Deforming semistable Galois representations

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ABSTRACT Let \( V \) be a \( p \)-adic representation of \( \text{Gal}(\bar{Q}/Q) \). One of the ideas of Wiles’s proof of FLT is that, if \( V \) is the representation associated to a suitable automorphic form (a modular form in his case) and if \( V \) is another \( p \)-adic representation of \( \text{Gal}(\bar{Q}/Q) \) “closed enough” to \( V \), then \( V \) is also associated to an automorphic form. In this paper we discuss which kind of local condition at \( p \) one should require on \( V \) and \( V' \) in order to be able to extend this part of Wiles’s methods.

Geometric Galois Representations (refs. 1 and 2; exp. III and VIII). Let \( Q \) be a chosen algebraic closure of \( Q \) and \( G = \text{Gal}(\bar{Q}/Q) \). For each prime number \( \ell \), we choose an algebraic closure \( Q_\ell \) of \( Q \) together with an embedding of \( Q \) into \( Q_\ell \) and we set \( G_\ell = \text{Gal}(\bar{Q}_\ell/Q) \subset G \). We choose a prime number \( p \) and a finite extension \( E_\ell \) of \( Q_p \).

An \( E \)-representation of a profinite group \( J \) is a finite dimensional \( E \) vector space equipped with a linear and continuous action of \( J \).

An \( E \)-representation \( V \) of \( G \) is said to be geometric if

(i) it is unramified outside of a finite set of primes;

(ii) it is potenially semistable at \( p \) (we will write \( \text{pst} \) for short).

The second condition implies that \( V \) is de Rham, hence Hodge-Tate, and we can define its Hodge-Tate numbers \( h' = h'(V) = \dim_Q (C_p(r)) \otimes_Q V^p \) where \( C_p(r) \) is the usual Tate twist of \( Q \).

Example: \( N \times X \) is a proper and smooth variety over \( Q \) and \( m \in N \), \( j \in Z \), then the \( p \)-adic representation \( H^m(X_\ell, Q(j)) \) is geometric.

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Moreover, we can be true. Loosely speaking, say that a geometric irreducible \( E \)-representation \( V \) of \( G \) is a Hecke representation if there is a finite \( Z \)-algebra \( A \), generated by Hecke operators acting on some automorphic representation space, equipped with a continuous homomorphism \( \rho : G \rightarrow GL_n(A) \), “compatible with the action of the Hecke operators,” such that \( V \) comes from \( A \) (i.e., is isomorphic to the one we get from \( p \) via a map \( A \rightarrow E \)). Then any geometric Hecke representation of \( G \) should come from algebraic geometry and any geometric irreducible representation should be Hecke.

At this moment, this conjecture seems out of reach. Nevertheless, for an irreducible two-dimensional representation of \( G \), to be geometric Hecke means to be a Tate twist of a representation associated to a modular form. Such a representation is known to come from algebraic geometry. Observe that the heart of Wiles’s proof of FLT is a theorem (6, th. 0.2) asserting that, if \( V \) is a suitable geometric Hecke \( E \)-representation of dimension 2, then any geometric \( E \)-representation of \( G \) which is “closed enough” to \( V \) is also Hecke.

It seems clear that Wiles’s method should apply in more general situations to prove that, starting from a suitable Hecke \( E \)-representation of \( G \), any “closed enough” geometric representation is again Hecke. The purpose of these notes is to discuss possible generalizations of the notion of “closed enough” and the possibility of extending local computations in Galois cohomology which are used in Wiles’s theorem. More details should be given elsewhere.

Deformations (7–9). Let \( \bar{Q}_p \) be the ring of integers of \( E, p \) a uniformizing parameter and \( k = \bar{Q}_p/\pi \bar{Q}_p \) the residue field.

Denote by \( c \) the category of local noetherian complete \( \bar{Q}_p \)-algebras with residue field \( k \) (we will simply call the objects of this category \( \bar{Q}_p \)-algebras).

Let \( J \) be a profinite group and \( \text{Rep}_A(J) \) the category of \( Z \)-modules of finite length equipped with a linear and continuous action of \( J \). Consider a strictly full subcategory \( D \) of \( \text{Rep}_A(J) \) stable under subobjects, quotients, and direct sums.

For \( A \in c \), an \( A \)-representation \( TD \) of \( J \) is an \( A \)-module of finite type equipped with a linear and continuous action of \( J \). We say that \( T \) lies in \( D \) if all the finite quotients of \( T \) viewed as \( Z \)-representations of \( J \) are objects of \( D \). The \( A \)-representations of \( J \) lying in \( D \) form a subcategory \( \text{Rep}_D(J) \) of \( \text{Rep}_A(J) \).

We say \( T \) is flat if it is flat (\( \otimes k \) free) as an \( A \)-module.

Fix \( a \) a flat \( (\otimes k) \)-representation of \( J \) lying in \( D \). For any \( A \in c \), let \( F(A) = F_{A,a}(A) \) be the set of isomorphism classes of flat \( A \)-representations \( T \) of \( J \) such that \( T/\pi T = u \). Set \( F_{A,a}(A) = F_{A,a}(A) \) the subset of \( F(A) \) corresponding to representations which lie in \( D \).

Proposition. If \( H^0(J, g_l(u)) \times k \) and \( \dim_k H^1(J, g_l(u)) < +\infty \) then \( F \) and \( F_{A,a} \) are representable.

Closed Enough to \( V \) Representations. We fix a geometric \( E \)-representation \( V \) of \( G \) (morally a “Hecke representation”).
We choose a $G$-stable $\mathcal{O}_E$-lattice $U$ of $V$ and assume $u = U/\pi U$ absolutely irreducible (hence $V$ is a fortiori absolutely irreducible).

We fix also a finite set of primes $S$ containing $p$ and a full subcategory $\mathcal{D}_p$ of $\text{Rep}_{\mathbb{Q}_p}(G_p)$, stable under subobjects, quotients, and direct sums.

For any $E$-representation $W$ of $G_p$, we say $W$ lies in $\mathcal{D}_p$ if it is a $G_p$-stable lattice in $\mathcal{D}_p$.

We say an $E$-representation of $G$ is of type $(S, \mathcal{D}_p)$ if it is unramified outside of $S$ and lies in $\mathcal{D}_p$.

Now we assume $V$ is of type $(S, \mathcal{D}_p)$. We say an $E$-representation $V'$ of $G$ is of type $(S, \mathcal{D}_p)$-close to $V$ if:

(i) given a $G$-stable lattice $U'$ of $V'$, then $U'/\pi U' = u$;

(ii) $V'$ is of type $(S, \mathcal{D}_p)$.

Then, if $Q_p$ denote the maximal Galois extension of $Q$ contained in $Q$ unramified outside of $S$, deformation theory applies.

The category $\text{Rep}_{\mathbb{Q}_p}(G_p)$ whose objects are $\mathcal{O}_E$-modules with a $G_p$-action equipped with a linear and continuous action of $G_p$. We get in this way an $A$-linear functor

$$U : MF^{[-p-1,0]}(A) \rightarrow \text{Rep}_{\mathbb{Q}_p}(G_p)$$

which is exact and faithful. Moreover, the restriction of $U$ to $MF^{[-p-1,0]}(A)$ is fully faithful. We call $\mathcal{D}_p \subset (E)$ the essential image.

Proposition. Let $V'$ be an $E$-representation of $G_p$. Then $V'$ lies in $\mathcal{D}_p$ if and only if the three following conditions are satisfied:

(i) $V'$ is crystalline (i.e., $V'$ is $p$-stabilized with conductor $N_V(p) = 1$);

(ii) $h(V') = 0$ if $r > 0$ or $r < -p + 1$;

(iii) $V'$ has no nonzero subobject $V''$ with $V''(-p + 1)$ unramified.

Moreover (11), if $X$ is a proper and smooth variety over $Q_p$, with good reduction and if for each $a \in N$ with $0 \leq a \leq p - 2$, $H^0_p(X, \mathcal{O}_E \otimes \mathcal{O}_U)$ is an object of $\mathcal{D}_p(E)$.

Remarks:

(i) Define $\mathcal{D}_p$ as the full subcategory of $\text{Rep}_{\mathbb{Q}_p}(G_p)$, whose representations which are isomorphic to the general fiber of a finite and flat group scheme over $\mathcal{O}_p$. If $p \neq 2$, $\mathcal{D}_p$ is a full subcategory stable under extensions of $\mathcal{D}_p$ (this is the essential image of $MF^{[-1,0]}(Q_p)$).

(ii) Deformations in $\mathcal{D}_p(E)$ don’t change Hodge type: if $V'/V''$ are $E$-representations of $G_p$, lying in $\mathcal{D}_p$ and if one can find lattices $U$ of $V$ and $U''$ of $V''$ such that $U/\pi U'' = U''/\pi U''$, then $h(V') = h(V'')$ for all $r \in Z$ (if $U/\pi U'' = U''/\pi U''$).

Computation of $H^0_p$. This can be translated in terms of the category $MF_{c,E}(G_p) : = MF^{[-p-1,0]}(G_p)$.

In $MF_{c,E}(G_p)$, define $H^0_p(Q_p, M)$ as being the $i^0$ derived functor of the functor $\text{Hom}_{MF_{c,E}}(G_p, -)$. These groups are the cohomology of the complex

$$\text{Fil}^0 M \rightarrow \text{Fil}^{i+1} M \rightarrow 0 \rightarrow \ldots$$

If we set $i_M = MF/\text{Fil}^i M$, this implies $i_M \subset i_M^{(i)}$. Hence, if $U$ is a $G_p$-stable lattice of an $E$-representation $V$ of $G_p$, and if, for any $i \in \mathbb{Z}$, $h_i = h_i(V)$, with obvious notations, we get $H^0_p(Q_p, \mathcal{O}_E) = \text{Fil}^{i-1} H^0_p(Q_p, \mathcal{O}_E)$. Moreover, there is $h^0_p$ deformation theory problem is smooth, hence $R_{U,G_p}^\infty (\mathcal{O}_E) = \text{Fil}^{i-1} H^0_p(Q_p, \mathcal{O}_E) = \text{Fil}^{i-1} H^0_p(Q_p, \mathcal{O}_E)(U) = (\mathcal{O}_E)^{i-1}$.

A Special Case. Of special interest is the case where $H^0_p(Q_p, \mathcal{O}_E)(U) = k$, which is equivalent to the representability of the functor $F_{U,G_p}^\infty$. In this case, $H^0_p(Q_p, \mathcal{O}_E)(U) = (\mathcal{O}_E)^{i-1}$ and $H^0_p(Q_p, \mathcal{O}_E)(U) = (\mathcal{O}_E)^i$. Moreover, because there is no $H^2$, the deformation problem is smooth, hence $R_{U,G_p}^\infty \mathcal{O}_E = \mathcal{O}_E[[X_0, X_1, X_2, \ldots, X_n]]$.

Example 2: $\mathcal{O}_E$ (the naive generalization of $\mathcal{D}_p(E)$ to the semistable case).

For any $\mathcal{O}_E$-algebra $A$, we can define the category $MF(A)$ whose objects consist of a pair $(M, N)$ with $M$ object of $MF(A)$ and $N : M \rightarrow M$ such that

(i) $N(M) \subset \text{Fil}^{i-1} M$,

(ii) $N^r = N^r \mathcal{O}_E$.

With an obvious definition of the morphisms, this is an abelian $A$-linear category and $MF(A)$ can be identified to the full subcategory of $MF(A)$ consisting of $M$ with $N = 0$.

We have an obvious definition of the category $MF^{[-p-1,0]}(A)$, there is a natural way to extend $U$ to a functor

$$U : MF^{[-p-1,0]}(A) \rightarrow \text{Rep}_{\mathbb{Q}_p}(G_p)$$

again exact and fully faithful. We call $\mathcal{D}_p(E)$ the essential image.

There is again a simple characterization of the category $\mathcal{D}_p(E)$ of $E$-representations of $G_p$ lying in $\mathcal{D}_p(E)$ as a suitable full
subcategory of the category of semistable representations with crystalline semisimplification. Moreover:

If \( p \neq 2 \), the category of semistable \( V \) values with \( h^0(V) = 0 \) if \( r \neq 0 \) is a full subcategory stable under extensions of \( \mathbb{V}_p^t(E) \). For \( 0 \leq r < p - 1 \), let \( \mathcal{D}_p^{ord,t} \) the full subcategory of \( \text{Rep}_{/F}(G_p) \) of \( T \) such that there is a filtration (necessarily unique)

\[
0 = F_{-1}T \subset F_0T \subset \cdots \subset F_rT \subset F_0T = T
\]

such that \( gr_iT(-i) \) is unramified for all \( i \); then \( \mathcal{D}_p^{ord,t} \) is a full subcategory of \( \mathbb{V}_p^t \) stable under extensions.

Again, in \( \mathbb{V}_p^t \), deformations don't change Hodge type. The conductor may change.

Computation of \( H_{\text{st}}^2(G_p, \mathbb{Q}(U)) \). As before, this can be translated in terms of the category \( MFN^t(E) \subset MFN^{t,p,i}0(E) \): if we define \( H_{MFN}(Q_{pM}, M) \) as the \( i \)-derived functor, in the category \( MFN^t(E) \), of the functor \( \text{Hom}_{MFN^t}(I(E), -) \), these groups are the cohomology of the complex

\[
\text{Fil}^rMF \to \text{Fil}^{-r}MF \oplus M \to M \to 0 \to \cdots
\]

(with \( x \mapsto (N(x), 1 - \phi^O)x \) and \( y, z \mapsto (1 - \phi^{-1})y - Nz \)). Again, in this case, \( H_{MFN}(Q_{pM}, M) = H_{MFN}(Q_{pM}, \text{End}_{/F}(E)) \). But, (i) the formula for the length is more complicated, and (ii) the (local) deformation problem is not always smooth.

**Example 3:** \( \mathbb{V}_p^t \) [the good generalization of \( \mathbb{V}_p^t \) to the semistable case, theory due to Breuil (12)].

Let \( S = \mathbb{Z}_{<1} \) be the divided power polynomial ring in one variable \( u \) with coefficients in \( \mathbb{Z}_p \). If \( v = u - p \), we have also \( S = \mathbb{Z}_{<1} \). Define:

(a) \( \text{Fil}^0S \) as the ideal of \( S \) generated by the \( v^m/m! \), for \( m \geq i \);

(b) \( \phi \) as the unique \( \mathbb{Z}_p \)-endomorphism such that \( \phi(u) = u^p \);

(c) \( N \) as the unique \( \mathbb{Z}_p \)-derivation from \( S \) to \( S \) such that \( N(u) = -u \).

For \( r \leq p - 1 \), \( \phi^r \text{Fil}^rS \to \text{Fil}^rS \) is defined by \( \phi(x) = p^{-r}\phi(x) \).

(ii) the (local) deformation problem is not always smooth.

Moreover, the simple objects of \( MF^{t-r|0}(k) \), \( MFN^{t-r|0}(k) \), and \( MF^{t-r|0}(k) \) are the same. Breuil extends \( U \) to \( MF^{t-r|0}(A) \) and proves that this functor is again exact and fully faithful. We call \( \mathcal{D}_p^t(A) \) the essential image.

Let \( V \) be an \( E \)-representation of \( G_p \). Breuil proves that, if \( V \) lies in \( \mathcal{D}_p^t \) then \( V \) is semistable and \( h^0(V) = 0 \) if \( m > 0 \) or \( m < -r \).

Conversely, it seems likely that if \( V \) satisfies these two conditions, \( V \) lies in \( \mathcal{D}_p^t \). This is true if \( r = 1 \), and it has been proven by Breuil if \( E = \mathbb{Q} \). And \( V \) is of dimension 2. More importantly, Breuil proved also

**(Proposition 13).** Let \( X \) be a proper and smooth variety over \( Q_p \). Assume \( X \) as semistable reduction and let \( r \in \mathbb{N} \) with \( 0 \leq r \leq p - 2 \); then \( H^r_{/\text{et}}(X_\ast, \mathbb{Z}/p^2) \) is an object of \( \mathbb{V}_p^t(\mathcal{Z}_p) \).

When working with \( \mathbb{V}_p^t \) deformation may change the Hodge type (the conductor also). The computation of \( H_{MFN}(Q_{pM}, \mathbb{Q}(U)) \) still reduces to a computation in \( MF^{t-r|0}(E) \) (or equivalently in \( \mathcal{M}_t \)). This computation becomes difficult in general but can be done in specific examples.

**Final Results.** Let \( L \) be a finite Galois extension of \( Q_p \) contained in \( Q_{pM} \). The ring of integers and \( e_L = e/L_\ast \).

(a) \( \text{MF}_p^t \), the full subcategory of \( \text{Rep}_{/F}(G_p) \) whose objects are representations which, when restricted to \( Gal(Q_p/Q_\ell) \), extends to a finite and flat group scheme over \( Q_\ell \). Then \( e_L \leq p - 1 \), an \( E \)-representation \( V \) lies in \( \mathcal{D}_p^t \) if and only if it becomes crystalline over \( L \) and \( h^0(V) = 0 \) for \( m \neq (0, -1) \). If \( e_L < p - 1 \), Conrad (14) defines an equivalence between \( \mathbb{V}_p^t \) and a nice category of filtered modules equipped with a Frobenius and an action of \( Gal(L/\mathbb{Q}) \). Using it, one can get the same kind of results as we described for \( \mathbb{V}_p^t \). For \( e_L = p - 1 \), the same thing holds if we require that the representation of \( Gal(Q_p/Q_\ell) \) extends to a connected finite and flat group scheme over \( Q_\ell \).

(b) More generally, Breuil's construction should extend to \( E \)-representations becoming semistable over \( L \) with \( h^0(V) = 0 \) if \( m > 0 \) or \( v(p - 1)e_L \leq -v(p - 1)e_L \) with a "grain de sel".

(c) Let \( \text{Rep}_{/F}(G_p) \) (resp. \( \text{Rep}_{/F}(G_p)_{/\text{et}} \)) be the category of \( E \)-representations of \( V \) of \( G_p \) becoming crystalline over \( L \) (resp. semistable) with \( h^0(V) = 0 \) if \( m > 0 \) or \( m < -r \). Let \( \mathcal{D}_p^{t,r|L} \) (resp. \( \mathcal{D}_p^{t|L} \)) be the full subcategory of \( \text{Rep}_{/F}(G_p) \) consisting of \( T \) \( p \)-stable, \( \mathcal{D}_p^{t,r|L}_G \) \( p \)-stably lines \( U \subset U \) of \( V \) such that \( U = U/U' \). I feel unhappy not being able to prove the following:

**Conjecture.** \( C_p^{t,r|L} \) (resp. \( C_p^{t|L} \)). Let \( V \) be a \( Q_p \)-representation of \( V \) lying in \( \mathcal{D}_p^{t,r|L} \) (resp. \( \mathcal{D}_p^{t|L} \)). Then \( V \) an object of \( \text{Rep}_{/F}(G_p)_{/\text{et}} \) (resp. \( \text{Rep}_{/F}(G_p)_{/\text{et}} \)).

The only cases I know \( C_p^{t,r|L} \) are \( r = 0, r = 1 \), and \( e_L \leq p - 1 \). The only cases I know \( C_p^{t|L} \) are \( r = 0, r = 1 \), and \( e_L \leq p - 1 \). Of course, each time we know the answer is yes, this implies that the category is semistable.

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