

Interpolating coefficient systems and p -ordinary cohomology of arithmetic groups

Günter Harder

1. Introduction

Let p be a prime, we assume $p > 2$. In this note we construct an interpolation of (p -)ordinary cohomology of arithmetic groups. Our main tool will be the construction of sheaves, which p -adically interpolate coefficient systems, which are obtained by highest weight representations. After that we construct a p -adic interpolations of the Hecke operators. The cohomology of arithmetic groups is defined as sheaf cohomology and no automorphic forms enter.

It is not clear to me to how the results of this note are related to the work of Hida, Ash and Stevens and other authors. In any case I think that the approach which I use here- namely interpolating sheaves and interpolating the Hecke operators- is quite direct and natural and looks different from what I see in the in the literature.

For me it is relevant that the boundedness theorem, which is proved at the end of this note, is exactly what I need to make further progress in the questions which are discussed in my notes kolloquium.pdf on my home page (see below) and a manuskript on rank-one Eisenstein cohomology [Ha-rank1], which will be available on the preprint server of the Erwin Schrödinger Institute (ESI). I will say a little bit more about these applications at the end of this note. Only in the last section the automorphic forms enter the stage.

For the discussion of the Hecke operators and also for some basic notions and notations used in this note I refer to my book project on "Cohomology of arithmetic groups" ([Ha-Coh],chap.2-6) which exists in preliminary form on my home-page:

"www.math.uni-bonn.de/people/harder/Manuscripts/buch/"

Part of this paper was prepared when I was visiting the Institute for Advanced Study in Princeton, where I also found the proof for the boundedness of torsion. The idea that the boundedness may be true and may be an interesting theorem came to me when I walked down the Strudlhofstiege in Vienna during my stay at the workshop on Automorphic Forms in 2006 at the Erwin Schrödinger Institute. I thank both institutions for their support.

2. The case GL_2

2.1 The coefficient systems:

Let \mathbb{C}_p the completion of the algebraic closure $\bar{\mathbb{Q}}_p$, let $O_{\mathbb{C}_p}$ be its ring of integers. We have the canonical homomorphism $r : \mathbb{Z}_p^\times \rightarrow \mathbb{F}_p^\times$ the kernel is the group $\mathbb{Z}_p^{(1)}$ of 1-units and the Teichmüller character provides a section $\omega : \mathbb{F}_p^\times \hookrightarrow \mathbb{Z}_p^\times$. It is defined by the two requirements: a) $\omega(x)$ is a $p - 1$ -th root of unity, and b) $x/\omega(x) = 1 + pl(x) \in \mathbb{Z}_p^{(1)}$.

We will forget the homomorphism r and we will write $\omega(x)$ for $\omega(r(x))$. Then \mathbb{Z}_p^\times is a direct product $\mathbb{F}_p^\times \times \mathbb{Z}_p^{(1)}$. We consider a pair (ν, α) where $\nu \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $\alpha \in O_{\mathbb{C}_p}$. We denote such a pair by $\chi = (\nu, \alpha)$. Any such χ defines a character

$$\begin{array}{rcl} \chi & : & \mathbb{Z}_p^\times \longrightarrow O_{\mathbb{C}_p}^\times \\ & & 1 + xp \mapsto (1 + xp)^\alpha \\ \chi & : & \omega(x) \mapsto \omega(x)^\nu. \end{array}$$

1

For any integer m we get

$$\chi^{[m]} : (\mathbb{Z}/p^m\mathbb{Z})^\times \longrightarrow (O_{\mathbb{C}_p}/p^m O_{\mathbb{C}_p})^\times.$$

Let $B = B_+$ is the Borel subgroup of upper triangular matrices. Its quotient by its unipotent radical U_+ is equal to the torus of diagonal matrices

$$T = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right\}.$$

We define the character $\tilde{\chi}^{[m]} : B_+(\mathbb{Z}/(p^m)) \rightarrow (O_{\mathbb{C}_p}/p^m O_{\mathbb{C}_p})^\times$, which is given by

$$\tilde{\chi}^{[m]} : \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto \chi^{[m]}(t_1).$$

We can consider the induced $\mathrm{Gl}_2(\mathbb{Z}/p^m\mathbb{Z})$ -module

$$I_{\tilde{\chi}^{[m]}} := \mathrm{Ind}_{B(\mathbb{Z}/p^m\mathbb{Z})}^{G(\mathbb{Z}/p^m\mathbb{Z})} \tilde{\chi}^{[m]} =$$

$$\left\{ f : G(\mathbb{Z}/p^m\mathbb{Z}) \rightarrow O_{\mathbb{C}_p}/p^m O_{\mathbb{C}_p} \mid f\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \tilde{\chi}^{[m]}\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix}\right) f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \right\}.$$

Of course as usual the group acts on this module by translations from the right, i.e. $R_x(f)(g) = f(gx)$. Sometimes it is convenient to consider the submodule of those functions which take their values in a subring of $R \subset O_{\mathbb{C}_p}$: This ring must receive the values of χ , i.e. we must require that $\alpha \in R$. Then we put $R_m = R/(p^m)$. Hence we get that $I_{\tilde{\chi}^{[m]}}$ is a free R_m -module of rank $\#\mathbb{P}^1(\mathbb{Z}/(p^m))$.

We define a submodule

$$\mathcal{P}_{\tilde{\chi}^{[m]}} = \left\{ f \in I_{\tilde{\chi}^{[m]}} \mid f(gg_\nu) \equiv f(g) \pmod{p^\nu} \text{ for all } g_\nu \equiv 1 \pmod{p^\nu} \right\}.$$

For this system of submodules $G(\mathbb{Z}/(p^m))$ we have equivariant homomorphisms

$$r_{\tilde{\chi}^{[m]}} : \mathcal{P}_{\tilde{\chi}^{[m]}} \longrightarrow \mathcal{P}_{\tilde{\chi}^{[m-1]}}.$$

The homomorphism $\mathrm{Gl}_2(\mathbb{Z}/(p^{m+1})) \rightarrow \mathrm{Gl}_2(\mathbb{Z}/(p^m))$ induces equivariant homomorphisms

$$i_{\tilde{\chi}^{[m]}} : I_{\tilde{\chi}^{[m]}} \longrightarrow I_{\tilde{\chi}^{[m+1]}} \otimes \mathbb{Z}/(p^m).$$

We choose an open compact subgroup $K_f = \prod_\ell K_\ell$ and assume that $K_p = \mathrm{Gl}_2(\mathbb{Z}_p)$. Let $S_{K_f}^G$ the associated modular curve (See Ha-Coh], chap.3,1. 2), it has an adelic description as

$$S_{K_f}^G = \mathrm{Gl}_2(\mathbb{Q}) \backslash \mathrm{Gl}_2(\mathbb{R}) / \mathrm{SO}(2) \times G(\mathbb{A}_f) / K_f.$$

Our $\mathcal{P}_{\tilde{\chi}^{[m]}}$ are $\mathrm{Gl}_2(\hat{\mathbb{Z}})$ modules and hence we get sheaves $\tilde{\mathcal{P}}_{\tilde{\chi}^{[m]}}$ on $S_{K_f}^G$. These sheaves can be considered as sheaves for the analytic topology on $S_{K_f}^G$ but also as sheaves for the etale topology on $S_{K_f}^G$. We get a projective system of sheaves on $S_{K_f}^G$, and we define

$$H^1(S_{K_f}^G, \tilde{\mathcal{P}}_{\tilde{\chi}}) := \varprojlim_{\frac{m}{2}} H^1(S_{K_f}^G, \mathcal{P}_{\tilde{\chi}^{[m]}}).$$

If n is an integer ≥ 0 then we can define the special character

$$\chi_n = (n \pmod{p-1}, n).$$

Let us denote by \mathcal{M}_n the module of homogenous polynomials in two variables X, Y with \mathbb{Z} -coefficients and with an action of $\mathrm{Gl}_2(\mathbb{Z})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X, Y) = P(aX + cY, bX + dY) \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{-n}.$$

We get a $\mathrm{Gl}_2(\mathbb{Z}/p^m)$ invariant homomorphism

$$\mathcal{M}_n/p^m \mathcal{M}_n \longrightarrow \mathcal{P}_{\tilde{\chi}_{-n}^{[m]}},$$

which sends a polynomial $P \in \mathcal{M}_n$ to the function

$$f_P\left(\begin{pmatrix} u & v \\ x & y \end{pmatrix}\right) = P(x, y) \det\left(\begin{pmatrix} u & v \\ x & y \end{pmatrix}\right)^{-n}$$

on $\mathrm{GL}_2(\mathbb{Z}/p^m)$. It will not be injective in general. This homomorphism induces a homomorphism

$$H^\bullet(S_{K_f}^G, \mathcal{M}_n/p^m \mathcal{M}_n) \longrightarrow H^\bullet(S_{K_f}^G, \mathcal{P}_{\tilde{\chi}_{-n}^{[m]}}).$$

In this case we may take $R_m = \mathbb{Z}/p^m \mathbb{Z}$.

2.2. The Hecke operators:

We want to construct Hecke operators $T(\alpha, u_\alpha)$ which act on these cohomology groups as endomorphisms.

I recall the construction of Hecke operators acting on the cohomology with coefficients as it is outlined in [Ha-Coh], chap. II in the section on Hecke operators. It has to be translated into the adelic language, but this is a minor point.

Let Γ be any arithmetic congruence group. Any Γ -module M gives us a coefficient system \tilde{M} and we study the cohomology groups $H^\bullet(\Gamma \backslash X, \tilde{M})$. To get Hecke operators acting on these cohomology groups we need two data.

a) An element $\alpha \in G(\mathbb{Q})$ -

The group $\Gamma(\alpha) = \alpha\Gamma\alpha^{-1} \cap \Gamma$ has finite index in Γ . We define a new $\Gamma(\alpha)$ -module $M^{(\alpha)}$, which is equal to M as an abelian group, but on which $\gamma \in \Gamma(\alpha)$ acts by

$$(\gamma, m) \mapsto (\alpha^{-1}\gamma\alpha)m.$$

The second datum is

b) A $\Gamma(\alpha)$ -homomorphism $u_\alpha : M^{(\alpha)} \rightarrow M$

Then such a pair (α, u_α) induces an endomorphism in the cohomology (See [Ha-Coh], chap. 3,2)

$$T^\bullet(\alpha, u_\alpha) : H^\bullet(\Gamma \backslash X, \tilde{M}) \rightarrow H^\bullet(\Gamma \backslash X, \tilde{M}).$$

It also induces an endomorphism on the cohomology with compact supports and the cohomology of the boundary of the Borel-Serre compactification. Of course it is compatible with the fundamental long exact sequence.

In our case we will take for α the element $t_{p^m} = \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}$ and we have to look for the possible choices of $u_\alpha = u_{t_{p^m}}$.

If for instance our M is one of the modules \mathcal{M}_n (this is a \mathbb{Z} -module) then we have essentially only one good choice for $u_{t_{p^m}}$ this is the so called canonical or classical choice in chap.2 loc.cit. We briefly recall how it looks in this case. The element $u_{t_{p^m}} \in \text{Gl}_2(\mathbb{Q})$ induces a linear map $\mathcal{M}_n \otimes \mathbb{Q} \rightarrow \mathcal{M}_n \otimes \mathbb{Q}$. This map sends $X^\nu Y^{n-\nu} \rightarrow p^{\nu m} p^{-nm} X^\nu Y^{n-\nu}$. This element is up to a scalar the unique homomorphism from $(\mathcal{M}_n \otimes \mathbb{Q})^{(t_{p^m})} \rightarrow \mathcal{M}_n \otimes \mathbb{Q}$. We have to multiply it by p^{nm} to get a homomorphism

$$u_{t_{p^m}}^{\text{class}} : \mathcal{M}_n^{(t_{p^m})} \rightarrow \mathcal{M}_n.$$

But if we pass to $\mathcal{M}_n \otimes \mathbb{Z}/(p^m)$, or if we consider one of the modules $\mathcal{P}_{\tilde{\chi}^{[m]}}$, $I_{\tilde{\chi}^{[m]}}$, then we will find many more such u_α . We always can consider the reduction of $u_{t_{p^m}}^{\text{class}}$ to $\mathcal{M}_n/p^a \mathcal{M}_n$, and then we also call this reduction $u_{t_{p^m}}^{\text{class}}$.

We will not consider all of them. For any χ and any m we want to construct a set

$$\mathbb{H}_\chi^{[m]} = T(\{t_{p^m}, u_{t_{p^m}}\})$$

which means to give a collection of $u_{t_{p^m}}$.

We assume that $n > 0$ and formulate certain requirements, that should be fulfilled by this families of operators.

i) We want to have diagrams

$$\begin{array}{ccc} \mathcal{P}_{\tilde{\chi}^{[m]}}^{(t_{p^m})} & \xrightarrow{u_{t_{p^m}}} & \mathcal{P}_{\tilde{\chi}^{[m]}} \\ \downarrow & & \downarrow \\ I_{\tilde{\chi}^{[m]}}^{(t_{p^m})} & \xrightarrow{u_{t_{p^m}}} & I_{\tilde{\chi}^{[m]}} \end{array}$$

i.e. they are defined on the I and restrict to the \mathcal{P} .

ii) We want them to form a projective system if we restrict them to the sheaves $\mathcal{P}_{\tilde{\chi}^{[m]}}$. This means that given an

$$u_{t_{p^{m+1}}} : \mathcal{P}_{\tilde{\chi}^{[m+1]}}^{(t_{p^{m+1}})} \rightarrow \mathcal{P}_{\tilde{\chi}^{[m+1]}}$$

we require that it "pushes down" to an $u_{t_{p^m}}$, i.e. we get diagrams

$$\begin{array}{ccc} \mathcal{P}_{\tilde{\chi}^{[m+1]}}^{(t_{p^{m+1}})} & \xrightarrow{u_{t_{p^{m+1}}}} & \mathcal{P}_{\tilde{\chi}^{[m+1]}} \\ \downarrow & & \downarrow \\ \mathcal{P}_{\tilde{\chi}^{[m]}}^{(t_{p^m})} & \xrightarrow{u_{t_{p^m}}} & \mathcal{P}_{\tilde{\chi}^{[m]}} \\ & 4 & \end{array}$$

and

$$\begin{array}{ccc} I_{\tilde{\chi}^{[m+1]}}^{(t_{p^{m+1}})} \otimes \mathbb{Z}/(p^m) & \xrightarrow{u_{t_{p^{m+1}}}} & I_{\tilde{\chi}^{[m+1]}} \otimes \mathbb{Z}/(p^m) \\ \uparrow & & \uparrow \\ I_{\tilde{\chi}^{[m]}}^{(t_{p^m})} & \xrightarrow{u_{t_{p^m}}} & I_{\tilde{\chi}^{[m]}} \end{array} .$$

Clearly $u_{t_{p^m}}$ is uniquely determined by $u_{t_{p^{m+1}}}$, if it exists. If we want to check the existence of $u_{t_{p^m}}$ we have to show that for any function $f \in I_{\tilde{\chi}^{[m]}}^{(t_{p^m})}$ the function $u_{t_{p^{m+1}}}(i_{\tilde{\chi}^{[m]}}(f))$ is constant on the fibers of $\mathrm{Gl}_2(\mathbb{Z}/(p^{m+1})) \rightarrow \mathrm{Gl}_2(\mathbb{Z}/(p^m))$ and therefore in the image of $i_{\tilde{\chi}^{[m]}}$. This is certainly the case if $u_{t_{p^{m+1}}}$ has the following additional property

$$u_{t_{p^{m+1}}}(I_{\tilde{\chi}^{[m+1]}}^{(t_{p^{m+1}})}) \subset \mathcal{P}_{\tilde{\chi}^{[m+1]}} \quad (*)$$

We get a projective system of Hecke operators $r_{m+1} : \mathbb{H}_{\tilde{\chi}}^{[m+1]} \rightarrow \mathbb{H}_{\tilde{\chi}}^{[m]}$, we do not require that the r_{m+1} are surjective. The set $\mathbb{H}_{\tilde{\chi}}^{[m]}$ depends only on $\chi^{[m]}$ as the notation indicates.

iii) We want to construct a principal operator operator $u^{\mathrm{princ}} = (\dots, u_{t_{p^{m+1}}}^{\mathrm{princ}}, u_{t_{p^m}}^{\mathrm{princ}}, \dots)$ in the projective system of $\mathbb{H}_{\tilde{\chi}}^{[m]}$, which has the following property:

For any $u_{t_{p^m}} \in \mathbb{H}_{\tilde{\chi}}^{[m]}$ we find an integer $a \in \mathbb{Z}_p$ such that

$$(u_{t_{p^m}} - au_m^{\mathrm{princ}})(I_{\tilde{\chi}^{[m]}}) \subset pI_{\tilde{\chi}^{[m]}}$$

iv) And finally we want: If $\chi = \chi_{-n}$ as above then the classical Hecke operator on $\mathcal{M}_n \otimes \mathbb{Z}/(p^m)$ extends to an operator on $\mathbb{H}_{\tilde{\chi}_{-n}}^{[m]}$.

Our first and in some sense main result will be the existence of such a system of Hecke operators. We will discuss this first in the case of Gl_2 in section 3.3. Actually this discussion will be much to detailed, we will see that we really have quite a lot of Hecke operators, many more than we need. But then it will be clear that (under some mild conditions) we have a suitable system Hecke operators for arbitrary reductive groups.

2.3. First consequences

The existence of such a system of sheaves together with the Hecke operators has interesting consequences.

A first consequence of these properties is the following: Let us denote the set of operators $u_{t_{p^m}}$ for which the above number a is zero modulo p by $\mathbb{J}_{\tilde{\chi}, m}$. Then it is clear that any composition of operators in $\mathbb{H}_{\tilde{\chi}}^{[m]}$ which contains more than m factors from $\mathbb{J}_{\tilde{\chi}, m}$ annihilates the cohomology $H^\bullet(S_{K_f}^G, I_{\tilde{\chi}^{[m]}})$. The same applies to the cohomology with compact supports and the cohomology of the boundary.

As in [Ha-Coh] we write H_γ^\bullet for the various variants of cohomology, i.e. the γ may indicate cohomology with compact supports, $!$ cohomology or cohomology without supports. We can apply the same argument to the various cohomology groups

$$H_\gamma^\bullet(S_{K_f}^G(\mathbb{C}), I_{\tilde{\chi}^{[m]}}/\mathcal{P}_{\tilde{\chi}^{[m]}}), H^\bullet(\partial S_{K_f}^G, I_{\tilde{\chi}^{[m]}}/\mathcal{P}_{\tilde{\chi}^{[m]}}).$$

Using the long exact sequence we get the same consequence for the cohomology of the subsheaves

$$H_{\mathbb{Z}}^{\bullet}(S_{K_f}^G, \mathcal{P}_{\bar{\chi}^{[m]}}), H^{\bullet}(\partial S_{K_f}^G(\mathbb{C}), \mathcal{P}_{\bar{\chi}^{[m]}})$$

but now we may need $2m$ factors.

Hence we see: If X is any of the above cohomology groups, then we can take the principal Hecke operator u_m^{princ} and consider the image $(u_m^{\text{princ}})^N(X)$ for a high power N . This becomes stationary and will be called $X_{\text{ord}}^{\text{princ}}$. The operator u_m^{princ} induces an isomorphism on $X_{\text{ord}}^{\text{princ}}$. We denote by X_{nilpt} the maximal submodule of X on which $u_{t_p^m}^{\text{princ}}$ acts nilpotently then we get a decomposition

$$X = X_{\text{nilpt}} \oplus X_{\text{ord}}^{\text{princ}}.$$

It is also clear that we can replace u_m^{princ} by any other operator $u_{t_p^m} = u_m^{\text{princ}} + v_m$ where $v_m \in \mathbb{J}_{\chi^{[m]}}$, then $u_{t_p^m}$ and u_m^{princ} will define the same submodule X_{nilpt} . We can also say that X_{nilpt} is the maximal submodule on which $\mathbb{H}_{\chi}^{[m]}$ acts nilpotently. Now we define

$$X_{\text{ord}} = X/X_{\text{nilpt}}.$$

This quotient has a section, if we restrict the module structure to just one operator $u_{t_p^m} = u_m^{\text{princ}} + v_m$, but for the full system of Hecke operators such a section will not exist in general.

It is clear that a homomorphism $X \rightarrow X_1$ which is compatible with the action of the Hecke algebra sends $X_{\text{ord}} \rightarrow X_{1,\text{ord}}$ and from an exact sequence of Hecke modules

$$X' \rightarrow X' \rightarrow X''$$

we get an exact sequence

$$X'_{\text{ord}} \rightarrow X'_{\text{ord}} \rightarrow X''_{\text{ord}}.$$

We consider the cohomology groups $H^{\bullet}(S_{K_f}^G, \mathcal{M}_n/p^m \mathcal{M}_n)$. We use the Hecke operator $T(t_p^m, u_{t_p^m}^{\text{class}})$ to define $H_{\text{ord}}^{\bullet}(S_{K_f}^G, \mathcal{M}_n/p^m \mathcal{M}_n)$. Now we get our first main result

Theorem 1 *We get a sequence of isomorphisms*

$$H_{\text{ord}}^{\bullet}(S_{K_f}^G, \mathcal{M}_n/p^m \mathcal{M}_n) \xrightarrow{\sim} H_{\text{ord}}^{\bullet}(S_{K_f}^G, \mathcal{P}_{\chi_n^{[m]}}) \xrightarrow{\sim} H_{\text{ord}}^{\bullet}(S_{K_f}^G, I_{\chi_n^{[m]}}).$$

The same holds for the cohomology with compact supports and the cohomology of the boundary.

Our previous considerations imply that the second homomorphism is an isomorphism. If we investigate the morphism $\mathcal{M}_n/p^m \mathcal{M}_n \rightarrow I_{\chi_n^{[m]}}$ a little bit more closely then we see easily that the kernel has no ordinary cohomology, i.e. $\mathbb{H}_{\chi_n^{[m]}}$ acts nilpotently on the cohomology of the kernel of this homomorphism (this will be justified further down (See 3.6)). But for the cokernel it is easy to see that the $u_{t_p^m}$ satisfy $u_{t_p^m}(I_{\chi_n^{[m]}}) \subset pI_{\chi_n^{[m]}} + \mathcal{M}_n/p^m \mathcal{M}_n$ and then it is also clear that $\mathbb{H}_{\chi_n^{[m]}}$ acts nilpotently on the cohomology of the cokernel.

Therefore the claim is proved: All the homomorphisms above are isomorphism.

I want to call these sheaves $\tilde{\mathcal{P}}_\chi$ *interpolating* sheaves, they have the property that for two such characters $\chi = (\nu, \alpha), \psi = (\nu_1, \beta)$ we have $\mathcal{P}_{\chi_{-n}^{[m]}} = \mathcal{P}_{\psi_{-n}^{[m]}}$ if $\nu = \nu_1$ and $\alpha \equiv \beta \pmod{p^{m-1}}$. It seems to be impossible to do find such congruences for the \mathcal{M}_n . But still we know that

$$H_{\text{ord}}^\bullet(S_{K_f}^G, \mathcal{M}_n/p^m \mathcal{M}_n) \xrightarrow{\sim} H_{\text{ord}}^\bullet(S_{K_f}^G, \mathcal{M}_{n_1}/p^m \mathcal{M}_{n_1})$$

if $n, n_1 > 0$ and $n \equiv n_1 \pmod{(p-1)p^{m-1}}$.

3. The construction of Hecke operators on interpolating coefficient systems

3.1 The case $m = 1$

We analyse the special case $m = 1$. We assume that $0 \leq n < p-1$ and consider the $\text{Gl}_2(\mathbb{F}_p)$ homomorphism

$$j_n : \mathcal{M}_n/p\mathcal{M}_n \rightarrow I_{\chi_{-n}^{[1]}}$$

under our assumption on n this is always an inclusion. Of course we can define this homomorphism j_n for any integer $n \geq 0$ and it is important to notice that the right hand side depends only on $n \pmod{p-1}$.

We want to compute the spaces of Hecke operators for these modules and investigate the spaces of Hecke operators which are compatible with j_n . We anticipate the first page of 3.3. We have to compute the spaces of $U_+(\mathbb{F}_p)$ coinvariants and $U_-(\mathbb{F}_p)$ invariants for our modules and then we have to determine the spaces of $T(\mathbb{F}_p)$ -invariant homomorphism between the coinvariants and the invariants. Of course we have to use the Bruhat-decomposition. Let w the nontrivial element in the Weyl group, let $u \in U_+(\mathbb{F}_p)$. We define functions Ψ_0 (resp. $\Psi_{w,u}$) $\in I_{\chi_{-n}^{[1]}}$ by the condition, that they are supported on $B_+(\mathbb{F}_p)$ (resp. $B_+(\mathbb{F}_p)wu$ (see 3.3).)

Let $\bar{\Psi}_e$ (resp. $\bar{\Psi}_w$) be the images of Ψ_0 (resp. $\Psi_{w,u}$) in the space of coinvariants, the element $\bar{\Psi}_{w,u}$ does not depend on u . The space of $U_-(\mathbb{F}_p)$ invariants is generated by the functions $\Phi_e(g) = \Phi_e(bu_-) = \chi^{[1]}(b)$ and $\Phi_w(g) = \Phi_e(bwu_-) = \Phi_e(bw) = \chi^{[1]}(b)$. Then an easy computation shows

$$\begin{array}{ccc} (\mathcal{M}_n/p\mathcal{M}_n)_{U_+(\mathbb{F}_p)} = \mathbb{F}_p Y^n & \rightarrow & (I_{\chi_n^{[1]}})_{U_+(\mathbb{F}_p)} = \mathbb{F}_p \bar{\Psi}_e \oplus \mathbb{F}_p \bar{\Psi}_w \\ \downarrow u_{t_{p^m}}^{class} & & \downarrow \\ (\mathcal{M}_n/p\mathcal{M}_n)^{U_-(\mathbb{F}_p)} = \mathbb{F}_p Y^n & \rightarrow & (I_{\chi_n^{[1]}})^{U_-(\mathbb{F}_p)} = \mathbb{F}_p \Phi_e \oplus \mathbb{F}_p \Phi_w \end{array}$$

It is easy to see that the $T(\mathbb{F}_p)$ rational points of our standard torus acts by the characters $t \mapsto t_1^{-n}$ on $\bar{\Psi}_e, \Phi_e$ and $t \mapsto t_2^{-n}$ on $\bar{\Psi}_w, \Phi_w$. Hence we see that for $n > 0$ the space of possible $u_{t_{p^m}}$ is of dimension two: We have the operator, -which will be called the *principal* operator later-,

$$u_{t_{p^m}}^{princ} : \bar{\Psi}_e \mapsto \Phi_e, \bar{\Psi}_w \mapsto 0$$

and a second one, which just does the opposite.

We observe that for for all n the the polynomial Y^n goes to $\bar{\Psi}_e$ in the module of coinvariants. If $n > 0$ the polynomial Y^n maps to Φ_e in the module of invariants

and hence we see that under the assumption $n > 0$ the principal operator is an extension of the classical operator. Especially we have that the classical Hecke operator induces the zero map on $H^\bullet(S_{K_f}^G, I_{\tilde{\chi}_n^{[1]}}/(\mathcal{M}_n/p\mathcal{M}_n))$.

Hence we see that for $n > 0$ we get isomorphisms

$$j_n^\bullet : H_{\text{ord}}^\bullet((S_{K_f}^G, \mathcal{M}_n/p\mathcal{M}_n) \xrightarrow{\sim} H_{\text{ord}}^\bullet(S_{K_f}^G, I_{\tilde{\chi}_n^{[1]}})$$

But we also see that this is not the case if $n = 0$. In this case Y^n is mapped to $\Phi_e + \Phi_w$ in the space of invariants. So we see that iii) is not valid. But since the torus action becomes trivial we see that the space of possible $u_{t_p^m}$ has dimension 4 and we have several extension of the classical operator.

We also can replace n by $n_1 = n + k(p - 1)$, where $n > 0$ is an integer. This does not change the module $I_{\tilde{\chi}_n^{[1]}}$. It is easy to see that then the homomorphism $\mathcal{M}_{n_1}/p\mathcal{M}_{n_1} \rightarrow I_{\tilde{\chi}_{-n}^{[1]}}$ is surjective, if $k > 0$. But then-still under the assumption $n > 0$ -the homomorphism j_{n_1} has a kernel and now the classical Hecke operator induces the trivial map on the cohomology of the kernel. Hence we get for any $n > 0$ as above and any $n_1 = n + k(p - 1)$ that the natural maps

$$H_{\text{ord}}^\bullet(S_{K_f}^G, \mathcal{M}_{n_1}/p\mathcal{M}_{n_1}) \rightarrow H_{\text{ord}}^\bullet(S_{K_f}^G, \mathcal{P}_{\tilde{\chi}_n^{[1]}}) \rightarrow H_{\text{ord}}^\bullet(S_{K_f}^G, I_{\tilde{\chi}_n^{[1]}})$$

are isomorphisms.

If $n = 0$ the situation is a little bit different. Let us look at the more general case that $n \equiv 0 \pmod{p-1}$. In this case we have

$$I_{\tilde{\chi}_n^{[1]}} = I_{\tilde{\chi}_0^{[1]}} = \text{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \mathbb{F}_p$$

and this module decomposes into the one dimensional subspace of constant functions and a p -dimensional space of functions which are orthogonal to the constant functions. It is clear that the constant functions are simply the given by the image of j_0 and the complement is given as the image of j_{p-1} . Then j_{p-1} induces an isomorphism on ordinary cohomology, but j_0 does not if we define "ordinary" with respect to the principal operator.

If we take for $n = p^m(p - 1)$ with $m > 0$, then j_n becomes surjective. Then we have that in the p -adic topology $n = p^m(p - 1) \rightarrow 0$ but we do not have $\mathcal{M}_n/p^a\mathcal{M}_n \rightarrow \mathcal{M}_0/p^a\mathcal{M}_0$.

3.2 More general group schemes

These considerations generalise. Let us consider a semi simple (or reductive) group over G/\mathbb{Q} . Let p be a prime let us assume that $G \times \mathbb{Q}_p$ is quasisplit and splits over an unramified extension E_p/\mathbb{Q}_p . Then we can extend G/\mathbb{Q} to a flat group scheme of finite type $\mathcal{G}/\text{Spec}(\mathbb{Z})$, which is reductive over an open subset V_p containing p . This open subset may shrink during the following considerations. We can choose a maximal torus $\mathcal{T}/V_p \subset \mathcal{G} \times_{\text{Spec}(\mathbb{Z})} V_p = \mathcal{G}_{V_p}$ whose extension $\mathcal{T} \times_{V_p} \text{Spec}(\mathbb{Z}_p)$ is contained in a Borel subgroup $\mathcal{B}/\text{Spec}(\mathbb{Z}_p)$. This means that we can find a Weyl chamber $\mathcal{C} \subset X^*(\mathcal{T})$ which is invariant under the action of the Galois group $\text{Gal}(E_p/\mathbb{Q}_p)$. We have a unique element w_0 in the Weyl group, which

sends this chamber to the opposite chamber $-\mathcal{C}$. Let \mathcal{U}_+ be the unipotent radical of \mathcal{B} , let $\mathcal{U}_- = \mathcal{U}_+^{w_0}$, it is the unipotent radical of the opposite Borel $\mathcal{B}_- \supset \mathcal{T}$.

Let us assume that the derived group $G^{(1)}$ is simply connected. Let $\mathcal{T}^{(1)} \subset \mathcal{T}$ be the torus $G^{(1)} \cap \mathcal{T}$. Let π be the set of simple positive roots, the Galois group $\text{Gal}(E_p/\mathbb{Q}_p)$ acts by permutations on π . The dominant fundamental weight corresponding to $\alpha \in \pi$ is denoted by γ_α . These dominant weights $\gamma_\alpha, \gamma_\beta, \dots$ are in $X^*(\mathcal{T}^{(1)})$. If $\lambda \in X^*(\mathcal{T})$, then its restriction to $\mathcal{T}^{(1)}$ is a linear combination $\sum n_\alpha \gamma_\alpha$, if all $n_\alpha \geq 0$ (or $\sum n_\alpha \gamma_\alpha \in \mathcal{C}$), then λ is a highest weight. The homomorphism $X^*(\mathcal{T}) \rightarrow X^*(\mathcal{T}^{(1)})$ is surjective, hence any $\sum n_\alpha \gamma_\alpha$ extends to a λ , which is also considered as a highest weight.

For any highest weight λ we have the highest weight module \mathcal{M}_λ . Here a few words of explanation seem to be in order. Our torus splits over an extension E/\mathbb{Q} which is non ramified at p . Let \mathcal{O}_E be its ring of integers, let $V'_p \subset \text{Spec}(\mathcal{O}_E)$ be the inverse image of V_p . Then we can extend the Borel subgroup $\mathcal{B}/\text{Spec}(\mathbb{Z}_p)$ to a Borel subgroup $\tilde{\mathcal{B}}/V'_p$, the weight $w_0(\lambda)$ defines a line bundle $\mathcal{L}_{w_0(\lambda)}$ on the flag variety of Borel subgroups $(\tilde{\mathcal{B}} \backslash \mathcal{G})_{V'_p}$ (see [De-Gr], Exp. XXII, 5.8). This line bundle is ample if and only if $n_\alpha > 0$ for all α . It has non trivial global sections if and only if $n_\alpha \geq 0$ for all α . The space of global sections $H^0((\tilde{\mathcal{B}} \backslash \mathcal{G})_{V'_p}, \mathcal{L}_{w_0(\lambda)})$ is a finitely generated projective $\mathcal{O}(V'_p)$ module on which the group scheme $\mathcal{G}_{V'_p}$ acts. By definition we get an action of $\mathcal{G}(\mathbb{Z}_p)$ on this module, it is easy to see that we can extend $H^0((\tilde{\mathcal{B}} \backslash \mathcal{G})_{V'_p}, \mathcal{L}_{w_0(\lambda)})$ to an \mathcal{O}_E module \mathcal{M}_λ on which we have an action of $\mathcal{G}(\mathcal{O}_E)$. This extension is not unique, but we are only interested in what happens at p , so this does not matter.

The module $H^0((\tilde{\mathcal{B}} \backslash \mathcal{G})_{V'_p}, \mathcal{L}_{w_0(\lambda)})$ has two specific elements. Let \mathcal{A} be the ring of regular functions of $\mathcal{G}_{V'_p}$, then we have by definition

$$H^0((\tilde{\mathcal{B}} \backslash \mathcal{G})_{V'_p}, \mathcal{L}_{w_0(\lambda)}) = \{f \in \mathcal{A} \mid f(bg) = w_0(\lambda)(b)f(g)\}$$

where b, g are elements in $\mathcal{B}(\mathcal{O}(V'_p)), \mathcal{G}(\mathcal{O}(V'_p))$.

The subsets $\mathcal{B} \cdot \mathcal{U}_-, \mathcal{B}w_0\mathcal{U}_+ \subset \mathcal{G}_{V'_p}$ are open and Zariski dense. The complement of these subsets is a divisor D_-, \tilde{D}_+ respectively whose irreducible components correspond to the simple roots. We can write (See proof of Satz 1.3.1 in [Ha-vK].)

$$D_- = \sum_{\alpha \in \pi} Y_\alpha^-, D_+ = \sum_{\alpha \in \pi} Y_\alpha^+.$$

Now it is well known that the two functions, which are defined on $\mathcal{B} \cdot \mathcal{U}_-$ and $\mathcal{B}w_0\mathcal{U}_+$ respectively, namely

$$e_{w_0(\lambda)}(bu_-) = w_0(\lambda)(b), e_\lambda(bw_0u_+) = w_0(\lambda)(b)$$

extend to regular functions on $\mathcal{G}_{V'_p}$. Hence they yield elements in $H^0((\tilde{\mathcal{B}} \backslash \mathcal{G})_{V'_p}, w_0(\mathcal{L}_\lambda))$ (At this point we need that the coefficients $n_\alpha \geq 0$)

More precisely we know that the divisor of zeroes of these two sections are given by

$$\text{Div}(e_{w_0(\lambda)}) = \sum_{\alpha} n_\alpha Y_\alpha^-, \text{Div}(e_\lambda) = \sum_{\alpha} n_\alpha Y_\alpha^+$$

and hence we see that $e_{w_0(\lambda)}$ vanishes on the complement of $\mathcal{B} \cdot \mathcal{U}_-$ if all the $n_\alpha > 0$, i.e. our weight is regular.

Moreover it is clear that the two sections are eigensections for the the action of the torus: For an element $t \in \mathcal{T}(R)$ we have

$$te_{w_0(\lambda)} = w_0(\lambda)(t)e_{w_0(\lambda)}, te_\lambda = \lambda(t)e_\lambda$$

The vector $e_{w_0(\lambda)}$ is invariant under the action of \mathcal{U}_- hence it is a highest weight vector with respect to the Borel subgroup $\mathcal{B}_- = \mathcal{B}^{w_0}$. It is a lowest weight vector for \mathcal{B} . An analogous statement holds for e_λ .

We will also consider the reduction mod p , i.e. the module $H^0((\tilde{\mathcal{B}} \backslash \mathcal{G})_{V'_p} \times \mathbb{F}_p, \mathcal{L}_{w_0(\lambda)})$. This is a module for the group $\mathcal{G}(\mathbb{F}_p)$. This module is irreducible if all coefficients satisfy $0 \leq n_\alpha \leq p - 1$ and is equal to $\mathcal{M}_\lambda/p\mathcal{M}_\lambda$. (See [Ja],

Now we consider the group $\mathcal{T}(\mathbb{Z}_p)$ it sits in an exact sequence (See notations in the introduction to chap. 3)

$$1 \rightarrow \mathcal{T}^{(1)}(\mathbb{Z}_p) \rightarrow \mathcal{T}(\mathbb{Z}_p) \rightarrow \mathcal{C}'(\mathbb{Z}_p) \rightarrow 1.$$

We consider continuous characters $\chi : \mathcal{T}(\mathbb{Z}_p) \rightarrow \mathcal{O}_{\mathbb{C}_p}^\times$, basically we are only interested in their restriction to $\mathcal{T}^{(1)}(\mathbb{Z}_p)$. The group $\mathcal{T}^{(1)}(\mathbb{Z}_p) \subset \mathcal{T}^{(1)}(\mathcal{O}_{E_p})$ and it is the subgroup of Galois invariant elements. The group

$$\mathcal{T}^{(1)}(\mathcal{O}_{E_p}) = \prod_{\alpha \in \pi} \mathcal{O}_{E_p}^\times = \{(\dots, x_\alpha, \dots)_{\alpha \in \pi} | x_\alpha \in \mathcal{O}_{E_p}^\times\}$$

the Galois group acts by $\sigma(\dots, x_\alpha, \dots) = (\dots, x_{\sigma(\alpha)}^\sigma, \dots)$. Hence $\mathcal{T}(\mathbb{Z}_p)$ is the subgroup of elements which satisfy $\sigma(x_\alpha) = x_{\sigma(\alpha)}$. This tells us that the torus $\mathcal{T}^{(1)}$ is a product over induced tori, the factors in this product correspond to the orbits of the Galois group on π . If we denote such an orbit by $\bar{\alpha}$ and if we choose representatives $\alpha \in \bar{\alpha}$ then this defines a subfield $E_\alpha \subset E_p$ such that $\text{Gal}(\bar{\mathbb{Q}}_p/E_\alpha)$ is the stabilizer of α . Since E_p is unramified, we know that E_α/\mathbb{Q}_p is cyclic of order r_α , where r_α is the length of the orbit. The Galois group $\text{Gal}(E_\alpha/\mathbb{Q}_p)$ is cyclic and generated by the Frobenius element σ . Then the factor corresponding to $\bar{\alpha}$ is denoted by $\mathcal{T}_{\bar{\alpha}}$ and we have $\mathcal{T}_{\bar{\alpha}} = R_{\mathcal{O}_{E_\alpha}/\mathbb{Z}_p}(\mathbb{G}_m)$.

Then

$$\mathcal{T}_{\bar{\alpha}}(\mathbb{Z}_p) = \mathcal{O}_{E_\alpha}^\times \subset \prod_{\sigma^i: i=0}^{r_\alpha-1} \mathcal{O}_{E_p}^\times$$

where the embedding is given by

$$x \mapsto (x, \sigma(x), \dots, \sigma^{r_\alpha-1}(x)).$$

To get our interpolating modules consider characters on $\mathcal{T}^{(1)}(\mathcal{O}_{E_p})$ and restrict them to $\mathcal{T}^{(1)}(\mathbb{Z}_p)$. We choose an embedding $\mathcal{O}_{E_p} \rightarrow \mathcal{O}_{\mathbb{C}_p}$ and then we put as before

$$\chi((\dots, x_\alpha, \dots)) = \prod \omega(x_\alpha)^{\nu_\alpha} \left(\frac{x_\alpha}{\omega(x_\alpha)} \right)^{z_\alpha}$$

where $z_\alpha \in \mathcal{O}_{\mathbb{C}_p}$ and $\nu_\alpha \in \mathbb{Z}$.

We restrict χ to $\mathcal{T}(\mathbb{Z}_p)$, more precisely we look at the restriction to the components. Clearly we have $\omega(\sigma(x_\alpha)) = \omega(x_\alpha)^p$ and hence the factor in front is

$$\prod_{\bar{\alpha}} \prod_{i=0}^{r_\alpha-1} (x_\alpha)^{\nu_{\sigma^i(\alpha)} p^i} = \prod_{\bar{\alpha}} x_\alpha^{\sum_i \nu_{\sigma^i(\alpha)} p^i} = \prod_{\bar{\alpha}} x_\alpha^{\nu_{\bar{\alpha}}}.$$

Hence we see that the $\bar{\alpha}$ component of the factor in front only depends on

$$\nu_{\bar{\alpha}} = \sum_i \nu_{\sigma^i(\alpha)} p^i \pmod{(p^{r_\alpha} - 1)}$$

Now we can define the induced modules $\mathcal{P}_{\chi^{[m]}} \subset I_{\chi^{[m]}}$ as before. At this moment we assume $m = 1$, then the z_α do not play a role, we have $\mathcal{P}_{\chi^{[1]}} = I_{\chi^{[1]}}$. We observe that by definition λ is a rational character on the torus $\mathcal{T} \times E$. Any such character defines a homomorphism $\chi_\lambda : \mathcal{T}(\mathbb{Z}_p) \rightarrow \mathcal{O}_{E_p}^\times \subset \mathcal{O}_{\mathbb{C}_p}^\times$. If we consider the reduction mod p then we get a homomorphism $\chi_\lambda^{[1]} : \mathcal{T}(\mathbb{F}_p) \rightarrow (\mathcal{O}_{E_p}/(p))^\times \subset (\mathcal{O}_{\mathbb{C}_p}/(p))^\times$. We have $\mathcal{O}_{E_p}/(p) = \mathbb{F}_{p^r}$. We want to write this homomorphism in the form above, we can forget the z_α

$$\chi_\lambda^{[1]}((\dots, x_\alpha, \dots)) = \prod \omega(x_\alpha)^{\nu_\alpha}.$$

Now we have to analyse the relation between the coefficients n_i in λ and the ν_α . It suffices to investigate what happens on the factors \mathcal{T}_α . We pick a simple root α and we consider its orbit $\alpha, \sigma(\alpha), \dots, \sigma^{r-1}(\alpha)$

Since E_p is the splitting field of the entire torus it can happen, that our root α is fixed under the action of the Galois group. Then we have

$$\mathcal{T}_\alpha(\mathbb{F}_p) = \mathbb{G}_m(\mathbb{F}_p) = \mathbb{F}_p^\times \subset \mathbb{G}_m(\mathbb{F}_{p^r}) = \mathbb{F}_{p^r}^\times$$

The component of λ corresponding to this root is a rational character $x \mapsto x^{n_\alpha}$, if we restrict this to $\mathbb{G}_m(\mathbb{F}_p)$ it depends only on $n_\alpha \pmod{(p-1)}$. On the other side ν_α is an integer $\pmod{(p^r-1)}$ but if we restrict this to $\mathbb{G}_m(\mathbb{F}_p)$ this restriction only depends on $\nu_\alpha \pmod{(p-1)}$.

Now we consider the other extreme case namely the length of the orbit is r . Then we have

$$\mathcal{T}_\alpha(\mathbb{F}_p) = \mathbb{F}_{p^r}^\times \subset \prod_{\sigma} \mathbb{F}_{p^r}^\times$$

and the embedding is given by

$$x \mapsto (x, x^p, \dots, x^{p^{r-1}}).$$

If we now have a highest weight component $\lambda_{\bar{\alpha}} = \sum_{\sigma^i} n_{\sigma^i(\alpha)} \gamma_{\sigma^i(\alpha)}$ then induces on $\mathcal{T}_\alpha(\mathbb{F}_p) = \mathbb{F}_{p^r}^\times$ the homomorphism

$$x \mapsto x^{\sum_i n_{\sigma^i(\alpha)} p^i} = \prod (x^{p^i})^{n_{\sigma^i(\alpha)}}$$

and this implies that for any $\nu_{\bar{\alpha}} \in \mathbb{Z}/(p^r-1)$ we can find coefficients $0 \leq n_{\sigma^i(\alpha)} \leq p-1$ such that

$$\sum_i n_{\sigma^i(\alpha)} p^i = \nu_{\bar{\alpha}} \pmod{(p^r-1)}.$$

Hence we see, that for any χ we can find a $\lambda = \sum_{\alpha} n_{\alpha} \gamma_{\alpha}$ such that $w_0(\lambda)|\mathcal{T}(\mathbb{F}_p) = \chi^{[1]}$ and then we get a homomorphism

$$j_{\lambda} : \mathcal{M}_{\lambda}/p\mathcal{M}_{\lambda} \rightarrow I_{\chi^{[1]}}.$$

We can define Hecke operators $T(t_{p^a}, u_{t_{p^a}})$: We choose an element $t_{p^a} \in T(\mathbb{Q}_p)$ such that for all positive simple roots $|\alpha(t_{p^a})|_p < 1$. (t_{p^a} is our $\alpha \in G(\mathbb{Q})$, sorry for the notation.) Then it is again clear that the possible $u_{t_{p^a}}$ are given by elements in

$$\text{Hom}_{\mathcal{T}(\mathbb{F}_p)}((\mathcal{M}_{\lambda}/p\mathcal{M}_{\lambda})_{\mathcal{U}_+(\mathbb{F}_p)}, (\mathcal{M}_{\lambda}/p\mathcal{M}_{\lambda})^{\mathcal{U}_-(\mathbb{F}_p)}), \text{Hom}_{\mathcal{T}(\mathbb{F}_p)}(I_{\chi^{[1]}})_{\mathcal{U}_+(\mathbb{F}_p)}, I_{\chi^{[1]}}^{\mathcal{U}_-(\mathbb{F}_p)})$$

The $\mathcal{T}(\mathbb{F}_p)$ -modules $I_{\chi^{[1]}, \mathcal{U}_+(\mathbb{F}_p)}, I_{\chi^{[1]}}^{\mathcal{U}_-(\mathbb{F}_p)}$ are easy to compute, if we use the Bruhat decomposition. We have the action of $\mathcal{U}_+(\mathbb{F}_p), \mathcal{U}_-(\mathbb{F}_p)$ on $\mathcal{B}(\mathbb{F}_p) \backslash \mathcal{G}(\mathbb{F}_p)$ we write

$$\mathcal{G}(\mathbb{F}_p) = \bigcup_w \mathcal{B}(\mathbb{F}_p) w \mathcal{U}_+(\mathbb{F}_p) = \mathcal{B}(\mathbb{F}_p) w_0 \mathcal{U}_+(\mathbb{F}_p) \cup \dots \cup \mathcal{B}(\mathbb{F}_p),$$

$$\mathcal{G}(\mathbb{F}_p) = \bigcup_w \mathcal{B}(\mathbb{F}_p) w \mathcal{U}_-(\mathbb{F}_p) = \mathcal{B}(\mathbb{F}_p) \mathcal{U}_-(\mathbb{F}_p) \cup \dots \cup \mathcal{B}(\mathbb{F}_p) w_0.$$

Clearly the module $I_{\chi^{[1]}}$ decomposes into direct sums under the action of the two unipotent radicals $\mathcal{U}_+(\mathbb{F}_p), \mathcal{U}_-(\mathbb{F}_p)$ according to the decompositions. Then we have the function $\bar{\Phi}_e$ supported on the smallest orbit $\mathcal{B}(\mathbb{F}_p)$, its image $\bar{\Phi}_e$ in $I_{\chi^{[1]}, \mathcal{U}_+(\mathbb{F}_p)}$ generates a copy of $\mathbb{F}_p = \mathbb{F}_p \bar{\Phi}_e$. We have the function $\Psi_e \in I_{\chi^{[1]}}^{\mathcal{U}_-(\mathbb{F}_p)}$ which has support in $\mathcal{B}(\mathbb{F}_p) \mathcal{U}_-(\mathbb{F}_p)$, i.e. it is given by $\Psi_e(bu_-) = \lambda(b)$. If we send $\bar{\Phi}_e$ to Ψ_e and all other summands to zero then this gives us an element in $\text{Hom}_{\mathcal{T}(\mathbb{F}_p)}(I_{\chi^{[1]}})_{\mathcal{U}_+(\mathbb{F}_p)}, I_{\chi^{[1]}}^{\mathcal{U}_-(\mathbb{F}_p)})$ and this is our principal operator $u_{t_{p^a}}^{\text{princ}}$.

If now all $n_{\alpha} > 0$ (this is the regularity condition), then we have seen that the function Ψ_e can be interpreted as the restriction of the \mathcal{U}_- highest weight $e_{w_0(\lambda)}$ vector to $\mathbb{G}(\mathbb{F}_p)$. It is clear that the classical operator $u_{t_{p^a}}^{\text{class}}$ sends $e_{w_0(\lambda)}$ to $e_{w_0(\lambda)}$ and the subspace $\mathbb{F}_p e_{w_0(\lambda)}$ is the image of $u_{t_{p^a}}^{\text{class}}$. Hence we see that the classical and the principal Hecke operator coincide in the case and under the above regularity condition we get an isomorphism

$$j_{\lambda}^{\bullet} : H_{\text{ord}}^{\bullet}(S_{K_f}^G, \mathcal{M}_{\lambda}/p\mathcal{M}_{\lambda}) \rightarrow H_{\text{ord}}^{\bullet}(S_{K_f}^G, I_{\chi^{[1]}}).$$

3.3. The case $m > 1$ for the group Gl_2 For this special group we give a very detailed discussion of the possible choices of operators $u_{t_{p^m}}$ for our various coefficient systems. The point is, that these modules are $\mathbb{Z}/(p^m)$ -modules and the action of Γ factors through the quotient $\text{Gl}_2(\mathbb{Z}/(p^m))$. Let us assume that now M is any $\text{Gl}_2(\mathbb{Z}/(p^m\mathbb{Z}))$ -module. We consider

$$\Gamma(t_{p^m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b \equiv 0 \pmod{p^m} \right\}$$

and its image in $\text{Gl}_2(\mathbb{Z}/(p^m\mathbb{Z}))$ is the Borel subgroup

$$B_-(\mathbb{Z}/p^m\mathbb{Z}) = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mid a, d \in (\mathbb{Z}/p^m\mathbb{Z})^\times, c \in \mathbb{Z}/p^m\mathbb{Z} \right\}$$

The group $\Gamma(t_{p^m})$ acts in two different ways on M we have the modules $M^{(t_{p^m})}$ and M . The action on M is the action induced by the inclusion $\Gamma(t_{p^m}) \subset \Gamma$. The module $M^{(t_{p^m})}$ is as an abelian group equal to M but the action of $\Gamma(t_{p^m})$ is the one where we include $\Gamma(t_{p^m})$ via the conjugation $\gamma \rightarrow t_{p^m}^{-1}\gamma t_{p^m}$ into Γ . To make it clear: An element $u_{t_{p^m}} \in \text{Hom}_{\Gamma(t_{p^m})}(M^{(t_{p^m})}, M)$ is a homomorphism $u_{t_{p^m}} : M \rightarrow M$ which satisfies

$$u_{t_{p^m}}(\alpha^{-m}\gamma t_{p^m} f) = \gamma u_{t_{p^m}}(f)$$

or in terms of matrices

$$u_{t_{p^m}} \left(\begin{pmatrix} a & b \\ p^m c & d \end{pmatrix} f \right) = \left(\begin{pmatrix} a & p^m b \\ c & d \end{pmatrix} u_{t_{p^m}} f \right).$$

Let $U_-(\mathbb{Z}/p^m\mathbb{Z}), U_+(\mathbb{Z}/p^m\mathbb{Z})$ be the two unipotent radicals of $B_-(\mathbb{Z}/p^m\mathbb{Z}), B_+(\mathbb{Z}/p^m\mathbb{Z})$. Then the module $M_{U_+(\mathbb{Z}/p^m\mathbb{Z})}$ of coinvariants and $M^{U_-(\mathbb{Z}/p^m\mathbb{Z})}$ of invariants become $T(\mathbb{Z}/p^m\mathbb{Z})$ modules and it is clear that

$$\text{Hom}_{\Gamma(t_{p^m})}(M^{(t_{p^m})}, M) \xrightarrow{\sim} \text{Hom}_{T(\mathbb{Z}/p^m\mathbb{Z})}(M_{U_+(\mathbb{Z}/p^m\mathbb{Z})}, M^{U_-(\mathbb{Z}/p^m\mathbb{Z})})$$

We have to understand the $T(\mathbb{Z}/(p^m))$ -modules $I_{\chi^{[m]}, U_+(\mathbb{Z}/(p^m))}$ and $I_{\chi^{[m]}}^{U_-(\mathbb{Z}/(p^m))}$. To do this we have to investigate the action of $U_\pm(\mathbb{Z}/(p^m))$ on

$$B_+(\mathbb{Z}/(p^m)) \backslash GL_2(\mathbb{Z}/(p^m)) = \mathbb{P}^1(\mathbb{Z}/(p^m)).$$

We write the elements of $\mathbb{P}^1(\mathbb{Z}/(p^m))$ in the form (a, b) and the group acts by multiplication from the right.

We consider the action of $U_+(\mathbb{Z}/(p^m))$. We see that $x_0 = (0, 1)$ is the fixed points for $B_+(\mathbb{Z}/(p^m))$. Let w be the non trivial element in the Weyl group, i.e. $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$(0, 1)w \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = (1, u)$$

this gives us the “big cell”

$$B_+(\mathbb{Z}/(p^m)) \cdot wU_+(\mathbb{Z}/(p^m)) \subset GL_2(\mathbb{Z}/(p^m)).$$

The remaining points are of the form

$$(v, 1) \quad \text{where} \quad v \equiv 0 \pmod{p}.$$

The group $U_+(\mathbb{Z}/(p^m))$ acts by

$$(v, 1) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = (v, 1 + uv) = (v(1 + uv)^{-1}, 1),$$

and hence we see that two elements v, v' are in the same orbit for the action of $U_+(\mathbb{Z}/(p^m))$ if and only if

$$\text{ord}(v) = \text{ord}(v') \quad \text{and} \quad v \equiv v' \pmod{p^{2 \text{ord}(v)}}.$$

The stabilizer of $(v, 1)$ in $U_+(\mathbb{Z}/(p^m))$ is the congruence subgroup

$$U_+^{m-2 \text{ord}(v)}(\mathbb{Z}/(p^m)) = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \equiv 0 \pmod{p^{m-2 \text{ord}(v)}} \right\}$$

which becomes the full group after $2 \text{ord}(v) \geq m$. We put

$$l(v) = \begin{cases} m - 2 \text{ord}(v) & \text{if } 2 \text{ord}(v) \leq m \\ 0 & \text{else} \end{cases}$$

then $p^{l(v)}$ is also the length of the orbit. We denote the orbits of $U_+(\mathbb{Z}/(p^m))$ on the set of v -s by \bar{v} .

For any of the orbits we choose a representative $(1, 0)$, $(v, 1)$ and write it

$$\begin{aligned} (1, u) &= (0, 1) \cdot w \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = (0, 1) \cdot \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix} \\ (v, 1) &= (0, 1) \cdot \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \end{aligned}$$

Then we consider the double cosets $\mathcal{X}_w = B(\mathbb{Z}/p^m\mathbb{Z})wU_+(\mathbb{Z}/p^m\mathbb{Z})$, the intermediate cosets $\mathcal{X}_{\bar{v}} = B(\mathbb{Z}/p^m\mathbb{Z}) \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} U_+(\mathbb{Z}/p^m\mathbb{Z})$, where \bar{v} runs over the orbits with $\text{ord}(\bar{v}) = 1, \dots, m-1$ and the again special orbit $\mathcal{X}_0 = B(\mathbb{Z}/p^m\mathbb{Z}) = B(\mathbb{Z}/p^m\mathbb{Z})U_+(\mathbb{Z}/p^m\mathbb{Z})$.

We group the orbits according to the number $\nu = \text{ord}(\bar{v})$ and write the decomposition into double cosets in decreasing order

$$\text{Gl}_2(\mathbb{Z}/(p^m)) = \mathcal{X}_w \cup \bigcup_{\bar{v}: \text{ord}(\bar{v})=1} \mathcal{X}_{\bar{v}} \cup \dots \cup \bigcup_{\bar{v}: \text{ord}(\bar{v})=\nu} \mathcal{X}_{\bar{v}} \cup \dots \cup \bigcup_{\bar{v}: \text{ord}(\bar{v})=m-1} \mathcal{X}_{\bar{v}} \cup B(\mathbb{Z}/p^m\mathbb{Z}).$$

In this decomposition the number ν goes up from zero to m , the number $p^{l(v)}$ start with p^m and drops to p^{m-2}, p^{m-4} until we reach the middle and then it becomes constant equal to one. Another number $\mu = \min(\nu, m - \nu)$ goes up from zero to $\lceil \frac{m}{2} \rceil$ and after that drops again in steps by one to zero.

We get a decomposition of $I_{\bar{\chi}^{[m]}}$ into $U_+(\mathbb{Z}/(p^m))$ submodules

$$I_{\bar{\chi}^{[m]}} = I_{\bar{\chi}^{[m]}}^{(w)} \oplus \bigoplus_{\nu=1}^{\nu=m-1} \bigoplus_{\bar{v}: \text{ord}(\bar{v})=\nu} I_{\bar{\chi}^{[m]}}^{(\bar{v})} \oplus I_{\bar{\chi}^{[m]}}^{(0)},$$

the submodules consist of functions which are supported on the orbits.

Now we define the functions

$$\begin{aligned} \Psi_{w,u}(g) &= \begin{cases} \chi^{[m]}(b) & \text{if } g = bwu \\ 0 & \text{else} \end{cases} \\ \Psi_v(g) &= \begin{cases} \Psi_v \left(b \cdot \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \right) = \chi^{[m]}(b) & \text{if } g \in B_+(\mathbb{Z}/(p^m)) \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \\ 0 & \text{else} \end{cases} \end{aligned}$$

which form a basis of $I_{\bar{\chi}^{[m]}}$. These functions are essentially like δ -functions. If we want to be completely consistent, we should denote these functions by $\Psi_{w,u}^{[m]}, \dots$, but as long we work on a fixed level we suppress the superscript.

It is clear that the image of the elements $\Psi_{w,u}$ in $I_{\bar{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))}$ is independent of u let us call it $\bar{\Psi}_w$. Two elements $\Psi_v, \Psi_{v'}$ have the same image in $I_{\bar{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))}$ if and only if v, v' they are conjugate under the action of $U_+(\mathbb{Z}/(p^m))$. This means that each orbit \bar{v} of v -s contributes by a cyclic R_m -module and hence we get a direct sum decomposition

$$I_{\bar{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))} = R_m \bar{\Psi}_w \oplus \bigoplus_{\bar{v}} R_m \bar{\Psi}_{\bar{v}},$$

where $\bar{\Psi}_{\bar{v}}$ is the image of any of the $\Psi_v, v \in \bar{v}$. The summands $R_m \bar{\Psi}_{\bar{v}}$ are not necessarily free R_m -modules. But by definition they are cyclic and hence we have to determine their annihilators.

To understand these annihilators we have to take into account that these elements $(v, 1)$ still have stabilizers in $U_+(\mathbb{Z}/(p^m))$, we described them further above. If we denote such a stabilizer by $U_+^{(v)}$, then it is clear that $U_+^{(v)}$ acts upon the free R_m -modules $R_m \Psi_v$ and

$$(R_m \Psi_v)_{U_+(\mathbb{Z}/(p^m))} \simeq R_m \bar{\Psi}_{\bar{v}}.$$

For $x_w = (0, 1)$ the stabilizer is trivial and we get

$$R_m \Psi_w = R_m \bar{\Psi}_w.$$

Now we saw that the stabilizer becomes bigger and bigger if $\text{ord}(v)$ goes from 1 to m and once $2 \text{ord}(v) \geq m$ we have $U_+^{(v)} = U_+(\mathbb{Z}/(p^m))$.

We have to find out how these stabilizers act upon

$$R_m \Psi_v.$$

This is easy: Since $uv^2 \equiv 0 \pmod{p^m}$ we have

$$\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1+uv)^{-1} & u \\ 0 & (1+uv) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

Hence we see that

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \Psi_v = \chi(1+uv) \Psi_v,$$

and the annihilator of $\bar{\Psi}_v$ in R_m is the ideal generated by the elements

$$\chi(1+uv) - 1 = (1+uv)^\alpha - 1 = \alpha uv + \dots.$$

If we take into account that u satisfies $u \equiv 0 \pmod{p^{m-2 \text{ord}(v)}}$, then this ideal is

$$(\alpha uv) = \left(\alpha p^{m-2 \text{ord}(v)} \right)$$

as long as we have $\text{ord}(v) < \frac{m}{2}$. So the ideal becomes bigger and bigger as long as $\text{ord}(v) < \frac{m}{2}$. But after that the stabilizer becomes $U_+(\mathbb{Z}/(p^m))$ and the ideal will be

$$\left(\alpha p^{\text{ord}(v)}\right) \quad \text{for} \quad \text{ord}(v) \geq \frac{m}{2}$$

and eventually for $v \equiv 0 \pmod{p^m}$ it becomes trivial. We defined

$$\mu = \min(\nu, m - \nu)$$

then we see that this ideal is $(\alpha p^{m-\mu})$.

We have

$$R_m \Psi_0 = R_m \bar{\Psi}_0.$$

Now we observe that the $R_m \bar{\Psi}_{\bar{v}}$ do not depend on the choice of a $v \in \bar{v}$, and

$$(R_m \Psi_v)_{U_+^{(v)}} \simeq R_m / (\alpha p^{m-\mu}) \Psi_{\bar{v}} = R_m \bar{\Psi}_{\bar{v}} \subset I_{\tilde{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))}.$$

This means that we have a direct sum decomposition

$$I_{\tilde{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))} = R_m \bar{\Psi}_w \oplus \left(\bigoplus_{\nu=1}^{\nu=m-1} \bigoplus_{\bar{v}: \text{ord}(\bar{v})=\nu} R_m \bar{\Psi}_{\bar{v}} \right) \oplus R_m \bar{\Psi}_0.$$

We recall that we have to understand the module of coinvariants as a $T(\mathbb{Z}/(p^m))$ -module. The summands are $T(\mathbb{Z}/(p^m))$ modules and we have to investigate the action of the torus $T(\mathbb{Z}/(p^m))$ on these modules.

The torus leaves the two outer terms invariant, it acts on $R_m \bar{\Psi}_w$ by

$$(\tilde{\chi}^{[m]})^w : t = \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto \chi^{[m]}(t_1),$$

which is the conjugate by the Weyl group of the character $\tilde{\chi}^{[m]}$. On $R_m \bar{\Psi}_0$ it acts by $\chi^{[m]}$.

The individual summands in the middle are not invariant. The point is that the torus acts on the set of orbits for $U_+(\mathbb{Z}/(p^m))$ and the orbits under the torus action are given by the numbers $\nu = \text{ord}(v)$ which vary from 1 to $m - 1$. We group the summands according to the order $\nu = \text{ord}(v)$ and consider the summand

$$\bigoplus_{\bar{v}: \text{ord}(\bar{v})=\nu} R_m \bar{\Psi}_{\bar{v}}$$

where \bar{v} runs over the orbits of $U_+(\mathbb{Z}/(p^m))$. This sum is invariant under the torus $T(\mathbb{Z}/(p^m))$. The stabilizer of an orbit $\mathcal{X}_{\bar{v}}$ is the torus

$$T^{(\mu)}(\mathbb{Z}/(p^m)) = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mid t_1/t_2 \equiv 1 \pmod{p^\mu} \right\}.$$

An easy calculation shows that the $T^{(\mu)}(\mathbb{Z}/(p^m))$ fixes the module $R_m \bar{\Psi}_{\bar{v}}$. The restriction of our character $\chi^{[m]}$ to $T^{(\mu)}(\mathbb{Z}/(p^m))$ induces a character

$$\tilde{\chi}^{[m,\mu]} : T^{(\mu)}(\mathbb{Z}/(p^m)) \rightarrow (R_m/(\alpha p^{m-\mu}))^\times,$$

by this character $T^{(\mu)}(\mathbb{Z}/(p^m))$ acts on $R_m \bar{\Psi}_{\bar{v}}$. If we choose a representative \bar{v} with $\text{ord}(\bar{v}) = \nu$, we find for the term in the middle

$$\bigoplus_{0 < \nu < m} \text{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m \bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m,\mu]}.$$

A completely analogous computation gives us the modules of invariants $I_{\tilde{\chi}^{[m]}}^{U_-(\mathbb{Z}/(p^m))}$.

We start with the list of double cosets

$$\begin{aligned} \mathcal{Y}_w &= B_+(\mathbb{Z}/p^m\mathbb{Z})wU_-(\mathbb{Z}/p^m\mathbb{Z}) = B_+(\mathbb{Z}/p^m\mathbb{Z})w \\ , \dots \mathcal{Y}_{\bar{v}} &= B_+(\mathbb{Z}/p^m\mathbb{Z}) \begin{pmatrix} 0 & -1 \\ 1 & v \end{pmatrix} U_-(\mathbb{Z}/p^m\mathbb{Z}) \dots, \\ \mathcal{Y}_0 &= B_+(\mathbb{Z}/p^m\mathbb{Z})U_-(\mathbb{Z}/p^m\mathbb{Z}) \end{aligned}$$

where in the middle the v runs over the elements in $p\mathbb{Z} \bmod p^m\mathbb{Z}$ and the \bar{v} are the equivalence classes with respect to the equivalence relation above. But this time we order them in descending order of $\text{ord}(\bar{v})$.

The two extremal terms are again easy. We have the $U_-(\mathbb{Z}/(p^m))$ invariant function Φ_- which is supported on the big cell

$$\Phi_-(bu) = \tilde{\chi}(b),$$

and the function Φ_0 which is supported on $B(\mathbb{Z}/(p^m))w$ (the smallest cell)

$$\Phi_0(bw) = \tilde{\chi}(b),$$

and we know that $T(\mathbb{Z}/(p^m))$ act by $\tilde{\chi}^{[m]}$ on $R_m\Phi_-$ and by $\tilde{\chi}^{[m],w}$ on $R_m\Phi_0$.

We investigate the terms in the middle. Again we denote the orbit of v under $U_-(\mathbb{Z}/(p^m))$ by \bar{v} . For any such an orbit we can take a representative

$$B_+(\mathbb{Z}/(p^m)) \cdot \begin{pmatrix} 0 & -1 \\ 1 & v \end{pmatrix}$$

and define a function Φ_v which is supported on the double coset $B_+(\mathbb{Z}/p^m\mathbb{Z}) \begin{pmatrix} 0 & -1 \\ 1 & v \end{pmatrix} U_-^{(v)}(\mathbb{Z}/p^m\mathbb{Z})$

$$\Phi_v \left(b \cdot \begin{pmatrix} 0 & -1 \\ 1 & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \right) = \tilde{\chi}^{[m]}(b).$$

The function Φ_v is not invariant under the stabilizer $U_-^{(v)}(\mathbb{Z}/(p^m))$ because it picks up a factor $\chi(1 - uv)^{-1}$ if we translate it by $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in U_-^{(v)}(\mathbb{Z}/(p^m))$. An easy computation gives the formula

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \Phi_v = \chi(1 - uv)^{-1} \Phi_{\frac{v}{(1-uv)}},$$

if $u \in U_-^{(v)}(\mathbb{Z}/p^m\mathbb{Z})$ then the subscript $\frac{v}{(1-uv)} = v$. We saw that the expressions $1 - \chi(1-uv)^{-1}$ with $u \in U_-^{(v)}(\mathbb{Z}/p^m\mathbb{Z})$ generate the ideal $(\alpha p^{m-\mu})$: The annihilator this ideal is a principal ideal $(\beta(\mu)) \subset R_m$. Clearly $\beta(\mu)$ divides p^μ . If α is a unit then we see that it is given by $(\beta(\mu)) = (p^\mu)$. The element $\beta(\mu)\Phi_v \in R_m\Phi_v$ is a generator for the $U_-^{(v)}(\mathbb{Z}/(p^m))$ - invariants in the space of the functions supported on $B_+(\mathbb{Z}/p^m\mathbb{Z}) \begin{pmatrix} 0 & -1 \\ 1 & v \end{pmatrix} U_-^{(v)}(\mathbb{Z}/p^m\mathbb{Z})$.

We define

$$\tilde{\Phi}_v = \sum_{u \in \bar{U}_-(\mathbb{Z}/(p^m))/U_-^{(v)}(\mathbb{Z}/(p^m))} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \Phi_v = \sum_{v' \in \bar{v}} \chi(1-uv)^{-1} \Phi_{\frac{v}{(1-uv)}},$$

here we parametrized the elements of the orbit \bar{v} by $v' = v/(1-uv)$.

Then we find that $R_m\beta(\mu)\tilde{\Phi}_v$ is the space of $U_-(\mathbb{Z}/(p^m))$ - invariant functions with support on the orbit \bar{v} containing v , this is actually a free $R_m/(\alpha p^{m-\mu})$ module. An easy calculation shows that $T^{(\mu)}(\mathbb{Z}/(p^m))$ acts again by $\tilde{\chi}^{[m,\mu]}$ on this summand. We can summarize

The coinvariants give

$$R_m\bar{\Psi}_w \oplus \bigoplus_{\nu=1}^{m-1} \text{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(\alpha p^{m-\mu})\bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m,\mu]} \oplus R_m\bar{\Psi}_0, \quad (\text{coinv})$$

and the invariants (we arrange them in the opposite order)

$$R_m\Phi_0 \oplus \bigoplus_{\nu=m-1}^1 \text{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(\alpha p^{m-\mu})(\beta(\mu)\tilde{\Phi}_v) \otimes \tilde{\chi}^{[m,\mu]} \oplus R_m\Phi_-, \quad (\text{inv})$$

where v runs over the a set of representatives of elements given order $\text{ord}(v) = \nu$, we simply will take $v = p^\nu$. Note that the extremal terms are also induced, but in this case $\mu = 0$ so the induction step is trivial.

Now it is easy to see how to construct Hecke operators on our coefficient systems.

First of all we have the operator

$$u_{t_{p^m}}^{(\text{princ})} : \Psi_0 \longmapsto \Phi_-$$

which sends all the other summands to zero. This is the *principal* Hecke operator. Since the function $\Phi_- \in \mathcal{P}_{\tilde{\chi}^{[m]}}$ we see that it induces the zero operator on the quotient $I_{\tilde{\chi}^{[m]}}/\mathcal{P}_{\tilde{\chi}^{[m]}}$. I claim that the system of principal operators satisfies (iii), hence it yields an element in the projective limit. To see this we observe that we have for any $f \in \mathcal{P}_{\tilde{\chi}^{[m+1]}}^{(t_{p^{m+1}})}$ the formula $u_{t_{p^{m+1}}}^{(\text{princ})}(f) = f(e_{m+1})\Phi_-^{(m+1)}$, were e_{m+1} is the identity in $\text{Gl}_2(\mathbb{Z}/(p^{m+1}))$ and were $\Phi_-^{(m+1)}$ is our function Φ_- but on the next higher level. We have to check that for an element $x \in \text{Gl}_2(\mathbb{Z}/(p^{m+1}))$ which congruent to e_{m+1} modulo p^m we have

$$u_{t_{p^{m+1}}}^{(\text{princ})}(R_x(f)) \equiv u_{t_{p^{m+1}}}^{(\text{princ})}(f) \pmod{p^m}.$$

This is clear from the definition of $\mathcal{P}_{\tilde{\chi}^{[m+1]}}$.

We can enlarge the supply of Hecke operators by adding linear combinations of correction terms

$$u_{t_{p^m}}^{(\nu, \nu')} : \text{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m / (\alpha p^{m-\mu}) \bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m, \mu]} \longrightarrow \text{Ind}_{T^{(\mu')}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m / (\alpha p^{m-\mu'}) \tilde{\Phi}_{v'} \otimes \tilde{\chi}^{[m, \mu']}$$

where $\text{ord}(v') > 0$ if $\nu = 0$ and where we may require that $\text{ord}(v) + \text{ord}(v') \leq m$.

We consider maps of the form $u_{t_{p^m}} = u_{t_{p^m}}^{(0)} + \sum u_{t_{p^m}}^{(\nu, \nu')}$, where

$$u_{t_{p^m}}^{(\nu, \nu')} \in \text{Hom}_{T(\mathbb{Z}/(p^m))}(\text{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} (R_m / (\alpha p^{m-\mu}) \bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m, \mu]}), \text{Ind}_{T^{(\mu')}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m / (\alpha p^{m-\mu'}) \tilde{\Phi}_{v'} \otimes \tilde{\chi}^{[m, \mu']}).$$

By Frobenius reciprocity this is

$$u_{t_{p^m}}^{(\nu, \nu')} \in \text{Hom}_{T^{(\mu')}(\mathbb{Z}/(p^m))}(\text{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} (R_m / (\alpha p^{m-\mu}) \bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m, \mu]}), R_m / (\alpha p^{m-\mu'}) \tilde{\Phi}_{v'} \otimes \tilde{\chi}^{[m, \mu']}).$$

We assume $\mu \leq \mu'$ then $T^{(\mu')}(\mathbb{Z}/(p^m)) \subset T^{(\mu)}(\mathbb{Z}/(p^m))$ and as a module $T^{(\mu)}(\mathbb{Z}/(p^m))$ -module we have

$$\text{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} (R_m / (\alpha p^{m-\mu}) \bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m, \mu]}) = \bigoplus_{\xi \in T(\mathbb{Z}/(p^m))/T^{(\mu)}(\mathbb{Z}/(p^m))} (R_m / (\alpha p^{m-\mu}) \tilde{\Psi}_{\xi \bar{v}} \otimes \tilde{\chi}^{[m, \mu]})$$

and hence we see that our module of homomorphisms is given by

$$\begin{aligned} & \bigoplus_{\xi \in T(\mathbb{Z}/(p^m))/T^{(\mu)}(\mathbb{Z}/(p^m))} \text{Hom}_{T^{(\mu')}(\mathbb{Z}/(p^m))} (R_m / (\alpha p^{m-\mu}) \bar{\Psi}_{\xi \bar{v}}, R_m / (\alpha p^{m-\mu'}) \tilde{\Phi}_{v'}) = \\ & \bigoplus_{\xi \in T(\mathbb{Z}/(p^m))/T^{(\mu)}(\mathbb{Z}/(p^m))} R_m / (\alpha p^{m-\mu}) \end{aligned}$$

We have an large supply of Hecke operators for the $I_{\tilde{\chi}^{[m]}}$. Not all of them are good because we also want, that they send $\mathcal{P}_{\tilde{\chi}^{[m]}}^{(t_{p^m})}$ to $\mathcal{P}_{\tilde{\chi}^{[m]}}$.

3.4 Discuss the further requirements formulated at the beginning of this section.

In the beginning of this section we discussed some requirements the set $\mathbb{H}_{\tilde{\chi}}^{[m]}$ of Hecke operators should fulfill.

The first step is to modify the situation slightly. We want to get rid of the dependence of α and this means that we pass to the quotient $R_m / (p^{m-\mu})$ on the left hand side and on the right hand side we replace the factor $\beta(\mu)$ by p^μ and hence we get a submodule. We only look at correction terms, which go from the quotient on the left to the submodule on the right. Hence we see that the quotients on the left get smaller and smaller if we go from right to left until we reach the middle. The analogous assertion holds for the submodules on the right. We still go one step further. After we pass the middle we continue with the drop rate, i.e. in the decomposition of the module of coinvariants we replace $p^{m-\mu}$ by p^ν , and in

the decomposition of the module of invariants we replace $\beta(\mu)$ by p^ν . Then we get a quotient of the coinvariants and a submodule of the invariants. go to a still smaller quotient on the left and a still smaller submodule on the right.

Then we look for operators between the $T(\mathbb{Z}/(p^m))$ -modules

$$I_{\tilde{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m)), \text{small}} = R_m/(p^0)\bar{\Psi}_w \oplus \bigoplus_{\nu=1}^{m-1} \text{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(p^\nu)\bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m, \mu]} \oplus R_m\bar{\Psi}_0,$$

which is a quotient of the coinvariants and

$$I_{\tilde{\chi}^{[m]}, \text{small}}^{U_+(\mathbb{Z}/(p^m))} = R_m/(p^0)p^m\Phi_0 \oplus \bigoplus_{\nu=m-1}^1 \text{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(p^{m-\nu})(p^\nu\tilde{\Phi}_v) \otimes \tilde{\chi}^{[m, \mu]} \oplus R_m\Phi_-$$

Now it is easily verified that the elements in (*smallinv*) are actually in $\mathcal{P}_{\tilde{\chi}^{[m]}}$.

Then it is clear that $T(\mathbb{Z}/(p^m))$ invariant homomorphisms from $I_{\tilde{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m)), \text{small}}$ to $I_{\tilde{\chi}^{[m]}, \text{small}}^{U_+(\mathbb{Z}/(p^m))}$ satisfy (i). The property (iii) has been verified above. This system of Hecke operators also satisfies (ii), because any operator $u_{t_{p^m}}$ satisfies the condition (*) and the resulting "push down" again satisfies (*). So we are left to show that the classical Hecke operator extends.

3.5 The extension of the classical Hecke operator to an element in $\mathbb{H}_\chi^{[m]}$

Now we consider the case $\chi = \chi_{-n}$ and the morphism

$$\mathcal{M}_n/p^m\mathcal{M}_n \longrightarrow \mathcal{P}_{\tilde{\chi}_{-n}^{[m]}} \subset I_{\tilde{\chi}_{-n}^{[m]}}.$$

We have the classical Hecke operator on the cohomology of the sheaf $\tilde{\mathcal{M}}_n$ which is given by the map $u_{t_{p^m}}^{\text{class}} : X^\nu Y^{n-\nu} \mapsto p^{m\nu} X^\nu Y^{n-\nu}$. On $\mathcal{M}_n/p^m\tilde{\mathcal{M}}_n$ it maps Y^n to Y^n and all other monomials go to zero.

The monomial Y^n has as image in the module of coinvariants

$$Y^n \mapsto \bar{f}_{Y^n} = \left(\sum_{u \in U_+(\mathbb{Z}/(p^m))} u^n \right) \Psi_w + \sum_{\bar{v}} p^{l(\bar{v})} \bar{\Psi}_{\bar{v}}$$

Under the homomorphism given by the projection to the last component $I_{\tilde{\chi}_{-n}^{[m]}} \rightarrow R_m\Phi_0$ we find $\bar{f}_{Y^n} \mapsto \Psi_0$. Any $T(\mathbb{Z}/p^m\mathbb{Z})$ -invariant homomorphism from $R_m\Phi_0$ -viewed as quotient of $I_{\tilde{\chi}_{-n}^{[m]}}$ to the submodule of small invariants gives us a Hecke operator. Such a homomorphism is given by

$$\Psi_0 \mapsto \sum_{v \in p\mathbb{Z}/(p^m)} v^n \cdot \Phi_v + \Phi_-$$

where the right hand side is the image of the polynomial Y^n in $\mathcal{P}_{\tilde{\chi}_{-n}^{[m]}}^{U_-(\mathbb{Z}/p^m)}$. This gives us the desired extension. Here we need that $n > 0$ this is the regularity condition.

3.6. The case of a general reductive group scheme.

At this point it turns out, that our previous considerations are much to detailed. What we actually need is that our system $\mathbb{H}_{\tilde{\chi}}^{[m]}$ contains the principal operator and in the case that $\chi = \chi_{\lambda}$ we only want to extend the classical operator. We will see that we have the same extension procedure to extend the classical Hecke operator in general. The regularity condition guarantees that we have the essential property iii) for this operator.

We give a few comments. I recall the situation in 3.2. Let $S = \text{Spec}(R)$ where R is a local ring, we think of the cases $R = \mathbb{Q}, \mathbb{Z}_{(p)}, \mathbb{Z}/p^m\mathbb{Z} \dots$. We consider a quasisplit group scheme G/S , let $\mathcal{B} \subset \mathcal{G}$ be a Borel subscheme and $\mathcal{T} = \mathcal{B}/\mathcal{U}$. We can split this quotient and write $\mathcal{B} = \mathcal{T} \times \mathcal{U}$. For any highest weight λ we consider the module $\mathcal{M}_{\lambda} = H^0(\mathcal{B} \backslash \mathcal{G}, \mathcal{L}_{w_0(\lambda)})$ this gives us a well defined representation of the group scheme \mathcal{G}/S . We can restrict this representation to the torus \mathcal{T} and this representation is semi simple, i.e. we have a decomposition into weight spaces

$$\mathcal{M}_{\lambda} = \bigoplus_{\mu} \mathcal{M}_{\lambda, \lambda - \mu} = \mathcal{M}_{\lambda, 0} \oplus \dots \oplus \mathcal{M}_{\lambda, w_0(\lambda)},$$

where μ runs over a finite set of weights of the form $\mu = \sum_{\alpha} n_{\alpha} \alpha$ where $n_{\alpha} \geq 0$. The weight spaces are free R -modules. Now we define the character $\chi_{\lambda} : \mathcal{T}(\mathbb{Z}_p) \rightarrow \mathcal{O}_{E_p}^{\times}$ by $\chi_{\lambda}(x) = w_0(\lambda)(x)$.

Then we have the family of $\mathcal{G}(\mathbb{Z}_p)$ -homomorphisms

$$\mathcal{M}_{\lambda}/p^m \mathcal{M}_{\lambda} \xrightarrow{g_m} \mathcal{P}_{\tilde{\chi}_{\lambda}^{[m]}} \rightarrow I_{\tilde{\chi}_{\lambda}^{[m]}}.$$

We consider the $\mathcal{B}(R)$ -homomorphism $\psi_{\lambda} : I_{\tilde{\chi}_{\lambda}^{[m]}} \rightarrow R$ which is given by evaluation at the identity element. This linear map sends $e_{w_0(\lambda)}$ to 1, hence we can conclude that the kernel of g_m is contained in $\bigoplus_{\mu > 0} \mathcal{M}_{\lambda, \lambda - \mu}$. This makes it clear that the classical operator vanishes on the kernel of g_m and hence we see that it also acts trivially on the cohomology of this kernel. (This point was left open in the discussion of the Main result) On the other hand the principal operator $u_{t_p^m}^{\text{princ}} : (I_{\tilde{\chi}_{\lambda}^{[m]}})_{\mathcal{U}_+(\mathbb{Z}/p^m\mathbb{Z})} \rightarrow (I_{\tilde{\chi}_{\lambda}^{[m]}})^{\mathcal{U}_-(\mathbb{Z}/p^m\mathbb{Z})}$ is the composition of ψ_{λ} and the homomorphism $R \rightarrow I_{\tilde{\chi}_{\lambda}^{[m]}}^{\mathcal{U}_-(\mathbb{Z}/p^m\mathbb{Z})}$ which sends 1 to the function Φ_- , which is supported on the big cell $\mathcal{B}(\mathbb{Z}/p^m\mathbb{Z})\mathcal{U}_-(\mathbb{Z}_p/p^m\mathbb{Z})$. Hence we see that the principal operator sends $e_{w_0(\lambda)}$ to Φ_- whereas the classical operator sends $e_{w_0(\lambda)}$ to itself, here we consider $e_{w_0(\lambda)}$ as an element in $I_{\tilde{\chi}_{\lambda}^{[m]}}^{\mathcal{U}_-(\mathbb{Z}/p^m\mathbb{Z})}$ or $\mathcal{P}_{\tilde{\chi}_{\lambda}^{[m]}}^{\mathcal{U}_-(\mathbb{Z}/p^m\mathbb{Z})}$. Then the regularity condition asserts that $e_{w_0(\lambda)}$ vanishes mod p on the cells which are different from the big cell and hence we have shown that

$$(u_{t_p^m}^{\text{princ}} - u_{t_p^m}^{\text{class}})(I_{\tilde{\chi}_{\lambda}^{[m]}}) \subset p(I_{\tilde{\chi}_{\lambda}^{[m]}})$$

and this establishes the validity in the requirements i) to iv) made in section 2.

4. Further consequences and generalizations

4.1. Boundedness of ordinary torsion

We return to 3.2. To any dominant weight λ we can attach the character $\chi_{w_0(\lambda)}$, and we get the $\mathcal{G}(\mathbb{Z}_p)$ invariant homomorphisms

$$\mathcal{M}_\lambda/p^m\mathcal{M}_\lambda \rightarrow \mathcal{P}_{\chi_\lambda^{[m]}} \rightarrow I_{\chi_\lambda^{[m]}}.$$

If we now assume regularity, namely $n_\alpha > 0$ for all α then we get isomorphisms

$$j_\lambda^\bullet : H_{\text{ord}}^\bullet(S_{K_f}^G, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda) \rightarrow H_{\text{ord}}^\bullet(S_{K_f}^G, I_{\chi^{[m]}}).$$

Here we define ordinary with respect to $\mathbb{H}_\chi^{[m]}$.

The space of characters χ is $\Omega \times (X^*(T) \otimes \mathcal{O}_{\mathbb{C}_p})$, where Ω is the finite set of $\nu = \sum_\alpha \nu_\alpha \gamma_\alpha$. This space contains the subspace $\Omega \times (X^*(T) \otimes \mathbb{Z}_p)$. It is easy to see that the characters χ_λ are dense in $\Omega \times (X^*(T) \otimes \mathbb{Z}_p)$.

I want sketch the proof of a theorem which is the consequence of the existence of the interpolating family.

Theorem 2 *If λ varies over all dominant weights which satisfy the regularity condition $\nu_\alpha \not\equiv 0 \pmod{p-1}$ for all α , then the order of the p -torsion of the ordinary cohomology groups $H_{\text{ord}}^\bullet(S_{K_f}^G, \mathcal{M}_\lambda)$ is bounded.*

Proof: We prove it by downwards induction on the degree of the cohomology. Let us fix a $\nu(\lambda)$ which is regular, the λ varies in this residue class, i.e. $\lambda = \sum n_\alpha \gamma_\alpha$ with $n_\alpha \geq 0$ and $n_\alpha \equiv \nu_\alpha \pmod{p-1}$. This is dense in the space of χ with the same given ν .

Since the cohomological dimension is finite, we find a q such that our assertion is true for all cohomology groups in degree $> q$. Take such a q and prove that the assertion is also true for the cohomology in degree q . We assume that the exponent of the cohomology is bounded by m_0 . For $m \geq m_0$ we consider the exact sequences

$$0 \rightarrow H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda) \otimes \mathbb{Z}/p^m \rightarrow H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda) \rightarrow H_{\text{ord}}^{q+1}(S_{K_f}^G, \mathcal{M}_\lambda)[p^m] \rightarrow 0$$

The right term of the sequence has an order, that it bounded independently of λ , we restrict to those λ for which this number is p^δ , one of the finite number of possibilities.

We know that $H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda) \otimes \mathbb{Z}_p$ is finitely generated we write it as a direct sum

$$H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda) = \mathbb{Z}^{b^q(\lambda)} + H^q(S_{K_f}^G, \mathcal{M}_\lambda)_{\text{tors}}.$$

Hence we obtain

$$p^{mb^q(\lambda)} \#(H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda)_{\text{tors}} \otimes \mathbb{Z}/p^m) p^\delta = \#H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda).$$

Now the key is a lemma concerning the growth of $\#H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda)$. We write the exact sequence

$$0 \rightarrow p\mathcal{M}_\lambda/p^{m+1}\mathcal{M}_\lambda \rightarrow \mathcal{M}_\lambda/p^{m+1}\mathcal{M}_\lambda \rightarrow \mathcal{M}_\lambda/p\mathcal{M}_\lambda \rightarrow 0$$

and the resulting long exact sequence for cohomology

$$\begin{aligned} \dots \rightarrow H_{\text{ord}}^{q-1}(S_{K_f}^G, \mathcal{M}_\lambda/p\mathcal{M}_\lambda) \xrightarrow{\delta^{q-1}} H_{\text{ord}}^q(S_{K_f}^G, p\mathcal{M}_\lambda/p^{m+1}\mathcal{M}_\lambda) \rightarrow H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda/p^{m+1}\mathcal{M}_\lambda) \\ \rightarrow H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda/p\mathcal{M}_\lambda) \xrightarrow{\delta^q} \dots \end{aligned}$$

The two extremal modules are finite dimensional vector spaces over \mathbb{F}_p and clearly the dimensions of the kernels of δ^{q-1}, δ^q can only drop if m goes up. Hence we find values $m_{q-1}(\lambda), m_q(\lambda)$ such that these kernels of δ^{q-1}, δ^q become constant if $m \geq m_{q-1}(\lambda), m_q(\lambda)$.

Now the functions $\lambda \rightarrow m_{q-1}(\lambda), \lambda \rightarrow m_q(\lambda)$ are continuous. I claim that we can extend these functions to continuous functions $\{\nu\} \times (X^*(T) \otimes \mathcal{O}_{\mathbb{C}_p})$. To see that this is the case we consider the exact sequence

$$0 \rightarrow \mathcal{P}_{\chi^{[m, \iota]}} \rightarrow \mathcal{P}_{\bar{\chi}^{[m+1]}} \rightarrow \mathcal{P}_{\chi^{[1]}} \rightarrow 0,$$

where by definition $\mathcal{P}_{\chi^{[m, \iota]}}$ is the kernel of the restriction, it is equal to the intersection $\mathcal{P}_{\bar{\chi}^{[m+1]}} \cap pI_{\bar{\chi}^{[m+1]}}$. We have a canonical homomorphism $r : \mathcal{P}_{\chi^{[m, \iota]}} \rightarrow I_{\bar{\chi}^{[m]}}$, which is given by $f = pg \mapsto g \pmod{p^m}$. It is clear that $\mathcal{P}_{\bar{\chi}^{[m]}} \subset r(\mathcal{P}_{\chi^{[m, \iota]}}) \subset I_{\bar{\chi}^{[m]}}$ and hence we get isomorphisms

$$H_{\text{ord}}^\bullet(S_{K_f}^G, \mathcal{P}_{\bar{\chi}^{[m]}}) \xrightarrow{\sim} H_{\text{ord}}^\bullet(S_{K_f}^G, \mathcal{P}_{\chi^{[m, \iota]}}) \xrightarrow{\sim} H_{\text{ord}}^\bullet(S_{K_f}^G, I_{\bar{\chi}^{[m]}}).$$

This gives us again a long exact sequence

$$\begin{aligned} \dots \rightarrow H_{\text{ord}}^{q-1}(S_{K_f}^G, \mathcal{P}_{\chi^{[1]}}) \xrightarrow{\delta^{q-1}} H_{\text{ord}}^q(S_{K_f}^G, \mathcal{P}_{\bar{\chi}^{[m]}}) \rightarrow H_{\text{ord}}^q(S_{K_f}^G, \mathcal{P}_{\bar{\chi}^{[m+1]}}) \\ \rightarrow H_{\text{ord}}^q(S_{K_f}^G, \mathcal{P}_{\chi^{[1]}}) \xrightarrow{\delta^q} \dots \end{aligned}$$

the space of all characters $\chi \in \{\nu\} \times (X^*(T) \otimes \mathbb{Z}_p)$. This allows us to extend the definition of m_{q-1}, m_q to all characters χ .

Especially we can consider the extension to the compact space $\{\nu\} \times (X^*(T) \otimes \mathbb{Z}_p)$ and hence m_{q-1}, m_q assume maximal values on this subspace. Let m_1 greater than this maximal value.

Let us now put $c^i(\lambda) = \dim H_{\text{ord}}^i(S_{K_f}^G, \mathcal{M}_\lambda/p\mathcal{M}_\lambda)$ and let $s(\lambda)$ (resp., $t(\lambda)$) be the minimal value of the dimension of $\ker(\delta^{q-1})$ (resp. δ^q). This is a finite number of values. Then we find for $m \geq m_1$

$$\#H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda/p^{m+1}\mathcal{M}_\lambda) = \#H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda)p^{c^{q-1}(\nu) - s(\lambda) + t(\lambda)}.$$

We can subdivide the space of λ with ν fixed into a finite number of regions where $s(\lambda), t(\lambda)$, and $c^q(\lambda)$ have a fixed value.

On such a region we insert the last formula into the formula above and remembering that also δ was fixed, we find

$$p^{c^{q-1}(\lambda)-s(\lambda)+t(\lambda)} = p^{b^q(\lambda)} \frac{\#(H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda)_{\text{tors}} \otimes \mathbb{Z}/p^{m+1})}{\#(H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda)_{\text{tors}} \otimes \mathbb{Z}/p^m)}.$$

For a fixed value of λ the ratio of orders of torsion groups on the right is equal to one if $m \gg 0$. Hence

$$c^{q-1}(\nu) - s(\lambda) + t(\lambda) = b^q(\lambda)$$

and the function $m \rightarrow \#(H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda)_{\text{tors}} \otimes \mathbb{Z}/p^{m+1})$ is constant for $m > m_1$. Since $\lambda \rightarrow \#(H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda/p^{m_1}\mathcal{M}_\lambda))$ assumes only finitely many values we have proved the theorem.

We can draw some further conclusions. From our last formula it follows that $\lambda \rightarrow b^q(\lambda)$ is locally constant, more precisely it depends only on $\lambda \pmod{p^{m_1}}$. We consider a $\chi \in \{\nu\} \times (X^*(T) \otimes \mathbb{Z}_p)$ (the ν component satisfies the regularity condition) and we approximate χ by a λ . Then we look at our exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda) \otimes \mathbb{Z}/p^m & \rightarrow & H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda) & \rightarrow & H_{\text{ord}}^{q+1}(S_{K_f}^G, \mathcal{M}_\lambda)[p^m] \rightarrow 0 \\ & & & & \downarrow & & \\ & & & & H_{\text{ord}}^q(S_{K_f}^G, \mathcal{P}_{\tilde{\chi}^{[m]}}) & & \end{array}$$

The vertical arrow is an isomorphism, the modules in the diagram stay the same if we modify λ into a better approximation of χ .

Hence we see that $H_{\text{ord}}^q(S_{K_f}^G, \mathcal{P}_{\tilde{\chi}^{[m]}})$ has a submodule S_m which is the image of the module on the left and which is isomorphic to $H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda) \otimes \mathbb{Z}/p^m$ and where the cokernel by this submodule is $H_{\text{ord}}^{q+1}(S_{K_f}^G, \mathcal{M}_\lambda)[p^m]$.

If we now take the projective limit then we see that for the terms on the right the projective limit is zero. This yields for the projective limit

$$\varprojlim S_m = \varprojlim H_{\text{ord}}^q(S_{K_f}^G, \mathcal{P}_{\tilde{\chi}^{[m]}}) = H_{\text{ord}}^q(S_{K_f}^G, \mathcal{P}_\chi).$$

This gives us for $\chi \in \{\nu\} \times (X^*(T) \otimes \mathbb{Z}_p)$ the structure

$$H_{\text{ord}}^q(S_{K_f}^G, \mathcal{P}_\chi) \xrightarrow{\sim} H_{\text{ord}}^q(S_{K_f}^G, \mathcal{M}_\lambda)_{\text{tors}} \oplus \mathbb{Z}_p^{b^q(\chi)}$$

where $b^q(\chi) = b^q(\lambda)$ and λ is approximating χ well enough.

Our regularity assumption is a little bit too strong. We also consider the case that $\nu_\alpha \equiv 0 \pmod{p-1}$. Our character χ is of the form

$$x \mapsto \prod_{\alpha} \omega(x_\alpha)^{\nu_\alpha} \left(\frac{x_\alpha}{\omega(x_\alpha)} \right)^{z_\alpha}.$$

If we have $\nu_\alpha = 0$ then we still can approximate χ by a weight $\lambda = \sum n_\alpha \gamma_\alpha$ where $n_\alpha = (p-1)n_\alpha^*$. Then our argument works provided we keep the n_α^* away from zero in \mathbb{Z}_p , i.e. we restrict to a region where $\text{ord}_p(n_\alpha) \leq M$ with a given constant M .

4.2. Denominators of Eisenstein classes.

At various occasions we discuss the relationship between denominators of Eisenstein classes and special values of L -functions. These special values enter in the constant term of Eisenstein series, which are representing cohomology classes (See [Ha -MM],3.1.4. ,3.2.7, [Ha-Cong], [Ha-Coh], chap.3. 5.5). In all these cases we avoid to discuss that there are two sources for the emergence of denominators.

The first source is given by the L -values occuring in the constant term, the second is p -torsion. To illustrate this problem we refer to [Ha-Cong]. On p. 8 we write a sequence

$$0 \rightarrow H_1^3(\Gamma \backslash \mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \rightarrow H^3(\Gamma \backslash \mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \rightarrow H^3(\partial(\overline{\Gamma \backslash \mathbb{H}_2}), \tilde{\mathcal{M}}_{4,7} \otimes R) \rightarrow 0,$$

and we assume exactness of this sequence. In our special situation we can assume that $R = \mathbb{Z}_{(41)}$, this means we localize at the prime 41. More or less by definition the exactness of this sequence follows if we know that

$$H_{c, \text{ord}}^4(\Gamma \backslash \mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) = 0.$$

This is very likely the case, but we have very few tools to investigate the torsion, except that we compute the cohomology explicitly. We need the exactness to carry out our speculative consideration on p.11. These considerations imply that the number 41 should divide the denominator of the Eisenstein class, because 41 divides a certain value of an L -function. Then this in turn implies congruences, which have been checked numerically.

But of course it may be that this assumption is not true and $H_{c, \text{ord}}^4(\Gamma \backslash \mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \neq 0$. Then the denominator of the Eisenstein class will also contain a factor which is a non trivial divisor of $\#H_{c, \text{ord}}^4(\Gamma \backslash \mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R)$. This is the second source for denominators.

But now our theorem 2 allows to interpolate our weight $\lambda = 4\gamma_\beta + 7\gamma_\alpha$ 41-adically. Then we may find L -values $\Lambda(\tilde{f}, 14 + (41 - 1) \cdot a \cdot 41^b) / \tilde{\Omega}_+$ which are divisible by arbitrarily high powers of 41. But since the torsion in $H_{c, \text{ord}}^4(\Gamma \backslash \mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R)$ stays bounded, such a divisibility should also create arbitrarily high denominators of the Eisenstein class and hence high congruences between Siegel and elliptic modular forms.

This question is related to another question, which is also speculative. The exposition will be a little bit vague.

What is the arithmetic meaning of the constant term of "cohomological" Eisenstein series?

We refer to [Coh], chap. 6 and [Ha-rank1]. If we consider rank one Eisenstein classes, i.e. we induce from "cuspidal" classes on a maximal parabolic subgroup, then the constant term has two summands: The first summand is basically our original class and the "second" term involves the L -function of our original form, this is the classical formula of Langlands.

It may happen, that the second term is "zero in cohomology" and this is the case, when it seems to influence the denominator of the Eisenstein class and produces mixed motives whose extension classes are related to this second term (See [Ha-MM],Chap VI, [Ha-Cong])

But the second case is also interesting. Here we get rationality results for special values of L -functions (See [Ha-rank1]). But the results in this note here also allow to draw some more arithmetical consequences. We refer to a special example that is treated In [Ha-rank1]. We consider the case of $G = \mathrm{Sl}_3/\mathbb{Q}$ the two maximal parabolic subgroups $P/\mathbb{Q}, Q/\mathbb{Q}$. Since we want to say something about integral cohomology, we fix a level, for simplicity we assume that K_f is the standard maximal compact subgroup, let $\mathcal{M} = \mathcal{M}_\lambda$ be a \mathbb{Z} - module of highest weight λ .

Since the Manin-Drinfeld principle is available for Gl_2/\mathbb{Q} we get a rational decomposition (See chap. 3, 5.5.2)

$$H^2(\partial S_{K_f}^G, \mathcal{M}_\mathbb{Q}) = H_!^2(\partial S_{K_f}^G, \mathcal{M}_\mathbb{Q}) \oplus H_{\mathrm{Eis}}^2(\partial S_{K_f}^G, \mathcal{M}_\mathbb{Q}).$$

Now we invert certain primes that allow congruences between $!$ -cohomology and Eisenstein cohomology on Gl_2 , i.e. primes dividing certain values $\zeta(-1-k)$, and get a ring $R = \mathbb{Z}[\dots, 1/l, \dots]$ and a decomposition

$$H^2(\partial S_{K_f}^G, \mathcal{M}_R) = H_!^2(\partial S_{K_f}^G, \mathcal{M}_R) \bigoplus H_{\mathrm{Eis}}^2(\partial S_{K_f}^G, \mathcal{M}_R).$$

Since the boundary has two strata, which correspond to the two maximal parabolic subgroups, we get

$$H^2(\partial S_{K_f}^G, \mathcal{M}_R) = H^2(\partial_P S_{K_f}^G, \mathcal{M}_R) \bigoplus H^2(\partial_Q S_{K_f}^G, \mathcal{M}_R)$$

We can find an extension $\mathbb{Q} \subset F \subset \mathbb{C}$ such that we can decompose $H_!^2(\partial S_{K_f}^G, \mathcal{M}_F)$ into absolutely irreducible modules and if we extend \mathcal{O}_F to a larger ring R_1 by inverting some more congruence primes we get (See [Ha-rank1],[Ha-Coh] chap. 3).

$$H_!^2(\partial S_{K_f}^G, \mathcal{M}_{R_1}) = \bigoplus_{\sigma_f} H_!^2(\partial_P S_{K_f}^G, \mathcal{M}_{R_1})(\sigma_f) \oplus H_!^2(\partial_Q S_{K_f}^G, \mathcal{M}_{R_1})(\Theta\sigma_f) =$$

$$\bigoplus_{\sigma_f} H_!^1(S_{K_f}^{M_\alpha}, \mathcal{M}(s_\beta \cdot \lambda)_{R_1})(\sigma_f) \oplus H_!^1(S_{K_f}^{M_\beta}, \mathcal{M}(s_\alpha \cdot \lambda)_{R_1})(\Theta\sigma_f).$$

We denote an individual summand by $H_!^2(\partial S_{K_f}^G, \mathcal{M}_{R_1})(\sigma_f)$. Notice that $\mathcal{M}(s_\alpha \cdot \lambda)$ sits in degree one.

To simplify the discussion we assume that only one summand occurs, this is the case if the corresponding space of holomorphic modular forms has dimension 1, i.e. if the number d in [Ha-rank1] has the value $d = 10, 14, 16, 18, 20, 24$. Hence our ring $R_1 = \mathbb{Z}[1/n]$, where n is a product of some small primes.

The Eisenstein intertwining operator is a map

$$H_!^1(S_{K_f}^{M_\alpha}, \mathcal{M}(s_\beta \cdot \lambda)_\mathbb{C})(\sigma_f) \rightarrow H^2(S_{K_f}^G, \mathcal{M}_\mathbb{C})$$

and if we compose the Eisenstein intertwining operator with the restriction to the boundary we get

$$r \circ E : H^1(S_{K_f}^{M_\alpha}, \mathcal{M}(s_\beta \cdot \lambda)_\mathbb{C})(\sigma_f) \rightarrow H^1(S_{K_f}^{M_\alpha}, \mathcal{M}(s_\beta \cdot \lambda)_\mathbb{C})(\sigma_f) \bigoplus H^1(S_{K_f}^{M_\beta}, \mathcal{M}(s_\alpha \cdot \lambda)_\mathbb{C})(\Theta\sigma_f).$$

The right hand side is an isotypical submodule under the Hecke algebra and the summands are one dimensional \mathbb{C} vector spaces. These vector spaces are defined over \mathbb{Q} and the image of this composition is given by (See [Ha-rank1])

$$\psi_f + \frac{L(\sigma, -1)}{L(\sigma, 0)} T_\infty^{\text{loc}}([\omega]) \otimes T_f^{\text{loc}}(\psi_f).$$

We explained in [Ha-rank1], that the local intertwining operator induces an isomorphism between the two summands. But now the summands are base extensions free R_1 modules of rank 1, hence we find a period $\Omega(\sigma_f)$ which is unique up to an element in R_1^\times such that

$$\Omega(\sigma_f) T_f^{\text{loc}} : H_!^2(\partial_P S_{K_f}^G, \mathcal{M}_{R_1})(\sigma_f) \xrightarrow{\sim} H_!^2(\partial_Q S_{K_f}^G, \mathcal{M}_{R_1})(\Theta\sigma_f).$$

Then the image $\text{Im}(H^2(S_{K_f}^G, \mathcal{M}_{R_1}) \rightarrow H^2(\partial S_{K_f}^G, \mathcal{M}_{R_1})) = H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_{R_1})$ intersected with the the isotypical submodule

$$H_!^2(\partial_P S_{K_f}^G, \mathcal{M}_{\mathbb{C}})(\sigma_f) \oplus H_!^2(\partial_Q S_{K_f}^G, \mathcal{M}_{\mathbb{C}})(\Theta\sigma_f) \cap H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_{R_1}) = H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_{R_1})(\sigma_f)$$

is a rank one R_1 submodule in the one dimensional \mathbb{Q} -vector space

$$\left\{ \psi_f + \frac{L(\sigma, -1)}{\Omega(\sigma_f)L(\sigma, 0)} (\Omega(\sigma_f) T_f^{\text{loc}}(\psi_f)) \mid \psi_f \in H^1(S_{K_f}^{M_\alpha}, \mathcal{M}(s_\beta \cdot \lambda))_F(\sigma_f) \right\}.$$

We noticed already in [Ha-rank1], that $\frac{L(\sigma, -1)}{\Omega(\sigma_f)L(\sigma, 0)} \in \mathbb{Q}$ because

$$\left\{ \psi_f + \frac{L(\sigma, -1)}{\Omega(\sigma_f)L(\sigma, 0)} (\Omega(\sigma_f) T_f^{\text{loc}}(\psi_f)) \right\} = H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_{\mathbb{Q}}).$$

We find

$$H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_{R_1}) \subset H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_F) \cap H_!^2(\partial S_{K_f}^G, \mathcal{M}_{R_1})(\sigma_f).$$

Now we encounter interesting problems. We have the boundary map

$$\delta : H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_{\mathbb{Q}}) \cap H_!^2(\partial S_{K_f}^G, \mathcal{M}_{R_1})(\sigma_f) \rightarrow H_c^3(S_{K_f}^G, \mathcal{M}_{R_1})$$

and by construction its kernel is $H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_{R_1})$ and its image is torsion. Hence we have an inclusion

$$\delta : (H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_F) \cap H_!^2(\partial S_{K_f}^G, \mathcal{M}_{R_1})(\sigma_f)) / H_{\text{global}}^2(S_{K_f}^G, \mathcal{M}_{R_1}) \hookrightarrow H_c^3(S_{K_f}^G, \mathcal{M}_{R_1})_{\text{tors}}.$$

Now we pick a prime p which is not among those we inverted. Let us assume that σ_f is ordinary at p . Then we can interpolate p -adically our weight $\lambda = n_\alpha \gamma_\alpha + n_\beta \gamma_\beta$, i.e. we consider weights $\tilde{\lambda} = \lambda + \mu$ where $\mu \equiv 0 \pmod{p-1}$, i.e.

$$\tilde{\lambda} = (n_\alpha + (p-1)z_\alpha)\gamma_\alpha + (n_\beta + (p-1)z_\beta)\gamma_\beta, z_\alpha, z_\beta \in \mathbb{N}.$$

We introduce the notation $\tilde{\mathcal{M}} = \mathcal{M}_{\tilde{\lambda}} \otimes \mathbb{Z}_p$ then we know

$$H_{\text{ord},!}^1(S_{K_f}^{M_\alpha}, \tilde{\mathcal{M}}(s_\beta \cdot \lambda)_{\mathbb{Z}_p})(\tilde{\sigma}_f), H_{\text{ord},!}^1(S_{K_f}^{M_\alpha}, \tilde{\mathcal{M}}(s_\alpha \cdot \lambda)_{\mathbb{Z}_p})(\Theta \tilde{\sigma}_f),$$

are free of rank one (they lie in a Hida family), and the local intertwining operator identifies

$$\Omega(\tilde{\sigma}_f)T_f^{\text{loc}} : H_{\dagger}^2(\partial_P S_{K_f}^G, \tilde{\mathcal{M}}_{\mathbb{Z}_p})(\sigma_f) \xrightarrow{\sim} H_{\dagger}^2(\partial_Q S_{K_f}^G, \tilde{\mathcal{M}}_{\mathbb{Z}_p})(\Theta \sigma_f).$$

Hence we can say that

$$H_{\dagger}^2(\partial_P S_{K_f}^G, \tilde{\mathcal{M}}_{R_1})(\sigma_f) \oplus H_{\dagger}^2(\partial_Q S_{K_f}^G, \tilde{\mathcal{M}}_{R_1})(\Theta \tilde{\sigma}_f) = \mathbb{Z}_p \oplus \mathbb{Z}_p.$$

Hence we can view $H_{\text{global}}^2(S_{K_f}^G, \tilde{\mathcal{M}}_{\mathbb{Q}_p})$ as a point in $\mathbb{P}^1(\mathbb{Q}_p)$, it is the point $c_{\text{global}} = (1, \frac{L(\tilde{\sigma}, -1)}{\Omega(\tilde{\sigma}_f)L(\tilde{\sigma}, 0)})$ it defines a line $H_{\text{global}}(\tilde{\sigma}_f) \subset \mathbb{Z}_p^2$ and we saw that

$$H_{\text{global}}(\tilde{\sigma}_f)/H_{\text{global}}^2(S_{K_f}^G, \tilde{\mathcal{M}}_{\mathbb{Z}_p})(\tilde{\sigma}_f) \hookrightarrow H_{c, \text{ord}}^3(S_{K_f}^G, \tilde{\mathcal{M}}_{\mathbb{Z}_p})_{\text{tors}}.$$

Now we vary μ and apply our theorem and get, that $H_{c, \text{ord}}^3(S_{K_f}^G, \tilde{\mathcal{M}}_{\mathbb{Z}_p})_{\text{tors}}$ is bounded.

It seems to be clear to me that the computation of the cohomology from a suitable Čech-Complex will yield that $H_{\text{global}}^2(S_{K_f}^G, \tilde{\mathcal{M}}_{\mathbb{Z}_p})$ depends p -adically on $\tilde{\mathcal{M}} = \mathcal{M}_{\tilde{\lambda}}$ and this combined with the boundedness will imply that

$$\tilde{\sigma}_f \rightarrow \frac{L(\tilde{\sigma}, -1)}{\Omega(\tilde{\sigma}_f)L(\tilde{\sigma}, 0)}$$

is p -adically analytic, provided σ_f is ordinary at p .

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Günter Harder
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
harder@mpim-bonn.mpg.de