

## Summer School on Serre's modularity conjecture

### Abstracts

#### First week

##### Monday 9

**1** : 9 :30 – Jean-Pierre Serre, *Modular memories*

**2** : 11 :00 – Jean-Marc Fontaine, *Introduction 1*

**3** : 16 :00 – Jean-Marc Fontaine, *Introduction 2*

**4** : 18 :00 – Tony Scholl, *Automorphic forms and  $\ell$ -adic representations 1*

##### Tuesday 10

**5** : 9 :00 – Christophe Breuil,  *$p$ -adic Hodge theory 1*

*Abstract* : Let  $K$  be a finite totally ramified extension of  $\text{Frac}(W(k))$  where  $k$  is a perfect field of characteristic  $p > 0$ . I introduce  $B_{HT}$ ,  $B_{dR}$ ,  $B_{st}$  and  $B_{cris}$ . Then I give the definitions and main properties of Hodge-Tate, de Rham, semi-stable and crystalline representations of  $\text{Gal}(\overline{K}/K)$ . I finish with a description of Weil-Deligne representations attached to potentially semi-stable (or de Rham) representations.

**6** : 11 :00 – Michael Harris, *Automorphic forms and  $\ell$ -adic representations 2*

*Abstract* : see <http://www.math.u-psud.fr/~fontaine/ecole-ete/textes/harris1.pdf>

**F1** : 16 :00 – Dick Gross, *Involutions of Lie algebras*

*Abstract* : I will discuss what seems to be the proper generalization of an odd two-dimensional representation of the Galois group of  $\mathbb{Q}$ , and connections to deformation theory and automorphic forms.

**7** : 17 :15 – Vincent Pilloni, *Deformation theory 1*

*Abstract* : We introduce the language and give the classical results from the theory of deformations : deformations and framed deformations, representability, tangent spaces computation via Galois cohomology, formal smoothness...

Then we give a technique, due to Mark Kisin, which allows us to study the general fiber of the deformation spaces : let  $\rho$  be a Galois representation in  $\text{GL}(\mathbb{F})$  with  $\mathbb{F}$  a finite field and let  $\mathcal{O}_E$  be an integer ring of a local field  $E$ , with residual field  $\mathbb{F}$ . Let  $\rho_E$  be a lifting of  $\rho$  to  $\mathbf{GL}(\mathcal{O}_E)$ . One can build the deformation theory for  $\rho_E \otimes E$  out of the deformation theory of  $\rho$ .

As an application we describe, after Kisin, the general fiber of the deformation scheme of a representation  $\rho : G \rightarrow \mathbf{GL}_2(\mathbb{F})$ , where  $\mathbb{F}$  is a finite extension of  $\mathbb{F}_p$  and  $G$  the absolute Galois group of a local field of residual characteristic  $\ell \neq p$ .

**Wednesday 11****8** : 9 :00 – Christophe Breuil, *p-adic Hodge theory 2*

*Abstract* : I state the various comparison theorems concerning  $p$ -adic cohomologies of a proper smooth  $K$ -scheme for  $K$  as in the previous talk. Then I give an idea of the "syntomic" proof. I finish with a brief description of the weight spectral sequence on log-crystalline cohomology.

**9** : 11 :00 – Brian Conrad, *Integral p-adic Hodge theory 1*

*Abstract* : In  $p$ -adic Hodge theory there are fully faithful functors from certain categories of  $p$ -adic Galois representations of the Galois group  $G_K$  of a  $p$ -adic field  $K$  to certain categories of semi-linear algebra structures on finite-dimensional vector spaces in characteristic 0. For example, semistable representations give rise to weakly admissible filtered  $(\phi, N)$ -modules, and Fontaine conjectured that this is an equivalence of categories. For many purposes (such as in Galois deformation theory with artinian coefficients) it is useful to have a finer theory in which  $p$ -adic vector spaces are replaced with lattices or torsion quotients. Fontaine and Laffaille gave such a theory in the early 1980's, but subject to restrictions on Hodge-Tate weights (depending on ramification in  $K$ ). The aim of these lectures on integral  $p$ -adic Hodge theory is to explain a more recent theory, due largely to the work of Breuil and Kisin, that has no ramification restrictions.

In this lecture we describe a certain category of vector bundles with connection over the open  $p$ -adic unit disk ( $\phi$ -modules with finite  $E$ -height, endowed with a suitable differential operator) and use Kedlaya's theory of slope filtrations (as a black box) to prove it is equivalent to the category of filtered  $(\phi, N)$ -modules that are effective (in the sense that the filtration jumps are in non-negative degrees). We also show that under this equivalence, the weak admissibility of a filtered  $(\phi, N)$ -module can be expressed in terms of the slopes of the corresponding bundle with connection.

**F2** : 16 :30 – Michael Harris, *Automorphic Galois representations***10** : 18 :00 – Brian Conrad, *Integral p-adic Hodge theory 2*

*Abstract* : This lecture introduces the category of  $\mathfrak{S}$ -modules, roughly an integral version of the category of bundles with connection from the previous lecture on integral  $p$ -adic Hodge theory, and we set up a fully faithful functor from the category of effective weakly admissible filtered  $(\phi, N)$ -modules to the isogeny category of  $\mathfrak{S}$ -modules (and we describe the essential image). In the reverse direction we construct a fully faithful functor from the category of  $\mathfrak{S}$ -modules into the category of  $G_{K_\infty}$ -stable lattices in semistable  $G_K$ -representations, where  $K_\infty/K$  is generated by compatible  $p$ -power roots of a uniformizer of  $K$ . As applications, we obtain a proof of Fontaine's conjecture that the natural fully faithful functor from semistable representations to weakly admissible modules is an equivalence and we deduce Breuil's conjecture that restriction from crystalline  $G_K$ -modules to the underlying  $G_{K_\infty}$ -modules is fully faithful. We also use  $\mathfrak{S}$ -modules to describe the category of all  $G_{K_\infty}$ -stable lattices in crystalline representations of  $G_K$ .

## Thursday 12

**11 : 9 :00** – Brian Conrad, *Integral  $p$ -adic Hodge theory 3*

*Abstract* : We introduce the concept of a  $p$ -divisible group, discuss several ways of working with them, and use the results from previous lectures on integral  $p$ -adic Hodge theory to deduce an elegant way to describe  $p$ -divisible groups and finite flat group schemes over  $O_K$  in terms of  $\mathfrak{S}$ -modules. We also discuss applications to torsion and lattice representations of  $G_K$  in the context of the earlier work of Fontaine and Laffaille, and we study the restriction from  $G_K$  to  $G_{K_\infty}$  for representations arising from finite flat group schemes over  $O_K$ .

**12 : 11 :00** – Henri Carayol, *Automorphic forms and  $\ell$ -adic representations 3*

**13 : 16 :00** – Florian Herzig, *Deformation theory 2*

**14 : 18 :00** – Mark Kisin, *Deformation theory 3*

*Abstract* : I will explain how to use the results explained in the talks of Brian Conrad to construct quotients of the universal deformation ring of a local Galois representation, which parametrize deformation which are potentially semi-stable with given Galois type and  $p$ -adic Hodge type.

## Friday 13

**15 : 9 :00** – Mark Kisin, *Deformation theory 4*

*Abstract* : Starting with the constructions of the previous lecture, I will explain the construction of certain spaces which live over deformation rings of local Galois representations and parametrize finite flat models of these Galois representations.

These spaces serve as a kind of resolution of the flat deformation ring, and can be used, in some cases, to understand the connected components of its generic fibre.

**16 : 11 :00** – Takeshi Saito, *Automorphic forms and  $\ell$ -adic representations 4*

*Abstract* : The compatibility between the global and local Langlands correspondences for the  $\ell$ -adic representation associated to a Hilbert modular form is explained in Carayol's lecture in the case  $\ell \neq p$ , under the assumption that the Galois representation is constructed geometrically. We deduce the compatibility for  $\ell = p$  from that for  $\ell \neq p$ . More precisely, we compare the traces and the monodromy operators for the  $p$ -adic and the  $\ell$ -adic representations, using the weight spectral sequences. In order to do it, we use a geometric construction of the Galois representations, in a more strict sense.

**F3 : 16 :30** – Sophie Morel, *Intersection cohomology of certain unitary Shimura varieties*

**17 : 18 :00** – Gebhard Böckle, *Deformation theory 5*

*Abstract* : The aim of this talk is to prove a lower bound on the dimension of universal deformation rings based on obstruction theory. Using the local computations of previous talks, this will show that the Krull dimension of universal deformation rings is at least

one in the following crucial case : Deformations of 2-dimensional odd mod  $p$  Galois representations over a totally real field with suitable semistability conditions at all places above  $p$ , fixed determinant, and some further conditions at the other finitely many primes where ramification is allowed.

The rather general proof rests on two recent ideas of M. Kisin : (a) One presents the global universal deformation ring relative to the tensor product of corresponding local deformation rings. This has the advantage that the local deformation rings need no longer be complete intersection rings for the method to work. They only need to be Cohen-Macaulay (b) One circumvents local non-representability issues by replacing deformations by framed deformations.

The talk will first recall the relevant rings and the cohomological notions needed. Then we give the proof of the basic lower dimension bound. A simple base change of rings will transport this bound into the setting relevant to the proof of Serre's conjecture. Applying results of previous talks the above-mentioned lower bound of 1 can easily be derived. If time permits, I shall also briefly explain the local deformation theory at the infinity primes for  $p = 2$ .

#### **Saturday 14**

**18** : 9 :00 – Jean-Pierre Labesse, *Automorphic forms and  $\ell$ -adic representations 5*

**19** : 11 :00 – Laurent Clozel, *Automorphic forms and  $\ell$ -adic representations 6*

#### **Second week**

#### **Monday 16**

**20** : 9 :00 – Ken Ribet, *Overview of the proof*

**21** : 11 :00 – David Savitt, *Integral  $p$ -adic Hodge theory 4*

**F4** : 16 :30 – Bruno Klingler, *On the Andre-Oort conjecture (joint with A. Yafaev)*

*Abstract* : Given a set  $X$  of special points on a Shimura variety  $S$ , the Andre-Oort conjecture predicts that any irreducible component of the Zariski-closure of  $X$  in  $S$  is a subvariety of Hodge type. We prove this conjecture under the generalized Riemann Hypothesis, or if  $X$  lie in one Hecke orbit.

**22** : 18 :00 – Fred Diamond, *Congruences 1*

*Abstract* : I will explain how to prove the existence of various types of congruences between modular forms needed for the proofs of modularity results. These include generalizations of results of Ribet (on congruences between classical modular forms of different levels) to the context of Hilbert modular forms as developed by Taylor, Skinner-Wiles and others (using forms on definite quaternion algebras and possibly allowing solvable base-change). I'll also explain refinements of Carayol's Lemma (on congruences

between forms with different character) in this context, including freeness of spaces of forms over certain group rings (as needed for the Taylor-Wiles argument).

### Tuesday 17

**23** : 9 :00 – Fred Diamond, *Congruences 2*

*Abstract* : continuation of **22**.

**24** : 11 :00 – Mark Kisin,  *$R^{red} = T$  theorems 1*

*Abstract* : I will explain the modified Taylor-Wiles method which allows one to prove a modularity lifting theorem whenever one can control the structure of certain local deformation rings.

**25** : 16 :00 – Haruzo Hida, *Hida theory for  $GL(2)$  over a totally real field*

*Abstract* : Let  $p$  be a prime and  $F$  be a totally real field. We describe the structure theory of the nearly-ordinary Hilbert modular  $p$ -adic Hecke algebra for  $F$ . In particular, if we fix a central character (without allowing the character to deform), we prove that the dimension (over  $\mathbb{Z}_p$ ) of the cuspidal part is  $[F : \mathbb{Q}]$  and the Eisenstein part is  $1 + \delta$  for the  $p$ -adic defect  $\delta$  of the Leopoldt conjecture. We may be able to touch the control theorem and the Galois representation into  $GL(2)$  with coefficients in the algebra.

**26** : 18 :00 – Mark Kisin,  *$R^{red} = T$  theorems 2*

*Abstract* : Continuation of **24**

### Wednesday 18

**27** : 9 :00 – Mark Kisin, *Skinner-Wiles*

*Abstract* : I will try to explain how one can think about the Skinner-Wiles theorem on modularity of ordinary representations which are residually reducible.

**28** : 11 :00 – Richard Taylor, *The weight and level in Serre's conjecture*

*Abstract* : I will explain how the weak form of Serre's conjecture implies the strong form. I will also prove some results of a similar nature over some totally real fields. (I will work with the smallest generalisation that will suffice for the purposes of this conference.) Rather than follow the original algebro-geometric arguments (of Mazur, Ribet, Gross, Coleman-Voloch etc) I will use a new method, which has been developed by Toby Gee in order to attack generalisations over totally real fields and which relies on  $R=T$  theorems and the use of base change. However it involves no algebraic geometry. The idea is to find a lifting with the right local behaviour and then prove this is modular. In fact it suffices to show that the lifting is modular after a soluble base change and we choose this base change to simplify the problem.

**F4** : 16 :30 – Vytautas Paskunas, *Towards mod  $p$  local Langlands correspondence for  $GL(2)$*

*Abstract* : This is a joint work with Christophe Breuil. Let  $F$  be a finite unramified extension of  $\mathbb{Q}_p$  and let  $\rho$  be a 2-dimensional continuous “generic”  $\overline{\mathbb{F}}_p$ -representation of  $\text{Gal}(\overline{\mathbb{Q}}_p/F)$ . We associate to  $\rho$  a family of smooth admissible  $\overline{\mathbb{F}}_p$ -representations of  $\text{GL}_2(F)$  such that the  $\text{GL}_2(\mathfrak{o}_F)$ -socle consists of Buzzard-Diamond-Jarvis weights associated to  $\rho$ . A preliminary version of our article is available at : <http://www.ihes.fr/~breuil/PUBLICATIONS/supersingular.pdf>

**29** : 16 :00 – Richard Taylor, *Potential modularity 1 : applications*

*Abstract* : I will state a potential version of Serre’s conjecture which asserts that an odd, irreducible mod  $p$  representation of the Galois group of  $\mathbb{Q}$  becomes modular over some totally real field. I will explain how this theorem can be used to prove that such a representation can be lifted to an element of a compatible system of  $\ell$ -adic representations and moreover how the types (i.e. restriction of the associated Weil-Deligne representations to the inertia group) can be chosen to be anything satisfying some obvious necessary conditions (i.e. that they are unramified almost everywhere and there is no local obstruction).

**Thursday 19**

**30** : 9 :00 – Ren Schoof, *Bounds for discriminants. Applications to abelian varieties and  $p$ -divisible groups*

**31** : 11 :00 –Richard Taylor, *Potential modularity 2 : proofs*

*Abstract* : I will explain the proof of the potential modularity theorem stated in talk 29.

**F6** : 16 :30 – Luis Dieulefait, *How to facet a gemstone : from potential modularity to modularity and other modular excursions*

*Abstract* :Starting from some of the applications of Taylor’s potential modularity result (those due to Khare, Wintenberger and the speaker), I will show that a combination with the results of Skinner-Wiles and those in Schoof’s talk gives a proof of the first few cases of the modularity conjectures of Fontaine-Mazur and Serre. (Remark : similar results were obtained independently by Khare-Wintenberger).

The second part of the talk will consist of a selection of other modularity results that are also based on the potential modularity and modularity lifting techniques. Some possible topics : applications to Tate conjecture in the 4-dimensional case and to the endoscopy conjecture for genus 2 Siegel modular forms; existence of Galois conjugates and applications to weight reduction in the proof of level 1 Serre’s conjecture; results of Kisin and Colmez and an iterated level reduction, etc.

**F7** : 18 :30 – Günter Harder,  *$p$ -adic interpolation of ordinary cohomology*

*Abstract* : see <http://www.math.u-psud.fr/~fontaine/ecole-ete/textes/harder.pdf>

**Friday 20**

**32** : 9 :00 – Jean-Pierre Wintenberger, *Proof of the conjecture 1*

**33** : 11 :00 – Chandrashekhhar Khare, *Proof of the conjecture 2*

**F8** : 14 :10 – Pierre Colmez, *Sur la correspondance de Langlands locale  $p$ -adique pour  $GL_2(\mathbb{Q}_p)$*

**F9** : 15 :30 – Matthew Emerton, *Serre's conjecture as a mod  $p$  Langlands conjecture*

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