

Nesterenko's criterion when the small linear forms oscillate

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Abstract. In this paper we generalize Nesterenko's criterion to the case where the small linear forms have an oscillating behaviour (for instance given by the saddle point method). This criterion provides both a lower bound for the dimension of the vector space spanned over the rationals by a family of real numbers and a measure of simultaneous approximation to these numbers (namely, an upper bound for the irrationality exponent if 1 and only one other number are involved). As an application, we prove an explicit measure of simultaneous approximation to $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$, using Zudilin's proof that at least one of these numbers is irrational.

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1. Introduction. To investigate Diophantine properties of real numbers ξ_1, \dots, ξ_r , a strategy is to construct small linear forms in $1, \xi_1, \dots, \xi_r$ with integer coefficients. This is the only known way to study this problem if ξ_1, \dots, ξ_r are, for instance, values of Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ at odd integers $s \geq 3$.

In more precise terms, linear forms $\ell_{0,n} + \ell_{1,n}\xi_1 + \dots + \ell_{r,n}\xi_r$ are constructed, with absolute value $\leq \alpha^{n+o(n)}$ as $n \rightarrow \infty$ and coefficients $\ell_{j,n} \in \mathbb{Z}$ such that $|\ell_{j,n}| \leq \beta^{n+o(n)}$ (where $0 < \alpha < 1 < \beta$). Then:

- If $\ell_{0,n} + \ell_{1,n}\xi_1 + \dots + \ell_{r,n}\xi_r \neq 0$ for infinitely many n , the subgroup $\mathbb{Z} + \mathbb{Z}\xi_1 + \dots + \mathbb{Z}\xi_r$ of \mathbb{R} is not discrete so that at least one number among ξ_1, \dots, ξ_r is irrational.
- If $\alpha\beta < 1$ and $\ell_{0,n} + \ell_{1,n}\xi_1 + \dots + \ell_{r,n}\xi_r \neq 0$ for infinitely many n , the subspace spanned over \mathbb{Q} by $1, \xi_1, \dots, \xi_r$ has dimension at least 3, so that $1, \xi_i$ and ξ_j are \mathbb{Q} -linearly independent for some i, j with $1 \leq i < j \leq r$.

An elementary proof of this result can be found in [10, Proposition 1, Section 2.2].

- If the linear forms are not too small, namely $\geq \alpha^{n+o(n)}$, a result of Nesterenko [12] implies the following two properties (see the remark after Theorem 1 below for a precise formulation):
 - (i) A lower bound $1 - \frac{\log \alpha}{\log \beta}$ on the dimension of the vector space spanned over the rationals by $1, \xi_1, \dots, \xi_r$. This linear independence criterion is one of the main tools in the proof [3, 13] that $\zeta(s) \notin \mathbb{Q}$ for infinitely many odd integers $s \geq 3$.
 - (ii) A measure of simultaneous approximation to ξ_1, \dots, ξ_r ; if $r = 1$, this is an upper bound $\mu(\xi_1) \leq 1 - \frac{\log \beta}{\log \alpha}$ on the irrationality exponent of ξ_1 . For instance, Apéry proved at the same time [2] that $\zeta(3) \notin \mathbb{Q}$ and $\mu(\zeta(3)) \leq 13.41\dots$

The assumption that the linear forms are not too small is very important here, and it cannot be omitted. It can be weakened: Nesterenko proves such results for linear forms with absolute values between $\alpha_1^{n+o(n)}$ and $\alpha^{n+o(n)}$, where $0 < \alpha_1 < \alpha < 1$, but the conclusion is then weaker too.

In this paper we generalize Nesterenko’s results (i) and (ii) (without weakening the conclusion) to the case where the linear forms behave essentially like $\alpha^{n+o(n)} \cos(n\omega + \varphi)$ with $\omega, \varphi \in \mathbb{R}$. This is an interesting situation because the saddle point method is often applied to obtain asymptotic estimates for the linear forms, and it typically produces this kind of behaviour. Our main result is the following.

Theorem 1. *Let $r \geq 1, \xi_1, \dots, \xi_r \in \mathbb{R}, \alpha, \beta, \omega, \varphi \in \mathbb{R}$. Assume that $0 < \alpha < 1, \beta > 1$, and either $\omega \not\equiv 0 \pmod{\pi}$ or $\varphi \not\equiv \frac{\pi}{2} \pmod{\pi}$. For any $n \geq 1$, let $\ell_{0,n}, \dots, \ell_{r,n} \in \mathbb{Z}$ be such that, as $n \rightarrow \infty$:*

$$\max_{0 \leq i \leq r} |\ell_{i,n}| \leq \beta^{n+o(n)}$$

and

$$|\ell_{0,n} + \ell_{1,n}\xi_1 + \dots + \ell_{r,n}\xi_r| = \alpha^{n+o(n)} (|\cos(n\omega + \varphi)| + o(1)). \tag{1}$$

Then:

- (i) We have $\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \xi_1, \dots, \xi_r) \geq 1 - \frac{\log \alpha}{\log \beta}$.
- (ii) For any $\kappa > 1 - \frac{\log \beta}{\log \alpha}$ and any $q, p_1, \dots, p_r \in \mathbb{Z}$ with $q > 0$ sufficiently large in terms of κ , we have

$$\max \left(\left| \xi_1 - \frac{p_1}{q} \right|, \dots, \left| \xi_r - \frac{p_r}{q} \right| \right) \geq \frac{1}{q^\kappa}.$$

When $(\omega, \varphi) = (0, 0)$ these are exactly Nesterenko’s above-mentioned results [12]. If $e^{i\omega}$ and $e^{i\varphi}$ are algebraic numbers, a very concise remark of Sorokin [15] (which we expand in Section 2.2) provides another proof of Theorem 1, based upon lower bounds for linear forms in logarithms.

As the proof shows, the cosine may be replaced in Theorem 1 (and also in Proposition 1 below) with any continuous periodic function, which vanishes only at finitely many points within each period.

Theorem 1 can be used in the following situation (see Section 3.1 for other possible applications). Zudilin has proved in [17] that among $\zeta(5), \zeta(7), \zeta(9)$, and $\zeta(11)$, at least one is irrational (refining upon Rivoal’s result [14]). He proceeds by constructing small linear forms in 1 and these numbers. He applies the saddle point method to prove the estimate (1) (see Section 2.3 below), and deduce that the linear form is non-zero for infinitely many n , thereby proving the irrationality of at least one number among $\zeta(5), \zeta(7), \zeta(9)$, and $\zeta(11)$. Using Theorem 1 we obtain a quantitative version of this result:

Theorem 2. *For any $q, p_5, p_7, p_9, p_{11} \in \mathbb{Z}$ with $q > 0$ sufficiently large we have:*

$$\max \left(\left| \zeta(5) - \frac{p_5}{q} \right|, \left| \zeta(7) - \frac{p_7}{q} \right|, \left| \zeta(9) - \frac{p_9}{q} \right|, \left| \zeta(11) - \frac{p_{11}}{q} \right| \right) \geq \frac{1}{q^{438.23}}.$$

In particular, in the (very unlikely) case where $\zeta(5), \zeta(7)$ and $\zeta(9)$ would be rational numbers, this implies $\mu(\zeta(11)) \leq 438.23$.

The proof of Theorem 1 (see Section 2.2) relies on applying Nesterenko’s results to a subsequence $\ell_{0,\psi(n)} + \ell_{1,\psi(n)}\xi_1 + \dots + \ell_{r,\psi(n)}\xi_r$, where ψ is given by the following proposition (with $\mathbb{N} = 1$).

Proposition 1. *Let $N \geq 1, \omega_1, \dots, \omega_N, \varphi_1, \dots, \varphi_N \in \mathbb{R}$. Assume there exist infinitely many integers n such that, for any $i \in \{1, \dots, N\}, n\omega_i + \varphi_i \not\equiv \frac{\pi}{2} \pmod{\pi}$. Then there exist $\varepsilon, \lambda > 0$ and an increasing function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \frac{\psi(n)}{n} = \lambda$ and, for any $n \in \mathbb{N}$ and any $i \in \{1, \dots, N\}, |\cos(\psi(n)\omega_i + \varphi_i)| \geq \varepsilon$.*

This proposition is a consequence of Kronecker–Weyl’s equidistribution theorem (see Section 2.1). The assumption that for infinitely many $n, n\omega_i + \varphi_i \not\equiv \frac{\pi}{2} \pmod{\pi}$ for any i is of course necessary because if ψ exists, all $n \in \psi(\mathbb{N})$ have this property. If $N = 1$, it is equivalent to $(\omega_1, \varphi_1) \not\equiv (0 \pmod{\pi}, \frac{\pi}{2} \pmod{\pi})$. For $N = 2$, it is equivalent to:

$$\begin{cases} (\omega_1, \varphi_1) \not\equiv (0 \pmod{\pi}, \frac{\pi}{2} \pmod{\pi}) \\ (\omega_2, \varphi_2) \not\equiv (0 \pmod{\pi}, \frac{\pi}{2} \pmod{\pi}) \\ (\omega_1, \omega_2, \varphi_1, \varphi_2) \not\equiv (\frac{\pi}{2} \pmod{\pi}, \frac{\pi}{2} \pmod{\pi}, \frac{\pi}{2} \pmod{\pi}, 0 \pmod{\pi}) \\ (\omega_1, \omega_2, \varphi_1, \varphi_2) \not\equiv (\frac{\pi}{2} \pmod{\pi}, \frac{\pi}{2} \pmod{\pi}, 0 \pmod{\pi}, \frac{\pi}{2} \pmod{\pi}) \end{cases} \tag{2}$$

In all Diophantine applications we have in view (see Section 3.2), N is fixed (and even $N = 1$ for the above-mentioned results), so that this assumption is not too difficult to check. It should be pointed out that in Proposition 1, no Diophantine condition is assumed on ω_i or φ_i : for instance, we don’t assume $\pi, \omega_1, \dots, \omega_N$ to be \mathbb{Q} -linearly independent. This is very useful because such a condition could be very difficult to check (whereas approximate values of these numbers are often easily computed, making it very easy to check conditions such as (2)).

2. Proofs.

2.1. Proof of Proposition 1. Let $p : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}, x \mapsto x \pmod{\mathbb{Z}}$ denote the canonical surjection. For $x, y \in \mathbb{R}$ with $x < y$, we denote by $[x, y] \subset \mathbb{R}/\mathbb{Z}$ the image

under p of the segment between x and y , that is the set of all $p(z)$ with $x \leq z \leq y$. Its normalized Haar measure is $\min(y - x, 1)$.

Let s be a positive integer. For $x_1, \dots, x_s, y_1, \dots, y_s \in \mathbb{R}$ such that $x_i < y_i$, let $K = \prod_{i=1}^s [x_i, y_i] \subset (\mathbb{R}/\mathbb{Z})^s$. Given $\vartheta_1, \dots, \vartheta_s \in \mathbb{R}$, we denote by \mathcal{N} the set of all $n \in \mathbb{N}$ such that $(n\vartheta_1 \bmod \mathbb{Z}, \dots, n\vartheta_s \bmod \mathbb{Z}) \in K$. Then Kronecker–Weyl’s equidistribution theorem (see [11, Chapter 1.6]) asserts that, if $1, \vartheta_1, \dots, \vartheta_s$ are \mathbb{Q} -linearly independent:

$$\lim_{k \rightarrow \infty} \frac{|\{n \in \mathcal{N}, n \leq k\}|}{k} = \prod_{i=1}^s \min(y_i - x_i, 1).$$

Denoting by $\psi_0(n)$ the n -th element of \mathcal{N} in increasing order (so that $\psi_0 : \mathbb{N} \rightarrow \mathcal{N}$ is an increasing bijection), this implies

$$\lim_{n \rightarrow \infty} \frac{\psi_0(n)}{n} = \prod_{i=1}^s \max((y_i - x_i)^{-1}, 1) \tag{3}$$

by letting $k = \psi_0(n)$ and taking reciprocals. Equation (3) will be the main tool in the proof of Proposition 1.

To illustrate the ideas in a simpler case, let us prove Proposition 1 first when $N = 1$. If ω_1/π is a rational number c/d , then it is enough to choose $\psi(n) = nd + a$ for a suitable $a \in \{1, \dots, d\}$. If ω_1/π is irrational, let $K = [-\frac{\varphi_1}{\pi} - \frac{1}{4}, -\frac{\varphi_1}{\pi} + \frac{1}{4}] \subset \mathbb{R}/\mathbb{Z}$ and \mathcal{N} be the set of all $n \in \mathbb{N}$ such that $\frac{n\omega_1}{\pi} \bmod \mathbb{Z} \in K$. Denoting by $\psi_0(n)$ the n -th element of \mathcal{N} , Eq. (3) yields $\lim_{n \rightarrow \infty} \frac{\psi_0(n)}{n} = 2$. Moreover for any n we have

$$|\cos(\psi_0(n)\omega_1 + \varphi_1)| = \left| \cos \left(\pi \left(\psi_0(n) \frac{\omega_1}{\pi} + \frac{\varphi_1}{\pi} \right) \right) \right| \geq \frac{\sqrt{2}}{2}.$$

This concludes the proof when $N = 1$.

Let us come now to the proof of Proposition 1 for any N , starting with a special case: we assume that $\frac{\omega_1}{\pi}, \dots, \frac{\omega_N}{\pi}$ are irrational numbers. Let E denote the \mathbb{Q} -subspace of \mathbb{R} spanned by $1, \frac{\omega_1}{\pi}, \dots, \frac{\omega_N}{\pi}$. Since $\frac{\omega_1}{\pi} \notin \mathbb{Q}$, we have $\dim_{\mathbb{Q}} E \geq 2$; let $(1, \vartheta_1, \dots, \vartheta_s)$ denote a basis of E , with $s \geq 1$. For any $i \in \{1, \dots, N\}$, we have $\frac{\omega_i}{\pi} = \sum_{j=0}^s r_{i,j} \vartheta_j$ with $r_{i,j} \in \mathbb{Q}$, where we let $\vartheta_0 = 1$. Let D be a positive integer such that $Dr_{i,j} \in \mathbb{Z}$ for any i, j . Then we have

$$|\cos(Dn\omega_i + \varphi_i)| = \left| \cos \left(\pi \left(\sum_{j=1}^s Dr_{i,j} n\vartheta_j + \frac{\varphi_i}{\pi} \right) \right) \right| \tag{4}$$

because $Dr_{i,0}n\vartheta_0 \in \mathbb{Z}$.

For any $i \in \{1, \dots, N\}$, let $\Delta_i \subset (\mathbb{R}/\mathbb{Z})^s$ be the set of all $(\sigma_1, \dots, \sigma_s)$ such that $(\sum_{j=1}^s Dr_{i,j}\sigma_j) + \frac{\varphi_i}{\pi} - \frac{1}{2} \in \mathbb{Z}$. Since $(Dr_{i,1}, \dots, Dr_{i,s}) \in \mathbb{Z}^s \setminus \{(0, \dots, 0)\}$ because $\frac{\omega_i}{\pi} \notin \mathbb{Q}$, Δ_i is a finite union of translated tori of dimension $s - 1$ and a proper compact subset of $(\mathbb{R}/\mathbb{Z})^s$. There exists a point $(z_1, \dots, z_s) \in \mathbb{R}^s$ and a (small) positive real number η such that $K = \prod_{i=1}^s [z_i - \eta, z_i + \eta] \subset (\mathbb{R}/\mathbb{Z})^s$ is disjoint from $\Delta_1 \cup \dots \cup \Delta_N$. Let \mathcal{N} be the set of all $n \in \mathbb{N}$ such that $(n\vartheta_1 \bmod \mathbb{Z}, \dots, n\vartheta_s \bmod \mathbb{Z}) \in K$, and $\psi_0(n)$ denote the n -th element of \mathcal{N}

(in increasing order). Since $1, \vartheta_1, \dots, \vartheta_s$ are \mathbb{Q} -linearly independent, Kronecker–Weyl’s equidistribution theorem (3) yields

$$\lim_{n \rightarrow \infty} \frac{\psi_0(n)}{n} = (2\eta)^{-s} > 0.$$

Moreover, since $K, \Delta_1, \dots, \Delta_N$ are compact subsets there exists $\eta' > 0$ such that, for any $n \in \mathbb{N}$ and any $i \in \{1, \dots, N\}$, $\|(\sum_{j=1}^s Dr_{i,j}\psi_0(n)\vartheta_j) + \frac{\varphi_i}{\pi} - \frac{1}{2}\|_{\mathbb{Z}} \geq \eta'$ (where $\|x\|_{\mathbb{Z}}$ is the distance of $x \in \mathbb{R}$ to \mathbb{Z}). Using Eq. (4), this provides $\varepsilon > 0$ such that $|\cos(D\psi_0(n)\omega_i + \varphi_i)| \geq \varepsilon$, and by letting $\psi(n) = D\psi_0(n)$ this concludes the proof of Proposition 1 if $\frac{\omega_1}{\pi}, \dots, \frac{\omega_N}{\pi} \notin \mathbb{Q}$.

Let us deduce the general case from this special case. Reordering the pairs (ω_i, φ_i) if necessary, we may assume that for some $N' \in \{0, \dots, N\}$ we have $\frac{\omega_1}{\pi}, \dots, \frac{\omega_{N'}}{\pi} \notin \mathbb{Q}$ and $\frac{\omega_{N'+1}}{\pi}, \dots, \frac{\omega_N}{\pi} \in \mathbb{Q}$. Let $d \geq 1$ be a common denominator of $\frac{\omega_{N'+1}}{\pi}, \dots, \frac{\omega_N}{\pi}$ and for any i , let \mathcal{E}_i be the set of all $k \in \mathbb{N}$ such that $k\omega_i + \varphi_i \equiv \frac{\pi}{2} \pmod{\pi}$. Then \mathcal{E}_i has at most one element for $i \leq N'$, and \mathcal{E}_i is a union of residue classes mod d for $i > N'$. By assumption $\mathbb{N} \setminus (\mathcal{E}_1 \cup \dots \cup \mathcal{E}_N)$ is infinite, so that there exists $a \in \mathbb{N}$ such that for any k sufficiently large with $k \equiv a \pmod{d}$, we have $k \notin \mathcal{E}_1 \cup \dots \cup \mathcal{E}_N$. For any $i > N'$, the number $|\cos((nd + a)\omega_i + \varphi_i)|$ is positive and independent from n (since $d\omega_i \in \pi\mathbb{Z}$). If $N' = 0$ this concludes the proof by letting $\psi(n) = nd + a$. Otherwise we apply the special case of Proposition 1 proved above to $\cos((nd + a)\omega_i + \varphi_i) = \cos(nd\omega_i + a\omega_i + \varphi_i)$ for $1 \leq i \leq N'$, that is with $\omega'_1 = d\omega_1, \dots, \omega'_{N'} = d\omega_{N'}, \varphi'_1 = a\omega_1 + \varphi_1, \dots, \varphi'_{N'} = a\omega_{N'} + \varphi_{N'}$. We obtain in this way an increasing function ψ_0 , and letting $\psi(n) = \psi_0(n)d + a$ concludes the proof of Proposition 1.

Remark 1. If $\frac{\omega_1}{\pi}, \dots, \frac{\omega_N}{\pi}$ are irrational numbers, applying Kronecker–Weyl’s theorem with more general subsets K enables one to obtain ψ_0 such that $\lim_{n \rightarrow \infty} \frac{\psi_0(n)}{n}$ is arbitrarily close to 1 (because this is the inverse of the measure of K). This leads to a control upon $\lambda = \lim_{n \rightarrow \infty} \frac{\psi(n)}{n}$ in terms of the common denominator D in this case. If $1, \frac{\omega_1}{\pi}, \dots, \frac{\omega_N}{\pi}$ are \mathbb{Q} -linearly independent then we can take $D = 1$, so that λ can be chosen arbitrarily close to 1. However we did not try to go any further in this direction (nor to get a lower bound for ε) because this is completely useless for the applications we have in view.

2.2. Proof of Theorem 1. Since $\omega \not\equiv 0 \pmod{\pi}$ or $\varphi \not\equiv \frac{\pi}{2} \pmod{\pi}$, there are infinitely many integers n such that $n\omega + \varphi \not\equiv \frac{\pi}{2} \pmod{\pi}$. Applying Proposition 1 (with $N = 1$) yields $\varepsilon, \lambda > 0$ and an increasing function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\psi(n) = \lambda n + o(n)$ as $n \rightarrow \infty$ and $|\cos(\psi(n)\omega + \varphi)| \geq \varepsilon$ for any n . Therefore Eq. (1) yields

$$|\ell_{0,\psi(n)} + \ell_{1,\psi(n)}\xi_1 + \dots + \ell_{r,\psi(n)}\xi_r| = \alpha^{\psi(n)(1+o(1))} = \alpha^{\lambda n + o(n)}$$

and we have also $\max_{0 \leq i \leq r} |\ell_{i,\psi(n)}| \leq \beta^{\lambda n + o(n)}$. Therefore Nesterenko’s results [12] (that is, the special case of Theorem 1 where $\omega = \varphi = 0$) apply to the sequence $\ell_{0,\psi(n)} + \ell_{1,\psi(n)}\xi_1 + \dots + \ell_{r,\psi(n)}\xi_r$, with α^λ and β^λ instead of α and β . This provides exactly the same conclusions (i) and (ii) because $\frac{\log \alpha^\lambda}{\log \beta^\lambda} = \frac{\log \alpha}{\log \beta}$.

Remark 2. Another proof of Theorem 1 under the additional assumption that $e^{i\omega}$ and $e^{i\varphi}$ are algebraic numbers is provided by the following remark of Sorokin [15, p. 823]:

$$\lim_{\substack{n \rightarrow \infty \\ n \in S}} |\cos(n\omega + \varphi)|^{1/n} = 1,$$

in this case, if $\omega \not\equiv 0 \pmod{\pi}$ or $\varphi \not\equiv \frac{\pi}{2} \pmod{\pi}$, where S is the set of all n such that $\cos(n\omega + \varphi) \neq 0$. Indeed, it follows from Gel'fond's lower bound for linear forms in logarithms (see [5, Theorem 4.1, p. 179]) that for any $\varepsilon > 0$ and any k, n sufficiently large in terms of ε , we have

$$|n\omega + \varphi - \frac{\pi}{2} - k\pi| \geq (1 - \varepsilon)^n$$

provided the left hand-side is non-zero.

2.3. Simultaneous approximation to zeta values. This subsection is devoted to a proof of Theorem 2. We use Zudilin's linear forms ([17]; see also [19] for further details), which can be written as

$$S_n = \frac{1}{2} \frac{\prod_{j=1}^{10} ((13 + 2j)n)!}{(27n)!^6} \sum_{k=1}^{\infty} \frac{d^2}{dt^2} \left((37n + 2t) \frac{(t-27n)_{27n}^3 (t+37n+1)_{27n}^3}{\prod_{j=1}^{10} (t+(12-j)n)_{(13+2j)n+1}} \right) \Big|_{t=k}, \tag{5}$$

where the second derivative is taken at $t = k$ and $(\alpha)_p = \alpha(\alpha+1) \cdots (\alpha+p-1)$ is Pochhammer's symbol. This sum can be written as a linear form

$$S_n = \ell_{0,n} + \ell_{5,n}\zeta(5) + \ell_{7,n}\zeta(7) + \ell_{9,n}\zeta(9) + \ell_{11,n}\zeta(11)$$

with rational coefficients $\ell_{i,n}$. Zudilin deduces from the saddle point method that

$$S_n = e^{-C_0 n + o(n)} |\cos(n\omega + \varphi) + o(1)|, \tag{6}$$

with $\omega \not\equiv 0 \pmod{\pi}$ and $C_0 = 227.58019641 \dots$,

and constructs a common denominator D_n of the rational numbers $\ell_{0,n}, \ell_{5,n}, \ell_{7,n}, \ell_{9,n}$, and $\ell_{11,n}$, such that

$$D_n = e^{C_1 n + o(n)} \text{ with } C_1 = 226.24944266 \dots$$

Then $D_n S_n$ is a linear form in $1, \zeta(5), \dots, \zeta(11)$, with integer coefficients; as $n \rightarrow \infty$, it tends to 0 because $\alpha = e^{C_1 - C_0} < 1$, and is non-zero for infinitely many n thanks to (6). This proves that among the four numbers $\zeta(5), \dots, \zeta(11)$, at least one is irrational.

To prove Theorem 2, we also need the following upper bound on the coefficients $\ell_{j,n}$, which can be proved along the same lines as Lemma 4 of [3]:

$$\max_{j \in \{0,5,7,9,11\}} |\ell_{j,n}| \leq 2^{513n + o(n)}, \text{ since } 3(27 + 37 + 27) + \sum_{j=1}^{10} (13 + 2j) = 513.$$

This allows us to apply Theorem 1 with $\beta = e^{C_1} 2^{513}$, so that $1 - \frac{\log \beta}{\log \alpha} = 438.22134 \dots$. This concludes the proof of Theorem 2.

3. Other possible applications.

3.1. Applications of the linear independence criterion. For any $s \geq 2$, let i_s denote the least odd integer $i \geq 3$ such that

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \dots, \zeta(i)) \geq s.$$

Since $\zeta(3), \zeta(5), \dots$ span an infinite-dimensional \mathbb{Q} -vector space [3, 13], i_s exists for any s . Apéry's result that $\zeta(3) \notin \mathbb{Q}$ means $i_2 = 3$; conjecturally, $i_s = 2s - 1$ for any s . Ball-Rivoal's construction yields an upper bound for i_s , which has been improved for small values of s , namely $i_3 \leq 139$ and $i_4 \leq 1961$ ([10], refining upon previous bounds due to Zudilin [18]).

Now let j_s denote the least odd integer $j \geq 5$ such that

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \zeta(5), \zeta(7), \dots, \zeta(j)) \geq s.$$

The trivial remark that $j_s \leq i_{s+1}$ yields $j_2 \leq 139$ and $j_3 \leq 1961$. Now Zudilin's result [17] is an important improvement of the former bound, namely $j_2 \leq 11$. The linear forms in $1, \zeta(5), \zeta(7), \dots, \zeta(j)$ he constructs (in the style of (5) above: see also [18]) have the asymptotics (1), with $\omega \not\equiv 0 \pmod{\pi}$ in general. Theorem 1 enables one to deduce from it an upper bound for j_s , which should be better (for a fixed value of s) than the one derived from the trivial bound $j_s \leq i_{s+1}$. In particular the bound $j_3 \leq 1961$ can probably be improved in this way (note however that for proving that at least three numbers in a family are \mathbb{Q} -linearly independent, Theorem 1 can be replaced with Proposition 1 of [10], as explained at the beginning of the introduction: a lower bound for the linear forms is not necessary in this case).

3.2. Further generalizations of Nesterenko's criterion. Nesterenko's results (i) and (ii) (with $\omega = \varphi = 0$ in Eq. (1)) have been generalized in several directions, namely:

- In the setting of algebraic number fields [4, 16].
- If the coefficients $\ell_{i,n}$ are known to have divisors $\delta_{i,n}$ such that $\delta_{i,n}$ divides $\delta_{i,n+1}$, then (under suitable assumptions on $\delta_{i,n}$) both (i) and (ii) can be improved (see [10] for (i), and [6] for a detailed study of (ii) with only one number, namely $r = 1$).
- If the linear forms $\ell_{0,n}X_0 + \ell_{1,n}X_1 + \dots + \ell_{r,n}X_r$ are small at several points $(\xi_0^{(i)}, \dots, \xi_r^{(i)})$: see [7].
- A lower bound can be obtained [12] for the distance of the point $(1 : \xi_1 : \dots : \xi_r) \in \mathbb{P}^r(\mathbb{R})$ to any rational subspace of dimension less than $1 - \frac{\log \alpha}{\log \beta}$. This bound, proved by induction on the dimension, is Nesterenko's original approach to deduce (i) from (ii) (see [9] for a proof of this deduction based on Dirichlet's pigeonhole principle).

In all these generalizations, Proposition 1 enables one to replace the asymptotic behaviour $\alpha^{n+\sigma(n)}$ with the more general oscillating formula (1). The proof is the same as for Theorem 1 (see Section 2.2), namely one applies the result to a subsequence given by Proposition 1. In the case [7] where the linear forms $\ell_{0,n}X_0 + \ell_{1,n}X_1 + \dots + \ell_{r,n}X_r$ are small at several points, the full generality of Proposition 1 is needed (and not only the case $N = 1$).

At last, it would be interesting to apply Proposition 1 to other results in Diophantine approximation. For instance, an analogous question about oscillating linear forms was asked in [8, Section 6], and answered by Adamczewski [1] using ideas similar to the ones used here (but without Kronecker–Weyl’s theorem); see however [9] for a complete answer to all questions asked in [8].

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