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Irrationality of values of L-functions of Dirichlet characters

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Abstract

In a recent paper with Sprang and Zudilin, the following result was proved: if a is large enough in terms of $\varepsilon > 0$, then at least $2^{(1-\varepsilon)\frac{\log a}{\log\log a}}$ values of the Riemann zeta function at odd integers between 3 and a are irrational. This improves on the Ball-Rivoal theorem, that provides only $\frac{1-\varepsilon}{1+\log 2}\log a$ such irrational values – but with a stronger property: they are linearly independent over the rationals.

In the present paper we generalize this recent result to both L-functions of Dirichlet characters and Hurwitz zeta function. The strategy is different and less elementary: the construction is related to a Padé approximation problem, and a generalization of Shidlovsky's lemma is used to apply Siegel's linear independence criterion.

We also improve the analogue of the Ball-Rivoal theorem in this setting: we obtain $\frac{1-\varepsilon}{1+\log 2}\log a$ linearly independent values $L(f,s)$ with $s \leq a$ of a fixed parity, when f is a Dirichlet character. The new point here is that the constant $1+\log 2$ does not depend on f.

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The purpose of this paper is to prove results of irrationality, or linear independence, of values of the Hurwitz ζ function or L-functions of Dirichlet characters. Both are generalizations of the Riemann ζ function, so we begin with a quick survey of the main results in this setting.

When $s \geq 2$ is even, $\zeta(s)\pi^{-s}$ is a non-zero rational number so that $\zeta(s)$ is transcendental. Apéry has proved [1] that $\zeta(3)$ is irrational, but there is no odd $s \geq 5$ for which $\zeta(s)$ is known to be irrational. The next breakthrough is due to Ball-Rivoal [2, 20]:

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \ldots, \zeta(a)) \geq \frac{1+o(1)}{1+\log 2}\log a \text{ as } a \to \infty,\ a \text{ odd}.$$

Here and throughout this introduction, $o(1)$ denotes any sequence that tends to 0 as $a \to \infty$. In this paper we mention only asymptotic results (namely, as $a \to \infty$) eventhough most results can be made explicit, and often refined, for small values of a. At last we mention the following recent result [11]:

$$\text{at least } 2^{(1-o(1))\frac{\log a}{\log\log a}} \text{ numbers among } \zeta(3), \zeta(5), \ldots, \zeta(a) \text{ are irrational,} \tag{0.1}$$

for a odd, $a \to \infty$.

The natural setting to generalize these results to values of the Hurwitz ζ function or L-functions of Dirichlet characters is the following. Let $T \geq 1$, and $f : \mathbb{Z} \to \mathbb{C}$ be such that $f(n+T) = f(n)$ for any n. We assume that f is not identically zero. Let $\varepsilon > 0$, and a be sufficiently large (in terms of T and ε). For $p \in \{0, 1\}$ consider the complex numbers

$$L(f, s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \text{ with } 2 \leq s \leq a \text{ and } s \equiv p \bmod 2. \tag{0.2}$$

If f is a Dirichlet character mod T then these are exactly the values of the associated L-function.

The restriction on the parity of s in (0.2) is needed in some cases to get rid of powers of π. Indeed, if f is a Dirichlet character then f is either even (i.e., $f(-n) = f(n)$) or odd (i.e., $f(-n) = -f(n)$), according to whether $f(-1)$ is equal to 1 or -1. If $s \geq 2$ has the same parity as f then $L(f, s)\pi^{-s}$ is a non-zero algebraic number (see for instance [18, Chapter VII, §2]) so that the numbers $L(f, s)$ for s with this parity are linearly independent over $\overline{\mathbb{Q}}$. Moreover, for any periodic map $f : \mathbb{Z} \to \mathbb{Q}$ which is either even or odd (and not identically zero), we also have $L(f, s)\pi^{-s} \in \overline{\mathbb{Q}}^*$ when s and f have the same parity (see [12]). In these situations, we prove new results on the numbers (0.2) only when p and f have opposite parities.

An interesting case where (in general) f is neither odd nor even is the following. Given $u \in \{1, \ldots, T-1\}$ we define f by $f(n) = 1$ if $n \equiv u \bmod T$, and $f(n) = 0$ otherwise. Then

$$L(f, s) = \sum_{k=0}^{\infty} \frac{1}{(kT+u)^s} = \frac{1}{T^s} \sum_{k=0}^{\infty} \frac{1}{(k+u/T)^s} = \frac{1}{T^s}\zeta(s, \frac{u}{T})$$

where $\zeta(s, \alpha)$ is the Hurwitz ζ function. Therefore the general setting (0.2) encompasses both values of the Hurwitz ζ function and values of L-functions of Dirichlet characters.

As far as we know, Apéry's theorem has never been generalized in this direction; the first natural conjecture in this respect is probably that Catalan's constant $L(\chi, 2)$ is irrational, where χ is the non-principal character mod 4. The Ball-Rivoal theorem has been generalized to the L-function of this character by Rivoal and Zudilin [21]: they have proved (0.3) below with $2 + \log 2$ instead of $T + \log 2$, eventhough $T = 4$. In the general setting of (0.2), Nishimoto has generalized the Ball-Rivoal theorem as follows [19]:

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}} \Big\{ L(f, s),\ 2 \leq s \leq a,\ s \equiv p \bmod 2 \Big\} \geq \frac{1+o(1)}{T + \log 2} \log a \text{ as } a \to \infty. \tag{0.3}$$

In the special case where $\sum_{n=1}^{T} f(n) \neq 0$ (which includes the Hurwitz ζ function but not L-functions of non-principal Dirichlet characters), this lower bound appears already in Nash' thesis [17]. The constant $T+\log 2$ in Eq. (0.3) has been refined to $T/2+\log 2$ in [8], provided f is a Dirichlet character and T is a multiple of 4. When f is the non-principal character mod 4, this gives as a special case the lower bound of Rivoal and Zudilin [21].

Our first result is that one may replace the constant $T+\log 2$ in Eq. (0.3) with $1+\log 2$, so that the lower bound is uniform in T and is the same as for the Riemann ζ function.

Theorem 1. *Let $T \geq 1$, and $f : \mathbb{Z} \to \mathbb{C}$ be such that $f(n+T) = f(n)$ for any n. Assume that f is not identically zero. Let $p \in \{0,1\}$, $\varepsilon > 0$, and a be sufficiently large (in terms of T and ε). Then*

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}} \Big\{ L(f,s),\ 2 \leq s \leq a,\ s \equiv p \bmod 2 \Big\} \geq \frac{1-\varepsilon}{1+\log 2} \log a.$$

Of course the same result holds without the restriction $s \equiv p \bmod 2$, but it is weaker and even trivial in some cases where f is even or odd (as noticed above).

In another direction, we generalize the recent result (0.1) to this setting.

Theorem 2. *Let $T \geq 1$, and $f : \mathbb{Z} \to \mathbb{C}$ be such that $f(n+T) = f(n)$ for any n. Assume that f is not identically zero. Let E be a finite-dimensional $\mathbb{Q}$-vector space contained in $\mathbb{C}$, $p \in \{0,1\}$, $\varepsilon > 0$, and a be sufficiently large (in terms of $\dim E$, T, and ε). Then among the numbers $L(f,s)$ with $2 \leq s \leq a$ and $s \equiv p \bmod 2$, at least*

$$2^{(1-\varepsilon)\frac{\log a}{\log\log a}}$$

do not belong to E.

Taking $E = \mathbb{Q}$ we obtain at least $2^{(1-\varepsilon)\frac{\log a}{\log\log a}}$ irrational values among the numbers $L(f,s)$. The dependence in a is much better than in the lower bound of Theorem 1; however we obtain only numbers outside E, and not $\mathbb{Q}$-linearly independent numbers.

Before explaining the strategy used in the proofs of Theorems 1 and 2, we would like to state the two main special cases of Theorem 2 explicitly.

Corollary 1. *Let χ be a Dirichlet character; put $p = 0$ is χ is odd, and $p = 1$ if χ is even. Let E be a finite-dimensional $\mathbb{Q}$-vector space contained in $\mathbb{C}$. Let $\varepsilon > 0$, and a be sufficiently large (in terms of χ, $\dim E$, and ε). Then among the numbers $L(\chi,s)$ with $2 \leq s \leq a$ and $s \equiv p \bmod 2$, at least $2^{(1-\varepsilon)\frac{\log a}{\log\log a}}$ do not belong to E.*

Corollary 2. *Let r be a positive rational number, and $p \in \{0,1\}$. Let E be a finite-dimensional $\mathbb{Q}$-vector space contained in $\mathbb{C}$. Let $\varepsilon > 0$, and a be sufficiently large (in terms of r, $\dim E$, and ε). Then among the numbers*

$$\zeta(s,r) = \sum_{n=0}^{\infty} \frac{1}{(n+r)^s}$$

with $2 \leq s \leq a$ and $s \equiv p \bmod 2$, at least $2^{(1-\varepsilon)\frac{\log a}{\log\log a}}$ do not belong to E.

Corollary 2 is new even for $r = 1$, i.e. for the Riemann ζ function: it is a refinement of (0.1). We would like to emphasize the fact that the proof of [11] *does not* give this result for $E \neq \mathbb{Q}$: a different approach is used here, proving linear independence and not only irrationality.

The proof of Theorems 1 and 2 is based on the strategy of [8]: we apply Siegel's linear independence criterion using a general version of Shidlovsky's lemma (namely Theorem 3, stated in §1.3 and proved in [8] following the approach of Bertrand-Beukers [4] and Bertrand [3]). This makes it necessary to relate the construction to a Padé approximation problem with essentially as many equations as the number of unknowns. In the present paper we adapt this strategy so as to include Sprang's arithmetic lemma [23, Lemma 1.4] and the elimination trick of [24, 23, 11]. The proofs of Theorems 1 and 2 are essentially the same, except for the choice of parameters. It is also possible to prove other results of the same flavour (see Theorem 4 at the end of §3.2, which implies both Theorem 2 and – up to a multiplicative constant – Theorem 1).

Our construction contains as a special case the one used in [11] to prove (0.1). We prove this in §3.3; as a byproduct, we relate the construction of [11] to a Padé approximation problem with essentially as many equations as the number of unknowns.

The structure of this paper is as follows. We gather in Proposition 1 the output of the Diophantine construction (see §1.1), and prove it in §1. Then we deduce Theorems 1 and 2 from Proposition 1 in §3 using Siegel's linear independence criterion (stated in §2).

1 Diophantine construction

In this section we gather the Diophantine part of the proof, namely the construction of linearly independent linear forms. We prove Proposition 1 stated in §1.1, from which we shall deduce in §3 the results stated in the introduction. The linear forms are constructed in §1.2 using series of hypergeometric type. We relate them in §1.4 to a Padé approximation problem, and then apply a general version of Shidlovsky's lemma (stated in §1.3). At last, arithmetic and asymptotic properties are dealt with in §1.5.

1.1 Statement of the result

Let a, r, N be positive integers such that $1 \leq r < \frac{a}{3N}$. Let $N \geq 1$, and $f : \mathbb{Z} \to \mathbb{C}$ be such that $f(m+N) = f(m)$ for any m. Assume that f is not identically zero. Let $p \in \{0,1\}$; put

$$L(f,j) = \sum_{m=1}^{\infty} \frac{f(m)}{m^j} \text{ for any } j \in \{2,\ldots,a\}.$$

Let also

$$\alpha = (4e)^{(a+1)/N}(2N)^{2r+2}r^{-(a+1)/N+4(r+1)} \text{ and } \beta = (2e)^{(a+1)/N}(r+1)^{2r+2}N^{2r+2}. \quad (1.1)$$

Proposition 1. *There exists a constant c_1, which depends only on a and N, with the following property. For any integer multiple n of N there exist integers $s_{k,i}$, with $1 \leq k \leq c_1$ and $2 \leq i \leq a+N$, such that:*

(*i*) *For any n sufficiently large, the subspace $\mathcal{F}$ of $\mathbb{R}^{a+N-1}$ spanned by the vectors ${}^t(s_{k,2}, \dots, s_{k,a+N})$, $1 \leq k \leq c_1$, is non-zero and does not depend on n.*

(*ii*) *For any k and any i we have $|s_{k,i}| \leq \beta^{n+o(n)}$ as $n \to \infty$.*

(*iii*) *For any k we have, as $n \to \infty$:*

$$\Big| 2(-1)^p \sum_{\substack{i=2 \\ i \equiv p \bmod 2}}^{a} s_{k,i} L(f,i) + \sum_{i=0}^{N-1} s_{k,a+1+i} f(i) \Big| \leq \alpha^{n+o(n)}. \tag{1.2}$$

From now on, the symbols $o(\cdot)$ will be intended as $n \to \infty$. Since $k \leq c_1$, these symbols can be made uniform with respect to k.

The integers $s_{k,i}$ depend also implicitly on n, a, r, N, f and p. Their values for $i \not\equiv p \bmod 2$ do not appear in the linear combinations of part (*iii*), but they could be of interest in other settings. Another feature of this construction is that for $i \leq a$, the integers $s_{k,i}$ depend only on n, a, r, N but not on f or p. Probably this could lead to variants of our results in the style of [14] or [7].

Remark 1. *In [8] a similar construction is made, where the matrix $[s_{k,i}]_{i,k}$ has rank $a + N - 1$ for n sufficiently large so that the subspace $\mathcal{F}$ of part (i) is equal to $\mathbb{R}^{a+N-1}$. In the present setting we make a different construction to incorporate Sprang's arithmetic lemma (see §1.2 below), and the matrix $[s_{k,i}]$ we obtain has rank less than $a + N - 1$ for some values of the parameters (see Remark 3 in §1.4): the subspace $\mathcal{F}$ in Proposition 1 is not always equal to $\mathbb{R}^{a+N-1}$.*

1.2 Construction of the linear forms

In this section we define the numbers $s_{k,i}$ of Proposition 1 (see Eq. (1.17)) and express the linear form of Eq. (1.2) in a more convenient way (see Lemma 1). We postpone until §1.5 the proof that $s_{k,i} \in \mathbb{Z}$.

As in §1.1 we let a, r, N be positive integers such that $1 \leq r < \frac{a}{3N}$. For any integer multiple n of N we let

$$F(t) = (n/N)!^{(a+1)-(2r+1)N} \frac{(t-rn)_{(2r+1)n+1}}{\prod_{h=0}^{n/N} (t+Nh)^{a+1}}$$

where $(\alpha)_p = \alpha(\alpha+1)\dots(\alpha+p-1)$ is the Pochhammer symbol. Note that each factor $t + Nh$ of the denominator appears also in the numerator, so that the poles $t = -Nh$ of this rational function only have order a. This rational function $F(t)$ is similar to that of [8], but central factors have been inserted in the numerator to apply Sprang's arithmetic lemma (see Remark 2 below).

In this section we follow the proof of [8], except for Eq. (1.19) which is specific to the function F we consider here. We let

$$S_0(z) = \sum_{t=n+1}^{\infty} F(-t)z^t \qquad \text{and} \qquad S_\infty(z) = \sum_{t=1}^{\infty} F(t)z^{-t} \tag{1.3}$$

for $z \in \mathbb{C}$ with $|z| = 1$; then both series are convergent since the degree $-d_0$ of F satisfies

$$d_0 := -\deg F = (a+1)(\frac{n}{N}+1) - (2r+1)n - 1 \geq 2. \tag{1.4}$$

We let $\omega = e^{2i\pi/N}$ and for any $\ell \in \{1, \dots, N\}$ we consider the (inverse) discrete Fourier transform of f, defined by

$$\widehat{f}(\ell) = \frac{1}{N}\sum_{\lambda=1}^{N} f(\lambda)\omega^{-\ell\lambda}. \tag{1.5}$$

We also let

$$\delta_n = (Nd_{n/N})^{a+1}N^{(a+1)n/N}, \text{ where } d_j = \mathrm{lcm}(1, 2, \dots, j).$$

The linear forms of Proposition 1 are given by the following lemma. The rational numbers $s_{k,i}$ will be constructed explicitly in the proof (see Eq. (1.17)), and we shall prove in §1.5 that they are integers.

Lemma 1. *For any $1 \leq k \leq d_0 - 1$ there exist rational numbers $s_{k,2}$, ..., $s_{k,a+N}$ such that*

$$\begin{aligned}
&\delta_n \sum_{\ell=1}^{N} \widehat{f}(\ell)\Big[\omega^{\ell(k-1)}S_0^{(k-1)}(\omega^\ell) + (-1)^p\omega^{\ell(1-k)}S_\infty^{(k-1)}(\omega^{-\ell})\Big] \\
&= 2(-1)^p \sum_{\substack{i=2 \\ i\equiv p \bmod 2}}^{a} s_{k,i}L(f,i) + \sum_{i=0}^{N-1} s_{k,a+1+i}f(i)
\end{aligned} \tag{1.6}$$

where $S^{(k-1)}$ is the $(k-1)$-th derivative of a function S.

Let us prove Lemma 1. The partial fraction expansion of F reads

$$F(t) = \sum_{h=0}^{n/N}\sum_{j=1}^{a} \frac{p_{j,h}}{(t+Nh)^j}$$

with rational coefficients $p_{j,h}$; we consider

$$P_j(z) = \sum_{h=0}^{n/N} p_{j,h}z^{Nh} \in \mathbb{Q}[z]_{\leq n} \text{ for any } j \in \{1, \dots, a\}.$$

Let $P_{1,j} = P_j$ for any $j \in \{1, \dots, a\}$, and define inductively $P_{k,j} \in \mathbb{Q}(z)$ by

$$P_{k,j}(z) = P'_{k-1,j}(z) - \frac{1}{z} P_{k-1,j+1}(z) \text{ for any } k \geq 2 \text{ and any } j \in \{1, \dots, a\}, \tag{1.7}$$

where $P_{k-1,a+1} = 0$ for any k. We let also[1]

$$U_1(z) = -\sum_{t=1}^{n} z^t \sum_{j=1}^{a} \sum_{h=0}^{\lfloor (t-1)/N \rfloor} \frac{p_{j,h}}{(Nh-t)^j} \in \mathbb{Q}[z]_{\leq n} \tag{1.8}$$

$$\text{and } V_1(z) = -\sum_{t=0}^{n-1} z^t \sum_{j=1}^{a} \sum_{h=\lceil (t+1)/N \rceil}^{n/N} \frac{p_{j,h}}{(Nh-t)^j} \in \mathbb{Q}[z]_{\leq n}, \tag{1.9}$$

and define U_k, V_k for any $k \geq 2$ by the recurrence relations

$$U_k(z) = U'_{k-1}(z) - \frac{1}{1-z} P_{k-1,1}(z), \tag{1.10}$$

$$V_k(z) = V'_{k-1}(z) + \frac{1}{z(1-z)} P_{k-1,1}(z). \tag{1.11}$$

Then for any $k \geq 1$ we have (as in [2, 10])

$$S_0^{(k-1)}(z) = U_k(z) + \sum_{j=1}^{a} P_{k,j}(z)(-1)^j \mathrm{Li}_j(z) \tag{1.12}$$

$$\text{and } S_\infty^{(k-1)}(z) = V_k(z) + \sum_{j=1}^{a} P_{k,j}(z) \mathrm{Li}_j(1/z). \tag{1.13}$$

Since $P_j(z) \in \mathbb{Q}[z^N]$ for any $j \in \{1, \dots, a\}$, Eq. (1.7) yields $P_{k,j} \in z^{1-k}\mathbb{Q}[z^N]$. This property is very important to us since we shall evaluate $P_{k,j}$ at N-th roots of unity. To evaluate in the same way the rational functions $U_k, V_k \in \mathbb{Q}[z, z^{-1}]$ for $k \leq d_0 - 1$, we write

$$z^{k-1} U_k(z) = \sum_{\lambda=0}^{N-1} z^\lambda U_{k,\lambda}(z) \text{ and } z^{k-1} V_k(z) = \sum_{\lambda=0}^{N-1} z^\lambda V_{k,\lambda}(z) \tag{1.14}$$

with $U_{k,\lambda}, V_{k,\lambda} \in \mathbb{Q}[z^N, z^{-N}]$. Then Eqns. (1.12) and (1.13) yield

$$z^{k-1} S_0^{(k-1)}(z) = \sum_{\lambda=0}^{N-1} z^\lambda U_{k,\lambda}(z) + \sum_{j=1}^{a} z^{k-1} P_{k,j}(z)(-1)^j \mathrm{Li}_j(z) \tag{1.15}$$

$$\text{and } z^{k-1} S_\infty^{(k-1)}(z) = \sum_{\lambda=0}^{N-1} z^\lambda V_{k,\lambda}(z) + \sum_{j=1}^{a} z^{k-1} P_{k,j}(z) \mathrm{Li}_j(1/z). \tag{1.16}$$

[1] There is a misprint in the formula that gives $U(z)$ in [8].

We may now define the coefficients $s_{k,i}$ for any $k \geq 1$ by:

$$\begin{cases} s_{k,i} = \delta_n P_{k,i}(1) \text{ for } 2 \leq i \leq a, \\ s_{k,a+1+\lambda} = \delta_n(U_{k,\lambda}(1) + (-1)^p V_{k,N-\lambda}(1)) \text{ for } 0 \leq \lambda \leq N-1, \end{cases} \tag{1.17}$$

where $V_{k,N} = V_{k,0}$; recall that $\delta_n = (Nd_{n/N})^{a+1} N^{(a+1)n/N}$ with $d_j = \mathrm{lcm}(1, 2, \dots, j)$. Since $P_{k,j}(z)$, $U_{k,\lambda}(z)$ and $V_{k,N-\lambda}(z)$ are polynomials with rational coefficients, the numbers $s_{k,2}$, $\dots$, $s_{k,a+N}$ are rational. We shall prove in Lemma 3 that they are integers, thanks to the factor δ_n. We also point out that $s_{k,i}$ is not defined for $i = 1$; actually $P_{k,1}(1) = 0$ for the values of k we are interested in (see (1.18) below).

To conclude the proof of Lemma 1, we shall evaluate Eqns. (1.15) and (1.16) at roots of unity. At the point 1 this is possible since, as in [8, §4.3],

$$\text{for any } k \leq d_0 - 1, \qquad P_{k,1}(1) = 0 \quad \text{and} \quad U_k, V_k \text{ do not have a pole at } z = 1. \tag{1.18}$$

Now let $k \leq d_0 - 1$, and $z \in \mathbb{C}$ be such that $|z| = 1$. Then Eqns. (1.12) to (1.16) hold, upon agreeing that the sums start at $j = 2$ if $z = 1$; this remark will be used below when z is a N-th root of unity.

Let Λ_k be the right hand side of Eq. (1.6). Using (1.18) the definition (1.17) of $s_{k,i}$ yields

$$\Lambda_k = 2\delta_n(-1)^p \sum_{\substack{1 \leq j \leq a \\ j \equiv p \bmod 2}} P_{k,j}(1) L(f,j) + \delta_n \sum_{\lambda=0}^{N-1} (U_{k,\lambda}(1) + (-1)^p V_{k,N-\lambda}(1)) f(\lambda).$$

Now Eq. (1.5) yields

$$\sum_{\ell=1}^{N} \widehat{f}(\ell)\omega^{m\ell} = f(m) \text{ for any } m \in \mathbb{Z}, \text{ so that } \sum_{\ell=1}^{N} \widehat{f}(\ell)\mathrm{Li}_j(\omega^\ell) = \sum_{m=1}^{\infty} \frac{f(m)}{m^j} = L(f,j) \text{ for any } j \leq a.$$

Therefore we have, since $V_{k,N} = V_{k,0}$:

$$\begin{aligned} \Lambda_k = \quad & \delta_n \sum_{j=1}^{a} P_{k,j}(1)((-1)^j + (-1)^p) \sum_{\ell=1}^{N} \widehat{f}(\ell)\mathrm{Li}_j(\omega^\ell) \\ & + \delta_n \sum_{\lambda=0}^{N-1} \Big[\Big(\sum_{\ell=1}^{N} \widehat{f}(\ell)\omega^{\ell\lambda} \Big) U_{k,\lambda}(1) + (-1)^p \Big(\sum_{\ell=1}^{N} \widehat{f}(\ell)\omega^{-\ell\lambda} \Big) V_{k,\lambda}(1) \Big]. \end{aligned}$$

Then Eqns. (1.15) and (1.16) yield, since $U_{k,\lambda}(z)$, $V_{k,\lambda}(z)$, and $z^{k-1}P_{k,j}(z)$ depend only on z^N and ω is a N-th root of unity:

$$\Lambda_k = \delta_n \sum_{\ell=1}^{N} \widehat{f}(\ell) \Big[\omega^{\ell(k-1)} S_0^{(k-1)}(\omega^\ell) + (-1)^p \omega^{\ell(1-k)} S_\infty^{(k-1)}(\omega^{-\ell}) \Big].$$

This concludes the proof of Lemma 1.

Remark 2. *The only difference here with the construction of [8] is that the rational function F has been modified to incorporate Sprang's arithmetic lemma [23, Lemma 1.4]. In our setting this choice of F leads to the following additional property, that will be used in §1.4:*

$$U_1(z) + V_1(z) \in \mathbb{Q}[z^N]. \tag{1.19}$$

To prove this property we notice that

$$U_1(z) + V_1(z) = -\sum_{t=1}^{n} z^t \sum_{j=1}^{a} \sum_{h \neq t/N} \frac{p_{j,h}}{(Nh-t)^j};$$

for any t which is not a multiple of N, the coefficient of z^t is $-F(-t) = 0$.

1.3 A general version of Shidlosvky's lemma

Let q be a positive integer, and $A \in M_q(\mathbb{C}(z))$. We fix[2] $P_1, \ldots, P_q \in \mathbb{C}[z]$ and $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ such that $\deg P_i \leq n$ for any i. Then with any solution $Y = {}^t(y_1, \ldots, y_q)$ of the differential system $Y' = AY$ is associated a remainder $R(Y)$ defined by

$$R(Y)(z) = \sum_{i=1}^{q} P_i(z) y_i(z).$$

Let Σ be a finite subset of $\mathbb{C} \cup \{\infty\}$, which may contain singularities of the differential system $Y' = AY$. For each $\sigma \in \Sigma$, let $(Y_j)_{j \in J_\sigma}$ be a family of solutions of $Y' = AY$ such that the functions $R(Y_j)$, $j \in J_\sigma$, are $\mathbb{C}$-linearly independent and belong to the Nilsson class at σ (i.e., have a local expression at σ as linear combination of holomorphic functions, with coefficients involving complex powers of $z - \sigma$ and integer powers of $\log(z - \sigma)$). We agree that $J_\sigma = \emptyset$ if $\sigma \notin \Sigma$, and define rational functions $P_{k,i} \in \mathbb{C}(z)$ for $k \geq 1$ and $1 \leq i \leq q$ by

$$\begin{pmatrix} P_{k,1} \\ \vdots \\ P_{k,q} \end{pmatrix} = \left(\frac{\mathrm{d}}{\mathrm{d}z} + {}^tA \right)^{k-1} \begin{pmatrix} P_1 \\ \vdots \\ P_q \end{pmatrix}. \tag{1.20}$$

These rational functions $P_{k,i}$ play an important role because they are used to differentiate the remainders (see [22, Chapter 3, §4]):

$$R(Y)^{(k-1)}(z) = \sum_{i=1}^{q} P_{k,i}(z) y_i(z). \tag{1.21}$$

The following result is proved in [8, Theorem 1.2].

[2]We shall check in §1.4 that the notation introduced in the present section is consistent with the one used earlier in this paper.

Theorem 3. *There exists a positive constant c_2, which depends only on A and Σ, with the following property. Assume that, for some $\alpha \in \mathbb{C}$:*

(*i*) *The differential system $Y' = AY$ has a basis of local solutions at α with coordinates in $\mathbb{C}[\log(z-\alpha)][[(z-\alpha)^e]]$ for some positive real number e.*

(*ii*) *We have*

$$\sum_{\sigma\in\Sigma}\sum_{j\in J_\sigma} \mathrm{ord}_\sigma(R(Y_j)) \geq (n+1)q - n\#J_\infty - \tau$$

for some τ with $0 \leq \tau \leq n - c_2$.

(*iii*) *All rational functions $P_{k,i}$, with $1 \leq i \leq q$ and $1 \leq k < \tau + c_2$, are holomorphic at $z = \alpha$.*

Then the matrix $[P_{k,i}(\alpha)]_{1\leq i\leq q, 1\leq k<\tau+c_2} \in M_{q,\tau+c_2-1}(\mathbb{C})$ has rank at least $q - \#J_\alpha$.

In the special case where $\Sigma = \{0\}$, $\#J_0 = 1$, Y_j is analytic at 0, and $\alpha \neq 0$ is not a singularity of the differential system $Y' = AY$, this result was used by Shidlovsky to prove the Siegel-Shidlovsky theorem on values of E-functions (see [22, Chapter 3, Lemma 10]). The functional part of Shidlovsky's lemma has been generalized by Bertand-Beukers [4] to the case where $\Sigma \subset \mathbb{C}$, $\#J_\sigma = 1$ for any σ, and all functions Y_j are obtained by analytic continuation from a single one, analytic at all $\sigma \in \Sigma$. Then Bertrand has allowed [3, Théorème 2] an arbitrary number of solutions at each σ, assuming that $\infty \notin \Sigma$ and the functions Y_j, $j \in J_\sigma$, are analytic at σ. The proof [8] of Theorem 3 follows the approach of Bertand-Beukers and Bertrand, based on differential Galois theory.

An important feature of Theorem 3 is that α may be a singularity of the differential system $Y' = AY$, and/or an element of Σ. Both happen in the present paper, where $\alpha = 1$ (see §1.4 where Theorem 3 is applied to prove Lemma 2). If $\alpha \notin \Sigma$ then $J_\alpha = \emptyset$ so that Theorem 3 yields a matrix of maximal rank q. On the other hand, if $\alpha \in \Sigma$ then the $\#J_\alpha$ linearly independent linear combinations of the rows of the matrix $[P_{k,i}(z)]_{i,k}$ corresponding to $R(Y_j)$, $j \in J_\alpha$, may vanish at α: the lower bound $q - \#J_\alpha$ is best possible. In the setting of §1.4 we have $\alpha = 1$ and $J_1 = \{1\}$ so that Theorem 3 yields $\mathrm{rk}[P_{k,i}(1)] \geq q-1$. Now (1.18) in the proof of Lemma 1 shows that $P_{k,1}(1) = 0$ for any $k < \tau + c_2$ (since $\tau + c_2 \leq d_0$ because τ and c_2 are independent from n whereas d_0 tends to ∞ with n). Therefore the matrix $[P_{k,i}(1)]_{1\leq i\leq q, 1\leq k<\tau+c_2}$ has rank equal to $q-1$. Removing the first row, which is identically zero, yields a matrix of rank $q-1$ equal to the number of rows.

1.4 Padé approximation and application of Shidlovsky's lemma

In this section we prove part (i) of Proposition 1 for the numbers $s_{k,i}$ constructed in §1.2.

Lemma 2. *Let $s_{k,i}$ be defined by Eq. (1.17). Then there exists a positive constant c_1 (which depends only on a and N) such that for any n sufficiently large, the subspace $\mathcal{F}$ of $\mathbb{R}^{a+N-1}$ spanned by the vectors ${}^t(s_{k,2}, \ldots, s_{k,a+N})$, $1 \leq k \leq c_1$, is non-zero and does not depend on n.*

The proof of Lemma 2 falls into 3 steps. To begin with, we construct a Padé approximation problem related to our construction, with essentially as many equations as the number of unknowns; notice that this problem is not the same as in [8], since the function F used in the construction is different. Then we apply a general version of Shidlovsky's lemma, namely Theorem 3 stated in §1.3. This provides a matrix P with linearly independent rows. At last, we relate the numbers $s_{k,i}$ to P by constructing a matrix M such that $[s_{k,i}]_{i,k} = MP$. The point is that M does not depend on n (whereas P and $[s_{k,i}]$ do). The subspace spanned by the columns of $[s_{k,i}]_{i,k}$ is the same as the one spanned by the columns of M: it does not depend on n.

Step 1: Construction of the Padé approximation problem.

We recall from §1.2 that

$$F(t) = (n/N)!^{(a+1)-(2r+1)N} \frac{(t-rn)_{(2r+1)n+1}}{\prod_{h=0}^{n/N}(t+Nh)^{a+1}},$$

$$S_0(z) = \sum_{t=n+1}^{\infty} F(-t)z^t = U_1(z) + \sum_{j=1}^{a} P_j(z)(-1)^j \mathrm{Li}_j(z),$$

$$\text{and } S_\infty(z) = \sum_{t=1}^{\infty} F(t)z^{-t} = V_1(z) + \sum_{j=1}^{a} P_j(z)\mathrm{Li}_j(1/z).$$

Since $P_j(z) \in \mathbb{C}[z^N]$ for any $j \in \{1,\ldots,a\}$, we have $P_j(\omega^\ell z) = P_j(z)$ for any $\ell \in \mathbb{Z}$. Therefore letting

$$R_{0,\ell}(z) = S_0(\omega^\ell z), \qquad R_{\infty,\ell}(z) = S_\infty(\omega^\ell z), \qquad \overline{P}_{0,\ell}(z) = U_1(\omega^\ell z), \qquad \overline{P}_{\infty,\ell}(z) = V_1(\omega^\ell z) \tag{1.22}$$

for any $\ell \in \{1,\ldots,N\}$, we have

$$R_{0,\ell}(z) = \overline{P}_{0,\ell}(z) + \sum_{j=1}^{a} P_j(z)(-1)^j \mathrm{Li}_j(\omega^\ell z) = O(z^{(r+1)n+1}), \quad z \to 0, \tag{1.23}$$

$$\text{and } R_{\infty,\ell}(z) = \overline{P}_{\infty,\ell}(z) + \sum_{j=1}^{a} P_j(z)\mathrm{Li}_j(\frac{1}{\omega^\ell z}) = O(z^{-rn-1}), \quad z \to \infty. \tag{1.24}$$

Moreover, recall that $d_0 = -\deg F = (a+1)(\frac{n}{N}+1) - (2r+1)n - 1$; Lemma 3 of [10] shows that

$$\sum_{j=1}^{a} P_j(z)(-1)^{j-1}\frac{(\log z)^{j-1}}{(j-1)!} = O((z-1)^{d_0-1}), \quad z \to 1.$$

Using again the fact that $P_j(\omega^{-\ell}z) = P_j(z)$, we obtain for any $\ell \in \{1,\ldots,N\}$:

$$R_{\omega^\ell}(z) := \sum_{j=1}^{a} P_j(z)(-1)^{j-1}\frac{(\log(\omega^{-\ell}z))^{j-1}}{(j-1)!} = O((z-\omega^\ell)^{d_0-1}), \quad z \to \omega^\ell. \tag{1.25}$$

The new point here, with respect to [8], is that Eq. (1.19) in Remark 2 shows that $\overline{P} = \overline{P}_{0,\ell} + \overline{P}_{\infty,\ell}$ does not depend on ℓ. Therefore Eq. (1.23) can be written as

$$R_{0,\ell}(z) = \overline{P}(z) - \overline{P}_{\infty,\ell}(z) + \sum_{j=1}^{a} P_j(z)(-1)^j \mathrm{Li}_j(\omega^\ell z) = O(z^{(r+1)n+1}), \quad z \to 0. \qquad (1.26)$$

We have obtained a Padé approximation problem with $(a+N+1)(n+1)$ unknowns, namely the coefficients of $\overline{P}(z)$, $P_j(z)$ for $1 \le j \le a$, and $\overline{P}_{\infty,\ell}$ for $1 \le \ell \le N$. Eqns. (1.24), (1.25) and (1.26) amount to

$$N((r+1)n+1) + N(d_0 - 1) + N((r+1)n+1) = (a+N+1)(n+1) - \tau$$

linear equations, where $\tau = a + 1 - aN$ is the difference between the number of unknowns and the number of equations. If $N = 1$ then $\tau = 1$: this is exactly the Padé approximation problem of [10, Theorem 1], which has a unique solution up to proportionality. However if $N \ge 2$ then $\tau < 0$: we have solved a linear system with (slightly) more equations than the number of unknowns.

Step 2: Application of Shidlovsky's lemma.

Let us introduce some notation to fit into the context of §1.3, and check the assumptions of Theorem 3. Let $q = a+N+1$, and $A \in M_q(\mathbb{C}(z))$ be the matrix of which the coefficients $A_{i,j}$ are given by:

$$\begin{cases} A_{i,i-1}(z) = \frac{-1}{z} \text{ for any } i \in \{2, \ldots, a\} \\ A_{1,a+1}(z) = \frac{1}{z} \\ A_{1,a+1+\ell}(z) = \frac{1}{z(1-\omega^\ell z)} \text{ for any } \ell \in \{1, \ldots, N\} \end{cases}$$

and all other coefficients are zero. We consider the following solutions of the differential system $Y' = AY$, with $1 \le \ell \le N$:

$$Y_{0,\ell}(z) = {}^t\Big(-\mathrm{Li}_1(\omega^\ell z), \mathrm{Li}_2(\omega^\ell z), \ldots, (-1)^a \mathrm{Li}_a(\omega^\ell z), 1, 0, \ldots, 0, -1, 0, \ldots, 0 \Big),$$

$$Y_{\infty,\ell}(z) = {}^t\Big(\mathrm{Li}_1(\frac{1}{\omega^\ell z}), \mathrm{Li}_2(\frac{1}{\omega^\ell z}), \ldots, \mathrm{Li}_a(\frac{1}{\omega^\ell z}), 0, 0, \ldots, 0, 1, 0, \ldots, 0 \Big),$$

$$Y_{\omega^\ell}(z) = {}^t\Big(1, -\log(\omega^{-\ell} z), \frac{(\log(\omega^{-\ell} z))^2}{2!}, \ldots, (-1)^{a-1} \frac{(\log(\omega^{-\ell} z))^{a-1}}{(a-1)!}, 0, 0, \ldots, 0 \Big)$$

where the coefficient -1 in $Y_{0,\ell}(z)$ (resp. 1 in $Y_{\infty,\ell}(z)$) is in position $a+1+\ell$.

We let $J_0 = \{(0,1), (0,2), \ldots, (0,N)\}$, $J_\infty = \{(\infty,1), (\infty,2), \ldots, (\infty,N)\}$, $J_{\omega^\ell} = \{\omega^\ell\}$ for $1 \le \ell \le N$, and $\Sigma = \{0, \infty\} \cup \{\omega^\ell, 1 \le \ell \le N\}$, so that we have a solution Y_j for each $j \in J_\sigma$, $\sigma \in \Sigma$.

We also let $P_{a+1}(z) = \overline{P}(z)$ (which is equal to $\overline{P}_{0,\ell}(z) + \overline{P}_{\infty,\ell}(z) = (U_1 + V_1)(\omega^\ell z)$ for any ℓ), and $P_{a+1+\ell}(z) = \overline{P}_{\infty,\ell}(z) = V_1(\omega^\ell z)$ for any $\ell \in \{1, \ldots, N\}$. Then we have polynomials $P_1(z)$, …, $P_q(z)$ of degree at most n, and with the notation of §1.3 the remainders associated with the local solutions Y_j, $j \in J_\sigma$, $\sigma \in \Sigma$, are exactly the functions that appear in the Padé approximation problem of Step 1: $R(Y_{0,\ell}) = R_{0,\ell}(z)$, $R(Y_{\infty,\ell}) = R_{\infty,\ell}(z)$, and $R(Y_{\omega^\ell}) = R_{\omega^\ell}(z)$ for any $\ell \in \{1, \dots, N\}$.

Since P_a is not the zero polynomial, we have $R_{\omega^\ell}(z) \neq 0$ for any ℓ; the functions $R_{0,1}(z)$, …, $R_{0,N}(z)$ (resp. $R_{\infty,1}(z)$, …, $R_{\infty,N}(z)$) are $\mathbb{C}$-linearly independent (see [8, Lemma 2]).

Eqns. (1.24), (1.25) and (1.26) yield $\mathrm{ord}_\infty(R_{\infty,\ell}(z)) \geq rn + 1$, $\mathrm{ord}_{\omega^\ell}(R_{\omega^\ell}(z)) \geq d_0 - 1$ and $\mathrm{ord}_0(R_{0,\ell}(z)) \geq (r+1)n + 1$ for any $\ell \in \{1, \dots, N\}$, so that

$$\sum_{\sigma \in \Sigma} \sum_{j \in J_\sigma} \mathrm{ord}_\sigma R_j(z) \geq (2r+1)Nn + N(d_0+1) = (n+1)q - nN - \tau \text{ with } \tau = a + 1 - aN;$$

here $q = a + N + 1$, and we recall that $d_0 = -\deg F = (a+1)(\frac{n}{N} + 1) - (2r+1)n - 1$. As above, τ is exactly the difference between the number of unknowns and the number of equations in the Padé approximation problem of Step 1.

The definition (1.20) of $P_{k,i}$ is consistent with the one given (for $i \leq a$) by Eq. (1.7) in §1.2. We have $\tau = a + 1 - aN$, so that for n sufficiently large $\tau + c_2 \leq d_0$ where c_2 is the constant given by Theorem 3. Therefore (1.18) shows that U_k and V_k are holomorphic at $z = 1$ for any $k < \tau + c_2$. Eqns. (1.7), (1.10) and (1.11) imply that they are holomorphic at all other roots of unity. Now Eqns. (1.20), (1.10) and (1.11) yield

$$P_{k,a+1}(z) = \omega^{\ell(k-1)}(U_k(\omega^\ell z) + V_k(\omega^\ell z)) \text{ and } P_{k,a+1+\ell}(z) = \omega^{\ell(k-1)} V_k(\omega^\ell z) \qquad (1.27)$$

for any $\ell \in \{1, \dots, N\}$, by induction on $k \geq 1$. Therefore all $P_{k,i}$, with $k < \tau + c_2$ and $1 \leq i \leq q$, are holomorphic at 1.

We have checked all assumptions of Theorem 3 for n sufficiently large: the matrix $[P_{k,i}(1)]_{1 \leq i \leq q, 1 \leq k < \tau + c_2}$ has rank at least $q - \#J_1 = q - 1$. Now (1.18) implies $P_{k,1}(1) = 0$ for any $k < \tau + c_2$, so that we may remove the first row: the matrix $P = [P_{k,i}(1)]_{2 \leq i \leq q, 1 \leq k < \tau + c_2}$ has rank $q - 1$, equal to its number of rows.

Step 3: Expression of $s_{k,i}$ in terms of P and conclusion.

We shall now compute a matrix M *independent from* n such that $[s_{k,i}]_{i,k} = MP$; recall that the coefficients $s_{k,i}$ and the matrix P depend on n.

To begin with, Eq. (1.14) with $z = \omega^\ell$ yields

$$\omega^{(k-1)\ell} U_k(\omega^\ell) = \sum_{\lambda=0}^{N-1} \omega^{\lambda\ell} U_{k,\lambda}(1) \text{ for any } \ell \in \mathbb{Z},$$

since $U_{k,\lambda}(z) \in \mathbb{Q}[z^N, z^{-N}]$. Therefore we have

$$U_{k,\lambda}(1) = \frac{1}{N} \sum_{\ell=1}^{N} \omega^{(k-1-\lambda)\ell} U_k(\omega^\ell) \text{ for any } 0 \leq \lambda \leq N-1, \qquad (1.28)$$

and the same relation holds with $V_{k,\lambda}$ and V_k. Using Eq. (1.27) we deduce that

$$V_{k,\lambda}(1) = \frac{1}{N}\sum_{\ell=1}^{N} \omega^{-\lambda\ell} P_{k,a+1+\ell}(1) \text{ for } 0 \leq \lambda \leq N-1,$$

and also for $\lambda = N$ since $V_{k,N} = V_{k,0}$, and

$$U_{k,\lambda}(1) = \begin{cases} -\frac{1}{N}\sum_{\ell=1}^{N} \omega^{-\lambda\ell} P_{k,a+1+\ell}(1) \text{ if } 1 \leq \lambda \leq N-1, \\ P_{k,a+1}(1) - \frac{1}{N}\sum_{\ell=1}^{N} P_{k,a+1+\ell}(1) \text{ if } \lambda = 0. \end{cases}$$

Therefore the definition (1.17) of $s_{k,i}$ can be translated as

$$s_{k,i} = \sum_{j=2}^{q} m_{i,j} P_{k,j}(1) \tag{1.29}$$

for any $2 \leq i \leq a+N$ and any $1 \leq k \leq d_0 - 1$, where the coefficients $m_{i,j}$ are defined for $2 \leq i \leq a+N$ and $2 \leq j \leq q = a+N+1$ by

$$\begin{cases} m_{i,i} = \delta_n \text{ for } 2 \leq i \leq a+1 \\ m_{a+1,a+1+\ell} = \frac{\delta_n}{N}((-1)^p - 1) \text{ for } 1 \leq \ell \leq N \\ m_{a+1+\lambda,a+1+\ell} = \frac{\delta_n}{N}((-1)^p \omega^{\lambda\ell} - \omega^{-\lambda\ell}) \text{ for } 1 \leq \ell \leq N \text{ and } 1 \leq \lambda \leq N-1 \\ m_{i,j} = 0 \text{ for all other pairs } (i,j). \end{cases} \tag{1.30}$$

Let us choose now the constant c_1 of Lemma 2; the same constant appears in Proposition 1. We take $c_1 = \tau + c_2 - 1$; this constant depends only on a and N. We consider the matrices $M = [m_{i,j}]_{2 \leq i \leq a+N, 2 \leq j \leq q}$ and $P = [P_{k,j}(1)]_{2 \leq j \leq q, 1 \leq k \leq c_1}$. Then Eq. (1.29) means that

$$[s_{k,i}]_{2 \leq i \leq a+N, 1 \leq k \leq c_1} = MP. \tag{1.31}$$

Both M and P have coefficients in $\mathbb{Q}(\omega)$; recall that the coefficients $s_{k,i}$ of MP are rational numbers, and we shall prove in §1.5 that they are integers. Let $\mathcal{F}$ denote the subspace of $\mathbb{R}^{a+N-1}$ spanned by the $q-1$ columns ${}^t(m_{2,j}, \ldots, m_{a+N,j})$ of M. Now assume that n is sufficiently large; then we have proved in Step 2 that the $q-1$ rows of P are linearly independent. Therefore Eq. (1.31) shows that $\mathcal{F}$ is equal to the subspace spanned by columns ${}^t(s_{k,2}, \ldots, s_{k,a+N})$ of the matrix $[s_{k,i}]_{i,k}$. Since M does not depend on n, neither does $\mathcal{F}$: this concludes the proof of Lemma 2.

Remark 3. *Let us prove that in Lemma 2, the subspace $\mathcal{F}$ is not always equal to $\mathbb{R}^{a+N-1}$ (i.e., that the matrix $[s_{k,i}]$ may have rank less than its number of rows, namely $a+N-1$). Consider the case where p and N are even (so that $\omega^{N/2} = -1$). Then the definition* (1.30) *of the matrix M in Step 3 above yields $m_{a+1+N/2,j} = 0$ for any j, so that Eq.* (1.29) *implies $s_{k,a+1+N/2} = 0$ for any k: the matrix $[s_{k,i}]_{i,k}$ has a zero row. This phenomenon does not occur in [8]; it comes from the new property* (1.19) *obtained in Remark 2. Indeed a direct* *proof that* $s_{k,a+1+N/2} = 0$ *can be obtained as follows, using Eqns.* (1.17), (1.28), *and* (1.27) *but not the matrix* M*:*

$$\begin{aligned} s_{k,a+1+N/2} &= \delta_n \Big(U_{k,N/2}(1) + V_{k,N/2}(1) \Big) \\ &= \frac{\delta_n}{N} \sum_{\ell=1}^{N} \omega^{(k-1-N/2)\ell} (U_k + V_k)(\omega^\ell) = \frac{\delta_n}{N} P_{k,a+1}(1) \sum_{\ell=1}^{N} (-1)^\ell = 0. \end{aligned}$$

1.5 Arithmetic and Asymptotic Properties

In this section we conclude the proof of Proposition 1 stated in §1.1, by proving parts (ii) and (iii) and the fact that the $s_{k,i}$ are integers. Recall that

$$\alpha = (4e)^{(a+1)/N} (2N)^{2r+2} r^{-(a+1)/N+4(r+1)} \text{ and } \beta = (2e)^{(a+1)/N} (r+1)^{2r+2} N^{2r+2}.$$

Lemma 3. *We have* $s_{k,i} \in \mathbb{Z}$ *for any* $i \in \{2, \dots, a+N\}$ *and any* $k \leq d_0 - 1$*, and as* $n \to \infty$*:*

$$\Big| 2(-1)^p \sum_{\substack{i=2 \\ i \equiv p \bmod 2}}^{a} s_{k,i} L(f,i) + \sum_{i=0}^{N-1} s_{k,a+1+i} f(i) \Big| \leq \alpha^{n+o(n)} \text{ and } \max_{2 \leq i \leq a+N} |s_{k,i}| \leq \beta^{n(1+o(1))}.$$

In this lemma and throughout this section, we denote by $o(1)$ any sequence that tends to 0 as $n \to \infty$; it usually depends also on a, r, N, and k (but the dependence in k is not significant since k is bounded by $d_0 - 1$, which depends only on n, a, r, N). We also recall that d_n is the least common multiple of 1, 2, …, n.

We shall prove two lemmas now; the deduction of Lemma 3 from these lemmas (using Lemma 1 proved in §1.2) is exactly the same as the proof of Proposition 1 in [8, §4.5]. Recall from §1.2 that

$$F(t) = (n/N)!^{(a+1)-(2r+1)N} \frac{(t - rn)_{(2r+1)n+1}}{\prod_{h=0}^{n/N} (t + Nh)^{a+1}} = \sum_{h=0}^{n/N} \sum_{j=1}^{a} \frac{p_{j,h}}{(t+Nh)^j}.$$

Lemma 4. *For any* $j \in \{1, \dots, a\}$ *and any* $h \in \{0, \dots, n/N\}$ *we have*

$$(N d_{n/N})^{a+1-j} N^{(a+1)n/N} p_{j,h} \in \mathbb{Z} \tag{1.32}$$

$$\text{and } |p_{j,h}| \leq \Big(2^{(a+1)/N} N^{2(r+1)-(a+1)/N} (r+1)^{2r+2} \Big)^{n(1+o(1))} \tag{1.33}$$

where $o(1)$ *is a sequence that tends to 0 as* $n \to \infty$ *and may depend also on* N*,* a*, and* r*.*

Proof of Lemma 4: We follow the approach of [5] by letting

$$
\begin{aligned}
F_0(t) &= \tfrac{(n/N)!}{\prod_{h=0}^{n/N}(t+Nh)} &&= \sum_{h=0}^{n/N} \frac{(-1)^h N^{-n/N}\binom{n/N}{h}}{t+Nh}, \\
G_i(t) &= \tfrac{(t-in/N)_{n/N}}{\prod_{h=0}^{n/N}(t+Nh)} &&= \sum_{h=0}^{n/N} \frac{(-1)^{h+n/N} N^{-n/N}\binom{n/N}{h}\binom{Nh+in/N}{n/N}}{t+Nh} \text{ for } 1 \le i \le rN, \\
H_i(t) &= \tfrac{(t+1+in/N)_{n/N}}{\prod_{h=0}^{n/N}(t+Nh)} &&= \sum_{h=0}^{n/N} \frac{(-1)^h N^{-n/N}\binom{n/N}{h}\binom{-Nh+(i+1)n/N}{n/N}}{t+Nh} \text{ for } 0 \le i \le (r+1)N-1.
\end{aligned}
$$

Then the partial fraction expansion of

$$F(t) = F_0(t)^{a+1-(2r+1)N} t G_1(t) \dots G_{rN}(t) H_0(t) \dots H_{(r+1)N-1}(t)$$

can be obtained by multiplying those of F_0, G_i and H_i using repeatedly the formulas $\frac{t}{t+Nh} = 1 - \frac{Nh}{t+Nh}$ and

$$\frac{1}{(t+Nh)(t+Nh')^\ell} = \frac{1}{N^\ell (h'-h)^\ell (t+Nh)} - \sum_{i=1}^{\ell} \frac{1}{N^{\ell+1-i}(h'-h)^{\ell+1-i}(t+Nh')^i} \quad (1.34)$$

with $h \neq h'$. The denominator of $p_{j,h}$ comes both from this formula (and this contribution divides $(Nd_{n/N})^{a+1-j}$) and from the denominators of the coefficients in the partial fraction expansions of F_0, G_i, H_i (which belong to $N^{-n/N}\mathbb{Z}$, so that $N^{(a+1)n/N}$ accounts for this contribution). This concludes the proof of (1.32).

On the other hand, bounding from above the coefficients of the partial fraction expansions of F_0, G_i, H_i yields

$$|p_{j,h}| \le n^{O(1)} N^{-(a+1)n/N} 2^{(a+1)n/N} \prod_{i=1}^{rN} \frac{(n+in/N)!}{(n/N)!(n+(i-1)n/N)!} \prod_{i=0}^{(r+1)N-1} \frac{((i+1)n/N)!}{(n/N)!(in/N)!}$$

where $O(1)$ is a constant depending only on a, r, N which can be made explicit (see [5] for details). Simplifying the products and using the bound $\frac{m!}{m_1! \dots m_c!} \le c^m$ valid when $m_1 + \dots + m_c = m$, one obtains

$$|p_{j,h}| \le n^{O(1)} (2/N)^{(a+1)n/N} \frac{((r+1)n)!^2}{n!(n/N)!^{(2r+1)N}} \le n^{O(1)} (2/N)^{(a+1)n/N} ((r+1)N)^{2(r+1)n}.$$

This concludes the proof of Lemma 4.

The proof of the following lemma is inspired by that of [23, Lemma 1.4]. Recall that U_1 and V_1 are defined in Eqns. (1.8) and (1.9), and that

$$\delta_n = (Nd_{n/N})^{a+1} N^{(a+1)n/N}.$$

Lemma 5. *The polynomials $\delta_n U_1(z)$ and $\delta_n V_1(z)$ have integer coefficients.*

Proof of Lemma 5: Recall from Eq. (1.8) that

$$U_1(z) = -\sum_{t=1}^{n} z^t \sum_{j=1}^{a} \sum_{h=0}^{\lfloor (t-1)/N \rfloor} \frac{p_{j,h}}{(Nh-t)^j}.$$

Assume that $\delta_n U_1(z)$ does not have integer coefficients. Then there exists $t \in \{1, \ldots, n\}$ such that

$$\sigma := \sum_{j=1}^{a} \sum_{h=0}^{\lfloor (t-1)/N \rfloor} \frac{(Nd_{n/N})^{a+1} N^{(a+1)n/N} p_{j,h}}{(Nh-t)^j} \notin \mathbb{Z}.$$

Let $p'_{j,h} = (Nd_{n/N})^{a+1-j} N^{(a+1)n/N} p_{j,h}$, which is an integer thanks to Lemma 4. Then we have

$$\sigma = \sum_{j=1}^{a} \sum_{h=0}^{\lfloor (t-1)/N \rfloor} \frac{d_{n/N}^j p'_{j,h}}{(h-t/N)^j}.$$

If N divides t then $\frac{t}{N} - h$ is a positive integer less than or equal to n/N, so that it divides $d_{n/N}$: this contradicts the assumption $\sigma \notin \mathbb{Z}$. Therefore N does not divide t, so that $F(-t) = 0$.

Since $\sigma \notin \mathbb{Z}$ there exists h_0 such that

$$\sum_{j=1}^{a} \frac{d_{n/N}^j p'_{j,h_0}}{(h_0 - t/N)^j} \notin \mathbb{Z}.$$

Now $F(-t) = 0$ so that

$$\sum_{j=1}^{a} \sum_{\substack{h=0 \\ h \neq h_0}}^{n/N} \frac{d_{n/N}^j p'_{j,h}}{(h-t/N)^j} = -\sum_{j=1}^{a} \frac{d_{n/N}^j p'_{j,h_0}}{(h_0 - t/N)^j} \notin \mathbb{Z}.$$

This rational number has negative p-adic valuation for some prime number p. Therefore there exist $h_1 \neq h_0$ and j_0, j_1 such that

$$v_p\Big(\frac{d_{n/N}^{j_1}}{(h_1 - t/N)^{j_1}}\Big) < 0 \text{ and } v_p\Big(\frac{d_{n/N}^{j_0}}{(h_0 - t/N)^{j_0}}\Big) < 0.$$

This implies

$$\min(v_p(h_1 - t/N), v_p(h_0 - t/N)) > v_p(d_{n/N})$$

so that $v_p(h_0 - h_1) > v_p(d_{n/N})$. This is a contradiction since $1 \leq |h_0 - h_1| \leq n/N$. This concludes the proof that $\delta_n U_1(z) \in \mathbb{Z}[z]$; the same proof works for $\delta_n V_1(z)$.

2 Siegel's linear independence criterion

The following criterion is based on Siegel's ideas (see for instance [6, p. 81–82 and 215–216], [16, §3] or [15, Proposition 4.1]).

Proposition 2. *Let $\theta_1, \ldots, \theta_q$ be complex numbers, not all zero. Let $\tau > 0$, and (Q_n) be a sequence of real numbers with limit $+\infty$. Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$, $K \geq 1$, and for any $n \in \mathcal{N}$ let $L^{(n)} = [\ell_{k,i}^{(n)}]_{1\leq i\leq q, 1\leq k\leq K}$ be a matrix with integer coefficients such that as $n \to \infty$ with $n \in \mathcal{N}$:*

$$\max_{i,k} |\ell_{k,i}^{(n)}| \leq Q_n^{1+o(1)}$$

$$\text{and} \max_{1\leq k\leq K} |\ell_{k,1}^{(n)}\theta_1 + \ldots + \ell_{k,q}^{(n)}\theta_q| \leq Q_n^{-\tau+o(1)}.$$

Assume also that the subspace $\mathcal{F}$ of $\mathbb{R}^q$ spanned by the columns ${}^t(\ell_{k,1}^{(n)}, \ldots, \ell_{k,q}^{(n)})$ of $L^{(n)}$ is non-zero and independent from $n \in \mathcal{N}$ (provided n is large enough). Then we have

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\theta_1, \ldots, \theta_q) \geq \tau + 1.$$

The usual version of this criterion (see for instance [9, Theorem 4]) is the same statement, but the assumption on $\mathcal{F}$ is replaced by the assumption that $L^{(n)}$ is invertible. The latter is stronger, since it is equivalent to asking $\mathcal{F} = \mathbb{R}^q$ for any n. Indeed if $\mathcal{F} = \mathbb{R}^q$ then $L^{(n)}$ has rank q: for each n we may extract q linearly independent columns of $L^{(n)}$, and obtain an invertible matrix to which [9, Theorem 4] applies. The point is that we shall apply Proposition 2 to the matrices $[s_{k,i}]$ constructed in Proposition 1, and the subspace $\mathcal{F}$ is not always equal to $\mathbb{R}^q$ (see Remark 3 in §1.4).

Let us prove Proposition 2 now. Denote by $\mathcal{F}$ the image of $L^{(n)}$, assumed to be independent from $n \in \mathcal{N}$ (provided n is large enough). Let $p = \dim \mathcal{F}$. Permuting $\theta_1, \ldots, \theta_q$ if necessary, we may assume that a system of linear equations of $\mathcal{F}$ is given by

$$x_t = \sum_{i=1}^{p} \mu_{t,i} x_i \text{ for } p+1 \leq t \leq q, \text{ with } \mu_{t,i} \in \mathbb{Q}. \tag{2.1}$$

We point out that the coefficients $\mu_{t,i}$ are rational numbers because the matrices $L^{(n)}$ have integer coefficients. Since ${}^t(\ell_{k,1}^{(n)}, \ldots, \ell_{k,q}^{(n)}) \in \mathcal{F}$ for any $1 \leq k \leq K$ and any $n \in \mathcal{N}$ sufficiently large, Eq. (2.1) yields

$$\sum_{i=1}^{p} \ell_{k,i}^{(n)}\theta_i = \sum_{i=1}^{p} \ell_{k,i}^{(n)}\Big(\theta_i + \sum_{t=p+1}^{q} \mu_{t,i}\theta_t\Big) = \sum_{i=1}^{p} \ell_{k,i}^{(n)}\theta_i' \tag{2.2}$$

upon letting $\theta_i' = \theta_i + \sum_{t=p+1}^{q} \mu_{t,i}\theta_t$ for $1 \leq i \leq p$. Moreover for any $n \in \mathcal{N}$ sufficiently large, we have $\mathrm{rk} L^{(n)} = \dim \mathcal{F} = p$ and Eq. (2.1) shows that the last $q-p$ rows of $L^{(n)}$ are linear combinations of the first p rows. Therefore the first p rows are linearly independent: the matrix $[\ell_{k,i}^{(n)}]_{1\leq i\leq p, 1\leq k\leq K}$ has rank p. Accordingly for each n there exist pairwise distinct integers $k_1, \ldots, k_p$ between 1 and K such that the matrix $M^{(n)} = [\ell^{(n)}_{k_j,i}]_{1\le i,j\le p}$ is invertible. Using Eq. (2.2) we may apply the usual version of Siegel's criterion (namely [9, Theorem 4]) to this matrix and deduce that

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\theta'_1, \ldots, \theta'_p) \ge \tau + 1.$$

Since $\theta'_i \in \mathrm{Span}_{\mathbb{Q}}(\theta_1, \ldots, \theta_q)$ for any $1 \le i \le p$, this concludes the proof of Proposition 2.

Remark 4. *The idea of applying the usual version of Siegel's criterion to numbers θ'_i defined as linear combinations of $\theta_1, \ldots, \theta_q$ appears also in [9] (see Proposition 2 in §6 and Eq. (9.1)). However the situation is different in that paper: the rows of the matrix P (see Step 2 in §1.4 above) are linearly dependent, which is not the case here.*

3 Deduction of Theorems 1 and 2 from Proposition 1

In this section we prove Theorems 1 and 2 stated in the introduction, and also a result that nearly contains both of them (namely Theorem 4 stated at the end of §3.2). At last, we show in §3.3 that the linear forms constructed in [11] are a special case of those studied here.

3.1 Proof of Theorem 1

Let f, T, p, ε, and a be as in the statement of Theorem 1; put $N = T$. We consider the complex numbers $\theta_1, \ldots, \theta_{a+N-1}$ given by:

$$\begin{cases} \theta_{i-1} = 2(-1)^p L(f,i) \text{ for } 2 \le i \le a \text{ with } i \equiv p \bmod 2, \\ \theta_{i-1} = 0 \text{ for } 2 \le i \le a \text{ with } i \not\equiv p \bmod 2, \\ \theta_{a+i} = f(i) \text{ for } 0 \le i \le N-1. \end{cases}$$

We apply Proposition 1 to each integer multiple n of N, and let $\ell^{(n)}_{k,i} = s_{k,i+1}$ for $1 \le i \le a+N-1$ and $1 \le k \le c_1$. Then we apply Siegel's linear independence criterion (namely Proposition 2 stated and proved in §2) with $q = a+N-1$, $Q_n = \beta^n$ and $\tau = -\frac{\log\alpha}{\log\beta}$ (so that $Q_n^{-\tau} = \alpha^n$), where α and β are defined in §1.1; we take for $\mathcal{N}$ the set of integer multiples of N. Therefore we obtain

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\Big(\{L(f,i),\, 2 \le i \le a \text{ and } i \equiv p \bmod 2\} \cup \{f(0), \ldots, f(N-1)\}\Big) \ge 1 - \frac{\log\alpha}{\log\beta}. \quad (3.1)$$

Taking a very large and r equal to the integer part of $\frac{a}{(\log(a))^2}$ concludes the proof of Theorem 1 since

$$1 - \frac{\log\alpha}{\log\beta} - N = \frac{1+\varepsilon_a}{1+\log 2}\log a \text{ where } \lim_{a\to+\infty} \varepsilon_a = 0;$$

here the shift of N in the lower bound comes from $f(0), \ldots, f(N-1)$ that appear in Eq. (3.1).

3.2 Proof of Theorem 2

Let f, T, E, and p be as in the statement of Theorem 2. Let $0 < \varepsilon < 1/4$ and a be sufficiently large with respect to ε, T, and $\dim E$. We denote by D the product of all primes less than or equal to $(1-3\varepsilon)\log a$ (such a product has asymptotically the largest possible number of divisors with respect to its size, see [13, Chapter XVIII, §1]). Then we have

$$\log D \ = \sum_{p \leq (1-3\varepsilon)\log a} \log p \leq (1-2\varepsilon)\log a$$

by the prime number theorem, i.e., $D \leq a^{1-2\varepsilon}$. We take for r the integer part of a^{ε}. At last, we let $N = DT$.

For any divisor d of $D = N/T$ and any $m \in \mathbb{Z}$, let $g_d(m) = f(m/d)$ if d divides m, and $g_d(m) = 0$ otherwise. Since f is T-periodic we have $g_d(m+N) = g_d(m)$ for any m.

We shall choose below an integer w_d for each divisor d of D; let

$$g = \sum_{d|D} w_d g_{D/d}.$$

We shall apply Proposition 1 to the N-periodic function g and obtain linear forms in the numbers

$$L(g,i) = \sum_{m=1}^{\infty} \frac{1}{m^i} \sum_{\substack{d|D \\ D|md}} w_d f(md/D) = \sum_{d|D} w_d \sum_{m' \geq 1} \frac{f(m')d^i}{m'^i D^i}$$

by letting $m' = md/D$. Therefore we have

$$L(g,i) = D^{-i}\Big(\sum_{d|D} w_d d^i\Big) L(f,i). \tag{3.2}$$

Notice that D has $\delta = 2^{\pi((1-3\varepsilon)\log a)}$ divisors, with

$$\log \delta = \pi((1-3\varepsilon)\log a)\log 2 \geq (1-4\varepsilon)(\log 2)\frac{\log a}{\log\log a}. \tag{3.3}$$

Assume that the number of values (0.2) which do not belong to E is less than δ. Let $2 \leq i_1 < i_2 < \ldots < i_{\delta-1} \leq a$ be integers such that if $L(f,i) \notin E$ and $i \equiv p \bmod 2$, $2 \leq i \leq a$, then $i = i_j$ for some j.

The homogeneous linear system

$$\sum_{d|D} w_d d^{i_j} = 0 \text{ for any } j \in \{1, \ldots, \delta-1\} \tag{3.4}$$

has δ unknowns w_d, where d ranges through the set $\mathcal{D}$ of divisors of D, and $\delta - 1$ equations. Therefore it has a non-zero integer solution $(w_d) \in \mathbb{Z}^{\mathcal{D}}$.

At this point, the integers w_d are chosen in [11] such that $\sum_{d|D} w_d d \neq 0$, using an elementary zero estimate (namely, a generalized Vandermonde determinant is non-zero). Here we do not need to make any such assumption: we just assume that $w_d \neq 0$ for at least one d. Indeed a (much more complicated) zero estimate is used in the present proof, namely Theorem 3.

Proposition 1 applies to the N-periodic function $g = \sum_{d|D} w_d g_{D/d}$ defined above. Using also Siegel's linear independence criterion as in §3.1 we obtain

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\Big(\{g(0), \dots, g(N-1)\} \cup \{L(g,i),\, 2 \leq i \leq a \text{ and } i \equiv p \bmod 2\}\Big) \geq 1 - \frac{\log \alpha}{\log \beta} \tag{3.5}$$

with

$$1 - \frac{\log \alpha}{\log \beta} \sim \frac{\log r}{1 + \log 2} \sim \frac{\varepsilon}{1 + \log 2} \log a$$

as $a \to \infty$ (recall that r is the integer part of a^ε).

On the other hand, the numbers that appear in the left hand side of (3.5) have the following properties:

- $g(0), \dots, g(N-1)$ belong to $\{0, f(0), f(1), \dots, f(T-1)\}$.
- For $2 \leq i \leq a$ with $i \equiv p \bmod 2$, $L(g,i)$ is zero if $i \in \{i_1, \dots, i_{\delta-1}\}$, and belongs to E otherwise, as Eqns. (3.2) and (3.4) show.

Therefore we have

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\Big(\{g(0), \dots, g(N-1)\} \cup \{L(g,i),\, 2 \leq i \leq a \text{ and } i \equiv p \bmod 2\}\Big) \leq T + \dim E. \tag{3.6}$$

Combining Eqns. (3.5) and (3.6) yields a contradiction provided a is large enough. This concludes the proof of Theorem 2.

Since $4(1 + \log 2) > 7$, the same proof (with ε replaced with $\varepsilon/4$ to take Eq. (3.3) into account) provides the following refinement of Theorem 2.

Theorem 4. *Let $T \geq 1$, and $f : \mathbb{Z} \to \mathbb{C}$ be such that $f(n+T) = f(n)$ for any n. Assume that f is not identically zero. Let $p \in \{0,1\}$, $0 < \varepsilon < 1$, and a be sufficiently large (in terms of T and ε). Let E be a finite-dimensional $\mathbb{Q}$-vector space contained in $\mathbb{C}$ with $\dim E < \frac{\varepsilon}{7} \log a$. Then among the numbers $L(f,s)$ with $2 \leq s \leq a$ and $s \equiv p \bmod 2$, at least*

$$2^{(1-\varepsilon)\frac{\log a}{\log \log a}}$$

do not belong to E.

Choosing $\varepsilon = 7/8$, this refinement implies that the numbers $L(f,s)$ with $2 \leq s \leq a$ and $s \equiv p \bmod 2$ are not all contained in such a subspace E: they span a $\mathbb{Q}$-vector space of dimension at least $\frac{1}{8} \log a$. Except for the multiplicative constant ($\frac{1}{8}$ instead of $\frac{1-o(1)}{1+\log 2}$), Theorem 1 follows as a corollary of Theorem 4.

3.3 Connection to the proof of [11]

In this section we show that the linear forms used in [11] to prove (0.1) are a special case of those studied in the present paper (namely in the proof of Theorem 2 with $f(m) = 1$, $T = r = k = 1$, and $p \equiv a \bmod 2$). Accordingly they are related to the Padé approximation problem stated in §1.4, in which the number of equations is essentially equal to the number of unknowns.

We keep the notation of the proof of Theorem 2 in §3.2, with $T = 1$ and $f(m) = m$ for any $m \in \mathbb{Z}$. In particular $N = D$ is the product of all primes less than or equal to $(1-3\varepsilon)\log a$. For any divisor d of D we have $g_d(m) = 1$ if d divides m, and $g_d(m) = 0$ otherwise. The function $g = \sum_{d|D} w_d g_{D/d}$ satisfies

$$g(m) = \sum_{\substack{d|D \\ D|md}} w_d \text{ for any } m \in \mathbb{Z}. \tag{3.7}$$

Now let n be an integer multiple of $2N = 2D$, and let $p \in \{0,1\}$ be such that $p \equiv a \bmod 2$. Then the rational function F satisfies the symmetry property of well-poised hypergeometric series:

$$F(-n-t) = (-1)^{(2r+1)n+1+(a+1)(\frac{n}{N}+1)} F(t) = (-1)^p F(t). \tag{3.8}$$

This is the key ingredient (since the Ball-Rivoal theorem) to get rid of even zeta values, when $p = 1$. In our approach where Nesterenko's linear independence criterion is replaced with Siegel's combined with Shidlovsky's lemma, this property cannot be used in the same way because it is destroyed when considering $S^{(k-1)}(z)$ for $k \geq 2$. Using both S_0 and S_∞ in constructing the linear forms (see §1.2) makes it possible to overcome this difficulty (as in [8]). With $k = 1$ this trick does not modify the linear forms we are interested in, since for any $\ell \in \mathbb{Z}$ we have using Eqns. (1.3) and (3.8) and the fact that N divides n:

$$\begin{aligned} S_0(\omega^\ell) + (-1)^p S_\infty(\omega^{-\ell}) &= \sum_{t=n+1}^{\infty} F(-t)\omega^{\ell t} + (-1)^p \sum_{t=1}^{\infty} F(t)\omega^{\ell t} \\ &= \sum_{t=1}^{\infty} \omega^{\ell t}\Big(F(-n-t) + (-1)^p F(t)\Big) \\ &= 2(-1)^p \sum_{t=1}^{\infty} F(t)\omega^{\ell t}. \end{aligned} \tag{3.9}$$

We are now in position to express differently the linear forms constructed in part *(iii)* of Proposition 1 from the map g given by Eq. (3.7), in the special case where $N = D$, n is a multiple of $2N$, $p \equiv a \bmod 2$, $r = 1$, and $k = 1$. Denote by Λ_n this linear form. Then we have using Lemma 1 and Eqns. (3.9) and (3.7):

$$\begin{aligned}
\delta_n^{-1}\Lambda_n &= \sum_{\ell=1}^{D} \widehat{g}(\ell)\Big(S_0(\omega^\ell) + (-1)^p S_\infty(\omega^{-\ell})\Big) \\
&= 2(-1)^p \sum_{t=1}^{\infty} F(t) \sum_{\ell=1}^{D} \widehat{g}(\ell)\omega^{\ell t} \\
&= 2(-1)^p \sum_{t=1}^{\infty} F(t) g(t) \\
&= 2(-1)^p \sum_{d|D} w_d \sum_{\substack{t\geq 1 \\ D|dt}} F(t) \\
&= 2(-1)^p \sum_{d|D} w_d \sum_{t'=1}^{\infty} F(Dt'/d) \\
&= 2(-1)^p \sum_{d|D} w_d \sum_{j=1}^{d} \sum_{m=1}^{\infty} F(mD + j\frac{D}{d}).
\end{aligned}$$

In the last expression the sum on $m \geq 1$ should have begun at $m = 0$, but this makes no difference since $F(jD/d) = 0$ for any $1 \leq j \leq d$. Now let $R(t) = F(Dt)$; then we have

$$\frac{(-1)^p}{2\delta_n}\Lambda_n = \sum_{d|D} w_d \sum_{j=1}^{d} \sum_{m=1}^{\infty} R(m + \frac{j}{d}). \tag{3.10}$$

Up to the normalizing factor $\frac{(-1)^p}{2\delta_n}$ these are exactly the linear forms $\widehat{r_n}$ used in [11] to prove (0.1). Indeed the following notation is used in [11] for $1 \leq j \leq D$ and $d|D$:

$$R_n(t) = D^{3Dn}\ n!^{s+1-3D}\ \frac{\prod_{j=0}^{3Dn}(t - n + \frac{j}{D})}{\prod_{j=0}^{n}(t+j)^{a+1}}, \qquad r_{n,j} = \sum_{m=1}^{\infty} R_n\Big(m + \frac{j}{D}\Big),$$

$$\widehat{r}_{n,d} = \sum_{j=1}^{d} r_{n,j\frac{D}{d}}, \qquad \widetilde{r}_n = \sum_{d|D} w_d\, \widehat{r}_{n,d}.$$

Now, up to the normalizing factor $R_n(t)$ is equal to the rational function $R(t) = F(Dt)$ so that Λ_n is equal to $\widetilde{r}_n$ using Eq. (3.10).

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