CONTROL THEOREMS FOR SELMER GROUPS OF NEARLY ORDINARY DEFORMATIONS

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Contents

1. Introduction

Let p be an odd prime and let us fix the embeddings of the algebraic closure $\bar{\mathbb{Q}}$ of the field of rationals into $\mathbb{C}$ and the algebraic closure $\bar{\mathbb{Q}}_p$ of the field of p-adic numbers. Let $\mathcal{T}$ be a p-adic family of rank two Galois representations, that is, $\mathcal{T}$ is a free module of rank two over a complete local noetherian ring R of characteristic zero with finite residue field of characteristic p on which the absolute Galois group G_F of a number field F acts continuously. We fix a Zariski-dense subset S of $\mathrm{Hom}_{\mathrm{cont}}(R, \bar{\mathbb{Q}}_p)$. We assume that the triple $(\mathcal{T}, R, S)$ satisfies the following properties:

(1) The action of G_F on $\mathcal{T}$ is unramified outside a finite set of places Σ.
(2) For each $\kappa \in S$, the G_F-representation $T_\kappa = \mathcal{T} \otimes_R \kappa(R)$ is a lattice of the p-adic étale realization V_κ is of a pure motive M_κ which is critical in the sense of [De79].
(3) For every $\kappa \in S$, the complex L-function $L(M_\kappa, s)$ is meromorphically continued to the whole complex plane.

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(4) There exists $\kappa \in S$ such $L(M_\kappa, s)$ is holomorphic at $s = 0$ and it does not vanish at $s = 0$.
(5) For each $\mathfrak{p}|p$, the action of $G_{F_\mathfrak{p}}$ on $\mathcal{T}$ is nearly ordinary, in the sense that the image of the decomposition group $G_{F_\mathfrak{p}}$ is contained in a Borel subgroup of $\mathrm{Aut}_R(\mathcal{T}) \cong GL_2(R)$.

Under these hypotheses, it was suggested in [Gre91, Gre94] (see also [Och06]) that there exists a non zero-divisor p-adic L-function $L_p(\mathcal{T}) \in R$ interpolating the special values $L(M_\kappa, 0)$ in the sense that, for all $\kappa \in S$, $\kappa(L_p(\mathcal{T}))$ is equal to $L(M_\kappa, 0)$ divided by a suitable complex period of M_κ and multiplied by Euler factors for all $\mathfrak{p}|p$.

Let $\mathcal{A}$ be the discrete G_F-representation $\mathcal{T} \otimes_R D_P(R)$ where D_P is the Pontrjagin dual. Similarly, for each $\kappa \in S$, we denote by A_κ the discrete Galois representation $T_\kappa \otimes_{\kappa(R)} D_P(\kappa(R)) = T_\kappa \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$.

Using the filtration obtained by the property (5), Greenberg [Gre94] has defined Selmer groups $\mathrm{Sel}_\mathcal{A}$ and Sel_{A_κ} as subgroups of a suitable Galois cohomology group with coefficient $\mathcal{A}$ and A_κ respectively. The module $D_P(\mathrm{Sel}_\mathcal{A})$ (resp. $D_P(\mathrm{Sel}_{A_\kappa})$) is known to be of finite type over R (resp. $\kappa(R)$). Further, the module $D_P(\mathrm{Sel}_\mathcal{A})$ (resp. $D_P(\mathrm{Sel}_{A_\kappa})$) is conjectured to be linked with $L_p(\mathcal{T})$ (resp. $L(M_\kappa, 0)$). In particular, if $L(M_\kappa, 0)$ is non-zero, it is conjectured in [BK90] that $D_P(\mathrm{Sel}_{A_\kappa})$ is torsion over $\kappa(R)$. In analogy with the case of $R = \mathbb{Z}_p$ and taking into account property () above, the R-module $D_P(\mathrm{Sel}_\mathcal{A})$ is conjectured to be torsion over R. More precisely, the Tamagawa Number Conjecture states that when $\kappa(L_p(\mathcal{T}))$ is non-zero, its p-part should be related to the length of $D_P(\mathrm{Sel}_{A_\kappa})$. When R is a normal ring, the Iwasawa Main Conjecture states that the characteristic ideal $\mathrm{char}_R\, D_P(\mathrm{Sel}_\mathcal{A})$ should be equal to the ideal generated by $L_p(\mathcal{T})$. (The Iwasawa Main Conjecture for such general deformations was first proposed by Greenberg in [Gre94] and was made precise by the second author through a detailed study in the case of Hida deformation [Och06]. See also [Och10] for a precise formulation obtained through such study.)

Combining these two conjectures with the interpolation property defining $L_p(\mathcal{T})$ suggests that both $D_P(\mathrm{Sel}_{A_\kappa})$ and $D_P(\mathrm{Sel}_\mathcal{A}) \otimes_R \kappa(R)$ should be linked with $\kappa(L_p(\mathcal{T}))$ and thus that the natural map

$$D_P(\mathrm{Sel}_\mathcal{A}) \otimes_R \kappa(R) \longrightarrow D_P(\mathrm{Sel}_{A_\kappa})$$

should be very close to an isomorphism when κ belongs to S. When such a property holds, we say that Selmer groups satisfy a control theorem at κ. Apart from Iwasawa original work on ideal class groups in $\mathbb{Z}_p$-extensions, the first historical setting in which such a theorem was proved was in [Maz72] for the family of G_F-representations $T_pA \otimes_{\mathbb{Z}_p} \Lambda$, where T_pA is the Tate module of a good ordinary abelian variety defined over F and $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$ is the complete group algebra of the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of F with its natural $\mathrm{Gal}(F_\infty/F)$-action. Subsequently, it was generalized to abelian varieties and more general p-adic Lie extensions in [Gre03] among others, to $\mathbb{Z}_p^r$-extensions for some abstract $R[G_F]$-modules in [Och00, Och01] and [Nek06, (8.10)]. It was also generalized to families of $G_\mathbb{Q}$-representations including nearly ordinary Hida families of modular forms in [Och01, Och06].

From now on, we assume that $\mathcal{T}$ arises from Hida theory of nearly ordinary automorphic representations on $GL_2(\mathbb{A}_F)$ of the ring of adeles $\mathbb{A}_F$ of a totally real number field F in the following sense. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ which contains all conjugates of F. In [Hid88, Hid89b], nearly ordinary Hecke algebras $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ of tame conductor $\mathcal{N}$ for GL_2 over totally real number fields were constructed. The algebra $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ is a noetherian ring which is finite torsion-free over an Iwasawa algebra Λ of $1+[F:\mathbb{Q}]+\delta_{F,p}$ variables where $\delta_{F,p}$ is the defect of Leopoldt's conjecture at p. In general, $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ is a semi-local algebra which has finitely many primes of height zero. Let $\mathcal{R}=\mathbf{H}_{\mathcal{N},\mathcal{O}}/\mathfrak{A}$ be the quotient of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ by $\mathfrak{A}$, one of such primes of height zero. $\mathcal{R}$ is a local domain which is also finite and torsion-free over the same Iwasawa algebra Λ. We call such a quotient $\mathcal{R}$ of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ a branch of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$. Let us denote by $\mathbb{F}_{\mathcal{R}}$ the finite residual field of $\mathcal{R}$. To $\mathcal{R}$ is attached a residual G_F-representation $\bar{\rho}$ on a vector space of rank two over $\mathbb{F}_{\mathcal{R}}$. For simplicity of exposition, we assume in this introduction that $\bar{\rho}$ is irreducible and that $\mathcal{R}$ is a regular ring (much of our results in fact require neither assumptions, see Section 2.1.3 and Proposition 3.1). By the method of pseudo-representation developed in [Wil88], we have G_F-representations $\mathcal{T}$ satisfying the properties (1)-(5) listed above. In particular, to each arithmetic specialization κ of $\mathcal{T}$ of appropriate weight is attached a nearly ordinary eigen cuspform $f=f_\kappa$ which is an eigenform under the action of $\mathcal{R}$ and κ is the map which sends a Hecke operator to the corresponding eigenvalue of f. On the other hand, for every nearly ordinary eigen cuspform f of tame conductor $\mathcal{N}$, the Galois representation of f is obtained this way at some arithmetic specialization $\kappa=\kappa_f$ of $\mathcal{R}$. We remark that families of G_F-representation satisfying the properties (1)-(5) should in fact all arise in this way according to [FM95].

In this article, we consider the Pontrjagin dual $D_P(\mathrm{Sel}_{\mathcal{A}})$ of $\mathrm{Sel}_{\mathcal{A}}$ as well as the second cohomology group $\tilde{H}^2_f(F,\mathcal{T})$ of the Selmer complex $\mathrm{R}\Gamma_f(F,\mathcal{T})$, as defined and studied in [Nek06], and prove control theorems for these objects. We remark that both styles of control theorems have their own use: Selmer groups are easier to link to special values of L-function whereas Selmer complexes satisfy convenient base change properties. The following theorem summarizes our results but we refer to Section 2.2 for definitions and to Propositions 3.5 and 3.6 for precise statements. We especially draw the attention of the reader to the fact that control theorems for two different type of Selmer groups are proved in the body of the text.

Main Theorem . *Let $\mathcal{R}$ be a branch (cf. Definition 2.5) of the nearly ordinary Hecke algebra and let $\mathcal{T}$ be the G_F-representation with coefficients in $\mathcal{R}$ constructed by Hida theory.*

For each arithmetic specialization $\kappa=\kappa_f$ of $\mathcal{R}$ associated with some eigen cuspform f of arithmetic weight (cf. Definition 2.3 for the definition of arithmetic specializations), the second cohomology group $\tilde{H}^2_f(F,\mathcal{T}_{\mathcal{P}_\kappa})$ (resp. $\tilde{H}^2_f(F,\mathcal{T}_{\mathcal{P}_\kappa}\otimes_{\mathcal{R}}\kappa(\mathcal{R}))$) of $\mathrm{R}\Gamma_f(F,\mathcal{T}_{\mathcal{P}_\kappa})$ (resp. $\mathrm{R}\Gamma_f(F,\mathcal{T}_{\mathcal{P}_\kappa})\otimes_{\mathcal{R}}\kappa(\mathcal{R})$) satisfies:

$$\tilde{H}^2_f(F,\mathcal{T}_{\mathcal{P}_\kappa})\otimes_{\mathcal{R}}\kappa(\mathcal{R})\overset{\sim}{\to}\tilde{H}^2_f(F,\mathcal{T}_{\mathcal{P}_\kappa}\otimes_{\mathcal{R}}\kappa(\mathcal{R})),\tag{1}$$

where $\mathcal{T}_{\mathcal{P}_\kappa}$ is the base extension of $\mathcal{T}$ by $\otimes_{\mathcal{R}}\mathcal{R}_{\mathcal{P}_\kappa}$ with $\mathcal{R}_\kappa$ the localization of $\mathcal{R}$ at $\mathcal{P}_\kappa$. We also have:

$$(D_P(\mathrm{Sel}_{\mathcal{A}})\otimes_{\mathcal{R}}\mathcal{R}_{\mathcal{P}_\kappa})\otimes_{\mathcal{R}}\kappa(\mathcal{R})\twoheadrightarrow D_P(\mathrm{Sel}_{\mathcal{A}_\kappa})\otimes_{\kappa(\mathcal{R})}\mathrm{Frac}(\kappa(\mathcal{R})).\tag{2}$$

Assume further that, for all $\mathfrak{p}|p$, the automorphic representation $\pi_{\kappa,\mathfrak{p}}$ at $\mathfrak{p}$ corresponding to V_κ is not a Steinberg representation. Then the surjection (2) is an isomorphism.

Under some conditions, we have also the strong control theorem for which we do not take the localization (cf. Sections 3.1 and 3.2). Thus Iwasawa Main Conjecture relating the Selmer group for $\mathcal{T}$ and the p-adic L-function mentioned earlier implies the Tamagawa Number Conjecture relating for the order of the Selmer group of f and the special value of the Hecke L-function for f at infinitely many arithmetic specializations κ_f of $\mathcal{T}$ by using our control theorems.

The classical lower bound for the domain of convergence of the L-functions $L(f,s)$ of Hilbert modular forms f due to Hecke shows that there exists critical arithmetic specializations V_κ of $\mathcal{T}$ such that $L(V_\kappa, 0)$ is non-zero, and that they form a Zariski-dense subset. Consequently, the Selmer groups we consider are expected to be torsion, and this is indeed known when $F = \mathbb{Q}$ by combining [Kat04, Theorem 14.2] and [Och01]. When $F \neq \mathbb{Q}$, to the best of the knowledge of the authors, even examples of this were yet unknown except the case with complex multiplication or the case obtained by base-change from $\mathbb{Q}$. Our control theorem shows that if there exists an arithmetic κ such that $D_P(\mathrm{Sel}_{A_\kappa})$ is torsion over $\kappa(\mathcal{R})$, then the $\mathcal{R}$-module $D_P(\mathrm{Sel}_{\mathcal{A}})$ is torsion. Thanks to the Euler system of Heegner points, $D_P(\mathrm{Sel}_{A_\kappa})$ is known to be torsion over $\kappa(\mathcal{R})$ under the hypotheses listed in Theorem 4.2. Thus we have:

Corollary . *Assume that there exists an ordinary eigen cuspform of weight* $(2, 2, \cdots, 2)$ *and an arithmetic specialization* κ_f *on* $\mathcal{R}$ *such that* f *satisfies all the following properties* (i)-(iv) :

- (i) *The Neben character of* f *is trivial.*
- (ii) *The representation* $\pi_{f,\mathfrak{p}}$ *is principal series at every* $\mathfrak{p}|p$.
- (iii) *The* L*-function* $L(f,s)$ *does not vanish at* $s = 1$.
- (iv) *One of the following condition holds:*
  - (a) $2 \nmid [F : \mathbb{Q}]$.
  - (b) *There exists a finite place* λ *of* F *such that* $\pi_{f,\lambda}$ *is not a principal series representation.*
  - (c) *The form* f *has no complex multiplication and there exists an element* $\sigma \in G_F$ *such that one of the eigenvalues of* σ *on* V_f *are* ± 1 *and the other is not equal to* ± 1.

Then, $D_P(\mathrm{Sel}_{\mathcal{A}})$ *is torsion over* $\mathcal{R}$.

Conjecturally, the hypotheses of Theorem 4.2 hold roughly for the half of the nearly ordinary families coming from Hida theory. Thus, though our work is far from proving that $D_P(\mathrm{Sel}_{\mathcal{A}})$ is always torsion over $\mathcal{R}$, it provides plenty of examples.

For the organization of the article, we recall the properties of representations of G_F with coefficients in Hecke algebra coming from Hida theory and introduce Selmer groups and Selmer complexes attached to them in the first half of the paper. The control theorems proved in the latter half follow from two main ideas: the base-change properties of complexes and the behavior of the monodromy of $\mathcal{T}$ at the places outside p. In order to deduce the corollary which insists that $D_P(\mathrm{Sel}_{\mathcal{A}})$ is often torsion, we make use of a recent result [Nek10, Theorem B] by Nekovář. We remark that S. Zhang and Y. Tian have also a related result (cf. [YZZ08, Theorem 1.4.1]).

1.1. **Notations.** Let F be a totally real field of degree d and let $p \geq 3$ be a rational prime. We denote by $\mathfrak{r}_F$ the ring of integers of F. Let $\delta_{F,p}$ be the defect of Leopoldt's conjecture for F at p, hence the rank of the maximal $\mathbb{Z}_p$-extension of F minus 1. For q a finite place of a finite extension of $\mathbb{Q}$, Fr_q means the geometric Frobenius morphism. We denote the ring of adeles $(F \otimes_{\mathbb{Q}} \mathbb{R}) \times (F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ by $\mathbb{A}_F$. Let I_F be the set of infinite places of F. When L is a field, we denote by G_L the absolute Galois group of L. For a complete local noetherian ring R, we denote by $D_M(\cdot)$ the functor of Matlis duality (cf. 2.2) which is the functor from the category of R-modules onto itself. We denote by $D_P(\cdot)$ the functor of Pontrjagin duality which is the functor from the category of locally compact topological abelian groups into itself. Throughout the paper, we fix the embeddings of the algebraic closure $\bar{\mathbb{Q}}$ of the field of rationals $\mathbb{Q}$ into $\bar{\mathbb{Q}}_p$ and $\mathbb{C}$. We fix also a finite extension K of $\mathbb{Q}_p$ which contains the Galois closure of F and we denote by $\mathcal{O}$ the ring of integers of K.

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2. Selmer structures of nearly ordinary Hecke algebra

2.1. Generalities on nearly ordinary Hecke algebras.

2.1.1. *Hilbert modular forms.* To an ideal $\mathcal{M}$ of $\mathfrak{r}_F$, we attach the standard compact open subgroups K_0, K_1 and K_{11} of $\mathrm{GL}_2(\mathfrak{r}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ as follows:

$$K_0(\mathcal{M}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathfrak{r}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) \middle| c \equiv 0 \bmod \mathcal{M} \right\}$$
$$K_1(\mathcal{M}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathcal{M}) \middle| d \equiv 1 \bmod \mathcal{M} \right\}$$
$$K_{11}(\mathcal{M}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathcal{M}) \middle| a, d \equiv 1 \bmod \mathcal{M} \right\}$$

Definition 2.1. *A weight $k = \sum_{\tau \in I_F} k_\tau \tau$ is an element of $\mathbb{Z}[I_F]$; an arithmetic weight is a weight such that $k_\tau \geq 2$ for all $\tau \in I_F$ and that the k_τ have constant parity; a parallel weight is an integral multiple of the weight $t = \sum_{\tau \in I_F} t_\tau \tau$ where $t_\tau = 1$ for all $\tau \in I_F$. Two weights are said to be equivalent if their difference is a parallel weight. With an arithmetic* *weight $k \in \mathbb{Z}[I_F]$ is associated a weight $v \in \mathbb{Z}[I_F]$ called the parallel defect of k which is defined to be a weight $v = \sum_{\tau \in I_F} v_\tau \tau$ satisfying $k + 2v \in \mathbb{Z}t$.*

We refer to [Shi78, Section 1], [Hid88, Section 2], [SW99, Section 3.1] or [Nek06, Section 12.3] for definitions and basic properties of holomorphic Hilbert cuspforms. For k an arithmetic weight and v its parallel defect, we denote by $S_{k,w}(U;\mathcal{O})$ the space of holomorphic cuspforms of weight (k,w) and level U with coefficient in $\mathcal{O}$, where U is a finite index subgroup of $GL_2(\mathfrak{r}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ containing $K_{11}(\mathcal{M})$ for some $\mathcal{M} \subset \mathfrak{r}_F$. Here, by the standard normalization found in the references above, we put $w = v + k - t$,

Remark 2.2. *In Definition* 2.1*, a parallel defect $v \in \mathbb{Z}[I_F]$ associated to $k \in \mathbb{Z}[I_F]$ is only well-defined as an element in $\mathbb{Z}[I_F]/\mathbb{Z}t$. For a given form f of weight (k,w), replacing a choice of v by $v+t$ is related to translating the Hecke L-function $L(f,s)$ of f to $L(f,s+1)$.*

To $f \in S_{k,w}(U;\mathcal{O})$ is naturally attached an automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_F)$ (cf. [Ge]). For an automorphic representation π on $\mathrm{GL}_2(\mathbb{A}_F)$, the largest U such that $(V_\pi)^U \neq 0$ is called the level of π, where V_π is the representation space of π. We also recall that for λ a finite place of F, the local automorphic representation $\pi_\lambda = \pi|_{\mathrm{GL}_2(F_\lambda)}$ of $\mathrm{GL}_2(F_\lambda)$ is either an irreducible principal series, a twisted Steinberg or a supercuspidal representation (see also [Ge] for such classification).

2.1.2. *The nearly ordinary Hecke algebra.* For any fixed integral ideal $\mathcal{N}$ of F which is prime to p and for any $s \in \mathbb{N}$, we have a natural action of

$$(3) \qquad \mathbf{G} = \varprojlim_t K_0(p^t)/K_{11}(p^t)\mathfrak{r}_F^\times \cong \left((\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \times (\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times\right)/\,\overline{\mathfrak{r}_F^\times}, \quad \begin{pmatrix} a & \\ & d \end{pmatrix} \mapsto (a,d)$$

on $S_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O})$ through $K_0(p^s)/K_{11}(p^s)\mathfrak{r}_F^\times \cong ((\mathfrak{r}_F/p\mathfrak{r}_F)^\times \times (\mathfrak{r}_F/p\mathfrak{r}_F)^\times)/\,\mathfrak{r}_F^\times$. Here, $(\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \times (\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ is naturally embedded in the diagonal torus of $GL_2(\mathfrak{r}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ and $\overline{\mathfrak{r}_F^\times}$ is the closure of the diagonal embedding of $\mathfrak{r}_F^\times$. We recall another presentation of $\mathbf{G}$ as follows:

$$(4) \qquad \mathbf{G} = \left((\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \times (\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times\right)/\,\overline{\mathfrak{r}_F^\times} = (\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \times \left((\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times/\,\overline{\mathfrak{r}_F^\times}\right),$$

where the last isomorphism is induced by the map $(a,d) \mapsto (a^{-1}d, a)$.

Note that the space $S_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O})$ has an action of the p-Hecke operator $T_0(p)$ which is normalized according to the parallel defect v. We denote by $S^{\mathrm{n.o}}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O}) \subset S_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O})$ the largest $\mathcal{O}$-submodule of $S_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O})$ on which $T_0(p)$ acts invertibly. A form $f \in S_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O})$ is called nearly ordinary if it belongs to $S^{\mathrm{n.o}}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O})$.

Let us denote the complete group algebra $\mathcal{O}[[\mathbf{G}/\mathbf{G}_{\mathrm{tors}}]]$ by $\Lambda_{\mathcal{O}}$. The algebra $\Lambda_{\mathcal{O}}$ is non-canonically isomorphic to the power-series algebra $\mathcal{O}[[X_1, \cdots, X_r]]$ with $r = 1 + \delta_{F,p} + d$.

Recall that, for any $n \in \mathbb{Z}[I_F]$, we have a natural character

$$(5) \qquad (\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \longrightarrow \bar{\mathbb{Q}}_p^\times, \quad x \mapsto x^n.$$

We denote by χ, the character

$$(6) \qquad (\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times/\,\overline{\mathfrak{r}_F^\times} \twoheadrightarrow \mathrm{Gal}(F(\mu_{p^\infty})/F) \xrightarrow{\chi_{\mathrm{cyc}}} \bar{\mathbb{Q}}_p^\times,$$

where the first equality is the canonical identification of $\mathrm{Gal}(F(\mu_{p^\infty})/F)$ with a quotient of $(\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \overline{\mathfrak{r}_F^\times}$ obtained via the class field theory.

Definition 2.3. (1) *For $(k, w) \in \mathbb{Z}[I_F] \times \mathbb{Z}$, an algebraic character $\kappa : \mathbf{G} \longrightarrow \bar{\mathbb{Q}}_p^\times$ of weight (k, w) is a character of the following form:*

$$\kappa: \quad \mathbf{G} = (\mathfrak{r}_F \otimes \mathbb{Z}_p)^\times \times \left((\mathfrak{r}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \overline{\mathfrak{r}_F^\times} \right) \longrightarrow \bar{\mathbb{Q}}_p$$
$$(a, z) \longmapsto \psi(a, z) \chi^{[n+2v]}(z) a^n$$

where ψ is a character of finite order and $[n + 2v]$ is the unique integer satisfying $n + 2v = [n + 2v]t$. (We recall that we have the relation $w = v + k - t$ and $k = n + 2t$.) An algebraic character $\kappa : \mathbf{G} \longrightarrow \bar{\mathbb{Q}}_p^\times$ of weight (k, w) is called an arithmetic character of weight (k, w) if its restriction to the subgroup of global units $\mathfrak{r}_F^\times \subset (\mathfrak{r}_F \otimes \mathbb{Z}_p)^\times$ is trivial.

(2) *An algebra homomorphism $\kappa \in \mathrm{Hom}_{\mathrm{cont}}(\Lambda_{\mathcal{O}}, \bar{\mathbb{Q}}_p)$ is called an algebraic specialization (resp. arithmetic specialization) of weight (k, w) if $\kappa|_{\mathbf{G}}$ is an algebraic character (resp. arithmetic character) of weight (k, w).*

A prime ideal $\mathcal{P} = \mathcal{P}_\kappa \subset \Lambda_{\mathcal{O}}$ which is defined to be the kernel of an algebraic specialization (resp. arithmetic specialization) κ of $\Lambda_{\mathcal{O}}$ is called an algebraic point (resp. arithmetic point). We denote by $\mathrm{Spec}^{\mathrm{alg}}(\Lambda_{\mathcal{O}})$ (resp. $\mathrm{Spec}^{\mathrm{arith}}(\Lambda_{\mathcal{O}})$) the subset of $\mathrm{Spec}(\Lambda_{\mathcal{O}})$ which consists of algebraic points (resp. arithmetic points).

(3) *If R is a finite $\Lambda_{\mathcal{O}}$-algebra, an algebra homomorphism $\kappa \in \mathrm{Hom}_{\mathrm{cont}}(R, \bar{\mathbb{Q}}_p)$ is called an algebraic specialization (resp. arithmetic specialization) of weight (k, w) if $\kappa|_{\Lambda_{\mathcal{O}}}$ is an algebraic specialization (resp. arithmetic specialization) of weight (k, w).*

A prime ideal $\mathcal{P} = \mathcal{P}_\kappa \subset R$ which is defined to be the kernel of an algebraic specialization (resp. arithmetic specialization) κ of R is called an algebraic point (resp. arithmetic point). We denote by $\mathrm{Spec}^{\mathrm{alg}}(R)$ (resp. $\mathrm{Spec}^{\mathrm{arith}}(R)$) the subset of $\mathrm{Spec}(R)$ which consists of algebraic points (resp. arithmetic points).

Let $k = \sum_{\tau \in I_F} k_\tau \tau \in \mathbb{Z}[I_F]$ be an arithmetic weight and let $v \in \mathbb{Z}[I_F]$ a parity defect of k (cf. Definition 2.1 and Remark 2.2). The nearly ordinary Hecke algebra $\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ of weight (k, w) and level $K_1(\mathcal{N}) \cap K_{11}(p^s)$ is defined to be the sub-algebra of $\mathrm{End}_{\mathcal{O}} \left(S^{\mathrm{n.o}}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O}) \right)$ generated by Hecke operators where $S^{\mathrm{n.o}}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ is the space of nearly ordinary cuspforms.

The $\mathcal{O}$-algebra $\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$ is finite and flat over $\mathcal{O}$. Let the nearly ordinary Hecke algebra $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ be the inverse limit with respect to s of the $\mathbf{H}_{2t,0}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$. We recall the following fundamental results of Hida.

Theorem 2.4. (1) *For any arithmetic weight $k \in \mathbb{Z}[I_F]$ and any parallel defect $v \in \mathbb{Z}[I_F]$ of k, the nearly ordinary Hecke algebra $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ is isomorphic to $\varprojlim_s \mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s); \mathcal{O})$.*

(2) *The nearly ordinary Hecke algebra $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ is a finite torsion-free $\Lambda_{\mathcal{O}}$-module, hence a semi-local ring.*

The proof of the theorem is found in the paper [Hid89a] by Hida which extends his earlier result [Hid88] on ordinary Hecke algebras to nearly ordinary Hecke algebras. Especially, the statement (1) of the above theorem is [Hid89a, Theorem 2.3] and the statement (2) is [Hid89a, Theorem 2.4].

Since $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ is a semi-local algebra, we introduce the following notation.

Definition 2.5. *Let $\mathfrak{A}$ be one of finitely many ideals of height zero in $\mathbf{H}_{\mathcal{N},\mathcal{O}}$. The algebra $\mathcal{R} = \mathbf{H}_{\mathcal{N},\mathcal{O}}/\mathfrak{A}$ for each ideal of height zero in $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ is called a branch of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$.*

Let k be an arithmetic weight and v its parallel defect. Since giving an eigen cuspform $f \in S^{\mathrm{n.o}}_{k,w}(K_1(\mathcal{N}p^s);\mathcal{O})$ is equivalent to giving a homomorphism

$$q_f : \mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O}) \twoheadrightarrow \mathbf{H}_{k,w}(K_1(\mathcal{N}p^s);\mathcal{O}) \longrightarrow \bar{\mathbb{Q}}_p \tag{7}$$

by sending a Hecke operator $T \in \mathbf{H}_{k,w}(K_1(\mathcal{N}p^s);\mathcal{O})$ to $a_1(f|_T)$, we have the following theorem.

Theorem 2.6. *Let us fix an ideal $\mathcal{N} \subset \mathfrak{r}_F$ prime to p.*

1. *Let $\mathcal{R} = \mathbf{H}_{\mathcal{N},\mathcal{O}}/\mathfrak{A}$ be a branch of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ in the sense of Definition 2.5. Then for any arithmetic weight k and any parallel defect v of k, and for any arithmetic specialization $\kappa : \mathcal{R} \longrightarrow \bar{\mathbb{Q}}_p$ of weight (k,w) (cf. Definition 2.3 for the definition of arithmetic specializations), there exists a unique nearly ordinary eigen cuspform $f_\kappa \in S^{\mathrm{n.o}}_{k,w}(K_1(\mathcal{N}p^s))$ for some s such that $\kappa(\mathcal{R})$ is canonically identified with $q_{f_\kappa}(\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O}))$. Here, we recall that $k = n + 2t$, $w = v + k - t$. By basics on the theory of newforms, it is not hard to see that the form f_κ is a new vector for every prime dividing $\mathcal{N}$.*
2. *For any arithmetic weight k and any parallel defect v of k, and for any nearly ordinary eigen cuspform $f \in S^{\mathrm{n.o}}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s))$ which is new at every prime dividing $\mathcal{N}$, there exists a unique ideal $\mathfrak{A} \subset \mathbf{H}_{\mathcal{N},\mathcal{O}}$ of height zero and a unique arithmetic specialization $\kappa_f : \mathcal{R} \longrightarrow \bar{\mathbb{Q}}_p$ on $\mathcal{R} = \mathbf{H}_{\mathcal{N},\mathcal{O}}/\mathfrak{A}$ of weight (k,w) such that $\kappa_f(\mathcal{R})$ is canonically identified with $q_f(\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O}))$.*

The statements of the above theorem are contained in [Hid89a, Theorem 2.4] if we count the perfect duality between the nearly ordinary Hecke algebra $\mathbf{H}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s);\mathcal{O})$ and the space of nearly ordinary forms $S^{\mathrm{n.o}}_{k,w}(K_1(\mathcal{N}) \cap K_{11}(p^s))$.

As indicated by the following lemma, a branch $\mathcal{R}$ of the nearly ordinary Hecke algebra $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ is often regular. Since we do not find a reference for this fact, we will also give the proof below.

Lemma 2.7. *Let $f \in S_{k,w}(K_1(\mathcal{N});\bar{\mathbb{Q}})$ be a normalized newform of weight (k,w) for some $\mathcal{N} \subset \mathfrak{r}_F$. If f is nearly ordinary at a prime number p, let $\mathfrak{m}$ be the maximal ideal of $\mathbf{H}_{k,w}(K_1(\mathcal{N});\mathcal{O})$ corresponding to the mod p Hecke eigen system for f. After a finite extension of $\mathcal{O}$ if necessary, the rings $\mathbf{H}_{k,w}(K_1(\mathcal{N});\mathcal{O})_{\mathfrak{m}}$ are regular local rings for almost all primes p such that f is nearly ordinary at p and the branch $\mathcal{R}$ of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ introduced in Theorem 2.6 (2) is a regular local ring.*

Proof. By the fact that $S_{k,w}(K_1(\mathcal{N});\bar{\mathbb{Q}})$ is of finite dimension, there can be only finitely many primes p such that a given newform $f \in S_{k,w}(K_1(\mathcal{N});\bar{\mathbb{Q}})$ is congruent to another newform $f' \in S_{k,w}(K_1(\mathcal{N});\bar{\mathbb{Q}})$ under the p-adic ideal of $\bar{\mathbb{Q}}$ fixed at the beginning of the article. Let p be a prime at which f is nearly ordinary and which is outside such finitely many congruence primes. By the duality between Hecke operators and modular forms, $\mathbf{H}_{k,w}(K_1(\mathcal{N});\mathcal{O})_{\mathfrak{m}}$ is isomorphic to a discrete valuation ring $\mathcal{O}$ if we make a finite flat extension of $\mathcal{O}$ such that the eigenvalues of f belongs to $\mathcal{O}$. Let $\mathcal{P}$ be the kernel of κ_f. The prime ideal $\Lambda_{\mathcal{O}} \cap \mathcal{P}$ being generated by a regular sequence contained in a system of parameters $(x_1, \cdots, x_r)$ in $\Lambda_{\mathcal{O}}$ with $r = 1 + \delta_{F,p} + d$, Theorem 2.6 implies that $\mathbf{H}_{k,w}(K_1(\mathcal{N});\mathcal{O})_{\mathfrak{m}} = \mathcal{R}/(x_1, \cdots, x_r)\mathcal{R}$ for some branch $\mathcal{R}$ of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$. Hence $\mathcal{R}$ has to be a regular local ring. This completes the proof. □

2.1.3. *Galois representations.* In this section, we recall the results on the Galois representations of Hilbert modular forms and the Galois deformations associated to deformations of Hilbert modular forms. More precisely, For a single (not necessarily nearly ordinary) Hilbert modular cuspform f, we recall the Galois representation for f (Theorem 2.8) and its local property (Theorem 2.9). When f is nearly ordinary at p, the form f (or its Hecke algebra) fits into the Hida family as explained before. Hence,we recall the Galois deformation associated to this Hida family (Theorem 2.10) and then describe its local property (Theorem 2.11).

We recall the following theorems (Theorem 2.8, Theorem 2.9) for the Galois representation associated to a Hilbert modular eigen cuspform constructed and studied by Carayol[Car86], Ohta[Oht82], Wiles[Wil88] and Taylor[Tay89] etc. generalizing earlier work by Deligne and Shimura for $F = \mathbb{Q}$. Especially, if our Hilbert modular forms satisfy the conditions (iv) (a) or (iv) (b) of Corollary in the introduction, we are reduced to studying the etale cohomology of certain Shimura curves associated to quaternion algebras over F by help of Jacquet-Langlands-Shimizu correspondence and the construction is similar to that of Deligne (see [Car86] and [Oht82] for this case). In the remaining case, we construct the Galois representation by the method of p-power congruences initiated by Shimura (see [Wil88] and [Tay89] for this case).

Theorem 2.8. *Let $f \in S_{k,w}(K_1(\mathcal{M});\bar{\mathbb{Q}}_p)$ be a normalized eigen cuspform of arithmetic weight k and let K be a finite extension of $\mathbb{Q}_p$ containing all Hecke eigen values for f. Then, there exists a continuous, irreducible G_F-representation $V_f \cong K^{\oplus 2}$ which is unramified outside $\mathcal{M}p$ and which verifies*

$$\det(1 - \mathrm{Fr}_\lambda X | V_f) = 1 - T_\lambda(f)X + S_\lambda(f)X^2 \tag{8}$$

for every $\lambda \nmid \mathcal{M}p$ where T_λ (resp. S_λ) is the Hecke operator induced by the coset class $K_1(\mathcal{M}) \begin{pmatrix} 1 & 0 \\ 0 & \varpi_\lambda \end{pmatrix} K_1(\mathcal{M})$ (resp. $K_1(\mathcal{M}) \begin{pmatrix} \varpi_\lambda & 0 \\ 0 & \varpi_\lambda \end{pmatrix} K_1(\mathcal{M})$) with a uniformizer ϖ_λ at λ and Fr_λ is (the conjugate class of) geometric frobenius element at λ.

The G_F-representation V_f is known to be irreducible and thus, by using Chebotarev density theorem, is characterized up to isomorphism by (8).

The following local property is known for the Galois representation of Hilbert modular forms.

Theorem 2.9. *Let $f \in S_{k,w}(K_1(\mathcal{M});\bar{\mathbb{Q}}_p)$ be a normalized eigen cuspform. We denote by $w_{\max}$ the maximum among the coefficients w_τ. Let V_f (resp. π_f) be the Galois representation (resp. automorphic representation) associated with f.*

(1) *Suppose that a prime λ of F are not over p. Then*
   (a) *The inertia group I_λ at λ acts on V_f through an infinite quotient if and only if $\pi_{f,\lambda}$ is a Steinberg representation. In this case, V_f has a unique filtration by graded pieces of dimension one:*
   $$0 \longrightarrow (V_f)_\lambda^+ \longrightarrow V_f \longrightarrow (V_f)_\lambda^- \longrightarrow 0$$
   which is stable under the decomposition group D_λ at λ. The inertia group I_λ acts on $(V_f)_\lambda^+$ (resp. $(V_f)_\lambda^-$) through a finite quotient of I_λ. An eigenvalue α of the action of a lift of Fr_λ *to G_{F_λ} on $((V_f)_\lambda^+$ (resp. $((V_f)_\lambda^-$) is an algebraic number satisfying $|\alpha|_\infty = (N_{F/\mathbb{Q}}\lambda)^{\frac{w_{\max}+1}{2}}$ (resp. $(N_{F/\mathbb{Q}}\lambda)^{\frac{w_{\max}-1}{2}}$).*
   (b) *If I_λ acts on V_f through a finite quotient, the action of I_λ is reducible if and only if $\pi_{f,\lambda}$ is a principal series. If I_λ acts on V_f through a finite quotient, an eigenvalue α of the action of a lift of* Fr_λ *to G_{F_λ} on V_f is an algebraic number satisfying $|\alpha|_\infty = (N_{F/\mathbb{Q}}\lambda)^{\frac{w_{\max}}{2}}$.*

(2) *Suppose that a prime $\mathfrak{p}$ of F is over p and that f is nearly ordinary at $\mathfrak{p}$. Then V_f has a unique filtration by graded pieces of dimension one:*
   $$0 \longrightarrow (V_f)_\mathfrak{p}^+ \longrightarrow V_f \longrightarrow (V_f)_\mathfrak{p}^- \longrightarrow 0$$
   which is stable under the decomposition group $D_\mathfrak{p}$ at $\mathfrak{p}$ and in which the Hodge-Tate weight of $(V_f)_\mathfrak{p}^+$ is greater than that of $(V_f)_\mathfrak{p}^-$.

In the above two theorems, we presented results for the p-adic Galois representation associated to a single Hilbert modular form f. In the two theorems below, we make these p-adic Galois representations into a big p-adic family which corresponds to a p-adic family of nearly ordinary modular forms for varying weights, so called the Hida family. Recall that the Hida family was introduced at Theorem 2.4 and Theorem 2.6 above (Note that we deformed nearly ordinary p-adic Hecke algebras for varying weights rather than p-adic modular forms. However, since the space of modular forms are linear dual to Hecke algebras, it is essentially the same thing as deforming modular forms.).

Though we might sometimes be able to construct such p-adic family of p-adic Galois representations by taking a limit of those for each finite level (as we did when we constructed the family of p-adic Hecke algebras), there is a method called the theory of pseudo-representations invented by Wiles[Wil88]. Roughly speaking, the theory of pseudo-representations say that, if odd p-adic Galois representations are constructed at a set of specializations $q_\alpha : R \longrightarrow \bar{\mathbb{Q}}_p$ when α varies with which $\{\mathrm{Ker}(q_\alpha)\}_\alpha$ is a Zarski dense subset of $\mathrm{Spec}(R)$ a big family of Galois representation over R which are specialized to given p-adic representations at each q_α as long as only the traces of every elements of the Galois group are interpolated over R. Though there are some new technical issues related to the complicated nature of Hilbert modular forms, the ideas explained here provide the following two theorems which is a family-version of the previous two theorems (cf. [Wil88] and [Hid89b]):

Theorem 2.10. *Let $\mathcal{R}$ be a branch of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$. Then there exists a finitely generated torsion-free $\mathcal{R}$-module $\mathcal{T}$ with continuous G_F-action which satisfies the following properties:*

(i) *The vector space $\mathcal{V} = \mathcal{T} \otimes_\mathcal{R} \mathcal{K}$ is of dimension two over $\mathcal{K}$ where $\mathcal{K}$ is the field of fractions of $\mathcal{R}$.*

(ii) *The representation of* G_F *on* $\mathcal{V}$ *is irreducible and is unramified outside the primes dividing* $\mathcal{N}p\infty$.

(iii) *For any arithmetic weight* (k, w) *and for any nearly ordinary eigen cuspform* $f \in S^{\text{n.o}}_{k,w}(K_1(\mathcal{N}p^s); \bar{\mathbb{Q}})$ *which appears on the branch* $\mathcal{R}$ *in the sense of Theorem* 2.6, $T_f = \mathcal{T} \otimes_{\mathcal{R}} \kappa_f(\mathcal{R})$ *is a lattice of the Galois representation* V_f *associated with* f *introduced in Theorem* 2.8.

We recall the following property for the Hida deformation $\mathcal{V}$ which is known by construction:

Theorem 2.11. *Let* $\mathcal{R}$ *be a branch of* $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ *and* $\mathcal{V}$ *the Galois representation over the field of fractions* $\mathcal{K}$ *of* $\mathcal{R}$ *introduced in Theorem* 2.10. *Then,*

(1) *For every prime* $\lambda \nmid \mathcal{N}p$, *we have:*

$$\det(1 - \mathrm{Fr}_\lambda X|\mathcal{V}) = 1 - T_\lambda X + S_\lambda X^2,$$

where T_λ *and* S_λ *are the Hecke operator on* $\mathcal{R}$ *at* λ *which is obtained as the limit of the Hecke operators in Theorem* 2.8 *at finite levels.*

(2) *For every prime* $\mathfrak{p}$ *of* F *over* p, *we have a canonical filtration obtained as the limit of the filtration given at Theorem* 2.9*:*

$$0 \longrightarrow \mathcal{V}^+_{\mathfrak{p}} \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}^-_{\mathfrak{p}} \longrightarrow 0$$

which is stable under the action of the decomposition group $D_{\mathfrak{p}}$ *at* $\mathfrak{p}$.

For later use, we summarize the setting and the assumptions which we consider:

Setting 2.12. *For an ideal* $\mathcal{N}$ *of* $\mathfrak{r}_F$ *which is prime to the prime number* p *fixed at the beginning, we have a Hida's nearly ordinary Hecke algebra* $\mathbf{H}_{\mathcal{N},\mathcal{O}}$. *We fix a branch* $\mathcal{R}$ *of* $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ *and a representation* $\mathcal{T}$ *as in Theorem* 2.10. *We assume that we have a* $D_{\mathfrak{p}}$*-stable filtration*

$$0 \longrightarrow \mathcal{T}^+_{\mathfrak{p}} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}^-_{\mathfrak{p}} \longrightarrow 0$$

by finite type $\mathcal{R}$*-modules* $\mathcal{T}^+_{\mathfrak{p}}$ *and* $\mathcal{T}^-_{\mathfrak{p}}$ *with continuous* $D_{\mathfrak{p}}$*-action which gives rise to the exact sequence*

$$0 \longrightarrow \mathcal{V}^+_{\mathfrak{p}} \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}^-_{\mathfrak{p}} \longrightarrow 0$$

in Theorem 2.11 *by taking the base extension to* $\mathcal{K}$.

In addition to Setting 2.12 *above, the following conditions are sometimes assumed in what follows* (*we discuss some of the relation between these conditions in the remark below*)*:*

(Reg-b) *The algebra* $\mathcal{R}$ *is a regular local ring* (*Regularity of Branch*).

(Gor) *The algebra* $\mathcal{R}$ *is Gorenstein ring* (*Gorenstein Property of Branch*).

(DS) $\mathcal{T}^+_{\mathfrak{p}}$ *and* $\mathcal{T}^-_{\mathfrak{p}}$ *are direct summands of* $\mathcal{T}$ *as* $\mathcal{R}$*-module.*

(Fr-w) *The representation* $\mathcal{T}$ *can be chosen to be free of rank two over* $\mathcal{R}$ (*Weak Freeness*).

(Fr-s) *The representation* $\mathcal{T}$ *can be chosen to be free of rank two over* $\mathcal{R}$ *and, for each* $\mathfrak{p}|p$, *the graded pieces* $\mathcal{T}^+_{\mathfrak{p}}$ *and* $\mathcal{T}^-_{\mathfrak{p}}$ *are both free of rank one over* $\mathcal{R}$ (*Strong Freeness*).

(Reg-g) *The residual representation at the maximal ideal* $\mathfrak{M}_{\mathcal{R}}$ *of* $\mathcal{R}$ *restricted to the decomposition group* $D_{\mathfrak{p}}$ *is an extension of two different characters of* $D_{\mathfrak{p}}$ *with values in* $(\mathcal{R}/\mathfrak{M}_{\mathcal{R}})^\times$ *for each* $\mathfrak{p}|p$ (*Galois Theoretic Regularity*). (*see* [MW86, §9] *for the fact that the residual representation independent of the choice of* $\mathcal{T}$ *exists.*)

(Ir) *The residual representation of G_F over $\mathcal{R}/\mathfrak{M}_{\mathcal{R}}$ is irreducible (Irreducibility of Residual Representation).*

Remark 2.13. (1) *By a refinement and a generalization of the theory of pseudo-representations due to Nyssen (cf.* [Nys96, Théorème 1]*), the condition* **(Ir)** *implies the condition* **(Fr-w)**.

(2) *Since $\mathcal{R}$ is a local ring, the conditions* **(Fr-w)** *and* **(DS)** *imply the condition* **(Fr-s)**.

(3) *The conditions* **(Fr-w)** *and* **(Reg-g)** *imply the condition* **(Fr-s)**.

(4) *In the doctoral thesis of the first author, it is shown that* **(Gor)** *implies* **(Fr-s)** *when the condition* (iv) (a) *or* (iv) (b) *of Corollary in Introduction is satisfied. In fact, under these conditions, our p-adic Galois representation is related to the etale cohomology of Shimura curves via the Jacquet-Langlands-Shimizu correspondence and this allows us to deduce deeper properties of our Galois representation. We also prove* **(DS)** *by using the connected-etale decomposition of the p-divisible groups over Shimura curves (The same implication and the same analysis are known for the case $F = \mathbb{Q}$ using the geometry of modular curves (cf.* [MW86]*)).*

We explain the implication of the statement (3). Let us define $\mathcal{T}_{\mathfrak{p}}^-$ to be the image of $\mathcal{T} \hookrightarrow \mathcal{V} \twoheadrightarrow \mathcal{V}_{\mathfrak{p}}^-$ and define $\mathcal{T}_{\mathfrak{p}}^+$ to be the kernel of $\mathcal{T} \twoheadrightarrow \mathcal{T}_{\mathfrak{p}}^-$. This fits into Setting 2.12. Then, by taking the base extension $\otimes_{\mathcal{R}}\mathcal{R}/\mathfrak{M}_{\mathcal{R}}$, we have:

$$0 \longrightarrow \mathcal{T}_{\mathfrak{p}}^+ \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{M}_{\mathcal{R}} \longrightarrow \mathcal{T} \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{M}_{\mathcal{R}} \longrightarrow \mathcal{T}_{\mathfrak{p}}^- \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{M}_{\mathcal{R}} \longrightarrow 0.$$

Note that the sequence is left-exact since $\mathcal{T}_{\mathfrak{p}}^-$ is a torsion-free $\mathcal{R}$-module. By the condition **(Fr-w)**, $\mathcal{T} \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{M}_{\mathcal{R}}$ is of dimension 2 over $\mathcal{R}/\mathfrak{M}_{\mathcal{R}}$. Thus, $\mathcal{T}_{\mathfrak{p}}^- \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{M}_{\mathcal{R}}$ is of dimension 1 or 2. On the other hand, since we have $\mathcal{T}_{\mathfrak{p}}^- \subset \mathcal{V}_{\mathfrak{p}}^-$ by construction, the characters with values in $(\mathcal{R}/\mathfrak{M}_{\mathcal{R}})^\times$ which appear in the Jordan-Hölder-Schreier components of $\mathcal{T}_{\mathfrak{p}}^-$ are unique. By the condition **(Reg-g)**, $\mathcal{T}_{\mathfrak{p}}^- \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{M}_{\mathcal{R}}$ is thus of dimension 1. Since $\mathcal{T}_{\mathfrak{p}}^- \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{M}_{\mathcal{R}}$ is a cyclic $\mathcal{R}$-module, $\mathcal{T}_{\mathfrak{p}}^-$ is also a cyclic $\mathcal{R}$-module by Nakayama's lemma. This implies that $\mathcal{T}_{\mathfrak{p}}^-$ is a free $\mathcal{R}$-module of rank 1 because $\mathcal{T}_{\mathfrak{p}}^-$ is torsion-free over $\mathcal{R}$. By the dimension counting argument, $\mathcal{T}_{\mathfrak{p}}^+ \otimes_{\mathcal{R}} \mathcal{R}/\mathfrak{M}_{\mathcal{R}}$ is of dimension 1 over $\mathcal{R}/\mathfrak{M}_{\mathcal{R}}$. Since $\mathcal{T}_{\mathfrak{p}}^+$ is a cyclic $\mathcal{R}$-module whose base extension $\mathcal{T}_{\mathfrak{p}}^+ \otimes_{\mathcal{R}} \mathcal{K}$ is of dimension 1 over $\mathcal{K}$, $\mathcal{T}_{\mathfrak{p}}^+$ is also a free $\mathcal{R}$-module of rank 1. This completes the proof of Remark 2.13 (3).

We discuss the local property of $\mathcal{V}$ at primes λ dividing $\mathcal{N}$ in the next subsection.

2.1.4. *Rigidity of automorphic types.*

Lemma 2.14. *Let us fix a branch $\mathcal{R}$ of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ and a representation $\mathcal{T}$ as in Theorem* 2.10. *Let κ_f and $\kappa_{f'}$ be arithmetic specializations on $\mathcal{R}$. At each prime λ dividing $\mathcal{N}$, the automorphic representations π_f and $\pi_{f'}$ associated to κ_f and $\kappa_{f'}$ have the same automorphic type. Especially, if $\pi_{f,\lambda}$ is Steinberg, $\pi_{f',\lambda}$ is also Steinberg.*

Proof. We discuss each of the following cases:

(a) The image of the inertia group I_λ in $\mathrm{Aut}_{\mathcal{R}}(\mathcal{T})$ is finite.
(b) The image of the inertia group I_λ in $\mathrm{Aut}_{\mathcal{R}}(\mathcal{T})$ is infinite.

We start from the case (a). For any arithmetic specialization κ on $\mathcal{R}$, every element of $\mathrm{Ker}[\mathrm{Aut}_{\mathcal{R}}(\mathcal{T}) \longrightarrow \mathrm{Aut}_{\kappa(\mathcal{R})}(T_\kappa)]$ is of infinite order for $T_\kappa = \mathcal{T} \otimes_{\mathcal{R}} \kappa(\mathcal{R})$. Hence, for two arithmetic specializations κ_f and $\kappa_{f'}$ of arithmetic weights (k, w) and (k', w') respectively, the image of I_λ on V_f and $V_{f'}$ are both isomorphic to the image of I_λ on $\mathrm{Aut}_{\mathcal{R}}(\mathcal{T})$. By the local Langlands correspondence proved in [Car86], when the inertia group I_λ for $\lambda | \mathcal{N}$ acts on V_f through a finite quotient, the action of I_λ is reducible (resp. irreducible) if and only if the local automorphic representation $\pi_{f,\lambda}$ is principal series (resp. supercuspidal) at λ. Hence, in the case (a), we conclude that the automorphic type does not change when arithmetic specializations κ_f varies.

Next, we consider the case (b). The kernels of arithmetic specializations κ_f forms a Zariski dense subset of $\mathrm{Spec}(\mathcal{R})$, Hence, there exists an arithmetic specialization κ_f such that the image of I_λ is infinite in the Galois representation V_f of rank two, thus the local automorphic representation $\pi_{f,\lambda}$ is Steinberg at λ. In order to show that $\pi_{f',\lambda}$ is Steinberg at every arithmetic specialization $\kappa_{f'}$ on $\mathcal{R}$ of any arithmetic weight (k, w), it suffices to show:

(9) The image of I_λ is infinite in the Galois representation $V_{f'}$ for every such $\kappa_{f'}$.

Since $\pi_{f,\lambda}$ is Steinberg, it is written as $\mathrm{St}_\lambda \otimes \phi$ where ϕ is a character of finite order. If $J \subset I_\lambda$ is the inertia subgroup of the F_ϕ which a finite extension of F_λ corresponding to the kernel of the local Galois character ϕ of F_{F_λ}, the image of J on V_f is represented by a non-trivial unipotent matrix $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $(* \neq 0)$. Thus, $(V_f)^J$ is of dimension one over the field of fractions of $\kappa_f(\mathcal{R})$. Take a totally real field F' which is a finite Galois extension over F such that

(i) The composite field $F'F_\lambda$ contains F_ϕ.
(ii) F'/F is a solvable extension.

By the solvable base change theorem, the representation of $G_{F'}$ on $\mathcal{T}$ is also a branch of the big nearly ordinary Hida deformation for Hilbert modular forms over F'. To show the property (9), we may replace F by F' and it suffices to show the same property as (9) for every primes of F' over λ. Without loss of generality, we may reduce the proof to the case where the open subgroup J above is equal to I_λ. Since λ does not divide (p), we have a unique subgroup $I'_\lambda \subset I_\lambda$ such that $I_\lambda / I'_\lambda \cong \mathbb{Z}_p$ by a well-known structure of the absolute Galois group of l-adic fields. We have the following claim:

Claim 2.15. *In the above setting, the group I'_λ acts trivially on $\mathcal{T}$.*

Assuming this Claim, we show that the action of $I_\lambda / I'_\lambda \cong \mathbb{Z}_p$ on $V_{f'}$ is non-trivial for every arithmetic specialization $\kappa_{f'}$. In fact, if the action of $I_\lambda / I'_\lambda \cong \mathbb{Z}_p$ on $V_{f'}$ is trivial for some arithmetic specialization $\kappa_{f'}$, the conductor of f' is not divisible by λ. By the fact that any eigen cuspform f' of any arithmetic weight (k, w) extends uniquely to a branch of Hida deformation, the conductor of every eigen cuspform f which appears in $\mathcal{T}$ has to be prime to λ. This is a contradiction to the fact that $\mathcal{T}$ has at least one arithmetic specialization κ_f for which π_f is Steinberg at λ. This shows that the automorphic type is constant for $\mathcal{T}$ at λ when it has a specialization which is Steinberg. Finally we finish the proof by completing the proof of the above claim. Recall that I'_λ has an extension as follows:

$$0 \longrightarrow P \longrightarrow I'_\lambda \longrightarrow \prod_{l \nmid p} \mathbb{Z}_l \longrightarrow 0,$$

where P is the wild inertia subgroup which is a pro-p group. We show that every element of the image of P on $\mathrm{Aut}(\mathcal{T})$ is of finite order. If there is $g \in P$ such that the image of g on $\mathrm{Aut}(\mathcal{T})$ is of infinite order, there exists an arithmetic specialization κ_f such that the image of g in $\mathrm{Aut}(V_f)$. This is a contradiction to the fact that P acts through a finite quotient on V_f for any eigen cuspform f. Since the prime-to-p part of the part of $\mathrm{Aut}_{\mathcal{R}}(\mathcal{T})$ is a finite group and $\prod_{l \nmid p} \mathbb{Z}_l$ has no non-trivial pro-p quotient, I'_λ acts on $\mathcal{T}$ through a finite quotient. Then, by the same argument with that of the case (a), the fact that I'_λ acts trivially on the specialization of $\mathcal{T}$ at κ_f implies the fact that I'_λ acts trivially on $\mathcal{T}$. Thus we complete the proof of the claim. □

Corollary 2.16. *For any arithmetic specialization κ_f on $\mathcal{R}$ associated with an eigen cuspform f of arithmetic weight (k, w), the natural map $H^0(I_\lambda, \mathcal{T}) \otimes_{\mathcal{R}} \kappa(\mathcal{R}) \longrightarrow H^0(I_\lambda, \mathcal{T} \otimes_{\mathcal{R}} \kappa(\mathcal{R}))$ is an isomorphism after taking the base extension $\otimes_{\kappa(\mathcal{R})}\mathrm{Frac}(\kappa(\mathcal{R}))$. Further, the above map is an isomorphism without taking the base extension $\otimes_{\kappa(\mathcal{R})}\mathrm{Frac}(\kappa(\mathcal{R}))$ if $\pi_{f,\lambda}$ is not Steinberg at λ.*

Remark 2.17. *The map $H^0(I_\lambda, \mathcal{T}) \otimes_{\mathcal{R}} \kappa(\mathcal{R}) \longrightarrow H^0(I_\lambda, \mathcal{T} \otimes_{\mathcal{R}} \kappa(\mathcal{R}))$ might not be an isomorphism if $\pi_{f,\lambda}$ is Steinberg at λ. We remark that the unipotent matrix is not necessarily triangulated for some choice of basis over an algebra R which is not a DVR nor a field. We have a unipotent matrix $\begin{pmatrix} 1-pX & -p^2 \\ X^2 & 1+pX \end{pmatrix}$ over $R = \mathbb{Z}_p[[X]]$ for such a example.*

2.2. **Selmer structures.** Our main result is concerned with the control theorem for (R, T) where R and T is the algebra $\mathcal{R}$ and the representation $\mathcal{T}$ summarized in Setting 2.12. However, most of the statements which appear in this subsection hold for more general couples (R, T). So we summarize the situation we consider in this subsection as follows:

Setting 2.18. *Let R be a complete local domain whose residue field $\mathbb{F}_R$ is finite and let T be a finitely generated torsion-free R-module with continuous action of G_F. We assume that the action of G_F is unramified outside a finite set Σ of places of F containing the set Σ_p of places of F above p. For each prime $\mathfrak{p}$ of F over p, we fix an R-direct summand $T^+_{\mathfrak{p}}$ of T which is stable under the action of the decomposition group $D_{\mathfrak{p}}$ at $\mathfrak{p}$.*

2.2.1. *Generalities.* Under Setting 2.18, we denote by F_Σ the maximal Galois extension of F unramified outside Σ. Let G be $\mathrm{Gal}(F_\Sigma/F)$ or G_{F_λ} for λ a place of F. Then the continuous cohomology groups $H^i(G, T)$ are finitely generated R-modules (cf. [Nek06, Proposition 4.2.3]). Let $C^\bullet_{\mathrm{cont}}(G, T)$ denote the complex of continuous cochains with values in T and $\mathrm{R}\,\Gamma(G, T)$ the corresponding object in the derived category. We write $H^i(F_\Sigma/F, T)$ and $H^i(F_\lambda, T)$ respectively for $H^i(\mathrm{Gal}(F_\Sigma/F), T)$ and $H^i(G_{F_\lambda}, T)$ and we use a similar notation for complexes.

Lemma 2.19. *Let us keep the situation of Setting 2.18. Let $\mathbf{x} = (x_1, \cdots, x_r)$ be an R-regular sequence. For $i \geq 1$, let $\mathbf{x}_i$ be $(x_1, \cdots, x_i)$ for $1 \leq i < r$. Then we have the following isomorphism if T is flat over R.*

$$\mathrm{R}\,\Gamma(G, T) \overset{\mathrm{L}}{\otimes} R/\mathbf{x}_i \xrightarrow{\sim} \mathrm{R}\,\Gamma(G, T/\mathbf{x}_i T)$$

Proof. By the assumption that T is flat over R, the exact sequence

$$0 \longrightarrow R/\mathbf{x}_i \xrightarrow{x_{i+1}} R/\mathbf{x}_i \longrightarrow R/\mathbf{x}_{i+1} \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow T/\mathbf{x}_i \xrightarrow{x_{i+1}} T/\mathbf{x}_i \longrightarrow T/\mathbf{x}_{i+1} \longrightarrow 0$$

hence an exact sequences of complexes:

$$0 \longrightarrow C^{\bullet}_{\mathrm{cont}}(G, T/\mathbf{x}_i) \longrightarrow C^{\bullet}_{\mathrm{cont}}(G, T/\mathbf{x}_i) \longrightarrow C^{\bullet}_{\mathrm{cont}}(G, T/\mathbf{x}_{i+1}) \longrightarrow 0.$$

□

We sum up the previous property by saying that $\mathrm{R}\,\Gamma(G, \cdot)$ descends perfectly. Let I be an injective hull of the residue field $\mathbb{F}_R$ and let D_M be the Matlis duality functor $D_M(\cdot) = \mathrm{Hom}_R(\cdot, I)$. Since $\mathbb{F}_R$ is finite, we have the following lemma (see [Nek06, §2.9] for the proof):

Lemma 2.20. *The Pontrjagin duality functor $D_P(\cdot) = \mathrm{Hom}_{\mathrm{cont}}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p)$ and the Matlis duality functor D_M coincide on the category of R-modules.*

Though the functor D_M and the functor D_P coincide to each other in the case where $\mathbb{F}_R$ is finite, we will sometimes distinguish two functors in what follows in order not to cause confusion for the use of the arguments of this article on forthcoming projects where we will consider the case with infinite residue field $\mathbb{F}_R$.

2.2.2. *Greenberg's Selmer groups.* Under the situation of Setting 2.18, we denote by A the discrete representation $T \otimes_R D_P(R)$. For $\mathfrak{p}|p$, let $A^+_{\mathfrak{p}}$ be $T^+_{\mathfrak{p}} \otimes_R D_P(R)$ and $A^-_{\mathfrak{p}}$ be $A/A^+_{\mathfrak{p}}$. Let $\mathrm{Sel}^{\mathrm{str}}_A$ and Sel_A be a strict Selmer group (resp. Selmer group) due to Greenberg which are defined by the following exact sequences:

$$0 \longrightarrow \mathrm{Sel}^{\mathrm{str}}_A \longrightarrow H^1(F_\Sigma/F, A) \longrightarrow \bigoplus_{\mathfrak{p}|p} H^1(G_{F_{\mathfrak{p}}}, A^-_{\mathfrak{p}}) \oplus \bigoplus_{\lambda \in \Sigma\backslash\Sigma_p} H^1(I_\lambda, A). \tag{10}$$

$$0 \longrightarrow \mathrm{Sel}_A \longrightarrow H^1(F_\Sigma/F, A) \longrightarrow \bigoplus_{\mathfrak{p}|p} H^1(I_{\mathfrak{p}}, A^-_{\mathfrak{p}}) \oplus \bigoplus_{\lambda \in \Sigma\backslash\Sigma_p} H^1(I_\lambda, A). \tag{11}$$

By [Gre94, §4, Proposition], their Pontrjagin duals are finite type R-modules.

As suggested by the notation, these modules do not depend on Σ. They fit in the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}^{\mathrm{str}}_A & \longrightarrow & H^1(F_\Sigma/F, A) & \longrightarrow & \bigoplus_{\mathfrak{p}|p} H^1(G_{F_{\mathfrak{p}}}, A^-_{\mathfrak{p}}) \oplus \bigoplus_{\lambda \in \Sigma\backslash\Sigma_p} H^1(I_\lambda, A) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Sel}_A & \longrightarrow & H^1(F_\Sigma/F, A) & \longrightarrow & \bigoplus_{\mathfrak{p}|p} H^1(I_{\mathfrak{p}}, A^-_{\mathfrak{p}}) \oplus \bigoplus_{\lambda \in \Sigma\backslash\Sigma_p} H^1(I_\lambda, A). \end{array}$$

By inflation-restriction and the snake lemma, there is thus an exact sequence:

$$0 \longrightarrow \mathrm{Sel}_A^{\mathrm{str}} \longrightarrow \mathrm{Sel}_A \longrightarrow \bigoplus_{\mathfrak{p}|p} H^1(G_{F_\mathfrak{p}}/I_\mathfrak{p}, (A_\mathfrak{p}^-)^{I_\mathfrak{p}}) \tag{12}$$

Thus, we obtain:

Lemma 2.21. *If* $\bigoplus_{\mathfrak{p}|p} H^1(G_{F_\mathfrak{p}}/I_\mathfrak{p}, (A_\mathfrak{p}^-)^{I_\mathfrak{p}})$ *is finite,* $\mathrm{Sel}_A^{\mathrm{str}}$ *coincides with* Sel_A *modulo finite group.*

Especially, if T_f is a nearly ordinary representation obtained by specializing Hida family $\mathcal{T}$ at an arithmetic specialization κ_f of $\mathcal{R}$ corresponding to a certain cuspform f, $A_f = T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ has a standard choice of $(A_f)_\mathfrak{p}^+$. By the local Langlands correspondence obtained in [Car86], if $\pi_{f,\mathfrak{p}}$ is not Steinberg, $H^1(G_{F_\mathfrak{p}}/I_\mathfrak{p}, ((A_f)_\mathfrak{p}^-)^{I_\mathfrak{p}}) = ((A_f)_\mathfrak{p}^-)^{I_\mathfrak{p}}/(\mathrm{Fr}_\mathfrak{p} - 1)((A_f)_\mathfrak{p}^-)^{I_\mathfrak{p}}$ is finite.

Corollary 2.22. *If* κ_f *is an arithmetic specialization on* $\mathcal{R}$ *for which the corresponding automorphic* $\pi_{f,\mathfrak{p}}$ *is non-Steinberg at every* $\mathfrak{p}|p$*, the kernel and the cokernel of the natural map* $D_P(\mathrm{Sel}_{A_f}) \longrightarrow D_P(\mathrm{Sel}_{A_f}^{\mathrm{str}})$ *are finite.*

2.2.3. *Selmer complexes.* In this subsection, let X be T or $T \otimes_R D_M(R)$ under the situation of Setting 2.18. For $\mathfrak{p}|p$, let $X_\mathfrak{p}^+$ be respectively $T_\mathfrak{p}^+$ or $T_\mathfrak{p}^+ \otimes_R D_M(R)$. We consider the local condition as follows for every finite prime q of F:

$$C_f^\bullet(F_q, X) = \begin{cases} C_{\mathrm{cont}}^\bullet(F_q, X_q^+) & \text{for } q|p, \\ C_{\mathrm{cont}}^\bullet(\mathrm{Fr}_q, X^{I_q}) & \text{for } q \nmid p. \end{cases}$$

Let $C_f^\bullet(F, X)$ be the complex of (co-)finite type R-modules:

$$C_f^\bullet(F, X) = \mathrm{Cone}\left(C_{\mathrm{cont}}^\bullet(F_\Sigma/F, X) \oplus \bigoplus_{q\in\Sigma} C_f^\bullet(F_q, X) \longrightarrow \bigoplus_{q\in\Sigma} C_{\mathrm{cont}}^\bullet(F_q, X) \right)[-1]. \tag{13}$$

Let $\mathrm{R}\Gamma_f(F, X)$ be the corresponding object in the derived category and let $\tilde{H}_f^i(F, X)$ be the i-th cohomology group of $\mathrm{R}\Gamma_f(F, X)$. According to [Nek06, (8.9.6.1)], Matlis duality induces an isomorphism of complexes

$$\mathrm{R}\Gamma_f(F, T) \xrightarrow{\sim} D_M(\mathrm{R}\Gamma_f(F, D_M(T)(1)))[-3].$$

inducing isomorphisms in cohomology:

$$\tilde{H}_f^i(F, T) \xrightarrow{\sim} D_M\left(\tilde{H}_f^{3-i}(F, D_M(T)(1))\right). \tag{14}$$

Since $\mathbb{F}_R$ is finite, $A := T \otimes_R D_P(R) = D_P(\mathrm{Hom}_R(T, R))$ is equal to $D_M(\mathrm{Hom}_R(T, R))$ by Lemma 2.20. Hence, we have the following lemma:

Lemma 2.23. *Under the situation of Setting* 2.18*, we have:*

$$\tilde{H}_f^2(F, T^*(1)) \cong D_P(\tilde{H}_f^1(F, A)),$$

where $T^* = \mathrm{Hom}_R(T, R)$ *and* $A = T \otimes_R D_P(R)$.

By definition, we also have the following lemma:

Lemma 2.24. *The following sequence is exact:*

$$0 \to \tilde{H}^0_f(F, A) \to H^0(F, A) \to \bigoplus_{\mathfrak{p}|p} H^0(F_\lambda, A^-_{\mathfrak{p}}) \to \tilde{H}^1_f(F, A) \to \mathrm{Sel}^{\mathrm{str}}_A \to 0. \tag{15}$$

Assume in addition that $H^0(F_\mathfrak{p}, A^-_\mathfrak{p}) = 0$ for all $\mathfrak{p}|p$. Then the above two lemmas imply that $\tilde{H}^2_f(F, T^*(1))$ is isomorphic to $D_P(\mathrm{Sel}^{\mathrm{str}}_A)$. The following proposition also plays an important role in the proof of control theorem.

Proposition 2.25. *Let us assume that R is regular in addition to Setting 2.18. Suppose also the following conditions:*

(i) *The representation $T \otimes_R \mathrm{Frac}(R)$ is irreducible as a representation of G_F.*

(ii) *The residual representation $T \otimes_R R/\mathfrak{M}_R$ is irreducible as a representation of G_F where $\mathfrak{M}$ is the maximal ideal of R.*

Then, the complex $\mathrm{R}\Gamma_f(F, T)$ is a complex of R-modules concentrated in degrees 1 and 2.

Proof. The complexes $\mathrm{R}\Gamma(F_\Sigma/F, T)$, $\mathrm{R}\Gamma(F_\lambda, T)$ for $\lambda|\mathcal{N}$ and $\mathrm{R}\Gamma_f(F_\mathfrak{p}, T)$ for $\mathfrak{p}|p$ are concentrated in degrees $[0, 2]$ because T and $T^+_\mathfrak{p}$ are free R-modules and as the p-cohomological dimension of $G_{F,\Sigma}$ and $G_{F_\mathfrak{p}}$ for all $\mathfrak{p}$ are bounded by 2. For $\lambda \nmid p$, $\mathrm{R}\Gamma(\mathrm{Fr}_\lambda, T^{I_\lambda})$ is a complex concentrated in degrees $[0, 1]$ since the pro-cyclic group generated by Fr_λ is of cohomological dimension one. This implies that the complex of R-modules $\mathrm{R}\Gamma_f(F, T)$ is concentrated in degrees $[0, 3]$. By the assumption (i), $\tilde{H}^0_f(F, T)$ vanishes. By the assumption (ii), $\tilde{H}^3_f(F, T) \cong D_M(\tilde{H}^0_f(F, D_M(T)(1))$ (cf. the equality (14)) vanishes. □

3. Control theorems

In this section, we prove control theorems for Selmer complexes and Selmer groups. We discuss the control theorem for Selmer complexes in 3.1. Then, we prove the control theorem for Selmer groups in 3.2 by comparing Selmer complexes and Selmer groups.

3.1. Control for Selmer complexes.

Proposition 3.1. *Let $\mathcal{R}$ be a branch of a Hida's nearly ordinary Hecke algebra $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ and let $\mathcal{T}$ be a G_F-representation over $\mathcal{R}$ as in Setting 2.12. Assume the conditions* **(Ir)**, **(Reg-b)** *and* **(Fr-s)**. *Then, for any arithmetic specialization κ_f on $\mathcal{R}$ associated with an eigen cuspform f of some arithmetic weight (k, w), the natural map:*

$$\mathrm{R}\Gamma_f(F, \mathcal{T}) \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} \kappa_f(\mathcal{R}) \longrightarrow \mathrm{R}\Gamma_f(F, T_f) \tag{16}$$

is an isomorphism after the base extension $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Further, the map:

$$\tilde{H}^2_f(F, \mathcal{T}) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R}) \longrightarrow \tilde{H}^2_f(F, T_f) \tag{17}$$

is a surjection with finite kernel. If we assume that π_f is not Steinberg at every $\lambda|\mathcal{N}$, this map is an isomorphism.

Proof. As $\mathcal{R}$ is a regular ring, there exists an $\mathcal{R}$-regular sequence $\mathbf{x} = (x_1, x_2, \cdots, x_r)$ generating $\mathcal{P}_{\kappa_f}$. By Lemma 2.19, the complexes $\mathrm{R}\Gamma(F_\Sigma/F, T)$, $\mathrm{R}\Gamma(F_\lambda, T)$ for $\lambda|\mathcal{N}$ and $\mathrm{R}\Gamma_f(F_\mathfrak{p}, T)$ for $\mathfrak{p}|p$ descend $\mathbf{x}$-perfectly. Hence, in order to prove the control theorem for the Selmer complex, we need to study the behavior of $\mathrm{R}\Gamma_f(F_\lambda, \mathcal{T})$ under the specialization at $\mathbf{x}$ for all λ dividing $\mathcal{N}$. By Lemma 2.14, for each $\lambda|\mathcal{N}$, the natural map $\mathcal{T}^{I_\lambda} \otimes \mathcal{R}/\mathbf{x} \longrightarrow (\mathcal{T}/\mathbf{x}\mathcal{T})^{I_\lambda}$ is a map between free-module over $\kappa_f(\mathcal{R}) \cong \mathcal{R}/\mathbf{x}$ which becomes an isomorphism after the base extension $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Hence, the natural map from the complex

$$\mathrm{R}\,\Gamma_f(F_\lambda, \mathcal{T}) \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} \kappa_f(\mathcal{R}) \overset{\sim}{\longrightarrow} [\mathcal{T}^{I_\lambda} \overset{\mathrm{Fr}_\lambda - 1}{\longrightarrow} \mathcal{T}^{I_\lambda}] \otimes_{\mathcal{R}} \kappa_f(\mathcal{R})$$

to the complex:

$$[T_\kappa^{I_\lambda} \overset{\mathrm{Fr}_\lambda - 1}{\longrightarrow} T_\kappa^{I_\lambda}] \overset{\sim}{\longrightarrow} \mathrm{R}\,\Gamma_f(F_\lambda, T_\kappa)$$

is an isomorphism after the base extension $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. This completes the proof of the control theorem of the Selmer complex. Since $\mathrm{R}\,\Gamma_f(F, \mathcal{T})$ is not supported at the degree greater than 2, the map $\tilde{H}^2_f(F, \mathcal{T}) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R}) \longrightarrow \tilde{H}^2_f(F, T_f)$ is a surjection whose kernel comes from the error at degree 1 of the specializations $\mathrm{R}\,\Gamma_f(F_\lambda, \mathcal{T}) \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} \kappa_f(\mathcal{R}) \longrightarrow \mathrm{R}\,\Gamma_f(F_\lambda, T_f)$ for every $\lambda|\mathcal{N}$. Thus, the statement (16) follows. ☐

3.2. **Control for Selmer groups.**

Proposition 3.2. *Let $\mathcal{R}$ be a branch of a Hida's nearly ordinary Hecke algebra $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ and let $\mathcal{T}$ be a G_F-representation over $\mathcal{R}$ as in Setting 2.12. Assume the conditions* **(Ir)**, **(Reg-b)** *and* **(Fr-s)**. *Then, for any arithmetic specialization κ_f on $\mathcal{R}$ associated with an eigen cuspform f of some arithmetic weight (k, w) whose local automorphic representation is principal series at every $\mathfrak{p}|p$, the natural map*

$$D_P(\mathrm{Sel}^{\mathrm{str}}_{\mathcal{A}}) \otimes \kappa_f(\mathcal{R}) \longrightarrow D_P(\mathrm{Sel}^{\mathrm{str}}_{A_f}) \tag{18}$$

is a surjection and has a finite kernel.

Proof. By definition, we have the following exact sequence:

$$0 \longrightarrow \bigoplus_{\mathfrak{p}|p} H^0(G_{F_\mathfrak{p}}, \mathcal{A}^-_\mathfrak{p}) \longrightarrow \tilde{H}^1_f(F, \mathcal{A}) \longrightarrow \mathrm{Sel}^{\mathrm{str}}_{\mathcal{A}} \longrightarrow 0.$$

Similarly, for $A_f = \mathcal{A}[\mathcal{P}_{\kappa_f}]$, we have:

$$0 \longrightarrow \bigoplus_{\mathfrak{p}|p} H^0(G_{F_\mathfrak{p}}, A^-_\mathfrak{p})[\mathcal{P}_{\kappa_f}] \longrightarrow \tilde{H}^1_f(F, A_f) \longrightarrow \mathrm{Sel}^{\mathrm{str}}_{A_f} \longrightarrow 0.$$

Taking Pontrjagin duals yields:

$$0 \longrightarrow D_P(\mathrm{Sel}^{\mathrm{str}}_{\mathcal{A}}) \longrightarrow \tilde{H}^2_f(F, \mathcal{T}^*(1)) \longrightarrow \bigoplus_{\mathfrak{p}|p} D_P(H^0(G_{F_\mathfrak{p}}, \mathcal{A}^-_\mathfrak{p})) \longrightarrow 0,$$

$$0 \longrightarrow D_P(\mathrm{Sel}^{\mathrm{str}}_{A_f}) \longrightarrow \tilde{H}^2_f(F, \mathcal{T}^*(1) \otimes_{\mathcal{R}} \kappa(\mathcal{R})) \longrightarrow \bigoplus_{\mathfrak{p}|p} D_P(H^0(G_{F_\mathfrak{p}}, (A_f)^-_\mathfrak{p})) \longrightarrow 0.$$

Put $M = \mathrm{Tor}_{\mathcal{R}}(\kappa_f(\mathcal{R}), \bigoplus_{\mathfrak{p}|p} D_P(H^0(G_{F_\mathfrak{p}}, \mathcal{A}_\mathfrak{p}^-)))$. Then the following sequence is exact:

$$M \longrightarrow D_P(\mathrm{Sel}_{\mathcal{A}}^{\mathrm{str}}) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R}) \longrightarrow \tilde{H}_f^2(F, \mathcal{T}^*(1)) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R})$$
$$\longrightarrow \left(\bigoplus_{\mathfrak{p}|p} D_P(H^0(G_{F_\mathfrak{p}}, \mathcal{A}_\mathfrak{p}^-))\right) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R}) \longrightarrow 0.$$

By definition, the last term of the above equation is equal to $\bigoplus_{\mathfrak{p}|p} D_P(H^0(G_{F_\mathfrak{p}}, (A_f)_\mathfrak{p}^-))$, which is a finite group since the local automorphic representation $\pi_{f,\mathfrak{p}}$ is principal series at every $\mathfrak{p}|p$. We complete the proof if we show that M is finite.

Let $\eta_\mathfrak{p}$ be the character with values in $\mathcal{R}^\times$ through which $G_{F_\mathfrak{p}}$ acts on $\mathcal{A}_\mathfrak{p}^-$ and let $\mathcal{I}_\mathfrak{p}$ be the ideal of $\mathcal{R}$ generated by $\eta_\mathfrak{p}(\sigma) - 1$ for all σ in $G_{F_\mathfrak{p}}$. Then we have the isomorphism $D_P(H^0(G_{F_\mathfrak{p}}, \mathcal{A}_\mathfrak{p}^-)) \cong \mathcal{R}/\mathcal{I}_\mathfrak{p}$. Because $\pi_{f,\mathfrak{p}}$ is not Steinberg, $(\mathcal{R}/\mathcal{I}_\mathfrak{p}) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R})$ is finite. We choose elements $(y_1, \cdots, y_b)$ of $\mathcal{R}$ generating $\mathcal{I}_\mathfrak{p}$ and such that y_b is not zero modulo $(\mathbf{x})$. Let a be $\binom{b}{2}$. The complex equal to the the tensor product of the free resolution of $\mathcal{R}/\mathcal{I}_\mathfrak{p}$ coming from the Koszul complex of $(y_1, \cdots, y_b)$ with $\kappa_f(\mathcal{R})$ ends with:

$$\cdots \longrightarrow \kappa_f(\mathcal{R})^{\oplus a} \xrightarrow{r_2} \kappa_f(\mathcal{R})^{\oplus b} \xrightarrow{r_1} \kappa_f(\mathcal{R}) \xrightarrow{r_0} 0.$$

Since the above sequence is obtained by taking the base extension $\otimes_{\mathcal{R}} \kappa_f(\mathcal{R})$ of the following free resolution and by replacing the last map by a zero map:

$$\cdots \longrightarrow \mathcal{R}^{\oplus a} \xrightarrow{\widetilde{r}_2} \mathcal{R}^{\oplus b} \xrightarrow{\widetilde{r}_1} \mathcal{R} \longrightarrow \mathcal{R}/\mathcal{I}_\mathfrak{p},$$

the module $\mathrm{Ker}(r_0)/\mathrm{Im}(r_1)$ is isomorphic to $(\mathcal{R}/\mathcal{I}_\mathfrak{p}) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R})$ and $\mathrm{Ker}(r_1)/\mathrm{Im}(r_2)$ is isomorphic to $\mathrm{Tor}_{\mathcal{R}}(\kappa_f(\mathcal{R}), \mathcal{R}/\mathcal{I}_\mathfrak{p})$. Since $(\mathcal{R}/\mathcal{I}_\mathfrak{p}) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R})$ is finite, the image of r_2 is at least of rank b, which implies that $\mathrm{Tor}_{\mathcal{R}}(\kappa_f(\mathcal{R}), \mathcal{R}/\mathcal{I}_\mathfrak{p})$ is finite. Hence M is finite and this completes the proof. □

Remark 3.3. *We remark that even when the residual representation on $\mathbb{F}_{\mathcal{R}}$ is reducible, the previous control property makes sense after a choice of lattice $\mathcal{T}$. The only difference in that case will be that the map* (18) *has a cokernel isomorphic to $\tilde{H}_f^3(F, \mathcal{T}^*(1))[\mathcal{P}_{\kappa_f}]$.*

Corollary 3.4. *Under the same hypotheses as Proposition* 3.2 *and using the same notation, the arithmetic specialization map:*

$$D_P(\mathrm{Sel}_{\mathcal{A}}) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R}) \longrightarrow D_P(\mathrm{Sel}_{A_f}) \tag{19}$$

is a surjection with finite kernel.

Proof. The map (22) is a surjection by absolute irreducibility of the residual representation. Recall that we have the following exact sequence by definition:

$$D_P\left(\bigoplus_{\mathfrak{p}|p} H^1(G_{F_\mathfrak{p}}/I_\mathfrak{p}, (\mathcal{A}_\mathfrak{p}^-)^{I_\mathfrak{p}})\right) \longrightarrow D_P(\mathrm{Sel}_{\mathcal{A}}) \longrightarrow D_P(\mathrm{Sel}_{\mathcal{A}}^{\mathrm{str}}) \longrightarrow 0.$$

We will show that $H^1(G_{F_\mathfrak{p}}/I_\mathfrak{p}, (\mathcal{A}_\mathfrak{p}^-)^{I_\mathfrak{p}})$ is trivial for every $\mathfrak{p}|p$. Let us recall the following group introduced at 2.1:

$$\mathbf{G} = (\mathfrak{r}_F \otimes_\mathbb{Z} \mathbb{Z}_p)^\times \times \left((\mathfrak{r}_F \otimes_\mathbb{Z} \mathbb{Z}_p)^\times / \overline{\mathfrak{r}_F^\times} \right) = \prod_{q|p} (\mathfrak{r}_F)_q^\times \times \left((\mathfrak{r}_F \otimes_\mathbb{Z} \mathbb{Z}_p)^\times / \overline{\mathfrak{r}_F^\times} \right),$$

where $(\mathfrak{r}_F)_q^\times$ the completion of $\mathfrak{r}_F$ at q. We define a quotient $\mathbf{G}'$ of $\mathbf{G}$ to be:

$$\mathbf{G}' = \prod_{q|p,\ q\neq\mathfrak{p}} (\mathfrak{r}_F)_\mathfrak{p}^\times \times \left((\mathfrak{r}_F \otimes_\mathbb{Z} \mathbb{Z}_p)^\times / \overline{\mathfrak{r}_F^\times} \right).$$

and the algebra $\mathcal{R}'$ to be $\mathcal{R} \otimes_{\mathbb{Z}_p[[\mathbf{G}]]} \mathbb{Z}_p[[\mathbf{G}']]$. We denote by $\widetilde{\alpha}_\mathfrak{p}$ the unique unramified character of $D_\mathfrak{p}$ with values in $\mathcal{R}'$ whose value $\widetilde{\alpha}_\mathfrak{p}(\mathrm{Fr}_\mathfrak{p})$ interpolates the eigenvalues of the action of $\mathrm{Fr}_\mathfrak{p}$ on $(A_f)_\mathfrak{p}^-$ for every arithmetic specialization κ_f with $\mathcal{P}_{\kappa_f} \subset \mathrm{Ker}[\mathcal{R} \twoheadrightarrow \mathcal{R}']$. Among such specializations, there are a lot of arithmetic specializations κ_f such that the eigenvalues of the action of $\mathrm{Fr}_\mathfrak{p}$ on $(A_f)_\mathfrak{p}^-$ is not trivial by the Ramanujan conjecture for Hilbert modular forms (which is already proved). Hence, the character $\widetilde{\alpha}_p$ in non-trivial. By the local property of Hida's Galois representation $\mathcal{T}$, we have:

$$(\mathcal{A}_\mathfrak{p}^-)^{I_\mathfrak{p}} \cong \begin{cases} 0 & \text{if } \eta_\mathfrak{p} \equiv 1 \bmod \mathfrak{M}_\mathcal{R} \\ D_P(\mathcal{R}'(\widetilde{\alpha}_\mathfrak{p})) & \text{otherwise.} \end{cases}$$

Hence we have shown that $\mathrm{H}^1(G_{F_\mathfrak{p}}/I_\mathfrak{p}, (\mathcal{A}_\mathfrak{p}^-)^{I_\mathfrak{p}})$ is trivial for every $\mathfrak{p}|p$, which implies that $\mathrm{Sel}_\mathcal{A}^{\mathrm{str}} = \mathrm{Sel}_\mathcal{A}$. On the other hand, by Corollary 2.22, the kernel and the cokernel of the natural map $D_P(\mathrm{Sel}_{A_f}) \longrightarrow D_P(\mathrm{Sel}_{A_f}^{\mathrm{str}})$ are finite for κ_f whose automorphic representation $\pi_{f,\mathfrak{p}}$ is non-Steinberg at every $\mathfrak{p}|p$. This completes the proof. □

3.3. **Weak Control for Selmer complexes and Selmer groups.** In Sections 3.1 and 3.2, we established the control theorem for Selmer complexes and Selmer groups assuming **(Ir)**, **(Reg-b)** and **(Fr-s)**. In this subsection, we prove weaker control theorems which controls the kernels and the cokernels of arithmetic specializations $\kappa = \kappa_f$ of arithmetic weight (k, w) only after localizing at $\ker(\kappa_f)$. The residual representation V_f of localized Galois representation is always irreducible as G_F-module and is regular as $D_\mathfrak{p}$-module. Hence, the results in this section and the next requires much less assumptions thanks to an ideal free lattice obtained by the same argument as Remark 2.13.

For a branch $\mathcal{R}$ of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ and for an arithmetic specialization $\kappa = \kappa_f$ on $\mathcal{R}$ associated with an eigen cuspform f of some arithmetic weight (k, w), we denote by $\mathcal{R}_{\mathcal{P}_\kappa}$ the localization of $\mathcal{R}$ at $\mathcal{P}_\kappa = \mathrm{Ker}(\kappa)$. For a G_F-representation $\mathcal{T}$ presented in Setting 2.12, we denote by $\mathcal{T}_{\mathcal{P}_\kappa}$ the base extension $\mathcal{T} \otimes_\mathcal{R} \mathcal{R}_{\mathcal{P}_\kappa}$. We recall the following facts:

(1) The algebra $\mathcal{R}_{\mathcal{P}_\kappa}$ is a regular local ring whose residue field is isomorphic to the field of fractions of $\kappa(\mathcal{R})$.
(2) The residual representation at the maximal ideal $\mathcal{P}_\kappa$ satisfies Galois theoretic regularity property as G_F-representation over a p-adic field $\mathrm{Frac}(\kappa(\mathcal{R}))$ which is the analogue of the Galois theoretic regularity property as G_F-representation over a finite field $\mathbb{F}_\mathcal{R}$ which was presented after Setting 2.12. Thus, the representation $\mathcal{T}_{\mathcal{P}_\kappa}$ is free of rank two over $\mathcal{R}_{\mathcal{P}_\kappa}$ and $\mathcal{T}_\mathfrak{p}^+$ and $\mathcal{T}_\mathfrak{p}^-$ become free of rank one over $\mathcal{R}_{\mathcal{P}_\kappa}$ after taking the base extension $\otimes_\mathcal{R} \mathcal{R}_{\mathcal{P}_\kappa}$ for every $\mathfrak{p}|p$.

Since $\mathcal{P}_\kappa$ is generated by a regular sequence in $\mathcal{R}_{\mathcal{P}_\kappa}$, we prove the following proposition by the same proof as that of Proposition 3.1:

Proposition 3.5. *Let $\mathcal{R}$ be a branch of a Hida's nearly ordinary Hecke algebra $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ and let $\mathcal{T}$ be a G_F-representation over $\mathcal{R}$ as in Setting* 2.12. *Then, for any arithmetic specialization $\kappa = \kappa_f$ on $\mathcal{R}$ associated with an eigen cuspform f of some arithmetic weight (k, w), the natural map:*

$$\mathrm{R}\Gamma_f(F, \mathcal{T}_{\mathcal{P}_\kappa}) \overset{\mathrm{L}}{\otimes}_{\mathcal{R}} \kappa(\mathcal{R}) \longrightarrow \mathrm{R}\Gamma_f(F, V_f) \tag{20}$$

is an isomorphism. Further, the map:

$$\tilde{H}^2_f(F, \mathcal{T}_{\mathcal{P}_\kappa}) \otimes_{\mathcal{R}} \kappa_f(\mathcal{R}) \longrightarrow \tilde{H}^2_f(F, V_f) \tag{21}$$

is an isomorphism.

The comparison obtained at Lemma 2.22 and Lemma 2.24 implies the following corollary:

Corollary 3.6. *Let us consider the same situation as Proposition 3.5. If the eigen cuspform f corresponding to $\kappa = \kappa_f$ on $\mathcal{R}$ is principal series at every $\mathfrak{p}|p$, the arithmetic specialization map:*

$$(D_P(\mathrm{Sel}_{\mathcal{A}}) \otimes_{\mathcal{R}} \mathcal{R}_{\mathcal{P}_\kappa}) \otimes_{\mathcal{R}} \kappa(\mathcal{R}) \longrightarrow D_P(\mathrm{Sel}_{A_f}) \otimes_{\kappa(\mathcal{R})} \mathrm{Frac}(\kappa(\mathcal{R})) \tag{22}$$

is an isomorphism.

4. Applications

4.1. **Examples of torsion Selmer groups.** Let $\mathcal{R}$ be a branch of $\mathbf{H}_{\mathcal{N},\mathcal{O}}$ and assume the situation as in Setting 2.12 (but we do not assume stronger conditions listed just after Setting 2.12 which were used in Sections 3.1 and 3.2). Under Setting 2.12, we propose the following conjecture.

Conjecture 4.1. $D_P(\mathrm{Sel}_{\mathcal{A}})$ *(resp. $\tilde{H}^2_f(F, \mathcal{T}^*(1))$) is a torsion $\mathcal{R}$-module.*

In some special cases, we will prove this below.

Theorem 4.2. *Assume also that there exists an arithmetic specialization $\kappa = \kappa_f$ such that V_f satisfies all the following properties* (i)-(iv)*:*

(i) *The cuspform f is of critical weight (k, w) with $k = 2t$ and $w = t$ and the Neben character of f is trivial.*
(ii) *The representation $\pi_{f,\mathfrak{p}}$ is principal series at every $\mathfrak{p}|p$.*
(iii) *The L-function $L(V_f, s)$ does not vanish at $s = 0$.*
(iv) *One of the following condition holds:*
  (a) $2 \nmid [F : \mathbb{Q}]$.
  (b) *There exists a finite place λ of F such that $\pi_{f,\lambda}$ is not a principal series representation.*
  (c) *The form f has no complex multiplication and there exists an element $\sigma \in G_F$ such that one of the eigenvalues of σ on V_f are ± 1 and the other is not equal to ± 1.*

Then $D_P(\mathrm{Sel}_{\mathcal{A}})$ (resp. $\tilde{H}^2_f(F, \mathcal{T}^*(1))$) is $\mathcal{R}$-torsion.

Proof. Let $\mathcal{K}$ be a finite extension of $\mathbb{Q}_p$ containing all Hecke eigenvalues of f. Under (iv) (a) or (iv) (b), there exists a compact quaternionic Shimura curve X defined over F such that the representation V_f is a direct summand of $H^1_{et}(X \times_F \bar{F}, \mathrm{Frac}(\mathcal{O}))$. Hence, by [FH95, Theorem B], there exists a totally imaginary quadratic extension K/F such that f does not have complex multiplication by K and such that the order of zero at $s=1$ of $L(V_f|_{G_K}, s)$ is of exact order 1 where $V_f|_{G_K}$ is the restriction of V_f to G_K. By [YZZ08, Theorem 1.3.1], then there exists a non torsion CM points on $X(K^{\mathrm{ab}})$ and this implies that $H^1_f(F, V_f)$ is trivial thanks to [Nek07, Theorem 3.2]. If we assume (iv) (c), then the form f satisfies the hypotheses of [Nek10, Theorem B]. Thus, $H^1_f(F, V_f)$ is also trivial.

Hence $\tilde{H}^2_f(F, T^*_f(1))$ is finite. By Proposition 3.5, the group:

$$\tilde{H}^2_f(F, \mathcal{T}^*_{\mathcal{P}_\kappa}(1)) \otimes_{\mathcal{R}} \kappa(\mathcal{R}) \cong \tilde{H}^2_f(F, V^*_f(1))$$

is trivial. Since $\tilde{H}^2_f(F, \mathcal{T}^*_{\mathcal{P}_\kappa}(1)) \cong \tilde{H}^2_f(F, \mathcal{T}^*(1)) \otimes_{\mathcal{R}} \mathcal{R}_{\mathcal{P}_\kappa}$, $\tilde{H}^2_f(F, \mathcal{T}^*(1)) \otimes_{\mathcal{R}} \mathcal{R}_{\mathcal{P}_\kappa}$ is trivial by Nakayama's lemma. This implies that $\tilde{H}^2_f(F, \mathcal{T}^*(1))$ is torsion over $\mathcal{R}$. By comparing the Selmer complex and the Selmer group, we also deduce that $D_P(\mathrm{Sel}_{\mathcal{A}})$ is torsion over $\mathcal{R}$. □

Remark 4.3. *In* [YZZ08]*, a proof of the vanishing of $H^1_f(F, V_f)$ by Y.Tian and S.Zhang is announced under* (iii) *(a) or (b) but without the hypothesis that the Neben character is trivial. Our result would then generalize likewise.*

Finally, we would emphasize that our work opens several possibilities for future research on the generalization of Iwasawa theory. Since the construction of $d+1$-variable analytic p-adic L-function is constructed in the article [DO] with which the second author is concerned, it makes sense to consider the Iwasawa main conjecture for a big Galois deformation obtained by nearly ordinary Hida deformation for Hilbert modular forms, in which we compare the characteristic ideal of the Selmer group or Selmer complex and the principal ideal generated by the analytic p-adic L-function (cf. [Och10] for precise statements). It will also be interesting to discuss the compatibility between the Iwasawa main conjecture and the Tamagawa number conjecture for each modular form especially under the formalism of Selmer complexes as discussed in [Nek06] and the doctoral thesis of the first author.

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