Infinitely many $\zeta(2n+1)$ are irrational (after T. Rivoal)

Stéphane Fischler

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This text consists in notes taken during a lecture given by Tanguy Rivoal at Institut Henri Poincaré (Paris), on May 29th 2000. It is meant to be an introduction to his paper [3], and to explain how Rivoal modified Nikishin's method [2] to prove his result.

1 Statement of the results

For odd integers $a \geq 3$, let $\delta(a)$ be the dimension (over $\mathbb{Q}$) of the $\mathbb{Q}$-vector space spanned by the real numbers 1, $\zeta(3)$, $\zeta(5)$, ..., $\zeta(a)$, where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is Riemann Zeta function. The main result is the following :

Theorem 1.1 *For every positive real number ε there is a constant $A(\varepsilon)$, effectively computable, such that for any odd integer $a \geq A(\varepsilon)$:*

$$\delta(a) \geq \frac{1-\varepsilon}{1+\log(2)} \log(a)$$

Corollary 1.2 *There are infinitely many odd integers $a_1, a_2, \ldots$, greater than or equal to 3, such that $1, \zeta(a_1), \zeta(a_2), \ldots$ are linearly independent over $\mathbb{Q}$.*

Moreover, the same method as the one used to prove Theorem 1.1 yields the following result :

Theorem 1.3 *We have $\delta(7) \geq 2$ and $\delta(169) \geq 3$.*

Apéry's result that $\zeta(3)$ is irrational (that is, $\delta(3) = 2$) is not obtained by the present method. But Theorem 1.3 shows there is at least one odd integer a, between 5 and 169, such that $\zeta(a)$ is irrational.

2 Earlier Works and New Ideas

2.1 Nikishin's Method

Nikishin studied linear independence of values of polylogarithms at rational points. His result is stated below (Theorem 2.2), in a special case.

For a positive integer a, the a-th polylogarithm L_a is defined, on the unit disk, by

$$L_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)^a} \tag{1}$$

With this definition, the first polylogarithm is $L_1(x) = -\frac{\log(1-x)}{x}$. We shall consider $L_a(\frac{1}{z})$, where z is a complex number of modulus greater than 1. As z tends to 1, $L_a(\frac{1}{z})$ tends to $\zeta(a)$ if $a \geq 2$. In the sequel, we fix a positive integer a, and we consider the polylogarithms $L_1, \ldots, L_a$.

Nikishin found explicit Padé approximations of the first kind for these polylogarithms. In more precise terms, for every positive integer n he looked for polynomials $P_{0,n}(X)$, $P_{1,n}(X)$, ..., $P_{a,n}(X)$, with rational coefficients, of degree at most n, such that the function

$$F_n(z) = P_{0,n}(z) + P_{1,n}(z)L_1(\frac{1}{z}) + \ldots + P_{a,n}(z)L_a(\frac{1}{z}) \tag{2}$$

vanishes at infinity with order at least $a(n+1)-1$ (i.e. $z^{a(n+1)-1}F_n(z)$ has a finite limit as z tends to infinity). This amounts to solving $a(n+1)+n-1$ linear equations, in $(a+1)(n+1)$ unknowns ; even if we ask for one more relation, namely $P_{0,n}(0) = 0$, there is a nontrivial solution and Nikishin constructs it explicitely. In particular, he shows that for this solution we have :

$$F_n(z) = \sum_{k=0}^{\infty} \frac{k(k-1)\ldots(k-a(n+1)+2)}{(k+1)^a(k+2)^a\ldots(k+n+1)^a} \frac{1}{z^k}$$

Nikishin proves the following estimates :

Proposition 2.1 *Let d_n be the least common multiple of the integers 1,2,...,n. Let $p_{i,n}(X) = a!d_n^a P_{i,n}(X)$ (for integers $i = 0, 1, \ldots, a$) and $f_n(z) = a!d_n^a F_n(z)$. Then the following holds for $i = 0, 1, \ldots, a$:*
- *The polynomial $p_{i,n}(X)$ has integer coefficients.*
- *For any complex number z with $|z| \geq 1$ we have $\log(|p_{i,n}(z)|) \leq n\log(\beta(z)) + o(n)$ as n tends to infinity, where $\beta(z) = |z|\,(4ea)^a$.*
- *If z is real and $z \leq -1$ then $\log(|f_n(z)|) = n\log(\alpha(z)) + o(n)$ as n tends to infinity, with $0 < \alpha(z) \leq e^a\,|z|^{-a}(1+\frac{1}{a})^{-a(a+1)}$.*

Using this Proposition, Nikhisin proves the following Theorem :

Theorem 2.2 *If z is a negative integer such that $|z| > (4a)^{a(a-1)}$ then the numbers 1, $L_1(\frac{1}{z})$, ..., $L_a(\frac{1}{z})$ are linearly independent over $\mathbb{Q}$.*

For values of z such that $|z| \leq (4a)^{a(a-1)}$, Nikishin does not prove any result. However, some partial statements may be derived from Nesterenko's criterion.

2.2 Nesterenko's Criterion

To prove Theorem 2.2, Nikishin actually constructs a series like F_n, and proves they are linearly independent. In the sequel, one series will suffice thanks to the following criterion of linear independence, due to Nesterenko [1] :

Theorem 2.3 *Let a be a positive integer, and $\theta_0, \ldots, \theta_a$ real numbers. For $n \geq 1$, let $\ell_n = p_{0,n}X_0 + \ldots + p_{a,n}X_a$ be a linear form with integer coefficients.*

Let α and β be real numbers, with $0 < \alpha < 1$ and $\beta > 1$.

Assume that the following estimates hold as n tends to infinity :
- $\log(|\ell_n(\theta_0, \ldots, \theta_a)|) = n\log(\alpha) + o(n)$.
- *For any i between 0 and a, $\log(|p_{i,n}|) \leq n\log(\beta) + o(n)$.*

Then $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\theta_0, \ldots, \theta_a) \geq 1 - \frac{\log(\alpha)}{\log(\beta)}$.

One may apply this criterion to Nikishin's construction explained in Section 2.1. Indeed, when $z = -1$, the right handside of Equation (2) is a linear relation, with rational coefficients, between 1, $\log(2)$, $\zeta(2)$, $\zeta(3)$,..., $\zeta(a)$ thanks to the elementary formula $L_a(-1) = (1 - 2^{1-a})\zeta(a)$. Using Proposition 2.1 and the Prime Number Theorem (which asserts that $\log(d_n)$ is equivalent to n as n tends to infinity), one obtains the following result :

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1, \log(2), \zeta(2), \zeta(3), \ldots, \zeta(a)) \geq 2$$

Of course, this statement is very weak and well known ; it contains nothing more than the irrationality of $\log(2)$, obtained for $a = 2$. However, this is a motivation for introducing new ideas.

2.3 Rivoal's Contribution

The first idea used in Rivoal's paper is to modify Nikishin's series F_n by introducing a new parameter r, which is an integer between 1 and a :

$$\tilde{F}_n(z) = n!^{a-2r} \sum_{k=0}^{\infty} \frac{k(k-1)\ldots(k-rn+1)}{(k+1)^a(k+2)^a\ldots(k+n+1)^a} \frac{1}{z^k}$$

This gives Padé-type approximations, i.e. polynomials $P_{0,n}(X)$, $P_{1,n}(X)$, ..., $P_{a,n}(X)$, of degree at most n, such that Equation (2) holds with $\tilde{F}_n$. Now the vanishing order of $\tilde{F}_n$ at infinity is only rn, but sharper estimates hold for $|P_{i,n}(z)|$. Applying Theorem 2.3 as above, and choosing r in a suitable way, Rivoal proves the following result :

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1, \log(2), \zeta(2), \zeta(3), \ldots, \zeta(a)) \geq c_1 \log(a) \tag{3}$$

for $a \geq c_2$, with positive absolute constants c_1 and c_2. The proof is not published, since (3) is a trivial consequence of Euler's formulae (proving that $\frac{\zeta(2k)}{\pi^{2k}} \in \mathbb{Q}$) and Lindemann's Theorem that π is transcendental. However, "only" one step is missing to prove Theorem 1.1 : to get rid of $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, ... in the left handside of (3).

The problem is to modify Nikishin's series F_n in a proper way. In the special case $a = 4$, K. Ball introduced the following series :

$$B_n(z) = n!^2 \sum_{k=0}^{\infty} (k+1+\frac{n}{2}) \frac{k(k-1)\ldots(k-(n-1))(k+n+2)(k+n+3)\ldots(k+2n+1)}{(k+1)^4\,(k+2)^4\,\ldots\,(k+n+1)^4} \frac{1}{z^k}$$

This series gives a linear combination of 1 and $\zeta(3)$. According to K. Ball, this could be generalized to $\zeta(5)$, and so on. This was done by Rivoal, who considered [4] the following series :

$$S_n(z) = n!^{a-2r} \sum_{k=0}^{\infty} \frac{k(k-1)\ldots(k-rn+1)(k+n+2)(k+n+3)\ldots(k+(r+1)n+1)}{(k+1)^a\,(k+2)^a\,\ldots\,(k+n+1)^a} \frac{1}{z^k} \tag{4}$$

where a is (as usual) a fixed positive integer, and r is a parameter to be chosen in a suitable way (see Section 3.4). We assume that the positive integer r is less than $\frac{a}{2}$; in this way, the series converges for any z of modulus at least 1.

3 Sketch of the Proof

3.1 Notation

To simplify the formulae, we shall use Pochammer symbol $(\alpha)_k$ defined, for a nonnegative integer k, by $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$ (with the convention $(\alpha)_0 = 1$). Then we can write

$$S_n(z) = \sum_{k=0}^{\infty} R_n(k) \frac{1}{z^k} \tag{5}$$

where

$$R_n(t) = n!^{a-2r} \frac{(t-rn+1)_{rn}(t+n+2)_{rn}}{(t+1)_{n+1}^a} \tag{6}$$

Since $r < \frac{a}{2}$, the rational fraction $R_n(t)$ can be written as

$$R_n(t) = \sum_{i=1}^{a} \sum_{j=0}^{n} \frac{c_{i,j,n}}{(t+j+1)^i} \tag{7}$$

where the rationals $c_{i,j,n}$ are given by

$$c_{i,j,n} = \frac{1}{(a-i)!} (\frac{d}{dt})^{a-i} (R_n(t)(t+j+1)^a)_{|t=-j-1} \tag{8}$$

3.2 Obtaining the Linear Relation

Define the following polynomials with rational coefficients :

$$P_{0,n}(X) = -\sum_{i=1}^{a} \sum_{j=1}^{n} c_{i,j,n} \sum_{k=0}^{j-1} \frac{1}{(k+1)^i} X^{j-k} \text{ and } P_{i,n}(X) = \sum_{j=0}^{n} c_{i,j,n} X^j \text{ for } 1 \le i \le a$$

Proposition 3.1 *The polynomials $P_{i,n}$ are Padé-type approximations ; more precisely, for $|z| > 1$:*

$$S_n(z) = P_{0,n}(z) + \sum_{i=1}^{a} P_{i,n}(z) L_i(\frac{1}{z}) \tag{9}$$

This Proposition results from the following computation, for $|z| > 1$:

$$\begin{aligned} S_n(z) &= \sum_{i=1}^{a} \sum_{j=0}^{n} c_{i,j,n} \sum_{k=0}^{\infty} \frac{1}{(k+j+1)^i} \frac{1}{z^k} && \text{by (5) and (7)} \\ &= \sum_{i=1}^{a} \sum_{j=0}^{n} c_{i,j,n} z^j (L_i(\frac{1}{z}) - \sum_{k=0}^{j-1} \frac{1}{(k+1)^i} \frac{1}{z^k}) && \text{by (1)} \\ &= \sum_{i=1}^{a} P_{i,n}(z) L_i(\frac{1}{z}) + P_{0,n}(z) \end{aligned}$$

The assumption $r < \frac{a}{2}$ implies that the series in Equation (4) converges for $z = 1$, thus defining $S_n(1)$. Consider the limit, as z tends to 1 (with z real and $z > 1$), of Equation (9). Noticing that $P_{1,n}(1)$ vanishes since it is the sum of residues of $R_n(t)$, we obtain the following linear relation :

$$S_n(1) = P_{0,n}(1) + \sum_{i=2}^{a} P_{i,n}(1)\zeta(i) \tag{10}$$

This is a first improvement : assuming $r < \frac{a}{2}$, we are able to take $z = 1$ in Equation (9) (whereas Nikishin had to take $z = -1$) ; consequently, $\log(2)$ has disappeared from the linear relation.

The second improvement is far more important :

Proposition 3.2 *Assume a is odd and n is even. Then $P_{i,n}(1) = 0$ for every even integer i between 2 and a, and the linear relation (10) reads :*

$$S_n(1) = P_{0,n}(1) + \sum_{\substack{i=3 \\ i \text{ odd}}}^{a} P_{i,n}(1)\zeta(i) \tag{11}$$

We shall deduce Proposition 3.2 from the following symmetry property of the coefficients $c_{i,j,n}$:

Lemma 3.3 *For any indices i and j we have :*

$$c_{i,n-j,n} = (-1)^{(n+1)a-i} c_{i,j,n}$$

Proposition 3.2 follows easily from this Lemma. Indeed, assuming a odd, n and i even, we have $c_{i,n-j,n} = -c_{i,j,n}$. Summing this equality for $j = 0, \ldots, n$ yields $P_{i,n}(1) = -P_{i,n}(1)$, thereby proving Proposition 3.2.

Let us now prove Lemma 3.3. Denote by $\Phi_{n,j}(t)$ the function $R_n(-t-1)(j-t)^a$, so that Equation (8) reads $c_{i,j,n} = \frac{(-1)^{a-i}}{(a-i)!}\Phi_{n,j}^{(a-i)}(j)$. Now we claim that $\Phi_{n,j}(t)$ and $\Phi_{n,n-j}(t)$ satisfy the following property for any t :

$$\Phi_{n,n-j}(n-t) = (-1)^{na}\Phi_{n,j}(t)$$

Lemma 3.3 follows directly from this claim, by differentiating $a - i$ times and letting $t = j$. Let us now prove the claim. We shall use three times the equality $(\alpha)_\ell = (-1)^\ell(-\alpha - \ell + 1)_\ell$, which holds for any nonnegative integer ℓ. We have :

$$\begin{aligned}
\Phi_{n,n-j}(n-t) &= R_n(t-n-1)(t-j)^a \\
&= n!^{a-2r}\frac{(t-(r+1)n)_{rn}(t+1)_{rn}}{(t-n)_{n+1}^a}(t-j)^a \qquad \text{by (6)} \\
&= n!^{a-2r}(-1)^{na}\frac{(-t+n+1)_{rn}(-t-rn)_{rn}}{(-t)_{n+1}^a}(j-t)^a \\
&= (-1)^{na}R_n(-t-1)(j-t)^a = (-1)^{na}\Phi_{n,j}(t)
\end{aligned}$$

3.3 Estimates

To apply Nesterenko's criterion, we need estimates similar to those of Proposition 2.1.

Proposition 3.4 *Assume a is odd and n is even. Let d_n be the least common multiple of the integers 1,2,…,n. Let $p_{i,n} = d_n^a P_{i,n}(1)$ (for integers $i = 0, 3, 5, 7, \ldots, a$) and $\ell_n = d_n^a S_n(1)$. Then the following holds for any $i \in \{0, 3, 5, 7, \ldots, a\}$:*

– *The rational* $p_{i,n}$ *is an integer.*
– *We have* $\log(|p_{i,n}|) \leq n\log(\beta) + o(n)$ *as* n *tends to infinity, where* $\beta = e^a(2r+1)^{2r+1}2^{a-2r}$.
– *We have* $\log(|\ell_n|) = n\log(\alpha) + o(n)$ *as* n *tends to infinity, with*
$0 < \alpha \leq e^a(2r+1)^{2r+1}\frac{((a+1)r)^{(a+1)r}(a-2r)^{a-2r}}{((r+1)a-r)^{(r+1)a-r}}$.

To prove the first two assertions, Rivoal follows Nikishin, using (8) and Cauchy's integral formula. For the third one, he uses the following formula (valid for $|z| \geq 1$) :

$$S_n(z) = \frac{((2r+1)n+1)!}{n!^{2r+1}} z^{(r+1)n+2} \int_{[0,1]^{a+1}} \left(\frac{\prod_{i=1}^{a+1} x_i^r(1-x_i)}{(z - x_1x_2\dots x_{a+1})^{2r+1}} \right)^n \frac{dx_1dx_2\dots dx_{a+1}}{(z-x_1x_2\dots x_{a+1})^2}$$

3.4 End of the proof

Applying Theorem 2.3 to the linear forms $\ell_{2n} = p_{0,2n} + \sum_{\substack{i=3 \\ i \text{ odd}}}^{a} p_{i,2n}\zeta(i)$ yields the following result (where $\delta(a)$ is defined in Section 1) :

Proposition 3.5 *Let* $a \geq 3$ *be an odd integer, and let* r *be an integer such that* $1 \leq r < \frac{a}{2}$. *Then*

$$\delta(a) \geq \frac{f(a,r)}{g(a,r)}$$

where

$$f(a,r) = (a-2r)\log(2) + ((r+1)a-r)\log((r+1)a-r) - ((a+1)r)\log((a+1)r) - (a-2r)\log(a-2r)$$

and

$$g(a,r) = a + (a-2r)\log(2) + (2r+1)\log(2r+1)$$

Actually, Rivoal gives a sharper upper bound for α in Proposition 3.4 ; this allows him to prove $\delta(169) > 2.001$ (by choosing $r = 10$). He claims this constant 169 can probably be refined, using more precise estimates for $|p_{i,n}|$ in Proposition 3.4.

For large values of a (and r), we have

$$f(a,r) = a\log(r) + O(a) + O(r\log(r))$$

and

$$g(a,r) = (1+\log(2))a + O(r\log(r))$$

We choose for r the integer part of $\frac{a}{(\log(a))^2}$; then $\delta(a) \geq \frac{a\log(a)(1+o(1))}{a(1+\log(2))(1+o(1))}$. This is exactly Theorem 1.1.

Références

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Stéphane Fischler
Département de Mathématiques et Applications
École Normale Supérieure
45, rue d'Ulm
75230 Paris Cedex 05, France
fischler@clipper.ens.fr