A density measure in rational approximation

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Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$. How small can $|\xi - \frac{p}{q}|$ (or $|q\xi - p|$) be in terms of q ?

Two exponents:

- the *irrationality exponent* $\mu(\xi)$ measures the *precision* of rational approximants (classical).
- the *density exponent* $\nu(\xi)$ measures the *regularity* (or *density*) of *sequences* of rational approximants (new).

Throughout the lecture, we assume $\xi \in \mathbb{R} \setminus \mathbb{Q}$.

Definition of the irrationality exponent $\mu(\xi)$

The *irrationality exponent* $\mu(\xi) \in \mathbb{R} \cup \{+\infty\}$ is such that the following property holds:

Let $\mu \in \mathbb{R}$. The equation

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{\mu}} \tag{1}$$

has

- infinitely many solutions (p, q) if $\mu < \mu(\xi)$,
- only finitely many solutions (p, q) if $\mu > \mu(\xi)$.

Lower bound: $\mu(\xi) \geq 2$.

ξ is said to be a *Liouville number* if $\mu(\xi) = +\infty$, that is if (1) has infinitely many solutions for any $\mu \in \mathbb{R}$. For instance:

$$\xi = \sum_{n=1}^{+\infty} \frac{1}{10^{n!}}$$

Definition of the density exponent $\nu(\xi)$

Let $\mathbf{q} = (q_n)_{n\geq 1}$ be an increasing sequence of positive integers. Let p_n be the closest integer to $q_n\xi$, and

$$\alpha_\xi(\mathbf{q}) = \limsup_{n\rightarrow+\infty} \frac{|q_{n+1}\xi - p_{n+1}|}{|q_n\xi - p_n|},$$

$$\beta(\mathbf{q}) = \limsup_{n\rightarrow+\infty} \frac{q_{n+1}}{q_n}.$$

The *density exponent* $\nu(\xi)$ is the infimum of

$$\log\sqrt{\alpha_\xi(\mathbf{q})\beta(\mathbf{q})}$$

as $\mathbf{q}$ runs through the set of all increasing sequences such that

$$\alpha_\xi(\mathbf{q}) < 1 \quad \text{and} \quad \beta(\mathbf{q}) < +\infty. \qquad (2)$$

If there is no sequence $\mathbf{q}$ satisfying (2), we let $\nu(\xi) = +\infty$.

Lower bound: $\nu(\xi) \geq 0$.

A glance at irrationality proofs

Many irrationality proofs go by constructing $\mathbf{q}$ such that

$$q_n\xi - p_n \to 0 \text{ as } n \to \infty, \text{ and}$$

$$q_n\xi - p_n \neq 0 \text{ for any } n.$$

Sometimes the stronger property

$$\alpha_\xi(\mathbf{q}) < 1 \quad \text{and} \quad \beta(\mathbf{q}) < +\infty$$

holds (e.g. in Apéry's proof that $\zeta(3) \notin \mathbb{Q}$).

$\nu(\xi) < \infty$

$\Leftrightarrow$ there exists $\mathbf{q}$ with

$\alpha_\xi(\mathbf{q}) < 1$ and $\beta(\mathbf{q}) < +\infty$

$\Leftrightarrow$ there exists a proof that $\xi \notin \mathbb{Q}$ in which the sequences (q_n) and $(|q_n\xi - p_n|)$ have geometrical behaviour

A simple case, to fix the ideas

Assume there is an increasing sequence $\mathbf{q}$ with

$$\nu(\xi) = \log \sqrt{\alpha_\xi(\mathbf{q})\beta(\mathbf{q})}$$

and also

$$\alpha_\xi(\mathbf{q}) = \lim_{n\to+\infty} \frac{|q_{n+1}\xi - p_{n+1}|}{|q_n\xi - p_n|} < 1,$$

$$\beta(\mathbf{q}) = \lim_{n\to+\infty} \frac{q_{n+1}}{q_n} < +\infty.$$

Then

$|q_n\xi - p_n| = (\alpha_\xi(\mathbf{q}) + o(1))^n$ and $q_n = (\beta(\mathbf{q}) + o(1))^n$,

so that

$q_n|q_n\xi - p_n| = (\alpha_\xi(\mathbf{q})\beta(\mathbf{q}) + o(1))^n = e^{(2\nu(\xi)+o(1))n}$.

In this case, $\nu(\xi)$ measures how fast $q_n|q_n\xi - p_n|$ grows with n.

If $\nu(\xi) > 0$ and n is sufficiently large then q_n can *not* be a convergent in the continued fraction expansion of ξ.

Now we come back to the general case

Generic behaviour

Recall that, for any $\xi \in \mathbb{R} \setminus \mathbb{Q}$,

$$2 \leq \mu(\xi) \leq +\infty,$$

$$0 \leq \nu(\xi) \leq +\infty.$$

Theorem 1 *For almost all $\xi \in \mathbb{R} \setminus \mathbb{Q}$ with respect to Lebesgue measure,*

$$\mu(\xi) = 2 \textit{ and } \nu(\xi) = 0.$$

General philosophy: If ξ comes in *"naturally"* (that is, is not constructed on purpose) then it should satisfy

$$\mu(\xi) = 2 \text{ and } \nu(\xi) = 0.$$

Liouville numbers

Recall that ξ is a *Liouville number* if $\mu(\xi) = +\infty$, that is if for any $\mu \in \mathbb{R}$ the equation

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has infinitely many solutions (p, q).

Theorem 2 *If* $\mu(\xi) = +\infty$ *then* $\nu(\xi) = +\infty$.

So we know that $\nu(\xi) = +\infty$ for some numbers ξ, for instance

$$\xi = \sum_{n=1}^{+\infty} \frac{1}{10^{n!}}$$

Continued fraction expansion

$$\xi = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

Convergents: the most precise rational approximants p/q. So $\mu(\xi)$ can be easily computed from the sequence (a_k).

But the sequence of convergents has not (in general) a geometric growth, so it can *not* be used to compute $\nu(\xi)$...except in one case:

Theorem 3 *If* $a_k \leq A$ *for any* k*, then*

$$\mu(\xi) = 2,$$
$$\nu(\xi) \leq \log\left(\frac{A+1}{\sqrt{A+2}}\right).$$

For a quadratic number ξ, the sequence (a_k) is ultimately periodic and

$$\nu(\xi) = 0.$$

So we know that $\nu(\xi) = 0$ for some numbers ξ.

The number e

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$$

$$\mu(e) = 2$$

We do not know *anything* on $\nu(e)$, except of course

$$0 \leq \nu(e) \leq +\infty.$$

Is it finite ? Is it zero ?

The number $\zeta(3)$

$$\zeta(3) = \sum_{n=1}^{+\infty} \frac{1}{n^3}$$

Theorem 4 (Apéry, 1978) $\zeta(3)$ *is irrational, and*

$$\mu(\zeta(3)) < 13.42$$

Further refinements by various authors. Best known bound (Rhin-Viola, 2001) :

$$\mu(\zeta(3)) < 5.52$$

Conjecture: $\mu(\zeta(3)) = 2$.

Apéry's construction yields

$$\nu(\zeta(3)) \le 3$$

The further refinements yield less precise upper bounds.

Conjecture: $\nu(\zeta(3)) = 0$.

Other hypergeometric constructions

For

$$\zeta(2) = \frac{\pi^2}{6} = \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

and for $\log(2)$, the situation is the same.

$$\nu(\zeta(2)) \leq 2$$

$$\nu(\log(2)) \leq 1$$

The number $\zeta(5)$ is not known to be irrational.

$$\nu(\pi) < 21$$

Algebraic numbers

If ξ is algebraic then:

- $\mu(\xi) \leq \deg(\xi)$ (Liouville)
- $\mu(\xi) = 2$ (Roth)

Theorem 5 (with Adamczewski) *If ξ is algebraic then $\nu(\xi)$ is finite.*

Conjecture: If ξ is algebraic then $\nu(\xi) = 0$.

Open questions

Does there exist a number ξ such that

$$0 < \nu(\xi) < +\infty \quad ?$$

Is it true that for any irrational *period* ξ we have

$$\nu(\xi) = 0 \quad ?$$