\right). \quad (3.3.4.1)$$

By construction, $Z(\mathfrak{a})_{\mathrm{Iw}}$ is free module of rank 1 which we may view as a submodule of

$$H^1_{\mathrm{et}}(\mathbb{Z}[1/\Sigma],\tilde H^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)},\mathbb{Z}_p)_{\mathfrak{m}_{\bar\rho}})\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}R(\mathfrak{a})_{\mathrm{Iw}}$$

after the trivialization of $\operatorname{Det}_{R(\mathfrak{a})_{\mathrm{Iw}}}M(\mathfrak{a})_{\mathrm{Iw}}/\pi_{\Sigma,\Sigma(\mathfrak{a})}(M_{\Sigma,\mathrm{Iw}})$ induced by

$$\operatorname{Det}_{R(\mathfrak{a})_{\mathrm{Iw}}}\left(\frac{M(\mathfrak{a})_{\mathrm{Iw}}}{\pi_{\Sigma,\Sigma(\mathfrak{a})}(M_{\Sigma,\mathrm{Iw}})}\right)\subset\operatorname{Det}_{\operatorname{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})}\left(\frac{M(\mathfrak{a})_{\mathrm{Iw}}}{\pi_{\Sigma,\Sigma(\mathfrak{a})}(M_{\Sigma,\mathrm{Iw}})}\right)\otimes\operatorname{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})\overset{\mathrm{can}}{\simeq}\operatorname{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})$$

Definition 3.3.14. *Let ψ be either a specialization $\psi:\mathbf{T}_{\Sigma,\mathrm{Iw}}\longrightarrow S$ with values in a reduced, flat $\mathbb{Z}_p$-algebra or in a torsion $\mathbb{Z}_p$-algebra or a specialization $\psi:R(\mathfrak{a})_{\mathrm{Iw}}\longrightarrow S$ with values in a domain. Assume in both cases that $\mathbf{z}_\psi\neq0$.*

In the first case, define Z_ψ to be $Z_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}S$. In the second case, there exists a specialization $\mathbf{T}_{\Sigma,\mathrm{Iw}}\longrightarrow S$ factoring through ψ. Define Z_ψ by

$$Z_\psi=\left(Z_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}S\right)\otimes_S\bigotimes_{\ell\in\Sigma}\operatorname{Det}_S\left(M_\psi/M_\psi^\Sigma\right).$$

We say that a specialization ψ is shimmering if it is as in definition 3.3.14. Note that shimmering specializations of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ may have values in artinian rings.

Definition 3.3.15. *Define $\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})$ to be the free $\mathbf{T}_{\Sigma,\mathrm{Iw}}$-module of rank 1*

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})=\operatorname{Det}^{-1}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}\operatorname{Det}^{-1}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}Z_{\Sigma,\mathrm{Iw}}.$$

Define $\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})$ to be the free $R(\mathfrak{a})_{\mathrm{Iw}}$-module of rank 1

$$\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})=\mathscr{X}(T(\mathfrak{a})_{\mathrm{Iw}})^{-1}\otimes_{R(\mathfrak{a})_{\mathrm{Iw}}}\operatorname{Det}^{-1}_{R(\mathfrak{a})_{\mathrm{Iw}}}Z(\mathfrak{a})_{\mathrm{Iw}}.$$

If ψ is a shimmering specialization of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$, define $\Delta_S(T_\psi)$ to be the S-module of rank 1

$$\Delta_S(T_\psi)=\operatorname{Det}_S^{-1}\mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma],T_\psi)\otimes_S\operatorname{Det}_S^{-1}Z_\psi.$$

If ψ is a shimmering specialization of $R(\mathfrak{a})_{\mathrm{Iw}}$, define $\Delta_S(T_\psi)$ to be the S-module of rank 1

$$\Delta_S(T_\psi)=\mathscr{X}(T_\psi)^{-1}\otimes\operatorname{Det}_S^{-1}Z_\psi.$$

For shimmering specializations ψ factoring through $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ or $R(\mathfrak{a}_x)_{\mathrm{Iw}}$, there are currently two potentially conflicting definitions of Z_ψ and $\Delta_{S_{\mathrm{Iw}}}(T_\psi)$. This conflict is resolved by the following theorem which shows that the fundamental lines $\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})$ and $\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})$ satisfy the base-change property forming one of the central part of the ETNC.

Theorem 3.3.16. *There are canonical isomorphisms*

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}},\psi}\mathbf{T}^x_{\Sigma,\mathrm{Iw}}\overset{\mathrm{can}}{\simeq}\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})$$

and

$$\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})\otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\psi(\mathfrak{a}_x)_{\mathrm{Iw}}}R(\mathfrak{a}_x)_{\mathrm{Iw}}\overset{\mathrm{can}}{\simeq}\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}}).$$

If ψ is a shimmering specialization, then there is a canonical isomorphism

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}S\overset{\mathrm{can}}{\simeq}\Delta_S(T_\psi)\text{ or }\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})\otimes_{R(\mathfrak{a})_{\mathrm{Iw}}}S\overset{\mathrm{can}}{\simeq}\Delta_S(T_\psi).$$

depending on whether ψ is a specialization of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ or $R(\mathfrak{a})_{\mathrm{Iw}}$.

Proof. The first assertion follows immediately from the compatibility with base-change of cohomology with compact support, the base-change property of algebraic determinants over local domains and the existence of the zeta morphisms (3.3.2.2) and (3.3.3.2).

We prove the second assertion for a specialization ψ of $R(\mathfrak{a})_{\mathrm{Iw}}$. Assume first that the equality

$$\operatorname{rank}_S T_\psi^{I_\ell}=\operatorname{rank}_{R(\mathfrak{a})_{\mathrm{Iw}}}T(\mathfrak{a})_{\mathrm{Iw}}^{I_\ell}$$

holds for all $\ell\in\Sigma^{(p)}$. As in the proof of proposition 3.3.10, proposition 3.1.6 then implies that $\mathscr{X}(T(\mathfrak{a})_{\mathrm{Iw}})\otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\psi}S$ is equal to $\mathscr{X}(T_\psi)$ and proposition 3.2.3 implies that $Z(\mathfrak{a})_{\mathrm{Iw}}\otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\psi}S$ is equal to Z_ψ. The claim is thus established in this case.

Now assume that there exists an $\ell\in\Sigma^{(p)}$ such that

$$\operatorname{rank}_S T_\psi^{I_\ell}>\operatorname{rank}_{R(\mathfrak{a})_{\mathrm{Iw}}}T(\mathfrak{a})_{\mathrm{Iw}}^{I_\ell}.$$

The discussion in the proof of proposition 3.1.6 then implies that $T(\mathfrak{a}_x)^{I_\ell}_{\mathrm{Iw}}$ is of rank 1 while T_ψ is unramified at ℓ. Since cohomology with compact support commutes with base-change and since $\psi(\delta_{\Sigma,\mathrm{Iw}})=\delta^{\Sigma}_{\psi,\mathrm{Iw}}$ by definition, the contribution of such ℓ to the definitions of $\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})$ and $\Delta_S(T_\psi)$ is purely local at ℓ. Hence, we may further assume that there is a single such ℓ. Denote by α the unique eigenvalue of $\mathrm{Fr}(\ell)$ acting on T_ψ such that α is the image through ψ of $\mathrm{Fr}(\ell)$ acting on $T(\mathfrak{a})_{\mathrm{Iw}}$ and denote by β its other eigenvalue. Then

$$\left(\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})\otimes_{R(\mathfrak{a}),\psi}S\right)\otimes_S\Delta_S(T_\psi)^{-1}$$

decomposes into

$$\left(\mathscr{X}_\ell(T(\mathfrak{a})_{\mathrm{Iw}})\otimes_{R(\mathfrak{a}),\psi}S\right)^{-1}\otimes_S\mathscr{X}_\ell(T_\psi)\overset{\mathrm{can}}{\simeq}\operatorname{Det}_S^{-1}S/\beta$$

and

$$\operatorname{Det}_S^{-1}(M(\mathfrak{a})_{\mathrm{Iw}}\otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\psi}S)\otimes\operatorname{Det}_S M_\psi.$$

According to proposition 3.2.3, $M(\mathfrak{a})_{\mathrm{Iw}}\otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\psi}S$ is a lattice inside M_ψ with index equal to β. Hence

$$\operatorname{Det}_S^{-1}(M(\mathfrak{a})_{\mathrm{Iw}}\otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\psi}S)\otimes\operatorname{Det}_S M_\psi\overset{\mathrm{can}}{\simeq}\operatorname{Det}_S S/\beta$$

and finally

$$\left(\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})\otimes_{R(\mathfrak{a}_x),\psi}S\right)\otimes_S\Delta_S(T_\psi)^{-1}\overset{\mathrm{can}}{\simeq}S.$$

Finally, if ψ is a shimmering specialization of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$, then $\mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma],T_{\Sigma,\mathrm{Iw}})\overset{\mathrm{L}}{\otimes}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}S$ is isomorphic to $\mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma],T_\psi)$ and $Z_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}S$ is equal to Z_ψ by definition. □

Remark: Though it follows from the definition of $\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})$, the compatibility of fundamental lines with shimmering specializations of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ which are not classical is far from formal. Consider for instance the commutative diagram

$$\begin{array}{ccc} \mathbf{T}_{\Sigma,\mathrm{Iw}} & & \\ {\scriptstyle\psi_x}\downarrow & \searrow^{\pi} & \\ \mathbf{T}^x_{\Sigma,\mathrm{Iw}} & \xrightarrow{\pi} & \mathbf{T}_{\Sigma,\mathrm{Iw}}/\mathfrak{m}^n \end{array}$$

in which x is a classical prime and π is the canonical projection (which is shimmering for large enough n). Then the compatibility of fundamental lines with ψ_x and π asserts that the reduction modulo $\mathfrak{m}^n$ of Kato's Euler system is equal to the image of the zeta element in completed cohomology of definition 3.3.1 and so ultimately relies on the $G_{\mathbb{Q},\Sigma}$-equivariant isomorphism

$$\tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)},\mathcal{O})_{\mathfrak{m}_{\bar\rho}}/\mathfrak{m}^n_{\mathcal{O}}\tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)},\mathcal{O})_{\mathfrak{m}_{\bar\rho}}\simeq\tilde{\tilde{H}}^1_{\mathrm{et}}(U_1(N(\Sigma))^{(p)},\mathcal{O}/\mathfrak{m}^n_{\mathcal{O}})_{\mathfrak{m}_{\bar\rho}}.$$

In particular, a variant of theorem 3.3.16 for the fundamental lines of [41, Conjecture 3.2], in which the role completed cohomology plays in this manuscript is played by $T_{\Sigma,\mathrm{Iw}}$ or $T(\mathfrak{a})_{\mathrm{Iw}}$ themselves, would be false for the natural specialization maps on $T_{\Sigma,\mathrm{Iw}}$ or $T(\mathfrak{a})_{\mathrm{Iw}}$. The proof of theorem 3.3.16 gives an indication of where to look for a counterexample: fix $\ell\nmid p$ a prime in Σ and consider the non-classical specialization corresponding to the intersection of two irreducible components of $\operatorname{Spec}\mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$ for which the generic rank of I_ℓ-invariants are distinct (see [29, Section 4.2.2] for a numerical example of this construction).

3.4 Statement of the conjectures

We are in position to state the Equivariant Tamagawa Number Conjectures with coefficients in Hecke algebras. In addition to the full form of the conjecture, we consider a weaker form whose main feature of interest is that it is amenable to proof.

3.4.1 Conjectures with coefficients in $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$

Conjecture 3.4.1. *The trivialization morphism is an isomorphism*

$$\mathrm{triv}_\Sigma:\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})\simeq\mathbf{T}^x_{\Sigma,\mathrm{Iw}}.$$

A weaker form of conjecture 3.4.1, which we refer to as the weak ETNC with coefficients in $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ is as follows.

Conjecture 3.4.2. *The trivialization morphism induces an inclusion*

$$\mathrm{triv}_\Sigma:\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})^{-1}\hookrightarrow\mathbf{T}^x_{\Sigma,\mathrm{Iw}}.$$

It follows at once from proposition 3.3.5 that conjectures 3.4.1 and 3.4.2 are compatible with change of allowable set Σ and specializations.

Proposition 3.4.3. *Let $\Sigma\subset\Sigma'$ be an inclusion of allowable sets of primes. Conjecture 3.4.1 (resp. 3.4.2) is true for Σ if it is true for Σ'. If conjecture 3.4.2 is true for Σ' and conjecture 3.4.1 is true for Σ then, conjecture 3.4.1 is true for Σ'.*

If conjecture 3.4.1 is true, then triv_ψ *induces an isomorphism*

$$\mathrm{triv}_\psi:\Delta_\psi\simeq S_{\mathrm{Iw}}$$

for all $\psi:\mathbf{T}^x_{\Sigma,\mathrm{Iw}}\longrightarrow S_{\mathrm{Iw}}$ as in proposition 3.3.5. If conjecture 3.4.2 is true and there exists ψ as above such that triv_ψ *is an isomorphism*

$$\mathrm{triv}_\psi:\Delta_\psi\simeq S_{\mathrm{Iw}}$$

then conjecture 3.4.1 is true for Σ.

Proof. If $\mathrm{triv}_{\Sigma'}$ is either an isomorphism $\Delta_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}(T^x_{\Sigma',\mathrm{Iw}}) \simeq \mathbf{T}^x_{\Sigma',\mathrm{Iw}}$ or an embedding

$$\Delta_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}(T^x_{\Sigma',\mathrm{Iw}})^{-1} \hookrightarrow \mathbf{T}^x_{\Sigma',\mathrm{Iw}},$$

then the commutativity of (3.3.2.6) and the fact that a local morphism sends units to units and non-units to non-units implies that triv_{Σ} is respectively an isomorphism or an embedding.

Likewise, if $\mathrm{triv}_{\Sigma'}$ is an embedding $\Delta_{\mathbf{T}^x_{\Sigma',\mathrm{Iw}}}(T^x_{\Sigma',\mathrm{Iw}})^{-1} \hookrightarrow \mathbf{T}^x_{\Sigma',\mathrm{Iw}}$ and triv_{Σ} is an isomorphism $\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}}) \simeq \mathbf{T}^x_{\Sigma,\mathrm{Iw}}$, then (3.3.2.6) implies that $\mathrm{triv}_{\Sigma'}$ is an isomorphism.

The other assertions are proved exactly in the same way by appealing to the (easier) commutativity of the diagram (3.3.2.5) in place of the commutativity of the diagram (3.3.2.6). □

3.4.2 Conjectures with coefficients in $R(\mathfrak{a}_x)_{\mathrm{Iw}}$

Conjecture 3.4.4. *The trivialization morphism is an isomorphism*

$$\mathrm{triv} : \Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}}) \simeq R(\mathfrak{a}_x)_{\mathrm{Iw}}.$$

A weaker form of conjecture 3.4.4, which we refer to the weak ETNC with coefficients in $R(\mathfrak{a}_x)_{\mathrm{Iw}}$ is as follows.

Conjecture 3.4.5. *The trivialization morphism induces an inclusion*

$$\mathrm{triv} : \Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})^{-1} \hookrightarrow R(\mathfrak{a}_x)_{\mathrm{Iw}}.$$

It follows from proposition 3.3.10 and proposition 3.3.11 that both the ETNC and the weak ETNC with coefficients in $R(\mathfrak{a}_x)_{\mathrm{Iw}}$ are compatible with specializations and change of Σ.

Proposition 3.4.6. *If conjecture 3.4.4 (resp. conjecture 3.4.5) is true, then* triv_{ψ} *induces an isomorphism* $\Delta_{S_{\mathrm{Iw}}}(T_{\psi}) \simeq S_{\mathrm{Iw}}$ *(resp. an embedding* $\Delta^{-1}_{S_{\mathrm{Iw}}} \hookrightarrow S_{\mathrm{Iw}}$*) for all specializations* ψ *as in proposition 3.3.10. If conjecture 3.4.5 is true and if there exists a* ψ *such that* triv_{ψ} *induces an isomorphism* $\Delta_{S_{\mathrm{Iw}}}(T_{\psi}) \simeq S_{\mathrm{Iw}}$*, then conjecture 3.4.4 is true.*

Conjecture 3.4.1 (resp. conjecture 3.4.2) implies conjecture 3.4.4 (resp. conjecture 3.4.5). If conjecture 3.4.2 and conjecture 3.4.4 hold, then conjecture 3.4.1 holds.

Proof. If triv is either an isomorphism $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}}) \simeq R(\mathfrak{a}_x)_{\mathrm{Iw}}$ or an embedding

$$\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})^{-1} \hookrightarrow R(\mathfrak{a}_x)_{\mathrm{Iw}}$$

then the commutativity of (3.3.3.5) and the fact that a local morphism sends units to units and non-units to non-units implies that triv_{ψ} is respectively an isomorphism or an embedding.

Likewise, if triv is an embedding $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})^{-1} \hookrightarrow R(\mathfrak{a}_x)_{\mathrm{Iw}}$ and triv_{ψ} is an isomorphism $\Delta_{\psi} \simeq S_{\mathrm{Iw}}$, then (3.3.3.5) implies that triv is an isomorphism.

If triv_{Σ} is either an isomorphism $\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}}) \simeq \mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ or an embedding

$$\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})^{-1} \hookrightarrow \mathbf{T}^x_{\Sigma,\mathrm{Iw}},$$

then proposition 3.3.11 implies in a similar way that triv is either an isomorphism

$$\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}}) \simeq R(\mathfrak{a}_x)_{\mathrm{Iw}}$$

or an embedding $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})^{-1} \hookrightarrow R(\mathfrak{a}_x)_{\mathrm{Iw}}$. Conversely, proposition 3.3.11 implies that $\Delta_{\mathbf{T}^x_{\Sigma,\mathrm{Iw}}}(T^x_{\Sigma,\mathrm{Iw}})$ is isomorphic to $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$ if triv_{Σ} realizes an embedding of one into the other and if triv is an isomorphism $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}}) \simeq R(\mathfrak{a}_x)_{\mathrm{Iw}}$. □

An interesting corollary of proposition 3.4.6 is the compatibility of conjecture 3.4.1 with modular specialization, and hence with conjecture 2.3.1. Note that it is not a mere restatement of proposition 3.4.3 for $\psi = \lambda$, as the image of $\mathbf{z}^x_{\Sigma,\mathrm{Iw}}$ through λ is not $\mathbf{z}(f_{\lambda})_{\mathrm{Iw}}$.

Corollary 3.4.7. *Let $\lambda : \mathbf{T}_{\Sigma}^{x} \longrightarrow \bar{\mathbb{Q}}_p$ be a modular specialization. If conjecture 3.4.1 is true, then conjecture 2.3.3 for λ is true. If conjecture 3.4.2 is true and conjecture 2.3.3 is true for λ then conjecture 3.4.1 is true.*

Proof. This follows formally by two applications of proposition 3.4.6, first to $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})$ and λ, then to $\Delta_{\mathbf{T}^{x}_{\Sigma,\mathrm{Iw}}}(T^{x}_{\Sigma,\mathrm{Iw}})$ and $\Delta_{R(\mathfrak{a}_x)_{\mathrm{Iw}}}(T(\mathfrak{a}_x)_{\mathrm{Iw}})$. ☐

3.4.3 Conjectures with coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and $R(\mathfrak{a})_{\mathrm{Iw}}$

The complex $\mathrm{R}\Gamma_c(\mathbb{Z}[1/\Sigma], T_{\Sigma,\mathrm{Iw}})$ is of perfect amplitude $[0,3]$ and has trivial cohomology in degree 3 so admits a presentation $[C_0 \longrightarrow C_1 \longrightarrow C_2]$. Hence, the determinant of the complex $[\mathbf{T}_{\Sigma,\mathrm{Iw}} \oplus C_0 \longrightarrow C_1 \longrightarrow C_2]$ is canonically isomorphic to $\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})$ through the identification between $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and $Z_{\Sigma,\mathrm{Iw}}$ sending 1 to $\mathbf{z}_{\Sigma,\mathrm{Iw}}$.

After tensor product with $Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})$, the complex $[\mathbf{T}_{\Sigma,\mathrm{Iw}} \oplus C_0 \longrightarrow C_1 \longrightarrow C_2]$ becomes quasi-isomorphic to $[Q(\mathbf{T}_{\Sigma,\mathrm{Iw}}) \longrightarrow Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})]$ in degree 0 and 1 and with zero differential map. Hence, there is a canonical injection

$$\mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[\mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \mathbf{T}_{\Sigma,\mathrm{Iw}}] \hookrightarrow \Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}}) \otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}} Q(T_{\Sigma,\mathrm{Iw}}) \tag{3.4.3.1}$$

obtained by composing $\mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[\mathbf{T}_{\Sigma,\mathrm{Iw}} \to \mathbf{T}_{\Sigma,\mathrm{Iw}}] \subset \mathrm{Det}_{Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})}[Q(\mathbf{T}_{\Sigma,\mathrm{Iw}}) \to Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})]$ with the canonical isomorphism $\mathrm{Det}_{Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})}[Q(\mathbf{T}_{\Sigma,\mathrm{Iw}}) \to Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})] \simeq \Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}}) \otimes Q(T_{\Sigma,\mathrm{Iw}})$.

Conjecture 3.4.8. *Inside $\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}}) \otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}} Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})$, there is an equality*

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}}) = \mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[\mathbf{T}_{\Sigma,\mathrm{Iw}} \xrightarrow{0} \mathbf{T}_{\Sigma,\mathrm{Iw}}]$$

in which the complex on the right-hand side is concentrated in degree 0 and 1 and is seen inside $\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}}) \otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}} Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})$ through (3.4.3.1).

The exact same constructions and arguments as in the beginning of this subsection but over $R(\mathfrak{a})_{\mathrm{Iw}}$ yield a canonical injection

$$\mathrm{Det}_{R(\mathfrak{a})_{\mathrm{Iw}}}[R(\mathfrak{a})_{\mathrm{Iw}} \xrightarrow{0} R(\mathfrak{a})_{\mathrm{Iw}}] \hookrightarrow \Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}}) \otimes \mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}}). \tag{3.4.3.2}$$

Conjecture 3.4.9. *Inside $\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}}} \mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})$, there is an equality*

$$\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}}) = \mathrm{Det}_{R(\mathfrak{a})_{\mathrm{Iw}}}[R(\mathfrak{a})_{\mathrm{Iw}} \xrightarrow{0} R(\mathfrak{a})_{\mathrm{Iw}}]$$

in which the complex on the right-hand side is concentrated in degree 0 and 1 and is seen inside $\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}}} \mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})$ through (3.4.3.2).

More generally, there is a canonical injection

$$\mathrm{Det}_S[S \xrightarrow{0} S] \hookrightarrow \Delta_S(T_\psi) \otimes Q(S)$$

for all shimmering specialization $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow S$ with values in reduced ring and a canonical injection

$$\mathrm{Det}_S[S \xrightarrow{0} S] \hookrightarrow \Delta_S(T_\psi) \otimes \mathrm{Frac}(S)$$

for all shimmering specializations $\psi : R(\mathfrak{a})_{\mathrm{Iw}} \longrightarrow S$.

Conjecture 3.4.10. *Let $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow S$ be a shimmering specialization with values in a reduced ring. Inside $\Delta_S(T_\psi) \otimes_S Q(S)$, there is an equality*

$$\Delta_S(T_\psi) = \mathrm{Det}_S[S \xrightarrow{0} S].$$

When there exists a classical point of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ with reducible local Galois representation, then the quotient $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$ is a shimmering specialization, so the previous conjecture applies in particular to $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$. For the convenience of the reader, we state it explicitly.

Conjecture 3.4.11. *Inside* $\Delta_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}}(T^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}}Q(\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}})$*, there is an equality*

$$\Delta_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}}(T^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}})=\mathrm{Det}_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}}[\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}\xrightarrow{0}\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}]$$

in which the complex on the right-hand side is concentrated in degree 0 and 1 and is seen inside $\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})$ *through* (3.4.3.1).

As in the previous subsections, we consider the weaker conjectures.

Conjecture 3.4.12. *Inside* $\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})$*, there is an inclusion*

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})^{-1}\subset\mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[\mathbf{T}_{\Sigma,\mathrm{Iw}}\xrightarrow{0}\mathbf{T}_{\Sigma,\mathrm{Iw}}]$$

in which the complex on the right-hand side is concentrated in degree 0 and 1 and is seen inside $\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})\otimes_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}Q(\mathbf{T}_{\Sigma,\mathrm{Iw}})$ *through* (3.4.3.1)*. Equivalently, there exists a non-zero divisor* $x\in\mathbf{T}_{\Sigma,\mathrm{Iw}}$ *such that the equality*

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})^{-1}=\mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[x\mathbf{T}_{\Sigma,\mathrm{Iw}}\xrightarrow{0}\mathbf{T}_{\Sigma,\mathrm{Iw}}]$$

holds.

Conjecture 3.4.13. *Inside* $\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})\otimes_{R(\mathfrak{a})_{\mathrm{Iw}}}\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})$*, there is an inclusion*

$$\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})^{-1}\subset\mathrm{Det}_{R(\mathfrak{a})_{\mathrm{Iw}}}[R(\mathfrak{a})_{\mathrm{Iw}}\xrightarrow{0}R(\mathfrak{a})_{\mathrm{Iw}}]$$

in which the complex on the right-hand side is concentrated in degree 0 and 1 and is seen inside $\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})\otimes_{R(\mathfrak{a})_{\mathrm{Iw}}}\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})$ *through* (3.4.3.2)*. Equivalently, there exists a non-zero* $x\in R(\mathfrak{a})_{\mathrm{Iw}}$ *such that the equality*

$$\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}})^{-1}=\mathrm{Det}_{R(\mathfrak{a})_{\mathrm{Iw}}}[xR(\mathfrak{a})_{\mathrm{Iw}}\xrightarrow{0}R(\mathfrak{a})_{\mathrm{Iw}}]$$

holds.

Conjecture 3.4.14. *Let* $\psi:\mathbf{T}_{\Sigma,\mathrm{Iw}}\longrightarrow S$ *be a shimmering specialization with values in a reduced ring. Inside* $\Delta_S(T_\psi)\otimes_S Q(S)$*, there is an inclusion*

$$\Delta_S(T_\psi)^{-1}\subset\mathrm{Det}_S[S\xrightarrow{0}S].$$

Equivalently, there exists a non-zero divisor $x\in S$ *such that the equality*

$$\Delta_S(T_\psi)^{-1}=\mathrm{Det}_S[xS\xrightarrow{0}S].$$

holds.

As their counterparts after specialization at a classical prime, these conjectures are compatible with $\psi(\mathfrak{a})_{\mathrm{Iw}}:\mathbf{T}_{\Sigma,\mathrm{Iw}}\longrightarrow R(\mathfrak{a})_{\mathrm{Iw}}$.

Proposition 3.4.15. *Conjecture 3.4.8 (resp. conjecture 3.4.12) implies conjecture 3.4.9 (resp. conjecture 3.4.13).*

Proof. The proof is similar to that of proposition 3.3.11, but easier. ☐

The analogue of proposition 3.4.6 for shimmering specializations also holds. Because we lack a genuine zeta morphism with coefficients in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$, it is not quite formal even at modular specializations.

Proposition 3.4.16. *Let* ψ *be a shimmering specialization with values in a reduced ring. Conjecture 3.4.8 (resp. conjecture 3.4.12) implies conjecture 3.4.10 (resp. conjecture 3.4.14) for* T_ψ*.*

Proof. We prove the statement for a shimmering specialization $\psi : R(\mathfrak{a})_{\mathrm{Iw}} \longrightarrow S$ as the proof of a shimmering specialization of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is similar but easier. By theorem 3.3.16, there is a canonical isomorphism $\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}}) \otimes_\psi S \overset{\mathrm{can}}{\simeq} \Delta_\psi$. It is thus enough to show that the trivialization triv_ψ of $\Delta_S(T_\psi)$ is the same as the trivialization induced by the equality

$$\Delta_{R(\mathfrak{a})_{\mathrm{Iw}}}(T(\mathfrak{a})_{\mathrm{Iw}}) = \mathrm{Det}_{R(\mathfrak{a})_{\mathrm{Iw}}}[R(\mathfrak{a})_{\mathrm{Iw}} \longrightarrow R(\mathfrak{a})_{\mathrm{Iw}}],$$

change of ring of coefficients to S and the canonical isomorphism between $\mathrm{Det}_S[S \longrightarrow S]$ and S. The first trivialization compares the relative positions of $\mathscr{X}(T_\psi)$ and the inverse of the determinant of S. The conjectures respectively assert that these two modules are equal or that the latter is included in the former. The second trivialization identifies $\mathscr{X}(T_\psi)$ with the determinant of S placed in degree 1 and the conjectures then asserts that $S\mathbf{z}_\psi$ is identified with S or that it inverse is included in it. ☐

3.4.4 Comparison with classical Iwasawa theory

For the convenience of the reader, we explain in this subsection the translation of conjecture 3.4.9 into the language of classical Iwasawa theory, as in subsection 1.1.1.

Proposition 3.4.17. *Assume that $R(\mathfrak{a})_{\mathrm{Iw}}$ is a regular local ring and that conjecture 3.4.13 holds. Then the $R(\mathfrak{a})_{\mathrm{Iw}}$-module $H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})$ is torsion and the $R(\mathfrak{a})_{\mathrm{Iw}}$-module $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})$ is of rank 1. There exists a class which we denote in a slight abuse of notation by $\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}$ in $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})$ such that*

$$\mathrm{char}_{R(\mathfrak{a})_{\mathrm{Iw}}} H^2(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}}) | \, \mathrm{char}_{R(\mathfrak{a})_{\mathrm{Iw}}} \left(H^1(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})/\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}\right) \qquad (3.4.4.1)$$

and satisfying statement (ii) *of corollary 1.1.4. If moreover conjecture 3.4.9 holds, then*

$$\mathrm{char}_{R(\mathfrak{a})_{\mathrm{Iw}}} H^2(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}}) = \mathrm{char}_{R(\mathfrak{a})_{\mathrm{Iw}}} \left(H^1(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})/\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}\right) \qquad (3.4.4.2)$$

and the class $\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}$ satisfies statement (ii) *of corollary 1.1.4 with the divisibility* (1.1.1.2) *replaced by the equality*

$$\mathrm{char}_{\Lambda_{\mathrm{Iw}}} H^2(\mathbb{Z}[1/p], T(f) \otimes \Lambda_{\mathrm{Iw}}) = \mathrm{char}_{\Lambda_{\mathrm{Iw}}} \left(H^1(\mathbb{Z}[1/p], T(f) \otimes \Lambda_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}}\right). \qquad (3.4.4.3)$$

Proof. Let $\mathfrak{p}$ be a prime ideal of $R(\mathfrak{a})_{\mathrm{Iw}}$ of height 1, necessarily principal as $R(\mathfrak{a})_{\mathrm{Iw}}$ is regular, and denote by $R_\mathfrak{p}$ the localization of $R(\mathfrak{a})_{\mathrm{Iw}}$ at $\mathfrak{p}$. Because $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})_\mathfrak{p}$ is included inside $H^1(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a})_{\mathrm{Iw}})_\mathfrak{p}$, it is $\mathfrak{p}$-torsion free and thus free. As in [43, Section 13.9], $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})_\mathfrak{p}$ is thus free of rank 1 and contains the image of $\mathbf{z}_{\Sigma,\mathrm{Iw}}$ through the specialization

$$\psi(\mathfrak{a})_\mathfrak{p} : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow R_\mathfrak{p}$$

divided by the product of Euler factors at primes of bad reduction

$$\prod_{\ell \in \Sigma^{(p)}} \mathrm{Eul}_\ell(T(\mathfrak{a})_{\mathrm{Iw}})_\mathfrak{p}.$$

As $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})$ is of finite projective dimension as $R(\mathfrak{a})_{\mathrm{Iw}}$-module by the Auslander-Buchsbaum and Serre theorem, the image of $\mathbf{z}_{\Sigma,\mathrm{Iw}}$ through the specialization $\psi(\mathfrak{a})$ divided by the product of Euler factors at primes of bad reduction

$$\prod_{\ell \in \Sigma^{(p)}} \mathrm{Eul}_\ell(T(\mathfrak{a})_{\mathrm{Iw}})_\mathfrak{p},$$

which we denote by $\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}$, actually belongs to $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})$. Lemma 3.3.12 then applies so conjecture 3.4.13 becomes the statement

$$\mathrm{char}_{R(\mathfrak{a})_{\mathrm{Iw}}} H^2(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}}) | \, \mathrm{char}_{R(\mathfrak{a})_{\mathrm{Iw}}} \left(H^1(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})/\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}\right).$$

while conjecture 3.4.9 becomes the statement

$$\mathrm{char}_{R(\mathfrak{a})_{\mathrm{Iw}}} H^2(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}}) = \mathrm{char}_{R(\mathfrak{a})_{\mathrm{Iw}}} \left(H^1(\mathbb{Z}[1/p], T(\mathfrak{a})_{\mathrm{Iw}})/\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}\right).$$

Any classical specialization $\psi_f : R(\mathfrak{a})_{\mathrm{Iw}} \longrightarrow \mathcal{O}_{\mathrm{Iw}}$ of $R(\mathfrak{a})_{\mathrm{Iw}}$ is shimmering so proposition 3.4.16 implies the truth of conjectures 3.4.14 or 3.4.10 for ψ_f depending on whether conjecture 3.4.13 or conjecture 3.4.9 holds. Denoting the image of $\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}$ by $\mathbf{z}(f)_{\mathrm{Iw}}$ as in corollary 1.1.4, lemma 3.3.12 again implies the divisibility

$$\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} H^2(\mathbb{Z}[1/p], T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}}) | \mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} \left(H^1(\mathbb{Z}[1/p], T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}}\right). \tag{3.4.4.4}$$

or the equality

$$\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} H^2(\mathbb{Z}[1/p], T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}}) = \mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} \left(H^1(\mathbb{Z}[1/p], T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}}\right). \tag{3.4.4.5}$$

Finally, the image of $\mathbf{z}(\mathfrak{a})_{\mathrm{Iw}}$ inside $H^1(\mathbb{Z}[1/p], T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}})$ coincides by construction with the class in [43, Theorem 12.5 (1)] so satisfies the rest of statement (ii) of corollary 1.1.4. □

The translation of conjecture 3.4.11 into corollary 1.1.3 is slightly more involved, as it requires a comparison of characteristic ideals of étale cohomology modules with Selmer groups.

Proposition 3.4.18. *Let $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$ be a normal, Gorenstein irreducible component of $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}$ and denote by $T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}}$ the $G_{\mathbb{Q},\Sigma}$-representation $T^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}\otimes_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}}\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$. Assume that conjecture 3.4.11 holds. Then there is an equality*

$$(L_p(\mathfrak{a})) = \left(\mathrm{char}_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}} \tilde{H}^2_f(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})\right) \tag{3.4.4.6}$$

between the ideal generated by the Mazur-Kitagawa two-variable p-adic L-function $L_p(\mathfrak{a}) \in \mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$ and the characteristic ideal of the second cohomology group of the Nekovář-Selmer complex $\mathrm{R}\Gamma_f(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})$ with Greenberg's local condition at p. Moreover, $\tilde{H}^2_f(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})$ satisfies the perfect control property

$$\left(\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} \tilde{H}^2_f(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})\right)\otimes_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}}\mathcal{O}_{\mathrm{Iw}} = \mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} \tilde{H}^2_f(G_{\mathbb{Q},\Sigma}, T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}}) \tag{3.4.4.7}$$

for all eigencuspform f attached to a classical prime of $\operatorname{Spec} T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}}[1/p]$.

Proof. First, recall that $T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}}$ admits a nearly-ordinary $G_{\mathbb{Q}_p}$-stable submodule $T(\mathfrak{a})^{\mathrm{ord},+}_{\mathrm{Iw}}$ which is free of rank 1 and which is co-free (necessarily of rank 1) under the assumption that $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$ is a Gorenstein ring. Let $\mathrm{R}\tilde{\Gamma}_f(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})$ be as in [58] the complex

$$\mathrm{Cone}\left(\mathrm{R}\Gamma(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})\oplus\bigoplus_{\ell\in\Sigma}\mathrm{R}\Gamma(G_{\mathbb{Q}_\ell}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})\longrightarrow\bigoplus_{\ell\in\Sigma}\mathrm{R}\Gamma_f(G_{\mathbb{Q}_\ell}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})\right)[-1]$$

where

$$\mathrm{R}\Gamma_f(G_{\mathbb{Q}_\ell}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}}) = \begin{cases}\mathrm{R}\Gamma_f(G_{\mathbb{Q}_p}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}}) = \mathrm{R}\Gamma(G_{\mathbb{Q}_p}, T(\mathfrak{a})^{\mathrm{ord},+}_{\mathrm{Iw}}) & \text{if } \ell = p,\\ \mathrm{R}\Gamma(G_{\mathbb{F}_\ell}, H^0(I_\ell, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})) & \text{if } \ell\nmid p.\end{cases}$$

By Matlis's duality theorem for Nekovář-Selmer complexes ([58, Theorem 8.9.9]), the usual comparison between Selmer modules and cohomology of Nekovář-Selmer complexes (for instance [58, Lemma (9.6.3)]) and the fact that $H^0(G_{\mathbb{Q}_p}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}}/T(\mathfrak{a})^{\mathrm{ord},+}_{\mathrm{Iw}})_{\mathfrak{p}}$ vanishes for all prime $\mathfrak{p}\in\operatorname{Spec}\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$ of height 1, there is an equality

$$\mathrm{char}_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}} \tilde{H}^2_f(G_{\mathbb{Q},\Sigma}, T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}}) = \mathrm{char}_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}}\left(\mathrm{Hom}(\mathrm{Sel}_{\mathrm{Gr}}(T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}}), \mathbb{Q}_p/\mathbb{Z}_p)\right)$$

where $\mathrm{Sel}_{\mathrm{Gr}}(T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})$ is the strict Selmer group of $T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}}$ in the sense of Greenberg (see for instance [61, Section 4] for precise definitions in this context).

Let $\mathfrak{p}$ be a height one prime of $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$ and denote by $R_{\mathfrak{p}}$ the corresponding localization of $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$; a discrete valuation ring by our assumption. As in the proof of proposition 3.4.17, there exists a class $\mathbf{z}(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw},\mathfrak{p}}$ in $H^1_{\mathrm{et}}(\mathbb{Z}[1/p],T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})_{\mathfrak{p}}$ and the arguments of [43, Section 17.13] (which are also reproduced in our precise context in [32, Section 11.2]) then show that the equality

$$\mathrm{length}_{R_{\mathfrak{p}}} H^2_{\mathrm{et}}(\mathbb{Z}[1/p],T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})_{\mathfrak{p}}=\mathrm{length}_{R_{\mathfrak{p}}} H^1_{\mathrm{et}}(\mathbb{Z}[1/p],T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})_{\mathfrak{p}}/\mathbf{z}(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw},\mathfrak{p}}$$

imply the equality

$$\mathrm{length}_{R_{\mathfrak{p}}} \tilde{H}^2_f(G_{\mathbb{Q},\Sigma},T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})_{\mathfrak{p}}=\mathrm{ord}_{R_{\mathfrak{p}}}\left(L_p(\mathfrak{a})_{\mathfrak{p}}\right)$$

where $L_p(\mathfrak{a})_{\mathrm{Iw}}\in\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$ is the Mazur-Kitagawa p-adic L-function as defined in [23, page 551].

As conjecture 3.4.11 implies by lemma 3.3.12 that

$$\mathrm{length}_{R_{\mathfrak{p}}} H^2_{\mathrm{et}}(\mathbb{Z}[1/p],T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})_{\mathfrak{p}}=\mathrm{length}_{R_{\mathfrak{p}}} H^1_{\mathrm{et}}(\mathbb{Z}[1/p],T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})_{\mathfrak{p}}/\mathbf{z}(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw},\mathfrak{p}}$$

for all height one prime $\mathfrak{p}$ of $\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}$, it implies the equality (3.4.4.6). Because any classical specialization is shimmering, conjecture 3.4.11 and proposition 3.4.16 further imply that conjecture 3.4.10 holds for any such specialization so, again by lemma 3.3.12, that the equality

$$\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}} H^2(\mathbb{Z}[1/p],T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}})=\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}}\left(H^1(\mathbb{Z}[1/p],T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}})/\mathbf{z}(f)_{\mathrm{Iw}}\right). \tag{3.4.4.8}$$

holds for any eigencuspform f attached to a classical point $x\in\mathbf{T}^{\mathrm{ord}}_{\Sigma}/\mathfrak{a}$. Once more by [43, Section 17.13], equation (3.4.4.8) implies the equality

$$\left(L_p^{\mathrm{cyc}}(f)\right)=\left(\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}}\tilde{H}^2_f(G_{\mathbb{Q},\Sigma},T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}})\right)$$

where $L_p^{\mathrm{cyc}}(f)$ is the one-variable cyclotomic p-adic L-function of f ([53, 81, 2]). Because

$$L_p(\mathfrak{a}) \bmod x=L_p^{\mathrm{cyc}}(f)$$

by construction, we finally conclude that

$$\left(\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}}\tilde{H}^2_f(G_{\mathbb{Q},\Sigma},T(\mathfrak{a})^{\mathrm{ord}}_{\mathrm{Iw}})\right)\otimes_{\mathbf{T}^{\mathrm{ord}}_{\Sigma,\mathrm{Iw}}/\mathfrak{a}}\mathcal{O}_{\mathrm{Iw}}=\mathrm{char}_{\mathcal{O}_{\mathrm{Iw}}}\tilde{H}^2_f(G_{\mathbb{Q},\Sigma},T(f)\otimes_{\mathcal{O}}\mathcal{O}_{\mathrm{Iw}}).$$

□

4 Main results and their proofs

4.1 The main theorem

4.1.1 Statement

Recall that p is an odd prime and put $p^*=(-1)^{(p-1)/2}p$.

Theorem 4.1.1. *Assume that $\bar{\rho}$ satisfies the following properties.*

1. *(a) The representation $\bar{\rho}|_{G_{\mathbb{Q}(\sqrt{p^*})}}$ is irreducible.*

   (b) If $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is an extension

   $$0\longrightarrow\chi\longrightarrow\bar{\rho}|_{G_{\mathbb{Q}_p}}\longrightarrow\psi\longrightarrow 0,$$

   then $\chi\psi^{-1}\neq 1$ and $\chi\psi^{-1}\neq\bar{\varepsilon}_{\mathrm{cyc}}$.

   (c) The order of the image of $\bar{\rho}$ is divisible by p.

Moreover, assume that the compact open subgroup $U^{(p)}$ satisfies the following property.

2. *If $\ell \nmid p$ belongs to $\Sigma \backslash \Sigma(\bar{\rho})$ then at least one of the following holds.*
   (a) *$\ell \not\equiv \pm 1 \bmod p$.*
   (b) *$\ell \equiv -1 \bmod p$ and for all modular specialization $x : \mathbf{T}_{\Sigma,\mathrm{Iw}}(U^{(p)}) \longrightarrow \bar{\mathbb{Q}}_p$, the restriction of ρ_x to $G_{\mathbb{Q}_\ell}$ is reducible.*
   (c) *$\ell \equiv 1 \bmod p$ and for all modular specialization $x : \mathbf{T}_{\Sigma,\mathrm{Iw}}(U^{(p)}) \longrightarrow \bar{\mathbb{Q}}_p$, the restriction of ρ_x to I_ℓ is scalar.*

Then conjecture 3.4.12 is true for $\mathbf{T}_{\Sigma,\mathrm{Iw}}$. Assume furthermore that the following condition holds.

3. *There exists a modular specialization $x : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \bar{\mathbb{Q}}_p$ attached to a newform f such that conjecture 2.3.3 holds for f.*

Then conjecture 3.4.8 is true for $\mathbf{T}_{\Sigma,\mathrm{Iw}}$.

We record the following corollaries.

Corollary 4.1.2. *Assume $\bar{\rho}$ satisfies assumption 1 and that $U^{(p)}$ satisfies assumption 2 of theorem 4.1.1. Let $\mathfrak{a}$ be a minimal prime of $\mathbf{T}_\Sigma^{\mathrm{new}}$ and x be a modular specialization of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$. Then conjecture 3.4.13 for $R(\mathfrak{a})_{\mathrm{Iw}}$, conjecture 3.4.2 for $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$, conjecture 3.4.5 for $R(\mathfrak{a}_x)_{\mathrm{Iw}}$ and conjecture 2.3.4 for $M(f_x)$ all hold.*

Proof. Combine the result of theorem 4.1.1 with theorem 3.3.16 and propositions 3.4.15 and 3.4.16. □

As announced after 2.3.4, the last statement of corollary 4.1.2 eliminates the error term which appears in [43, Theorem 12.5] in the presence of an exceptional zero of the p-adic L-function.
The next corollary records a consequence of theorem 4.1.1 in classical Iwasawa theory.

Corollary 4.1.3. *Assume $\bar{\rho}$ satisfies assumption 1 of theorem 4.1.1 and that there exists a modular specialization $x : \operatorname{Spec} \mathbf{T}_{\Sigma,\mathrm{Iw}}(U^{(p)})[1/p] \longrightarrow \bar{\mathbb{Q}}_p$ such that conjecture 2.3.3 holds for $M(f_x)$. Then conjecture 2.3.3 holds for all classical points of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$.*

Note that there are no requirement on the level in this corollary. Compared to the main theorem of [23], notice also that the hypotheses of ordinarity and more significantly of vanishing μ-invariant have been removed. We postpone the proof of this corollary to section 4.2.3 below.

Corollary 4.1.4. *Assume $\bar{\rho}$ satisfies assumption 1a of 4.1.1 and that $U^{(p)}$ satisfies assumption 2 of theorem 4.1.1. Assume moreover that either of the following conditions hold.*

1. *There exists $\ell || N(\bar{\rho})$, the semisimplification of the representation $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is isomorphic to $\chi \oplus \psi$ with $\chi\psi^{-1} \neq 1$ and $\chi\psi^{-1} \neq \bar{\varepsilon}_{\mathrm{cyc}}$ and $\det \bar{\rho}$ is unramified outside p.*

2. *There exists $\ell || N(\bar{\rho})$ and there exists a modular specialization $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow \mathbb{Q}_p$ such that f_ψ belongs to $S_2(\Gamma_0(N))$ and verifies $a_p(f_\psi) = 0$.*

Let $\mathfrak{a}$ be a minimal prime of $\mathbf{T}_\Sigma^{\mathrm{new}}$ and x be a modular specialization of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ attached to an eigencuspform f. Then conjecture 3.4.8 for $\mathbf{T}_{\Sigma,\mathrm{Iw}}$, conjecture 3.4.9 for $R(\mathfrak{a})_{\mathrm{Iw}}$, conjecture 3.4.1 for $\mathbf{T}^x_{\Sigma,\mathrm{Iw}}$, conjecture 3.4.4 for $R(\mathfrak{a}_x)_{\mathrm{Iw}}$ and conjecture 2.3.3 for f all hold.

Proof. Under the assumptions of the corollary, hypothesis 3 of theorem 4.1.1 holds by theorem 2.3.2. □

In plain language, corollary 4.1.4 asserts that under the four first hypotheses of theorem 4.1.1 and either of its own hypotheses, all the conjectures considered in this manuscript are true. In particular, the Iwasawa Main Conjecture is then true for all modular point in $\mathbf{T}_{\Sigma,\mathrm{Iw}}$.

4.1.2 Examples

In this subsection, we give numerical examples of Hecke algebras $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ attached to various $\bar{\rho}$ which satisfies all four hypotheses of theorem 4.1.1 and of classical points of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$, which are thus known to satisfy conjecture 2.3.3 though they do not satisfy the hypotheses of theorem 2.3.2. In these examples, it is easy to check that the image of $\bar{\rho}$ is $\mathrm{GL}_2(\mathbb{F}_p)$ and that $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible (this claim can be verified by explicit computation but also follows for instance from the fact that $\bar{\rho}$ is the residual representation attached to an elliptic curves with good supersingular reduction at p and such that there exists $\ell \nmid p$ dividing exactly once $N(\bar{\rho})$). Hence, hypothesis 1 of theorem 4.1.1 is satisfied. The $U^{(p)}$ we consider are of the form $U_0(N)$ for some N and if $\ell \nmid N(\bar{\rho})$ belongs to $\Sigma(U^{(p)})$, then $\ell \not\equiv 1 \bmod p$ and $\ell || N$. Consequently, any modular specialization of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is attached to an automorphic representation of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ which is principal series or Steinberg at $\ell \equiv -1 \bmod p$ so hypothesis 2 of theorem 4.1.1 is satisfied. In order to verify that $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ satisfies hypothesis 3 of this theorem, we find a classical point $x \in \operatorname{Spec} \mathbf{T}_{\Sigma,\mathrm{Iw}}$ for which $H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_{x,\mathrm{Iw}})$ is computable and is equal to zero. Then $\mathbf{z}(f_x)_{\mathrm{Iw}}$ generates $H^1_{\mathrm{et}}(\mathbb{Z}[1/p], T_{x,\mathrm{Iw}})$ and the divisibility (2.3.3.1) is an equality.

In order to find classical points of $\operatorname{Spec} \mathbf{T}_{\Sigma,\mathrm{Iw}}$ for which $H^2_{\mathrm{et}}(\mathbb{Z}[1/p], T_{x,\mathrm{Iw}})$ is necessarily non-trivial, and hence for which corollary 4.1.2 does not reduce to the divisibility (2.3.3.1), we then make systematic use of [67, Theorem 1]. More precisely, we consider pairs of congruent newforms f, g attached to classical points x, y of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ of level N and $N\ell$ respectively and such that the Euler factor at ℓ of f evaluated at 1 is not a p-adic unit whereas all the Euler factors at primes dividing $N\ell$ of g evaluated at 1 are p-adic units. Under our hypotheses, the special value $L_{\{p\}}(f, \chi, 1)/\Omega_f$ where χ is a conductor of finite p-power order is an algebraic integer so $L_{\Sigma}(f, \chi, 1)/\Omega_f$ is not a p-adic unit (as it is multiplied by the Euler factor of f at ℓ). This in turn implies that $H^2_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T(f)_{\mathrm{Iw}})$ does not vanish. As $H^2_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T(f)_{\mathrm{Iw}}) \otimes_{\Lambda_{\mathrm{Iw}}} \Lambda_{\mathrm{Iw}}/\mathfrak{m}$ is isomorphic to $H^2_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T(g)_{\mathrm{Iw}}) \otimes_{\Lambda_{\mathrm{Iw}}} \Lambda_{\mathrm{Iw}}/\mathfrak{m}$, we deduce that $H^2_{\mathrm{et}}(\mathbb{Z}[1/\Sigma], T(g)_{\mathrm{Iw}}$ is not trivial. This in turns imply that $L_{\Sigma}(g, \chi, 1)/\Omega_g$ is not a p-adic unit, and so then is also $L_{\{p\}}(g, \chi, 1)/\Omega_g$ as the Euler factors intervening in the quotient of these two special values are by assumptions all p-adic units. This forces $H^2(\mathbb{Z}[1/p], T(g)_{\mathrm{Iw}})$ to be non-trivial.

The newforms f_4 and h_4 in the 3-adic and 7-adic paragraphs below are defined over the number fields $K_3 = \mathbb{Q}[X]/P_3$ of degree 15 and $K_7 = \mathbb{Q}[X]/P_7$ of degree 26 respectively. The polynomials P_3 and P_7 are given in appendix A.4. We warn the interested reader that the computation of the coefficients of the modular forms f_4 and h_4 below and the verification that they indeed have finite non-zero slope at some prime above i in the field K_i ($i = 3, 7$) is somewhat computationally intensive.

A 3-adic example: Consider

$$\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_3)$$

the only $G_{\mathbb{Q}}$-representation of Serre weight 2 and level 40 satisfying

$$\mathrm{tr}(\bar{\rho}(\mathrm{Fr}(\ell))) = \begin{cases} -1 & \text{if } \ell = 7, \\ 1 & \text{if } \ell = 11, \\ 1 & \text{if } \ell = 13, \\ -1 & \text{if } \ell = 17, \\ 1 & \text{if } \ell = 19, \\ 1 & \text{if } \ell = 23. \end{cases}$$

Put $N = 1640 = 2^3 \times 5 \times 41$. Here follows some newforms attached to classical points of $\operatorname{Spec} \mathbf{T}_{\Sigma,\mathrm{Iw}}(U_0(N))[1/p]$ (for which $\Sigma = \{2, 3, 5, 41\}$).

1. The modular form $f_1 \in S_2(\Gamma_0(40))$ with q-expansion starting with

$$q + q^5 - 4q^7 - 3q^9 + 4q^{11} - 2q^{13} + 2q^{17} \cdots \in S_2(\Gamma_0(40)).$$

The abelian variety attached to f_1 in the Jacobian of $X_0(40)$ is the elliptic curve

$$E_1 : y^2 = x^3 - 7x - 6,$$

which has good supersingular reduction at 3 with $a_3(f_1) = 0$.

2. The modular form $f_2 \in S_4(\Gamma_0(40))$ with q-expansion starting with

$$q - 6q^3 - 5q^5 - 34q^7 + 9q^9 + 16q^{11} + 58q^{13} + 30q^{15} - 70q^{17} + \cdots \in S_4(\Gamma_0(40)),$$

which has finite, non-zero slope at 3.

3. The modular form $f_3 \in S_2(\Gamma_0(1640))$ with q-expansion starting with

$$q + q^5 + 2q^7 - 3q^9 - 2q^{11} - 2q^{13} - 4q^{17} - 2q^{19} + \cdots \in S_2(\Gamma_0(1640)).$$

The abelian variety attached to f_3 in the Jacobian of $X_0(1640)$ is the elliptic curve

$$E_4 : y^2 = x^3 - 31307x - 1717706,$$

which has good supersingular reduction at 3.

4. Among the newforms in $S_4(\Gamma_0(1640))$, there exists exactly one, which we denote by f_4, such that $a_{41}(f_4)$ is equal to 41 and which has finite, non-zero slope at a single prime above 3. This newform is attached to a classical point of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and has coefficients in the ring of integers of $\mathbb{Q}[x]/P_3$ with P_3 equal to

$$x^{15} - 269x^{13} + 98x^{12} + 27795x^{11} + \cdots + 960245792x^3 - 3558446016x^2 + 1598326848x + 886331520.$$

Put $N' = 69160 = 2^3 \times 5 \times 7 \times 13 \times 19$ and $\Sigma' = \{2, 3, 5, 7, 13, 19\}$. Here follows some newforms attached to classical points of $\operatorname{Spec} \mathbf{T}_{\Sigma',\mathrm{Iw}}(U_0(N'))[1/p]$ (this level does not satisfy assumption 2 of theorem 4.1.1).

5. The modular form $f_5 \in S_2(\Gamma_0(520))$ with q-expansion starting with

$$q + \sqrt{6}q^3 + q^5 + 2q^7 + 3q^9 + (-2 + \sqrt{6})q^{11} - q^{13} + \sqrt{6}q^{15} + (2 - 2\sqrt{6})q^{17} + \cdots \in S_2(\Gamma_0(520)),$$

with has finite, non-zero slope at the prime $(3 + \sqrt{6})$ above 3 in $\mathbb{Q}[\sqrt{6}]$.

6. The modular form $f_6 \in S_2(\Gamma_0(760))$ with q-expansion starting with

$$q + 3q^3 + q^5 - q^7 + 6q^9 + 4q^{11} + q^{13} + 3q^{15} - 7q^{17} - q^{19} + \cdots\cdots \in S_2(\Gamma_0(760)).$$

The abelian variety attached to f_6 in the Jacobian of $X_0(760)$ is the elliptic curve

$$E_4 : y^2 = x^3 - 67x + 926,$$

which has good supersingular reduction at 3 with $a_3(f_4) \neq 0$.

7. The modular form $f_7 \in S_4(\Gamma_0(280))$ with q-expansion starting with

$$q + \frac{1}{2}xq^3 - 5q^5 + 7q^7 + \frac{1}{4}(x^2 - 108)q^9 + \frac{1}{12}(-x^2 + 44x + 144)q^{11} + \cdots \in S_4(\Gamma_0(280))$$

which has finite, non-zero slope at the prime ideal $(3, (x^2 - 14x - 96)/48)$ above 3 in the ring of integers of $\mathbb{Q}[x]/P$ with

$$P = x^3 - 4x^2 - 236x + 192.$$

It is easy to check that conjecture 2.3.3 is true for f_1 (the algebraic special value of the L-function at 1 for the trivial character is a 3-adic unit). It follows that conjecture 3.4.8 is true for $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and that conjecture 2.3.3 is true for all f_i (none of these forms satisfy the hypotheses of the main theorems of [76, 82]). The Euler factor of f_1 at 41 evaluated at 1 is congruent to 0 modulo 3 whereas the Euler factor of f_4 at 41 evaluated at 1 is congruent to 2 modulo 3. Both the special values $L_{\{2,3,5,41\}}(f_1,\chi,1)$ and $L_{\{2,3,5,41\}}(f_4,\chi,1)$ (for χ of finite 3-power order) are computed through the image of $\mathbf{z}_{\{2,3,5,41\},\mathrm{Iw}}$ and all Euler factors of f_1 and f_4 at $\{2,5\}$ are 3-adic units. This entails that $L_{\{3\}}(f_4,\chi,1)/\Omega_{f_4}$ has to be a non-unit modulo 3 (as it indeed is). Note that the key hypotheses of theorem 2.3.2 are either that f be ordinary or that it be of weight 2 and with $a_p(f)$ equal to 0. The form f_4 satisfies none of these hypotheses.

A 5-adic example: Consider

$$\bar{\rho}: G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_5)$$

the only $G_{\mathbb{Q}}$-representation of Serre weight 2 and level 34 satisfying

$$\mathrm{tr}(\bar{\rho}(\mathrm{Fr}(\ell))) = \begin{cases} -2 & \text{if } \ell = 3, \\ 1 & \text{if } \ell = 7, \\ 1 & \text{if } \ell = 11, \\ 2 & \text{if } \ell = 13, \\ 1 & \text{if } \ell = 19, \\ 0 & \text{if } \ell = 23. \end{cases}$$

Put $N = 58174 = 2 \times 17 \times 29 \times 59$. Here follows some newforms attached to classical points of $\mathrm{Spec}\,\mathbf{T}_{\Sigma,\mathrm{Iw}}(U_0(N))[1/p]$ (for which $\Sigma = \{2,5,17,29,59\}$).

1. The newform $g_1 \in S_2(\Gamma_0(34))$ with q-expansion starting with
$$q + q^2 - 2q^3 + q^4 - 2q^6 - 4q^7 + q^8 + q^9 + 6q^{11} \cdots \in S_2(\Gamma_0(34)).$$
The abelian variety attached to g_1 in the Jacobian of $X_0(34)$ is the elliptic curve
$$E_1 : y^2 + xy = x^3 - 3x + 1,$$
which has good supersingular reduction at 5.
2. The newform $g_2 \in S_6(\Gamma_0(34))$ with q-expansion starting with
$$q - 4q^2 + (3\sqrt{69} - 3)q^3 + 16q^4 + (4\sqrt{69} - 18)q^5 + (-12\sqrt{69} + 12)q^6 + (\sqrt{69} - 1)q^7 - 64q^8 + \cdots \in S_6(\Gamma_0(34)),$$
which has finite, non-zero slope at the prime ideal $(-7+\sqrt{69})/2$ above 5 in the ring of integers of $\mathbb{Q}[\sqrt{69}]$.
3. The newform $g_3 \in S_2(\Gamma_0(986))$ with q-expansion starting with
$$q + q^2 + (x-1)q^3 + q^4 + \frac{1}{93}(-4x^6 + 37x^5 - 49x^4 - 293x^3 + 525x^2 + 527x - 531)q^5 + \cdots \in S_2(\Gamma_0(986)),$$
which has finite, non-zero slope at the prime $(5, x+1)$ above 5 in the ring of integers of $\mathbb{Q}[x]/P$ with
$$P = x^7 - 10x^6 + 25x^5 + 35x^4 - 192x^3 + 112x^2 + 156x - 117.$$
4. The newform $g_4 \in S_2(\Gamma_0(2006))$ with q-expansion starting with
$$q + q^2 + \frac{1}{2}(2+\sqrt{5})q^3 + \frac{1}{2}(-5-\sqrt{5})q^5 + \frac{1}{2}(2+\sqrt{5})q^6 + q^7 + q^8 + \cdots \in S_2(\Gamma_0(2006)),$$
which has finite, non-zero slope at the prime $(\sqrt{5})$ above 5 in the ring of integers of $\mathbb{Q}[\sqrt{5}]$.

It is easy to check that conjecture 2.3.3 is true for g_1 (the algebraic special value of the L-function at 1 for the trivial character is a 5-adic unit). It follows that conjecture 3.4.8 is true for $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and that conjecture 2.3.3 is true for all g_i (the forms g_i for $i \neq 1$ do not satisfy the hypotheses of the main theorems of [76, 82]).

The Euler factor of g_1 at the primes 29 and 59 evaluated at 1 vanish modulo 5 so, as in the previous example, theorem 4.1.1 further entails that $L_{\{5\}}(g_3, \chi, 1)/\Omega_{g_3}$ and $L_{\{5\}}(g_4, \chi, 1)/\Omega_{g_4}$ (for χ of finite 5-power order) are not 5-adic units.

A 7-adic example: Consider

$$\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_7)$$

the only $G_{\mathbb{Q}}$-representation of Serre weight 2 and level 48 satisfying

$$\mathrm{tr}(\bar{\rho}(\mathrm{Fr}(\ell))) = \begin{cases} -2 & \text{if } \ell = 5, \\ 3 & \text{if } \ell = 11, \\ -2 & \text{if } \ell = 13, \\ 2 & \text{if } \ell = 17, \\ -3 & \text{if } \ell = 19, \\ 1 & \text{if } \ell = 23. \end{cases}$$

Put $N = 1056336 = 2^4 \times 3 \times 59 \times 373$. Here follows some newforms attached to classical points of $\operatorname{Spec} \mathbf{T}_{\Sigma,\mathrm{Iw}}(U_0(N))[1/p]$ (for which $\Sigma = \{2, 3, 7, 59, 373\}$).

1. The newform $h_1 \in S_2(\Gamma_0(48))$ with q-expansion starting with

   $$q + q^3 - 2q^5 + q^9 - 4q^{11} - 2q^{13} - 2q^{15} \cdots \in S_2(\Gamma_0(48)).$$

   The abelian variety attached to h_1 in the Jacobian of $X_0(48)$ is the elliptic curve

   $$E_1 : y^2 = x^3 + x^2 - 4x - 4,$$

   which has good supersingular reduction at 7.

2. The newform $h_2 \in S_8(\Gamma_0(48))$ with q-expansion starting with

   $$q - 27q^3 + 110q^5 - 504q^7 + 729q^9 - 3812q^{11} + 9574q^{13} - 2970q^{15} + 26098q^{17} + \cdots \in S_8(\Gamma_0(48)),$$

   which has finite, non-zero slope at 7.

3. The newform $h_3 \in S_2(2832)$ with q-expansion starting with

   $$q + q^3 + (x+1)q^5 + (-x^2/2 - x/2 + 3)q^7 + q^9 + 3q^{11} + (-x^2 - 3x - 5) + \cdots \in S_2(\Gamma_0(2832))$$

   which has finite, non-zero slope at the prime ideal $(-x^2/2 - x/2 + 3)$ above 7 in the ring of integers of $\mathbb{Q}[x]/(x^3 + 3x^2 - 6x - 4)$.

4. The newform $h_4 \in S_2(17904)$ which is the only one with coefficients in the ring of integers of $\mathbb{Q}[x]/P_7$ with

   $$P_7 = x^{26} + 58x^{25} + 1528x^{24} + 24066x^{23} + 250201x^{22} + \cdots + 1393890560x - 422379776.$$

   The form h_4 has finite, non-zero slope for some prime above 7 in $\mathbb{Q}[x]/P_7$.

Put $N' = 11472 = 2^4 \times 3 \times 239$ and $\Sigma' = \{2 \times 3 \times 7 \times 239\}$. Here follows a newform attached to a classical points of $\operatorname{Spec} \mathbf{T}_{\Sigma',\mathrm{Iw}}(U_0(N'))[1/p]$ (this level does not satisfy assumption 2 of theorem 4.1.1).

5. The newform $h_5 \in S_2(11472)$ with q-expansion starting with

$$q + q^3 + (2\sqrt{2} - 1)q^5 + q^9 + (3 - 2\sqrt{2})q^{11} + (4 - 2\sqrt{2})q^{13} + \cdots \in S_2(11472)$$

which has infinite slope at the prime $(-1 + 2\sqrt{2})$ above 7 in the ring of integers of $\mathbb{Q}[\sqrt{2}]$.

It is easy to check that conjecture 2.3.3 is true for h_1 (the algebraic special value of the L-function at 1 for the trivial character is a 7-adic unit). It follows that conjecture 3.4.8 is true for $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and that conjecture 2.3.3 is true for all h_i (none of these forms satisfy the hypotheses of [76, 82]). Because the Euler factor of h_1 at 373 evaluated at 1 vanishes modulo 7, it is likely that $L_{\{7\}}(h_4, \chi, 1)/\Omega_{h_4}$ (for χ of finite 7-power order) is not a 7-adic unit, though we have not checked this computationally.

4.2 Proof of the main theorem

We give the proof of theorem 4.1.1. Alongside the compatibility of the ETNC with specializations and change of levels, the crucial ingredient in the proof is the existence of a Taylor-Wiles system in the sense of [84, 80] as further refined in [18, 30, 6, 19, 47].

4.2.1 The Taylor-Wiles system of fundamental lines

Henceforth, we consistently assume that $\bar{\rho}$ satisfies hypothesis 1 of theorem 4.1.1; namely, we assume that it satisfies assumptions 4.2.1 and 4.2.2 below.

Assumption 4.2.1. *The $G_{\mathbb{Q}(\sqrt{p^*})}$-representation $\bar{\rho}$ is absolutely irreducible.*

Assumption 4.2.2. *If $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is an extension*

$$0 \longrightarrow \chi \longrightarrow \bar{\rho}|_{G_{\mathbb{Q}_p}} \longrightarrow \psi \longrightarrow 0,$$

then $\chi\psi^{-1} \neq 1$ and $\chi\psi^{-1} \neq \bar{\varepsilon}_{\mathrm{cyc}}$.

Under assumptions 4.2.1 and 4.2.2, the main results of [84, 80, 19, 6] show that, for any admissible set of primes Σ, the ring $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is a complete intersection ring of dimension 4 isomorphic to the universal deformation ring parametrizing deformations with coefficients into artinian $\mathbb{Z}_p$-algebras and unramified outside of Σ.

Definition 4.2.3. *An allowable compact open subgroup $U^{(p)}$ is $\bar{\rho}$-minimal (or minimal for short) if for all modular specialization $x : \mathbf{T}_{\Sigma,\mathrm{Iw}}(U^{(p)}) \longrightarrow \bar{\mathbb{Q}}_p$, the $G_{\mathbb{Q},\Sigma}$-representation ρ_x attached to x is a minimal deformation of $\bar{\rho}$ in the sense of definition [30, Definitions 3.3 and 3.32] (in particular $\Sigma(U^{(p)}) = \Sigma(\bar{\rho})$).*

The following definitions and lemma summarize the part of the Taylor-Wiles system machinery that we require.

Definition 4.2.4. *A Taylor-Wiles system is a set $\{B, R, M, (R_n, M_n)_{n\in\mathbb{N}}\}$ whose elements satisfy the following properties.*

1. *The ring B is a complete, local, flat $\mathcal{O}$-algebra of dimension $d + 1$.*
2. *The ring R is a B-algebra and M is a non-zero R-module.*
3. *There exists $(h, j) \in \mathbb{N}^2$ such that for all $n \in \mathbb{N}$, there are maps of $\mathcal{O}$-algebras*

$$\mathcal{O}[[y_1, \cdots, y_{h+j}]] \longrightarrow R_n \longrightarrow R.$$

Denote by $\mathbf{B}$ the $h + j + 1$-dimensional B-algebra $B[[x_1, \cdots, x_{h+j-d}]]$.

4. *For all $n \in \mathbb{N}$, R_n is a quotient of $\mathbf{B}$.*

5. *For all $n \in \mathbb{N}$, there is a surjective map $R_n \longrightarrow R$ of B-algebras whose kernel is the ideal $(y_1 \cdots, y_h)R_n$.*

6. *For all $n \in \mathbb{N}$, there is a surjective map of R_n-modules $M_n \longrightarrow M$ whose kernel is $(y_1 \cdots, y_h)M_n$.*

7. *For all $n \in \mathbb{N}$, the annihilator $\mathfrak{b}_n$ of M_n in $\mathcal{O}[[y_1, \cdots, y_{h+j}]]$ is included in the ideal*

$$\left((1+y_1)^{p^n} - 1, \cdots, (1+y_h)^{p^n} - 1\right)$$

and M_n is a finite $\mathcal{O}[[y_1, \cdots, y_{h+j}]]/\mathfrak{b}_n$-module free of rank s.

Let $\mathbf{TW} = \{B, R, M, (R_n, M_n)_{n\in\mathbb{N}}\}$ be a Taylor-Wiles system and let n be an integer. According to property 7 of definition 4.2.4, M_n is of depth at least $h+j$ as $\mathcal{O}[[y_1, \cdots, y_{h+j}]]/\mathfrak{b}_n$-module so M is of depth at least j as $\mathcal{O}[[y_{h+1}, \cdots, y_{h+j}]]$-module and so is a free $\mathcal{O}[[y_{h+1} \cdots, y_{h+j}]]$-module, necessarily of rank s.

For $(m, n) \in \mathbb{N}^2$ a pair of integers satisfying $m \geq 1$ and $n \geq m$, define r_m to be the integer $smp^m(h+j)$ and $\mathfrak{c}_m$ to be the ideal

$$\left(\varpi_{\mathcal{O}}^m, (1+y_1)^{p^m} - 1, \cdots, (1+y_h)^{p^m} - 1, y_{h+1}^{p^m}, \cdots, y_{h+j}^{p^m}\right)\mathcal{O}[[y_1, \cdots, y_{h+j}]].$$

Let $\mathfrak{m}_{R_n}^{(r_m)}$ be the ideal of R_n generated by r_m-th powers of elements of $\mathfrak{m}_{R_n}$. Let $D_{m,n}$ be the ring $R_n/\left(\mathfrak{c}_m R_n + \mathfrak{m}_{R_n}^{(r_m)}\right)$ and let $L_{m,n}$ be the quotient $M_n/\mathfrak{c}_m M_n$. Because $\mathfrak{m}_{R_n}^{(r_m)} M_n$ is included in $\mathfrak{c}_m M_n$ ([47, page 1158]), the quotient $L_{m,n}$ is endowed with a structure of $D_{m,n}$-module.

Definition 4.2.5. *Let $\mathbf{TW} = \{B, R, M, (R_n, M_n)\}$ be a Taylor-Wiles system with M of rank s over $\mathcal{O}[[y_{h+1}, \cdots, y_{h+j}]]$. For $n \geq m \geq 1$ two integers, a patching datum $\mathbf{PD}$ of level (n, m) attached to $\mathbf{TW}$ is a multiplet*

$$\left(D_{m,n}, L_{m,n}, \psi_{m,n}, \pi_{m,n}^{(1)}, \pi_{m,n}^{(2)}, \pi_{m,n}^{(3)}\right)$$

satisfying the following properties.

1. *The map ψ_m is a morphism of local $\mathcal{O}$-algebras from $\mathcal{O}[[y_1, \cdots, y_{h+j}]]$ to $D_{m,n}$.*

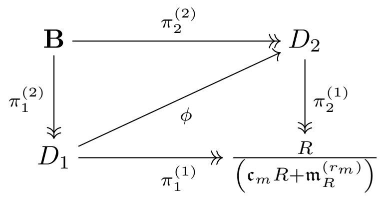
2. *The map $\pi_{m,n}^{(1)}$ is a surjective map of B-algebras $D_{m,n} \longrightarrow R/\left(\mathfrak{c}_m R + \mathfrak{m}_R^{(r_m)}\right)$.*

3. *The map $\pi_{m,n}^{(2)}$ is a surjective map of B-algebras $\mathbf{B} \longrightarrow D_{m,n}$.*

4. *The map $\pi_{m,n}^{(3)}$ is a surjective map of $\mathbf{B}$-modules $L_{m,n} \longrightarrow M/\mathfrak{c}_m M$.*

For reasons of brevity, we often refer to a patching datum $(D, L, \psi, \pi^{(1)}, \pi^{(2)}, \pi^{(3)})$ of some level (m, n) simply by the couple (D, L).

Two patching data $(D_1, L_1, \psi_1, \pi_1^{(1)}, \pi_1^{(2)}, \pi_1^{(3)})$ and $(D_2, L_2, \psi_2, \pi_2^{(1)}, \pi_2^{(2)}, \pi_2^{(3)})$ both of level m are isomorphic if there exists a pair (ϕ, ψ) of morphisms such that ϕ is a morphism of $\mathbf{B}$-algebras making the diagram



commutative and ψ is a surjective morphism of D_1-modules making the diagram

$$\begin{array}{ccc} L_1 & \overset{\pi_1^{(3)}}{\twoheadrightarrow} & M/\mathfrak{c}_m M \\ {\scriptstyle \psi}\big\downarrow & \nearrow_{\pi_2^{(3)}} & \\ L_2 & & \end{array}$$

commutative.

Let $\mathbf{TW}$ be a Taylor-Wiles system with associated patching datum $\mathbf{PD}$. The cardinality of the ring $D_{m,n}$ is bounded by the cardinality of $\mathbf{B}/\mathfrak{m}^{(r_m)}\mathbf{B}$ which is independent of n. Hence, there exists a subsequence $(Q(n))_{n\in\mathbb{N}}$ such that for all $n \geq m \geq 1$, the patching datum $(D_{m,Q(n)}, L_{m,Q(n)})$ is isomorphic to $(D_m, L_m) \overset{\text{def}}{=} (D_{m,Q(m)}, L_{m,Q(m)})$. Let $(D_m)_{m\geq 1}$ and $(L_m)_{m\geq 1}$ be the projective systems with transition maps induced by the isomorphisms of patching data

$$\Big(D_{m+1}/(\mathfrak{c}_m D_{m+1} + \mathfrak{m}_{D_{m+1}}^{(r_m)}), L_{m+1}/\mathfrak{c}_m L_{m+1}\Big) \simeq (D_m, L_m).$$

and let R_∞ and L_∞ be their inverse limits. We call $(D_m, L_m)_{m\geq 1}$ the patched system attached to the patching datum $\mathbf{PD}$ and (R_∞, L_∞) the limit object of the patched system attached to $\mathbf{PD}$.

The following lemma is presumably well-known. Because our setting and goals are not exactly those of the literature, we recall it for the convenience of the reader.

Lemma 4.2.6. *Let* $\mathbf{TW} = \{B, R, M, (R_n, M_n)_{n\in\mathbb{N}}\}$ *be a Taylor-Wiles system whose patched system has limit objects* (R_∞, L_∞) *and let* $\mathfrak{a}_\infty$ *be a minimal prime ideal of* R_∞. *Then* $R_\infty/\mathfrak{a}_\infty$ *is isomorphic to* $\mathbf{B}/\mathfrak{a} = B/\mathfrak{a}[[x_1, \cdots, x_{h+j-d}]]$ *for some minimal prime ideal* $\mathfrak{a}$ *of* B *and thus is a regular local ring if and only if* $B/\mathfrak{a}$ *is one.*

Proof. Recall that according to property 1 of definition 4.2.4, B is a local ring of dimension d. Hence, the ring $\mathbf{B}$ is a local of dimension $h + j + 1$ and for any minimal prime ideal $\mathfrak{a}$ of B, the quotient $\mathbf{B}/\mathfrak{a}$ is a local domain of dimension $h + j + 1$ which is a regular local ring if and only if $B/\mathfrak{a}$ is one. It is thus enough to prove that there exists a minimal prime ideal $\mathfrak{a}$ such that $R_\infty/\mathfrak{a}_\infty$ is isomorphic to $\mathbf{B}/\mathfrak{a}$.

For all integer $m \geq 1$, there exists by construction a surjection $\pi_m^{(2)} : \mathbf{B} \twoheadrightarrow D_m$ and hence a surjection $\pi_\infty^{(2)} : \mathbf{B} \twoheadrightarrow R_\infty$. Let $\mathfrak{a}_\infty$ be a minimal prime ideal of R_∞ and let $\mathfrak{a}$ be a minimal prime ideal of B such that $\pi_\infty^{(2)}$ induces a surjection from $\mathbf{B}/\mathfrak{a}$ onto $R_\infty/\mathfrak{a}_\infty$. The maps

$$\psi_m : \mathcal{O}[[y_1, \cdots, y_{h+j}]] \longrightarrow D_m$$

for $m \geq 1$ induce a map $\psi_\infty : \mathcal{O}[[y_1, \cdots, y_{h+j}]] \longrightarrow R_\infty$. According to the remark following definition 4.2.4, L_∞ is finite and free (of rank s) for this $\mathcal{O}[[y_1, \cdots, y_{h+j}]]$-module structure. Hence, the depth of L_∞ as $\mathcal{O}[[y_1, \cdots, y_{h+j}]]$-module is $h + j + 1$ by the Auslander-Buchsbaum and Serre theorem. The depth of a finitely generated module is invariant by change of local noetherian rings so

$$\operatorname{depth}_{\mathcal{O}[[y_1,\cdots,y_{h+j}]]} L_\infty = \operatorname{depth}_{R_\infty} L_\infty.$$

The depth of a finitely generated module over a local noetherian ring is less than the dimension of the ring so

$$h + j + 1 = \operatorname{depth}_{R_\infty} L_\infty \leq \dim R_\infty = \dim R_\infty/\mathfrak{a}_\infty \leq \dim \mathbf{B}/\mathfrak{a} = \dim \mathbf{B} = h + j + 1.$$

In particular $\dim R_\infty/\mathfrak{a}_\infty = \dim \mathbf{B}/\mathfrak{a}$. The morphism $\mathbf{B}/\mathfrak{a} \twoheadrightarrow R_\infty/\mathfrak{a}_\infty$ is thus a surjection from a complete, local, noetherian domain to a complete, local ring of the same dimension and hence an isomorphism. □

By Chebotarev's density theorem, there exist a set X whose elements are the empty set and finite sets Q of common cardinality $r > 0$ of rational primes $q \notin \Sigma$ such that, for all $n \in \mathbb{N}$, the set

$$X_n \overset{\text{def}}{=} \{Q \in X | \forall q \in Q,\ q \equiv 1 \bmod p^n \text{ and } \bar{\rho}(\text{Fr}(q)) \text{ has distinct eigenvalues}\}$$

is infinite. For $Q \in X$, we put $R_Q = \mathbf{T}_{\Sigma\cup Q,\text{Iw}}$ and $\Delta_Q = \Delta_{R_Q}(T_{\Sigma\cup Q,\text{Iw}})$. The injection

$$\text{triv}_Q : \Delta_Q \hookrightarrow Q(R_Q)$$

of conjecture 3.4.8 (for the set of primes $\Sigma \cup Q$) induces an isomorphism

$$\Delta_Q^{-1} \simeq \text{Det}_{R_Q}[y_Q R_Q \longrightarrow x_Q R_Q]$$

which depends on a choice of a pair $(x_Q, y_Q) \in R_Q^2$ of non zero-divisors and where the complex in the right-hand side is placed in degree -1 and 0

Proposition 4.2.7. *Assume that $U^{(p)}$ is $\bar{\rho}$-minimal. Then there exists a Taylor-Wiles system $\{\mathcal{O}, R_\varnothing, \Delta_{\Sigma,\text{Iw}}(T_{\Sigma,\text{Iw}}), (R_n, \Delta_n)_{n\in\mathbb{N}}\}$ satisfying the following properties*

1. *The integer j of definition 4.2.4 is equal to 3.*
2. *For all $n \in \mathbb{N}$, the ring R_n is of the form R_{Q_n} for some $Q_n \in X_n$.*
3. *For all $n \in \mathbb{N}$, let $Q_n \in X_n$ be the set such that $R_n = R_{Q_n}$. Then the R_n-module Δ_n is Δ_{Q_n}.*

Proof. First, we assume that, in addition to our ongoing hypotheses, the following statement holds : if $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible, then it arises after a twist by a character and extension of scalars from the $G_{\mathbb{Q}_p}$-module attached to the generic fiber of a $\mathbb{Z}_p$-scheme in $\bar{\mathbb{F}}_q$-vector spaces of type (p, p) (in the sense of [66]). Then the results of [84, 80] as improved in [18, 30] show that there exists a Taylor-Wiles system $\{\mathcal{O}, R^s, M^s, (R_n^s, M_n^s)_{n\in\mathbb{N}}\}$ where R^s, R_n^s are suitable classical, local Hecke algebra of finite weight and level and M^s, M_n^s are suitable Hecke-modules coming from the cohomology of modular curves. For this Taylor-Wiles system, the integers h, j of definition 4.2.4 are respectively equal to $2r$ and to zero. Moreover, M^s is in that case a free R^s-module and, for all $n \in \mathbb{N}$, M_n^s is a free R_n^s-module. Because the axioms satisfied by Taylor-Wiles system on M^s and M_n^s depend only on the actions of R^s and R_n^s, there is then no loss of generality in assuming that the rank of M^s and of the M_n^s as modules over R^s and R_n^s respectively is equal to 1. As fundamental lines are free module of rank 1, we get a Taylor-Wiles system $\{\mathcal{O}, R^s, \Delta, (R_n^s, \Delta_n)_{n\in\mathbb{N}}\}$ in which Δ (resp. Δ_n) is the fundamental line attached to the $G_{\mathbb{Q},\Sigma}$-representation with coefficients in R^s (resp. to the $G_{\mathbb{Q},\Sigma\cup Q_n}$-representation with coefficients in R_n^s).

Next, we relax our supplementary assumption that $\bar{\rho}|G_{\mathbb{Q}_p}$ is flat in the sense above if it is irreducible. In order to establish the existence of a Taylor-Wiles system $\{\mathcal{O}, R^s, \Delta, (R_n^s, \Delta_n)_{n\in\mathbb{N}}\}$ of the same form as above, only the case of irreducible $\bar{\rho}|G_{\mathbb{Q}_p}$ remains so we may further make this assumption. Twisting if necessary, we may further assume that $\bar{\rho}|G_{\mathbb{Q}_p}$ is the image through the Fontaine-Laffaille functor of a Fontaine-Laffaille module in the sense of [25] with coefficients in $\bar{k}$ having filtration indices 0 and $k-1$ with $2 \leq k \leq p$. The existence of a Taylor-Wiles of the form $\{\mathcal{O}, R^s, M^s, (R_n^s, M_n^s)_{n\in\mathbb{N}}\}$ then follows from [19, Lemma 3.3 and 3.4] (again, the integers h, j of this Tayor-Wiles system are equal to $2r$ and zero). As in the first paragraph of the proof, we then deduce that there exists a Taylor-Wiles system of the form $\{\mathcal{O}, R^s, \Delta, (R_n^s, \Delta_n)_{n\in\mathbb{N}}\}$.

It remains to establish the existence of a Taylor-Wiles system

$$\{\mathcal{O}, R_\varnothing, \Delta_{\Sigma,\text{Iw}}(T_{\Sigma,\text{Iw}}), (R_n, \Delta_n)_{n\in\mathbb{N}}\}$$

as in the proposition from the existence of the Taylor-Wiles system $\{\mathcal{O}, R^s, \Delta, (R_n^s, \Delta_n)_{n\in\mathbb{N}}\}$ in finite weight and level. According to [6, Theorem 4.1 and Lemma 4.3], the ring R^s (resp. R_n^s) is a quotient of a certain Hecke ring R^η (resp. R_n^η) by an ideal generated by a regular sequence of length at most 2. See [6, Section 2 and 4] for the definition. Denote by Δ^{η} (resp. Δ^{η}_n) the fundamental line of the quotient T^{η} (resp. T^{η}_n) of $T_{\Sigma,\mathrm{Iw}}$ with coefficients in R^{η} (resp. in R^{η}_n). Then the set $\{\mathcal{O}, R^{\eta}, \Delta^{\eta}, (R^{\eta}_n, \Delta^{\eta}_n)_{n\in\mathbb{N}}\}$ is a Taylor-Wiles system (with $h = 2r$ and $j = 2$). Finally, $R_{\varnothing}$ (resp. R_n) is by [6, Proposition 2.6] a finite, flat group-algebra over $R^{\eta}[[X]]$ (resp. $R^{\eta}_n[[X]]$), where the supplementary formal variable X corresponds to the cyclotomic variable. Hence, we finally get that $\{\mathcal{O}, R_{\varnothing}, \Delta_{\Sigma,\mathrm{Iw}}(T_{\Sigma,\mathrm{Iw}}), (R_n, \Delta_n)_{n\in\mathbb{N}}\}$ is a Taylor-Wiles system with $h = 2r + s$ for some integer s and $j = 3$. □

Denote by $(R_{\infty}, \Delta_{\infty})$ the limit objects of the patching datum of the Taylor-Wiles system of the previous proposition.

Proposition 4.2.8. *Suppose that the assumptions 2 of theorem 4.1.1 holds. Then there exists a Taylor-Wiles system* $\{B, R^{\square}_{\varnothing}, \Delta^{\square}_{\Sigma,\mathrm{Iw}}, (R^{\square}_n, \Delta^{\square}_n)_{n\in\mathbb{N}}\}$ *satisfying the following properties*

1. *The integer j of definition 4.2.4 is equal to* 3.
2. *For $\ell \in \Sigma$, let R_{ℓ} be the universal framed deformation ring of $\bar{\rho}|G_{\mathbb{Q}_{\ell}}$. The ring B is the completed tensor product $\widehat{\bigotimes_{\ell}} R_{\ell}$ over $\mathcal{O}$ of the local (framed) deformation rings R_{ℓ}. Each irreducible component of B is a regular local ring of dimension* $6 + 4|\Sigma^{(p)}|$.
3. *The ring $R^{\square}_{\varnothing}$ is the universal framed deformation ring $R^{\square}_{\Sigma,\mathrm{Iw}}(\bar{\rho})$ representing framed deformation of $\bar{\rho}$ unramified outside Σ and is thus isomorphic to* $\mathbf{T}_{\Sigma,\mathrm{Iw}}[[Y_1, Y_2, Y_3]]$
4. *For all $n \in \mathbb{N}$, the ring $R^{\square}_n$ is the universal framed deformation ring $R^{\square}_{Q_n}$ with $Q_n \in X_n$.*
5. *For all $n \in \mathbb{N}$, let $Q_n \in X_n$ be the set such that $R_n = R_{Q_n}$. Then the $R^{\square}_n$-module $\Delta^{\square}_n$ is* $\Delta_{Q_n} \otimes_{R_{Q_n}} R^{\square}_{Q_n}$.

Proof. We first show that there exists a Taylor-Wiles system $\{B, R^{\square}_{\varnothing}, M^{\square}, (M^{\square}_n, R^{\square}_n)\}$ as in the proof of [47, Theorem (3.4.11)]. Here $M^{\square}$ and $M^{\square}_n$ are the extension of scalars to $R^{\square}_{\varnothing}$ and $R^{\square}_n$ of suitable Hecke-modules M and M_n over $R_{\varnothing}$ and R_n respectively. This amounts to checking as in [47, Proposition (3.2.5)] that there exists sets of prime Q_n of common cardinality such that there exists an integer r such that for all Q_n, $R^{\square}_n$ is a quotient of $R^{\square}_{\varnothing}[[X_1, \cdots, X_r]]$. The same proof as in [47, Proposition (3.2.5)] shows that this holds for $|Q_n| = \dim_k H^1(G_{\mathbb{Q},\Sigma}, \mathrm{ad}^0\,\bar{\rho}(1))$ and $r = |Q_n| + |\Sigma| - 2$. Observe furthermore that $M^{\square}$ (resp. $M^{\square}_n$) is free of finite rank over $R^{\square}$ (resp. $R^{\square}_n$). Indeed, under our ongoing hypotheses, it follows from the standard Taylor-Wiles method and the complete intersection and freeness criterion of [30, Section 2.2] that M and M_n are free modules over $R_{\varnothing}$ and R_n respectively. Replacing $M^{\square}$ and $M^{\square}_n$ by free sub-modules of rank 1 stable under the action of $R^{\square}$ and $R^{\square}_n$ respectively then yields as in the proof of proposition 4.2.7 a Taylor-Wiles system $\{B, R^{\square}_{\varnothing}, \Delta^{\square}_{\Sigma,\mathrm{Iw}}, (R^{\square}_n, \Delta^{\square}_n)_{n\in\mathbb{N}}\}$

Next, we prove that every irreducible component of B is a regular, local ring. It is enough to prove that irreducible components of R_{ℓ} are regular, local rings. If $\ell|p$ and $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible, then R_{ℓ} is a power-series ring of relative dimension 5 over $\mathcal{O}$ by [65, Theorem 4.1] (in the flat case) and [19, Proposition 2.2] (in the general irreducible case). If $\ell|p$ and $\bar{\rho}|G_{\mathbb{Q}_p}$ is reducible, then the universal framed nearly-ordinary deformation ring R^{ord}_p is a power-series ring of relative dimension 3 by [75, Lemma 2.2]. By [5, Corollary 7.4], R_p is of relative dimension 5 over $\mathcal{O}$ and the kernel of the surjection from R_p onto R^{ord}_p is generated by two elements. These two elements thus necessarily form a regular sequence and so R_p is a regular local ring of relative dimension 5. Now we assume that $\ell \nmid p$. According to [74, Section 5], an irreducible component X of R_{ℓ} is a regular, local ring of dimension 5 except in two cases: if $\ell \equiv -1 \bmod p$, $\bar{\rho}|I_{\ell}$ is trivial and the Weil-Deligne representation attached to a point of X corresponds to a supercupsidal representation of $\mathrm{GL}_2(\mathbb{Q}_{\ell})$ or if $\ell \equiv 1 \bmod p$, $\bar{\rho}|I_{\ell}$ is trivial and the Weil-Deligne representation attached to a point of X corresponds to a ramified irreducible principal series $\pi(\chi, \psi)$ or to a Steinberg representation. In these two cases, ℓ belongs to $\Sigma \backslash \Sigma(\bar{\rho})$ and the restriction of ρ_x to $G_{\mathbb{Q}_{\ell}}$ (resp. I_{ℓ}) is irreducible (resp. non-scalar). They are thus excluded by our assumption that hypothesis 2 of theorem 4.1.1 holds.

Finally, each prime $\ell \in \Sigma^{(p)}$ contributes 4 to the relative dimension of B over $\mathcal{O}$ and R_p contributes 5 to the relative dimension of B over $\mathcal{O}$ so $\dim B = 6 + 4|\Sigma^{(p)}|$. $\square$

Denote by $(R_\infty^\square, \Delta_\infty^\square)$ the limit objects of the patching datum of the Taylor-Wiles system of the previous proposition.

4.2.2 Proof of the main theorem in the minimally ramified case

In this subsection, we conduct the Euler systems and Taylor-Wiles systems argument in the minimally ramified case, so we assume that $U^{(p)}$ is $\bar{\rho}$-minimal and consider the patched system $(D_n, L_n)_{n\geq 1}$ attached to the Taylor-Wiles system $\{\mathcal{O}, R_\varnothing, \Delta_{\Sigma,\mathrm{Iw}}(T_{\Sigma,\mathrm{Iw}}), (R_n, \Delta_n)_{n\in\mathbb{N}}\}$ of proposition 4.2.7. Strictly speaking, the proof presented in this subsection is subsumed by the proof in the general case so its reading is not logically necessary. Nevertheless, it may help discerning the logic of the argument

The following lemma roughly states that conjecture 3.4.12 is false, it remains false after suitable level-raising.

Lemma 4.2.9. *Suppose that in the isomorphism*

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})^{-1} \simeq \mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[y\mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow x\mathbf{T}_{\Sigma,\mathrm{Iw}}] \tag{4.2.2.1}$$

induced by triv_Σ*, y cannot be chosen in $\mathbf{T}_{\Sigma,\mathrm{Iw}}^\times$. Then there exist a sequence $a \in \mathbb{N}^\mathbb{N}$ and choices of trivializations*

$$\mathrm{triv}_{a_n} : \Delta_{a_n}^{-1} \simeq \mathrm{Det}_{R_{a_n}}[y_{a_n}R_{a_n} \longrightarrow x_{a_n}R_{a_n}]$$

such that $x_{a_n}/y_{a_n} \notin R_{a_n}$ for all $n \in \mathbb{N}$ and such that the diagram

$$\begin{array}{ccc} \Delta_{a_m}^{-1} \otimes_{R_{a_m}} D_{a_m} & \longrightarrow & \mathrm{Det}_{D_{a_m}}[y_{a_m}D_{a_m} \longrightarrow x_{a_m}D_{a_m}] \\ \downarrow & & \downarrow \\ \Delta_{a_n}^{-1} \otimes_{R_{a_n}} D_{a_n} & \longrightarrow & \mathrm{Det}_{D_{a_n}}[y_{a_n}D_{a_n} \longrightarrow x_{a_n}D_{a_n}] \end{array} \tag{4.2.2.2}$$

is commutative.

Proof. Suppose more generally that Σ and Σ' are admissible set of primes and let Σ'' be their union. Let J and J' be two ideals of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ and $\mathbf{T}_{\Sigma',\mathrm{Iw}}$ respectively such that $\mathbf{T}_{\Sigma,\mathrm{Iw}}/J$ and $\mathbf{T}_{\Sigma',\mathrm{Iw}}/J'$ are isomorphic to a ring R. Fix a choice of $(x,y) \in \mathbf{T}_{\Sigma,\mathrm{Iw}}$ and $(x',y') \in \mathbf{T}_{\Sigma',\mathrm{Iw}}$ such that triv_Σ and $\mathrm{triv}_{\Sigma'}$ respectively induce isomorphisms

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})^{-1} \simeq \mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[y\mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow x\mathbf{T}_{\Sigma,\mathrm{Iw}}]$$

and

$$\Delta_{\mathbf{T}_{\Sigma',\mathrm{Iw}}}(T_{\Sigma',\mathrm{Iw}})^{-1} \simeq \mathrm{Det}_{\mathbf{T}_{\Sigma',\mathrm{Iw}}}[y'\mathbf{T}_{\Sigma',\mathrm{Iw}} \longrightarrow x'\mathbf{T}_{\Sigma',\mathrm{Iw}}].$$

Assume moreover that x, y do not belong to J and that x', y' do not belong to J'. For $* = \varnothing, '$ or $''$, define

$$\phi_{\Sigma^*} = \frac{y^*}{x^*}\mathrm{triv}_{\Sigma^*}.$$

Equivalently, this corresponds to the trivialization of

$$[y^*\mathbf{T}_{\Sigma^*,\mathrm{Iw}} \longrightarrow x^*\mathbf{T}_{\Sigma^*,\mathrm{Iw}}]$$

obtained by identifying the two terms of the complex. Consider the diagram

$$\begin{array}{ccccccc} & & \Delta_{\mathbf{T}_{\Sigma',\mathrm{Iw}}} & \xrightarrow{\phi_{\Sigma'}} & \mathbf{T}_{\Sigma',\mathrm{Iw}} & & \\ & \overset{\pi_{\Sigma'',\Sigma'}}{\nearrow} & & \nearrow & & \overset{\mathrm{mod}\ J'}{\searrow} & \\ \Delta_{\mathbf{T}_{\Sigma'',\mathrm{Iw}}} & \xrightarrow{\phi_{\Sigma''}} & \mathbf{T}_{\Sigma'',\mathrm{Iw}} & & & & R \\ & \underset{\pi_{\Sigma'',\Sigma}}{\searrow} & & \searrow & & \underset{\mathrm{mod}\ J}{\nearrow} & \\ & & \Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}} & \xrightarrow{\phi_{\Sigma}} & \mathbf{T}_{\Sigma,\mathrm{Iw}} & & \end{array} \quad (4.2.2.3)$$

in which rings are viewed as free modules of rank 1 over themselves. Each arrow then sends a generator of its source to a generator of its target. Consequently, if there is an isomorphism

$$\Delta_{\mathbf{T}_{\Sigma'',\mathrm{Iw}}}(T_{\Sigma'',\mathrm{Iw}})^{-1} \simeq \mathrm{Det}_{\mathbf{T}_{\Sigma'',\mathrm{Iw}}}[y''\mathbf{T}_{\Sigma'',\mathrm{Iw}} \longrightarrow x''\mathbf{T}_{\Sigma'',\mathrm{Iw}}] \quad (4.2.2.4)$$

in which y'' is in $\mathbf{T}^{\times}_{\Sigma'',\mathrm{Iw}}$, then there exist isomorphisms

$$\Delta_{\mathbf{T}_{\Sigma^*,\mathrm{Iw}}}(T_{\Sigma^*,\mathrm{Iw}})^{-1} \simeq \mathrm{Det}_{\mathbf{T}_{\Sigma^*,\mathrm{Iw}}}[y^*\mathbf{T}_{\Sigma^*,\mathrm{Iw}} \longrightarrow x^*\mathbf{T}_{\Sigma^*,\mathrm{Iw}}]$$

with $y^* \in \mathbf{T}^{\times}_{\Sigma^*,\mathrm{Iw}}$ for $* = \varnothing,','',$. By contraposition, if there is no isomorphism

$$\Delta_{\mathbf{T}_{\Sigma^*,\mathrm{Iw}}}(T_{\Sigma^*,\mathrm{Iw}})^{-1} \simeq \mathrm{Det}_{\mathbf{T}_{\Sigma^*,\mathrm{Iw}}}[y^*\mathbf{T}_{\Sigma^*,\mathrm{Iw}} \longrightarrow x^*\mathbf{T}_{\Sigma^*,\mathrm{Iw}}] \quad (4.2.2.5)$$

with $y^* \in \mathbf{T}^{\times}_{\Sigma^*,\mathrm{Iw}}$ for $* = \varnothing$ (resp $* ='$), then there is no isomorphism (4.2.2.4) with $y'' \in \mathbf{T}^{\times}_{\Sigma'',\mathrm{Iw}}$ and so no isomorphism (4.2.2.5) for $* ='$ (resp. $* = \varnothing$) with $y^* \in \mathbf{T}^{\times}_{\Sigma^*,\mathrm{Iw}}$.

We now return to the claims of the lemma. The commutativity of diagram (4.2.2.2) can always be arranged by construction of the set $\{D_m\}_{m\in\mathbb{N}}$ of patched rings. If, as in the lemma, we assume in addition that in

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})^{-1} \simeq \mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[y\mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow x\mathbf{T}_{\Sigma,\mathrm{Iw}}], \quad (4.2.2.6)$$

the element y cannot be chosen in $\mathbf{T}^{\times}_{\Sigma,\mathrm{Iw}}$, then for $N \in \mathbb{N}$ large enough, x and y do not belong to $\mathfrak{m}^N_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}$. Hence, for m large enough, x and y do not belong to the ideal $\mathfrak{c}_m R_{Q(m)} + \mathfrak{m}^{(r_m)}_{R_{Q(m)}}$. If two integers n_1, n_2 are large enough, the reasoning of the first part of the proof thus applies to the rings $R_{n_1} = \mathbf{T}_{\Sigma\cup Q_{n_1},\mathrm{Iw}}$ and $R_{n_2} = \mathbf{T}_{\Sigma\cup Q_{n_2},\mathrm{Iw}}$ and to the ideals $\left(\mathfrak{c}_{m_i} R_{n_i} + \mathfrak{m}^{(r_{m_i})}_{R_{n_i}}\right)$ for $i \in \{1,2\}$ and m_i as in the definition of the patched system. This yields a sequence $(a_n)_{n\in\mathbb{N}}$ such that the Δ_{a_n} verify the desired properties. □

Replacing the Taylor-Wiles system $\{\mathcal{O}, R_\varnothing, \Delta_{\Sigma,\mathrm{Iw}}(T_{\Sigma,\mathrm{Iw}}), (R_n, \Delta_n)_{n\in\mathbb{N}}\}$ with the Taylor-Wiles system $\{\mathcal{O}, R_\varnothing, \Delta_{\Sigma,\mathrm{Iw}}(T_{\Sigma,\mathrm{Iw}}), (R_{a_n}, \Delta_{a_n})_{n\in\mathbb{N}}\}$ if necessary, we get a system of fundamental lines $(\Delta_n)_{n\in\mathbb{N}}$ with, for each n, a specified isomorphism

$$\Delta_n^{-1} \simeq \mathrm{Det}_{R_n}[y_n R_n \longrightarrow x_n R_n]$$

compatible with the isomorphisms $D_{n+1}/(\mathfrak{c}_n D_{n+1} + \mathfrak{m}^{(r_n)}_{D_{n+1}}) \overset{\sim}{\longrightarrow} D_n$ and in particular such that y_n belongs to the maximal ideal of R_n for all n. By construction, the limit object Δ_∞ of the patching datum of this Taylor-Wiles system is endowed with a specified isomorphism (which depends highly on all our previous choices)

$$\Delta_\infty^{-1} \simeq \mathrm{Det}_{R_\infty}[y_\infty R_\infty \longrightarrow x_\infty R_\infty]$$

with y_∞ in the maximal ideal of R_∞ and $y_\infty \nmid x_\infty$. The aim of the following lemma is to show that this implies the equivalent of conjecture 3.4.12 is false for some specializations with values in mixed characteristic regular rings of dimension 2.

Lemma 4.2.10. *If there is an isomorphism*

$$\Delta_{\infty}^{-1} \simeq \mathrm{Det}_{R_\infty}[y_\infty R_\infty \longrightarrow x_\infty R_\infty]$$

with y_∞ in the maximal ideal of R_∞ and $y_\infty \nmid x_\infty$, then there exists an admissible set of primes Σ', a discrete valuation ring S finite and flat over $\mathbb{Z}_p$ and a specialization

$$\psi : \mathbf{T}_{\Sigma',\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$$

satisfying the following properties.

1. *The specialization factors through a single irreducible component of $\mathbf{T}_{\Sigma',\mathrm{Iw}}$.*
2. *The class $\mathbf{z}_\psi$ is not zero.*
3. *The group $H^2(G_{\mathbb{Q}_p}, T_\psi)$ is finite.*
4. *The trivialization morphism triv_ψ does not induce an inclusion $\Delta_{S_{\mathrm{Iw}}}(T_\psi)^{-1} \hookrightarrow S_{\mathrm{Iw}}$.*

Proof. The ring R_∞ is a power-series ring of Krull dimension at least 4 equal to the inverse limit of a projective system of quotients of R_n with surjective transition maps so it admits a local morphism $\psi_0 : R_\infty \longrightarrow S_0$ satisfying the following properties.

1. The ring S_0 is a principal artinian quotient of a $\mathbb{Z}_p$-flat discrete valuation ring of residual characteristic p.
2. Neither $\psi_0(y_\infty)$ nor $\psi_0(x_\infty)$ are zero and
$$\max\{n \in \mathbb{N} | \psi_0(x_\infty) \in \mathfrak{m}_{S_0}^n\} < \max\{n \in \mathbb{N} | \psi_0(y_\infty) \in \mathfrak{m}_{S_0}^n\}.$$
3. There exists a surjection $R_n \twoheadrightarrow S_0$ for some $n \in \mathbb{N}$.

Denote by X the formal variable corresponding to cyclotomic deformations of $\bar{\rho}$. Then there exists formal variables $Y_1, \cdots, Y_m$ and a regular sequence $(f_1, \cdots, f_{m-2})$ in $\mathcal{O}[[Y_1, \cdots, Y_m]]$ such that R_n is a complete intersection ring of the form $\mathcal{O}[[X, Y_1, \cdots, Y_m]]/(f_1, \cdots, f_{m-2})$. By the smoothness of $\mathcal{O}[[Y_1, \cdots, Y_m]]$, there exists a complete intersection ring of relative dimension zero over $\mathcal{O}$ and a morphism $\psi_1 : R_n \longrightarrow S_1[[X]]$ such that the diagram

$$\begin{array}{ccc} & & S_1[[X]] \\ & \overset{\psi_1}{\nearrow} & \downarrow \pi \\ R_n & \xrightarrow{\psi_0} & S_0 \end{array}$$

is commutative. Denote by S the normalization of S_1 and denote by ψ the natural morphism $\psi : R_n \longrightarrow S_{\mathrm{Iw}}$. Suppose that the image of y_n in S_{Iw} divides the image of x_n in S_{Iw}. Then $\psi_0(y_\infty) = \pi \circ \psi(y_n)$ divides $\psi_0(x_\infty) = \pi \circ \psi(x_n)$; in contradiction with the definition of S_0. In particular, for any choice of $(x_\psi, y_\psi) \in S_{\mathrm{Iw}}^2$ such that there is an isomorphism

$$\Delta_S(T_\psi)^{-1} \simeq \mathrm{Det}_S[y_\psi S \longrightarrow x_\psi S],$$

the map triv_ψ does not induce an inclusion $\Delta_{S_{\mathrm{Iw}}}(T_\psi)^{-1} \hookrightarrow S_{\mathrm{Iw}}$.

It remains to show that there exists such a ψ satisfying in addition the requirements 1, 2 and 3 of the lemma. The union of S_{Iw}-valued points ψ of $\mathrm{Spec}\,\mathbf{T}_{\Sigma',\mathrm{Iw}}$ such that either $\mathbf{z}_\psi$ is a divisor of zero, or $H^2(G_{\mathbb{Q}_p}, T_\psi)$ is infinite or ψ factors through several distinct irreducible components is contained in a subscheme of codimension greater than 1 whereas the requirement 4 is open. Hence, there exists ψ as in the statement of the lemma. □

Proof of theorem 4.1.1 in the minimally ramified case. Recall that we are assuming that $\bar{\rho}$ satisfies assumption 1 4.2.1 and 4.2.2 and that $U^{(p)}$ is $\bar{\rho}$-minimal. Further assume by way of contradiction that triv_{Σ} does not induce an inclusion $\Delta_{\Sigma,\mathrm{Iw}}(T_{\Sigma,\mathrm{Iw}})^{-1} \hookrightarrow \mathbf{T}_{\Sigma,\mathrm{Iw}}$. According to lemma 4.2.9 and 4.2.10, there then exists an admissible set Σ', a discrete valuation ring finite and flat over $\mathbb{Z}_p$ and a specialization $\psi : \mathbf{T}_{\Sigma',\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$ such that $\mathbf{z}_{\psi}$ is not zero, the group $H^2(G_{\mathbb{Q}_p}, T_{\psi})$ is finite and triv_{ψ} does not induce an inclusion $\Delta_{S_{\mathrm{Iw}}}(T_{\psi})^{-1} \hookrightarrow S_{\mathrm{Iw}}$.

By assumption, the image of $\bar{\rho}$ acts irreducibly on $\bar{\mathbb{F}}_p^2$ and is of order divisible by p so contains a subgroup conjugate to $\mathrm{SL}_2(\mathbb{F}_q)$ for some $q = p^n$ and hence an element $\bar{\sigma} \neq \mathrm{Id}$ which is unipotent. Let σ be a lift of $\bar{\sigma}$ to $\rho_{\psi}(G_{\mathbb{Q}})$. Then the kernel of $\sigma - 1$ is strictly included in T_{ψ} and its cokernel is dimension 1 after tensor product with $S/\mathfrak{m}_S$. Hence, the cokernel of $\sigma - 1$ is not torsion and is generated by a single element, so it is free of rank 1. The representation T_{ψ}/XT_{ψ} thus satisfies hypotheses ($\mathrm{i}_{\mathrm{str}}$), ($\mathrm{ii}_{\mathrm{str}}$) and ($\mathrm{iv}_{\mathfrak{p}}$) of [42, Theorem 0.8]. As T_{ψ}/XT_{ψ} is absolutely irreducible and not abelian, the conclusion of [42, Proposition 8.7] also holds. As establishing this proposition is the sole function of hypothesis (iii) of [42, Theorem 0.8], the conclusion of this theorem holds for T_{ψ}/XT_{ψ}. As $H^2(G_{\mathbb{Q}_p}, T_{\psi})$ is finite, its localization any $\mathfrak{p} \in \operatorname{Spec} S_{\mathrm{Iw}}$ of height 1 vanishes and so there is a canonical isomorphism between its determinant and S_{Iw}. Hence, there is a canonical isomorphism

$$\mathrm{Det}_{S_{\mathrm{Iw}}} \mathrm{R}\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], T_{\psi}) \otimes_{S_{\mathrm{Iw}}} M_{\psi,\mathrm{Iw}} \overset{\mathrm{can}}{\simeq} \mathrm{Det}_{S_{\mathrm{Iw}}}[S_{\mathrm{Iw}} \longrightarrow x_{S_{\mathrm{Iw}}} S_{\mathrm{Iw}}]. \tag{4.2.2.7}$$

Denote by $\psi^* : R(\mathfrak{a})_{\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$ the specialization through which ψ factors. As S_{Iw} is a regular local ring, the left-hand side of (4.2.2.7) is canonically isomorphic to $\Delta_{S_{\mathrm{Iw}}}(T_{\psi^*})$ and the isomorphism (4.2.2.7) coincides with the trivialization of conjecture 3.4.14. In particular, taking tensor product with $\mathrm{Frac}(S_{\mathrm{Iw}})$ and identifying the determinants of acyclic complexes with $\mathrm{Frac}(S_{\mathrm{Iw}})$ induces an isomorphism

$$\Delta_{S_{\mathrm{Iw}}}(T_{\psi^*})^{-1} \simeq \mathrm{Det}_{S_{\mathrm{Iw}}}[S_{\mathrm{Iw}} \longrightarrow x_{S_{\mathrm{Iw}}} S_{\mathrm{Iw}}]. \tag{4.2.2.8}$$

Removing Euler factors then shows that triv_{ψ} induces an injection $\Delta_{S_{\mathrm{Iw}}}(T_{\psi})^{-1} \hookrightarrow S_{\mathrm{Iw}}$, in contradiction with the properties of ψ.

Hence, it is possible to choose $y \in \mathbf{T}_{\Sigma,\mathrm{Iw}}^{\times}$ in equation (4.2.2.1) and the first assertion of theorem 4.1.1 is established. We make a choice of (x, y) in equation 4.2.2.1 with $y \in \mathbf{T}_{\Sigma,\mathrm{Iw}}^{\times}$. If moreover assumption 3 of theorem 4.1.1 holds, then by theorem 3.3.16 there exists a classical point $z \in \operatorname{Spec} \mathbf{T}_{\Sigma,\mathrm{Iw}}[1/p]$ such that the image of x in $\mathbf{T}_{\Sigma,\mathrm{Iw}}^{z}$ is a unit. Hence, x is an element of $\mathbf{T}_{\Sigma,\mathrm{Iw}}^{\times}$ and the second assertion of 4.1.1 thus holds. □

4.2.3 Proof of the main theorem in the general case

In this subsection, we relax our assumption that $U^{(p)}$ be $\bar{\rho}$-minimal and consider the patched system $(D_n, L_n)_{n\geq 1}$ attached to the Taylor-Wiles system $\{B, R_{\varnothing}^{\square}, \Delta_{\Sigma,\mathrm{Iw}}^{\square}, (R_n^{\square}, \Delta_n^{\square})_{n\in\mathbb{N}}\}$ of proposition 4.2.8.

The following lemma is the equivalent of lemma 4.2.9 (note however that contrary to its counterpart in the minimally ramified case, it does not have a transparent arithmetic meaning).

Lemma 4.2.11. *Suppose that in the isomorphism*

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})^{-1} \simeq \mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[y\mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow x\mathbf{T}_{\Sigma,\mathrm{Iw}}] \tag{4.2.3.1}$$

induced by triv_{Σ}*, y cannot be chosen in $\mathbf{T}_{\Sigma,\mathrm{Iw}}^{\times}$. Then there exist a sequence $a \in \mathbb{N}^{\mathbb{N}}$ and choices of trivializations*

$$\mathrm{triv}_{a_n} : \left(\Delta_{a_n}^{\square}\right)^{-1} \simeq \mathrm{Det}_{R_{a_n}^{\square}}[y_{a_n} R_{a_n}^{\square} \longrightarrow x_{a_n} R_{a_n}^{\square}]$$

such that $x_{a_n}/y_{a_n} \notin R^{\square}_{a_n}$ *for all* $n \in \mathbb{N}$ *and such that the diagram*

$$\begin{array}{ccc} \left(\Delta^{\square}_{a_m}\right)^{-1} \otimes_{R^{\square}_{a_m}} D_{a_m} & \longrightarrow & \mathrm{Det}_{D_{a_m}}[y_{a_m}D_{a_m} \longrightarrow x_{a_m}D_{a_m}] \\ \downarrow & & \downarrow \\ \left(\Delta^{\square}_{a_n}\right)^{-1} \otimes_{R^{\square}_{a_n}} D_{a_n} & \longrightarrow & \mathrm{Det}_{D_{a_n}}[y_{a_n}D_{a_n} \longrightarrow x_{a_n}D_{a_n}] \end{array}$$

is commutative.

Proof. Because $R^{\square}_{\varnothing}$ is the power-series ring $\mathbf{T}_{\Sigma,\mathrm{Iw}}[[Y_1, Y_2, Y_3]]$, if in the isomorphism

$$\Delta_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}(T_{\Sigma,\mathrm{Iw}})^{-1} \simeq \mathrm{Det}_{\mathbf{T}_{\Sigma,\mathrm{Iw}}}[y\mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow x\mathbf{T}_{\Sigma,\mathrm{Iw}}] \tag{4.2.3.2}$$

induced by triv_{Σ}, y cannot be chosen in $\mathbf{T}^{\times}_{\Sigma,\mathrm{Iw}}$, then in the trivialization

$$\left(\Delta^{\square}\right)^{-1} \simeq \mathrm{Det}_{R^{\square}_{\varnothing}}[yR^{\square}_{\varnothing} \longrightarrow xR^{\square}_{\varnothing}]$$

induced from trivialization (4.2.3.2) by tensor product over $R_{\varnothing} = \mathbf{T}_{\Sigma,\mathrm{Iw}}$ with $R^{\square}_{\varnothing}$, it is not possible to choose y a unit in $R^{\square}_{\varnothing}$.

The proof of lemma 4.2.9 then relies exclusively on the compatibility of $\Delta_{\Sigma,\mathrm{Iw}}$ with arbitrary change of ring of coefficients so can be reproduced *verbatim.* □

As in the minimal case, we replace the Taylor-Wiles system $\{B, R^{\square}_{\varnothing}, \Delta^{\square}_{\Sigma,\mathrm{Iw}}, (R^{\square}_n, \Delta^{\square}_n)_{n\in\mathbb{N}}\}$ with the Taylor-Wiles system $\{B, R^{\square}_{\varnothing}, \Delta^{\square}_{\Sigma,\mathrm{Iw}}, (R^{\square}_{a_n}, \Delta^{\square}_{a_n})_{n\in\mathbb{N}}\}$ if necessary and deduce the existence of an isomorphism

$$\left(\Delta^{\square}_{\infty}\right)^{-1} \simeq \mathrm{Det}_{R^{\square}_{\infty}}[y_{\infty}R^{\square}_{\infty} \longrightarrow x_{\infty}R^{\square}_{\infty}]$$

with y_{∞} in the maximal ideal of $R^{\square}_{\infty}$ and $x_{\infty}/y_{\infty} \notin R^{\square}_{\infty}$.

Lemma 4.2.12. *If there is an isomorphism*

$$\left(\Delta^{\square}_{\infty}\right)^{-1} \simeq \mathrm{Det}_{R^{\square}_{\infty}}[y_{\infty}R^{\square}_{\infty} \longrightarrow x_{\infty}R^{\square}_{\infty}] \tag{4.2.3.3}$$

with y_{∞} *in the maximal ideal of* $R^{\square}_{\infty}$ *and* $x_{\infty}/y_{\infty} \notin R^{\square}_{\infty}$*, then there exists an admissible set of primes* Σ'*, a discrete valuation ring finite and flat over* $\mathbb{Z}_p$ *and a specialization*

$$\psi : \mathbf{T}_{\Sigma',\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$$

satisfying the following properties.

1. *The specialization factors through a single irreducible component of* $\mathbf{T}_{\Sigma',\mathrm{Iw}}$.
2. *The class* $\mathbf{z}_{\psi}$ *is not zero.*
3. *The group* $H^2(G_{\mathbb{Q}_p}, T_{\psi})$ *is finite.*
4. *The trivialization morphism* triv_{ψ} *does not induce an inclusion* $\Delta_{S_{\mathrm{Iw}}}(T_{\psi})^{-1} \hookrightarrow S_{\mathrm{Iw}}$.

Proof. Because $R^{\square}_{\infty}$ is reduced, if there exists a trivialization 4.2.3.3 with $x_{\infty}/y_{\infty} \notin R^{\square}_{\infty}$, there exists a minimal prime ideal $\mathfrak{a}_{\infty}$ of $R^{\square}_{\infty}$ and trivialization

$$\left(\Delta^{\square}_{\infty} \otimes_{R^{\square}_{\infty}} R^{\square}_{\infty}/\mathfrak{a}_{\infty}\right)^{-1} \simeq \mathrm{Det}_{R^{\square}_{\infty}/\mathfrak{a}_{\infty}}[y_{\infty}R^{\square}_{\infty}/\mathfrak{a}_{\infty} \longrightarrow x_{\infty}R^{\square}_{\infty}/\mathfrak{a}_{\infty}] \tag{4.2.3.4}$$

such that the images $\tilde{x}_{\infty}$ and $\tilde{y}_{\infty}$ (which are both non-zero) do not satisfy that $\tilde{x}_{\infty}/\tilde{y}_{\infty}$ belongs to $R^{\square}_{\infty}/\mathfrak{a}_{\infty}$. Because every irreducible component is a regular local ring, $R^{\square}_{\infty}/\mathfrak{a}_{\infty}$ is a regular local ring by lemma 4.2.6. The ring $R^{\square}_{\infty}/\mathfrak{a}_{\infty}$ is a regular local ring of Krull dimension at least 10 equal to an irreducible component of an inverse limit of a projective system of quotients $R^{\square}_n$ with surjective transition maps so it admits a local morphism $\psi_0 : R^{\square}_{\infty} \longrightarrow S_0$ satisfying the following properties.

1. The ring S_0 is a principal artinian quotient of a $\mathbb{Z}_p$-flat discrete valuation ring of residual characteristic p.

2. Neither $\psi_0(y_\infty)$ nor $\psi_0(x_\infty)$ are zero and

$$\max\{n \in \mathbb{N} | \psi_0(x_\infty) \in \mathfrak{m}_{S_0}^n\} < \max\{n \in \mathbb{N} | \psi_0(y_\infty) \in \mathfrak{m}_{S_0}^n\}.$$

3. There exists a surjection $R_n^\square \twoheadrightarrow S_0$ for some $n \in \mathbb{N}$.

Forgetting the choice of basis induces a surjection $R_n \twoheadrightarrow S_0$ after which the proof becomes the exact reproduction of the proof of lemma 4.2.10. □

Proof of theorem 4.1.1 in the general case. Assume by way of contradiction that triv_Σ does not induce an inclusion $\Delta_{\Sigma,\mathrm{Iw}}(T_{\Sigma,\mathrm{Iw}})^{-1} \hookrightarrow \mathbf{T}_{\Sigma,\mathrm{Iw}}$. According to lemma 4.2.11 and 4.2.12, there then exists an admissible set Σ', a discrete valuation ring finite and flat over $\mathbb{Z}_p$ and a specialization $\psi : \mathbf{T}_{\Sigma',\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$ such that $\mathbf{z}_\psi$ is not zero, the group $H^2(G_{\mathbb{Q}_p}, T_\psi)$ is finite and triv_ψ does not induce an inclusion $\Delta_{S_{\mathrm{Iw}}}(T_\psi)^{-1} \hookrightarrow S_{\mathrm{Iw}}$. The end of the proof is then exactly as in the minimally ramified case. □

We conclude with the proof of corollary 4.1.3.

Proof of corollary 4.1.3. Let Λ be the regular, local ring $\mathcal{O}[[X_1, X_2, X_3]]$. Under our ongoing assumptions on $\bar{\rho}$, the ring $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is a complete intersection Λ-algebra of relative dimension zero hence a finite, Cohen-Macaulay Λ-module. So $\mathbf{T}_{\Sigma,\mathrm{Iw}}$ is free of finite rank d as Λ-module. We view $T_{\Sigma,\mathrm{Iw}}$ and $Z_{\Sigma,\mathrm{Iw}}$ as free Λ-modules of finite ranks $2d$ and d respectively. As in definition 3.3.15, we put

$$\Delta_\Lambda(T_{\Sigma,\mathrm{Iw}}) = \mathrm{Det}_\Lambda^{-1} \mathrm{R}\,\Gamma_c(\mathbb{Z}[1/\Sigma], T_{\Sigma,\mathrm{Iw}}) \otimes_\Lambda \mathrm{Det}_\Lambda^{-1} Z_{\Sigma,\mathrm{Iw}}$$

and observe that (3.4.3.1) induces a trivialization morphisme

$$\mathrm{triv}_\Lambda : \Delta_\Lambda(T_{\Sigma,\mathrm{Iw}}) \otimes_\Lambda \mathrm{Frac}(\Lambda) \simeq \mathrm{Frac}(\Lambda).$$

Let S be a discrete valuation ring finite and flat over $\mathbb{Z}_p$ and le $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$ be a specialization such that $\mathbf{z}_\psi$ is not zero and such that the group $H^2(G_{\mathbb{Q}_p}, T_\psi)$ is finite. As in the proof of theorem 4.1.1 in the minimally ramified case, the $G_{\mathbb{Q},\Sigma}$-representation $(T_\psi, \rho_\psi, S_{\mathrm{Iw}})$ satisfies the hypotheses $(\mathrm{i}_{\mathrm{str}})$, $(\mathrm{ii}_{\mathrm{str}})$ and $(\mathrm{iv}_{\mathfrak{p}})$ of [42, Theorem 0.8] and so the trivialization map triv_ψ induces an embedding $\Delta_{S_{\mathrm{Iw}}}(T_\psi)^{-1} \hookrightarrow S_{\mathrm{Iw}}$. Let S' be the discrete valuation ring generated by the image of Λ through ψ. Viewing T_ψ as a module over the discrete valuation ring S'_{Iw} generated by the image of Λ through ψ, we deduce that the trivialization isomorphism $\left(\Delta_{S'_{\mathrm{Iw}}}(T_\psi)^{-1}\right) \simeq \mathrm{Frac}\,(S'_{\mathrm{Iw}})$ also induces an embedding $\Delta_{S'_{\mathrm{Iw}}}(T_\psi)^{-1} \hookrightarrow S'_{\mathrm{Iw}}$. If triv_Λ does not induce an embedding $\Delta_\Lambda(T_{\Sigma,\mathrm{Iw}})^{-1} \hookrightarrow \Lambda$, there exists such a specialization with $\Delta_{S'_{\mathrm{Iw}}}(T_\psi)^{-1} \not\hookrightarrow S'_{\mathrm{Iw}}$; a contradiction.

Suppose as in the hypotheses of the corollary that there furthermore exists a modular point $\psi : \mathbf{T}_{\Sigma,\mathrm{Iw}} \longrightarrow S_{\mathrm{Iw}}$ such that triv induces an isomorphism $\Delta_{S_{\mathrm{Iw}}}(T_\psi) \simeq S_{\mathrm{Iw}}$. The the restriction of ψ to Λ composed with extension of scalar to S_{Iw} also induces $\Delta_{S_{\mathrm{Iw}}}(T_\psi) \simeq S_{\mathrm{Iw}}$ and so the embedding $\Delta_\Lambda(T_{\Sigma,\mathrm{Iw}})^{-1} \hookrightarrow \Lambda$ identifies $\Delta_\Lambda(T_{\Sigma,\mathrm{Iw}})^{-1}$ with Λ. Thus the trivialization morphism induces an identification of $\Delta_{S'_{\mathrm{Iw}}}(T_\psi)$ with S'_{Iw} for all point of Λ which induces a modular point of $\mathbf{T}_{\Sigma,\mathrm{Iw}}$. □

A Appendices

A.1 The determinant functor

Let Λ be a ring. A complex of Λ-modules is perfect if and only if it is quasi-isomorphic to a bounded complex of projective Λ-modules. A Λ-module is perfect if it is a perfect complex, that is to say and if and only if it has finite projective dimension over Λ. If Λ is a local noetherian ring, then all bounded complexes of Λ-modules are perfect if (and only if) Λ is regular by the theorem of Auslander-Buchsbaum and Serre.

The determinant functor $\operatorname{Det}_\Lambda(-)$ of [49] (see also [17]) is a functor

$$\operatorname{Det}_\Lambda P = \left(\bigwedge_{\Lambda}^{\operatorname{rank}_\Lambda P} P, \operatorname{rank}_\Lambda P \right)$$

from the category of projective Λ-modules (with morphisms restricted to isomorphisms) to the symmetric monoidal category of graded invertible Λ-modules (with morphisms restricted to isomorphisms). The determinant functor admits an extension to a functor from the category of perfect complexes of Λ-modules with morphisms restricted to quasi-isomorphisms to the category of graded invertible Λ-modules by setting

$$\operatorname{Det}_\Lambda C^\bullet = \bigotimes_{i\in\mathbb{Z}} \operatorname{Det}_\Lambda^{(-1)^i} C^i. \tag{A.1.1}$$

This extension is the unique up to canonical isomorphism which satisfies the properties of [49, Definition I]. In particular, $\operatorname{Det}_\Lambda(-)$ commutes with derived tensor product; there is a canonical isomorphism between $\operatorname{Det}_\Lambda(0)$ and $(\Lambda, 0)$ and there exists a canonical isomorphism

$$\iota_\Lambda(\alpha,\beta) : \operatorname{Det}_\Lambda C_2^\bullet \overset{\text{can}}{\simeq} \operatorname{Det}_\Lambda C_1^\bullet \otimes_\Lambda \operatorname{Det}_\Lambda C_3^\bullet \tag{A.1.2}$$

compatible with base-change whenever

$$0 \longrightarrow C_1^\bullet \overset{\alpha}{\longrightarrow} C_2^\bullet \overset{\beta}{\longrightarrow} C_3^\bullet \longrightarrow 0$$

is a short exact sequence of complexes. If Λ is reduced, $\operatorname{Det}_\Lambda(-)$ further extends to the derived category of perfect complexes of Λ-modules with morphisms restricted to quasi-isomorphisms and (A.1.2) extends to distinguished triangles. The unit object $(\Lambda, 0)$ of this category is simply written Λ.

A.2 Étale and Galois cohomology

A finite p-ring Λ is a ring which is of finite cardinality as set and such that for every element $x \in \Lambda$, there exists $n \in \mathbb{N}$ such that $p^n x = 0$. A compact p-ring Λ is a compact inverse limit of finite p-rings. A p-ring with p inverted Λ is a ring $A[1/p]$ with A a compact p-ring.

For Λ a finite p-ring killed by a power of p and X a noetherian scheme, let $D(X, \Lambda)$ be the derived category of perfect complexes of sheaves of Λ-modules on X_{et}. A complex of sheaves $\mathscr{F}$ in $D(X, \Lambda)$ is said to be smooth if its cohomology satisfies the following properties.

1. The cohomology sheaves $H^i(X_{\text{et}}, \mathscr{F})$ are constructible ([15, Définition 4.3.2]) for all $i \in \mathbb{Z}$ and zero for i outside a finite range.
2. For all $x \in X$, the stalk $\mathscr{F}_{\bar{x}}$ at $x \in X$ is a perfect complex of Λ-modules.

Let $D_{\text{ctf}}(X, \Lambda)$ be the full sub-category of $D(X, \Lambda)$ whose objects are the smooth complexes of étale sheaves. Suppose now that Λ is a compact p-ring. As in [41, Section 3.1], a smooth complex of étale sheaves $\mathscr{F}$ of Λ-modules on X is by definition a projective system of smooth complexes of sheaves

$$(\mathscr{F})_{I\in U} \in (D_{\text{ctf}}(X, \Lambda/I))_{I\in U}$$

indexed by the set U of open ideals of Λ partially ordered by inclusion such that for all pairs $I \supset J$ of open ideals, the transition map $p_{I,J} : \mathscr{F}_J \longrightarrow \mathscr{F}_I$ factors through an isomorphism $\pi_{I,J} : \mathscr{F}_J \overset{\mathrm{L}}{\otimes}_{\Lambda/J} \Lambda/I \simeq \mathscr{F}_I$ verifying $\pi_{I,I} = \operatorname{Id}$ and $\pi_{J,K} \circ \pi_{I,J} = \pi_{I,K}$ if $I \supset J \supset K$. The category $D_{\text{ctf}}(X, \Lambda)$ is then the category of projective systems of smooth complexes of étale sheaves. Finally, suppose that Λ is a p-ring with p-inverted, so is of the form $A[1/p]$ where A is a compact p-ring. Then $D_{\mathrm{ctf}}(X,\Lambda)$ is the quotient of the category $D_{\mathrm{ctf}}(X,A)$ by the subcategory of p-torsion complex of sheaves (see [15, Section 2.8,2.9]). When Λ is a finite product of finite extensions of $\mathbb{Q}_p$, this recovers the definition of [41, Section 3.1.2 (3)].

Let Λ and Λ' be p-rings as above. Let $\mathscr{F} \in D_{\mathrm{ctf}}(X,\Lambda)$ be a smooth complex of étale sheaves and let $\mathscr{G}$ be the sheaf $\mathscr{F} \overset{\mathrm{L}}{\otimes}_\Lambda \Lambda'$ in $D_{\mathrm{ctf}}(X,\Lambda')$. The functor $\mathrm{R}\,\Gamma_{\mathrm{et}}(X,-)$ being triangulated and way-out, $\mathrm{R}\,\Gamma_{\mathrm{et}}(X,\mathscr{F})$ is a perfect complex of Λ-modules and there is a canonical isomorphism

$$\mathrm{R}\,\Gamma_{\mathrm{et}}(X,\mathscr{F}) \overset{\mathrm{L}}{\otimes}_\Lambda \Lambda' \overset{\mathrm{can}}{\simeq} \mathrm{R}\,\Gamma_{\mathrm{et}}(X,\mathscr{G}) \tag{A.2.1}$$

inducing a canonical isomorphism

$$(\mathrm{Det}_\Lambda \mathrm{R}\,\Gamma_{\mathrm{et}}(X,\mathscr{F})) \otimes_\Lambda \Lambda' \overset{\mathrm{can}}{\simeq} \mathrm{Det}_{\Lambda'} \mathrm{R}\,\Gamma_{\mathrm{et}}(X,\mathscr{F} \overset{\mathrm{L}}{\otimes}_\Lambda \Lambda'). \tag{A.2.2}$$

In the body of the manuscript, the only X we have cause to consider arise in the following way. Let $F/\mathbb{Q}$ be a finite extension of $\mathbb{Q}$ with ring of integers $\mathscr{O}_F$ and let Σ be a finite set of primes of $\mathscr{O}_F$ containing the primes above p. Let X be either $\mathrm{Spec}\,\mathscr{O}_F[1/\Sigma]$, or $\mathrm{Spec}\,\mathbb{R}$ or $\mathrm{Spec}\,F_v$ for v a finite prime (which may divide p) and denote by G the group $\pi_1^{\mathrm{et}}(X)$ (which is thus either $G_{F,\Sigma}$, or $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, or G_{F_v}). In that case, if $M^\bullet$ is a perfect complex of Λ-modules which are ind-admissible $\Lambda[G_{F,\Sigma}]$-modules (in the sense of [58, Section 3.3]), then $M^\bullet$ gives rise to a smooth étale sheaf $\mathscr{F}$ on X and $\mathrm{R}\,\Gamma_{\mathrm{et}}(X,\mathscr{F})$ is canonically isomorphic to the complex $\mathrm{R}\,\Gamma(G,M^\bullet)$ computing continuous group cohomology. Indeed, this is true if Λ is a finite p-ring and the general result reduces to this case as both $\mathrm{R}\,\Gamma_{\mathrm{et}}(X,-)$ and $\mathrm{R}\,\Gamma(G,-)$ are triangulated and way-out.

We note that when $X=\mathrm{Spec}\,\mathscr{O}_F[1/\Sigma]$, the complex $\mathrm{R}\,\Gamma_{\mathrm{et}}(X,\mathscr{F})$ need not be perfect if $\mathscr{F}$ is only known to belong to $D(X,\Lambda)$ and that, as 2 is invertible in Λ, the functor $H^0_{\mathrm{et}}(\mathrm{Spec}\,\mathbb{R},-)=H^0(\mathrm{Gal}(\mathbb{C}/\mathbb{R}),-)$ sends perfect complexes to perfect complexes and commutes with change of ring of coefficients.

A.3 Selmer complexes

For X equal to $\mathrm{Spec}\,\mathscr{O}_F[1/\Sigma]$, the complex $\mathrm{R}\,\Gamma_c(X,\mathscr{F})$ of étale cohomology with compact support outside p is defined to be the complex

$$\mathrm{Cone}\left(\mathrm{R}\,\Gamma_{\mathrm{et}}(X,\mathscr{F}) \oplus \bigoplus_{v|p} \mathrm{R}\,\Gamma_{\mathrm{et}}(\mathrm{Spec}\,F_v,\mathscr{F}) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{R}\,\Gamma_{\mathrm{et}}(\mathrm{Spec}\,F_v,\mathscr{F})\right)[-1].$$

Note that, contrary to normal usage, we do not impose a condition at primes dividing p. The complex $\mathrm{R}\,\Gamma_c(X,\mathscr{F})$ of Λ-modules satisfies

$$\mathrm{R}\,\Gamma_c(X,\mathscr{F}) \overset{\mathrm{L}}{\otimes}_\Lambda \Lambda' \overset{\mathrm{can}}{\simeq} \mathrm{R}\,\Gamma_c(X,\mathscr{G}) \tag{A.3.1}$$

if $\mathscr{G}=\mathscr{F} \overset{\mathrm{L}}{\otimes}_\Lambda \Lambda'$. Hence the determinant of compactly supported cohomology is defined if $\mathscr{F}$ is in $D_{\mathrm{ctf}}(X,\Lambda)$ and satisfies

$$(\mathrm{Det}_\Lambda \mathrm{R}\,\Gamma_c(X,\mathscr{F})) \otimes_\Lambda \Lambda' \overset{\mathrm{can}}{\simeq} \mathrm{Det}_{\Lambda'} \mathrm{R}\,\Gamma_c(X,\mathscr{F} \overset{\mathrm{L}}{\otimes}_\Lambda \Lambda'). \tag{A.3.2}$$

If $\mathscr{F}^\bullet$ is a complex of ind-admissible $\Lambda[G_{F,\Sigma}]$-representations $\mathscr{F}^\bullet$, the Nekovář-Selmer complex $\mathrm{R}\,\Gamma_f(G_{F,\Sigma},\mathscr{F}^\bullet)$ of $\mathscr{F}^\bullet$ is the object

$$\mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{F,\Sigma},\mathscr{F}^\bullet) \oplus \bigoplus_{v\in\Sigma^{(p)}} C^\bullet_f(G_{F_v},\mathscr{F}^\bullet) \longrightarrow \bigoplus_{v\in\Sigma^{(p)}} C^\bullet_{\mathrm{cont}}(G_{F_v},\mathscr{F}^\bullet)\right)[-1]$$

viewed in in the derived category and in which $C_f^\bullet(G_{F_v},\mathscr{F}^\bullet)$ is defined as in [58, Chapter 7]. When $\mathscr{F}$ belongs to $D_{\mathrm{ctf}}(X,\Lambda)$ and arises from a complex of ind-admissible $\Lambda[G_{F,\Sigma}]$-representations $\mathscr{F}^\bullet$, there is a canonical isomorphism

$$\mathrm{R}\,\Gamma_f(G_{F,\Sigma},\mathscr{F}^\bullet)\overset{\mathrm{can}}{\simeq}\mathrm{R}\,\Gamma_{\mathrm{et}}(\operatorname{Spec}\mathbb{Z}[1/p],\mathscr{F}).$$

This follows from the excision sequence in étale cohomology if Λ is finite and, as above, the general case is reduced to the finite case as $\mathrm{R}\,\Gamma_f(G_{F,\Sigma},-)$ and $\mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/\Sigma],-)$ are triangulated and way-out.

A.4 The polynomials of section 4.1.2

$$\begin{aligned}P_3 =&x^{15}-269x^{13}+98x^{12}+27795x^{11}-22052x^{10}-1385007x^9\\&+1763658x^8+33697748x^7-60675152x^6-347894604x^5\\&+838659848x^4+960245792x^3-3558446016x^2\\&+1598326848x+886331520\end{aligned}$$

$$\begin{aligned}P_7 =&x^{26}+58x^{25}+1528x^{24}+24066x^{23}+250201x^{22}\\&+1777238x^{21}+8485597x^{20}+24131426x^{19}+14289099x^{18}\\&-194507055x^{17}-852543623x^{16}-1353592301x^{15}\\&+1212201450x^{14}+9280787596x^{13}+14283381200x^{12}\\&-3899190818x^{11}-41570797764x^{10}-44310276332x^9\\&+20464624672x^8+75496919254x^7+35942792436x^6\\&-35420299000x^5-37588576512x^4-903997728x^3\\&+8639549952x^2+1393890560x-422379776\end{aligned}$$

References

[1] *Périodes p-adiques*. Société Mathématique de France, Paris, 1994. Papers from the seminar held in Bures-sur-Yvette, 1988, Astérisque No. 223 (1994).

[2] Yvette Amice and Jacques Vélu. Distributions p-adiques associées aux séries de Hecke. *Astérisque*, 24-25:119–131, 1975.

[3] Joël Bellaïche and Gaëtan Chenevier. Families of Galois representations and Selmer groups. *Astérisque*, (324):xii+314, 2009.

[4] Spencer Bloch and Kazuya Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[5] Gebhard Böckle. Demuškin groups with group actions and applications to deformations of Galois representations. *Compositio Math.*, 121(2):109–154, 2000.

[6] Gebhard Böckle. On the density of modular points in universal deformation spaces. *Amer. J. Math.*, 123(5):985–1007, 2001.

[7] Christophe Breuil and Peter Schneider. First steps towards p-adic Langlands functoriality. *J. reine angew. Math.*, 610:149–180, 2007.

[8] D. Burns and M. Flach. On Galois structure invariants associated to Tate motives. *Amer. J. Math.*, 120(6):1343–1397, 1998.

[9] Henri Carayol. Sur les représentations l-adiques associées aux formes modulaires de Hilbert. *Ann. Sci. École Norm. Sup. (4)*, 19(3):409–468, 1986.

[10] Henri Carayol. Sur les représentations galoisiennes modulo l attachées aux formes modulaires. *Duke Math. J.*, 59(3):785–801, 1989.

[11] Henri Carayol. Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet. In *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, volume 165 of *Contemp. Math.*, pages 213–237. Amer. Math. Soc., Providence, RI, 1994.

[12] William Casselman. On some results of Atkin and Lehner. *Math. Ann.*, 201:301–314, 1973.

[13] Gaëtan Chenevier. The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 221–285. Cambridge Univ. Press, Cambridge, 2014.

[14] Pierre Colmez. La conjecture de Birch et Swinnerton-Dyer p-adique. *Astérisque*, (294):ix, 251–319, 2004.

[15] P. Deligne. *Cohomologie étale.* Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

[16] Pierre Deligne. Formes modulaires et représentations ℓ-adiques. In *Séminaire Bourbaki, 21ème année (1968/69), Exp. No. 355*, pages 139–172. Springer, Berlin, 1969.

[17] Pierre Deligne. Le déterminant de la cohomologie. In *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, volume 67 of *Contemp. Math.*, pages 93–177. Amer. Math. Soc., Providence, RI, 1987.

[18] Fred Diamond. On deformation rings and Hecke rings. *Ann. of Math. (2)*, 144(1):137–166, 1996.

[19] Fred Diamond, Matthias Flach, and Li Guo. The Tamagawa number conjecture of adjoint motives of modular forms. *Ann. Sci. École Norm. Sup. (4)*, 37(5):663–727, 2004.

[20] Matthew Emerton. p-adic L-functions and unitary completions of representations of p-adic reductive groups. *Duke Math. J.*, 130(2):353–392, 2005.

[21] Matthew Emerton. A local-global compatibility conjecture in the p-adic Langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$. *Pure Appl. Math. Q.*, 2(2, Special Issue: In honor of John H. Coates. Part 2):279–393, 2006.

[22] Matthew Emerton. On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.*, 164(1):1–84, 2006.

[23] Matthew Emerton, Robert Pollack, and Tom Weston. Variation of Iwasawa invariants in Hida families. *Invent. Math.*, 163(3):523–580, 2006.

[24] Jean-Marc Fontaine. Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate. *Ann. of Math. (2)*, 115(3):529–577, 1982.

[25] Jean-Marc Fontaine and Guy Laffaille. Construction de représentations p-adiques. *Ann. Sci. École Norm. Sup. (4)*, 15(4):547–608 (1983), 1982.

[26] Jean-Marc Fontaine and Bernadette Perrin-Riou. Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 599–706. Amer. Math. Soc., Providence, RI, 1994.

[27] Olivier Fouquet. Euler systems in the Iwasawa theory of ordinary modular forms. In *Algebraic Number Theory and Related Topics 2008*, RIMS Kôkyûroku Bessatsu, B19, pages 255–276. Res. Inst. Math. Sci. (RIMS), Kyoto, 2010.

[28] Olivier Fouquet. Dihedral Iwasawa theory of nearly ordinary quaternionic automorphic forms. *Compos. Math.*, 149(3):356–416, 2013.

[29] Olivier Fouquet. p-adic properties of motivic fundamental lines. Accepted to *Journal de l'École polytechnique.*, 2016.

[30] Kazuhiro Fujiwara. Deformation rings and Hecke algebras in the totally real case, 1999. Preprint, 99pp.

[31] Takako Fukaya and Kazuya Kato. A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory. In *Proceedings of the St. Petersburg Mathematical Society. Vol. XII*, volume 219 of *Amer. Math. Soc. Transl. Ser. 2*, pages 1–85, Providence, RI, 2006. Amer. Math. Soc.

[32] Takako Fukaya and Kazuya Kato. On conjectures of Sharifi. Preprint, 2012.

[33] Fernando Q. Gouvêa and Barry Mazur. On the density of modular representations. In *Computational perspectives on number theory (Chicago, IL, 1995)*, volume 7 of *AMS/IP Stud. Adv. Math.*, pages 127–142. Amer. Math. Soc., Providence, RI, 1998.

[34] Ralph Greenberg. Iwasawa theory for motives. In *L-functions and arithmetic (Durham, 1989)*, volume 153 of *London Math. Soc. Lecture Note Ser.*, pages 211–233. Cambridge Univ. Press, Cambridge, 1991.

[35] Alexander Grothendieck. *Dix exposés sur la cohomologie des schémas.* Advanced Studies in Pure Mathematics, Vol. 3. North-Holland Publishing Co., Amsterdam, 1968.

[36] Haruzo Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.

[37] Benjamin Howard. The Heegner point Kolyvagin system. *Compos. Math.*, 140(6):1439–1472, 2004.

[38] Benjamin Howard. Iwasawa theory of Heegner points on abelian varieties of GL_2 type. *Duke Math. J.*, 124(1):1–45, 2004.

[39] Hervé Jacquet and Joseph A. Shalika. A non-vanishing theorem for zeta functions of GL_n. *Invent. Math.*, 38(1):1–16, 1976/77.

[40] Kazuya Kato. Iwasawa theory and p-adic Hodge theory. *Kodai Math. J.*, 16(1):1–31, 1993.

[41] Kazuya Kato. Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}. I. In *Arithmetic algebraic geometry (Trento, 1991)*, volume 1553 of *Lecture Notes in Math.*, pages 50–163. Springer, Berlin, 1993.

[42] Kazuya Kato. Euler systems, Iwasawa theory, and Selmer groups. *Kodai Math. J.*, 22(3):313–372, 1999.

[43] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, (295):ix, 117–290, 2004. Cohomologies p-adiques et applications arithmétiques. III.

[44] Kazuya Kato. Iwasawa theory and generalizations. In *International Congress of Mathematicians. Vol. I*, pages 335–357. Eur. Math. Soc., Zürich, 2007.

[45] Kazuya Kato, Masato Kurihara, and Takeshi Tsuji. Local Iwasawa theory of Perrin-Riou and syntomic complexes,. Preprint, 1997.

[46] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[47] Mark Kisin. Moduli of finite flat group schemes, and modularity. *Ann. of Math. (2)*, 170(3):1085–1180, 2009.

[48] Koji Kitagawa. On standard p-adic L-functions of families of elliptic cusp forms. In *p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, volume 165 of *Contemp. Math.*, pages 81–110. Amer. Math. Soc., Providence, RI, 1994.

[49] Finn Faye Knudsen and David Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div". *Math. Scand.*, 39(1):19–55, 1976.

[50] Shin-ichi Kobayashi. Iwasawa theory for elliptic curves at supersingular primes. *Invent. Math.*, 152(1):1–36, 2003.

[51] Viktor Kolyvagin. Euler systems. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 435–483. Birkhäuser Boston, Boston, MA, 1990.

[52] Ron Livné. On the conductors of mod l Galois representations coming from modular forms. *J. Number Theory*, 31(2):133–141, 1989.

[53] Ju. I. Manin. Non-Archimedean integration and p-adic Jacquet-Langlands L-functions. *Uspehi Mat. Nauk*, 31(1(187)):5–54, 1976.

[54] Barry Mazur. Rational points of abelian varieties with values in towers of number fields. *Invent. Math.*, 18:183–266, 1972.

[55] Barry Mazur. On the arithmetic of special values of L functions. *Invent. Math.*, 55(3):207–240, 1979.

[56] Barry Mazur. Deforming Galois representations. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 385–437. Springer, New York, 1989.

[57] Barry Mazur and Karl Rubin. Kolyvagin systems. *Mem. Amer. Math. Soc.*, 168(799):viii+96, 2004.

[58] Jan Nekovář. Selmer complexes. *Astérisque*, (310):559, 2006.

[59] Louise Nyssen. Pseudo-représentations. *Math. Ann.*, 306(2):257–283, 1996.

[60] Tadashi Ochiai. Euler system for Galois deformations. *Ann. Inst. Fourier (Grenoble)*, 55(1):113–146, 2005.

[61] Tadashi Ochiai. On the two-variable Iwasawa main conjecture. *Compos. Math.*, 142(5):1157–1200, 2006.

[62] Bernadette Perrin-Riou. Fonctions L p-adiques des représentations p-adiques. *Astérisque*, (229):198, 1995.

[63] Bernadette Perrin-Riou. Systèmes d'Euler p-adiques et théorie d'Iwasawa. *Ann. Inst. Fourier (Grenoble)*, 48(5):1231–1307, 1998.

[64] Robert Pollack. On the p-adic L-function of a modular form at a supersingular prime. *Duke Math. J.*, 118(3):523–558, 2003.

[65] Ravi Ramakrishna. On a variation of Mazur's deformation functor. *Compositio Math.*, 87(3):269–286, 1993.

[66] Michel Raynaud. Schémas en groupes de type $(p,\dots,p)$. *Bull. Soc. Math. France*, 102:241–280, 1974.

[67] Kenneth A. Ribet. Raising the levels of modular representations. In *Séminaire de Théorie des Nombres, Paris 1987–88*, volume 81 of *Progr. Math.*, pages 259–271. Birkhäuser Boston, Boston, MA, 1990.

[68] David E. Rohrlich. L-functions and division towers. *Math. Ann.*, 281(4):611–632, 1988.

[69] Karl Rubin. *Euler systems*, volume 147 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study.

[70] A. J. Scholl. Motives for modular forms. *Invent. Math.*, 100(2):419–430, 1990.

[71] Jean-Pierre Serre. Classes des corps cyclotomiques (d'après K. Iwasawa). In *Séminaire Bourbaki, Vol. 5*, pages Exp. No. 174, 83–93. Soc. Math. France, Paris, 1958.

[72] Jean-Pierre Serre. Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Duke Math. J.*, 54(1):179–230, 1987.

[73] Jean-Pierre Serre and John Tate. Good reduction of abelian varieties. *Ann. of Math. (2)*, 88:492–517, 1968.

[74] Jack Shotton. Local deformation rings for GL_2 and a Breuil-Mézard conjecture when $\ell \neq p$. *Algebra Number Theory*, 10(7):1437–1475, 2016.

[75] C. M. Skinner and A. J. Wiles. Residually reducible representations and modular forms. *Inst. Hautes Études Sci. Publ. Math.*, (89):5–126 (2000), 1999.

[76] Christopher Skinner and Eric Urban. The Iwasawa main conjectures for GL_2. *Invent. Math.*, 195(1):1–277, 2014.

[77] Florian Sprung. Iwasawa theory for elliptic curves at supersingular primes: a pair of main conjectures. *J. Number Theory*, 132(7):1483–1506, 2012.

[78] Florian Sprung. The Iwasawa Main Conjecture for elliptic curves at odd supersingular primes. 2015.

[79] Richard Taylor. On Galois representations associated to Hilbert modular forms. *Invent. Math.*, 98(2):265–280, 1989.

[80] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.

[81] M.M. Visik. Non-Archimedean measures connected with Dirichlet series. *Math. USSR, Sb.*, 28:216–228, 1976.

[82] Xin Wan. Iwasawa Main Conjecture for Supersingular Elliptic curves. 2015.

[83] Andrew Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988.

[84] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

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