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A natural finite element for axisymmetric problem

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conjoint work with

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1) Axi-symmetric model problem

2) Axi-Sobolev spaces

3) Discrete formulation

4) Numerical results for an analytic test case

5) About Clément's interpolation

6) Numerical analysis

7) Conclusion

Motivation : solve the Laplace equation in a axisymmetric domain
Find a solution of the form $u(r, z)\exp(i\,\theta)$
Change the notation : $x \equiv z\,,\ \ y \equiv r$

Consider the meridian plane Ω of the axisymmetric domain
$\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_{\rm D}} \cup \overline{\Gamma_{\rm N}}\,,\, \Gamma_0 \cap \Gamma_{\rm D} = \emptyset\,,\, \Gamma_0 \cap \Gamma_{\rm N} = \emptyset\,,\, \Gamma_{\rm D} \cap \Gamma_{\rm N} = \emptyset,$
Γ_0 is the intersection of $\overline{\Omega}$ with the "axis" $y = 0$

Then the function u is solution of

$$-\frac{\partial^2 u}{\partial x^2} - \frac{1}{y}\frac{\partial}{\partial y}\left(y\,\frac{\partial u}{\partial y}\right) + \frac{u}{y^2} = f \qquad \text{in}\,\Omega$$

Boundary conditions : $\quad u = 0 \quad \text{on}\ \Gamma_{\rm D}\,, \qquad \dfrac{\partial u}{\partial n} = g \quad \text{on}\ \Gamma_{\rm N}$

Test function v null on the portion $\Gamma_{\rm D}$ of the boundary
Integrate by parts relatively to the measure $y \, {\rm d}x \, {\rm d}y$.

Bilinear form $$a(u\,,\,v) \,=\, \int_\Omega y\,\nabla u \bullet \nabla v \,\,{\rm d}x\,{\rm d}y \,+\, \int_\Omega \frac{u\,v}{y}\,\,{\rm d}x\,{\rm d}y$$

Linear form $$< b\,,\,v > = \int_\Omega f\,v\,y\,\,{\rm d}x\,{\rm d}y \,+\, \int_{\Gamma_{\rm N}} g\,v\,\,y\,{\rm d}\gamma\,.$$

Two notations:
$$u_{\surd}(x\,,\,y) \,=\, \frac{1}{\sqrt{y}}\,u(x\,,\,y)\,, \qquad u^{\surd}(x\,,\,y) \,=\, \sqrt{y}\,\,u(x\,,\,y)\,, \quad (x\,,\,y) \in \Omega\,.$$

Sobolev spaces:
$${\rm L}^2_a(\Omega) \,=\, \{v : \Omega \longrightarrow {\rm I\!R}\,,\,\, v^{\surd} \in {\rm L}^2(\Omega)\}$$
$${\rm H}^1_a(\Omega) \,=\, \{v \in {\rm L}^2_a(\Omega)\,,\,\, v_{\surd} \in {\rm L}^2(\Omega)\,,\, (\nabla v)^{\surd} \in ({\rm L}^2(\Omega))^2\}$$
$${\rm H}^2_a(\Omega) \,=\, \left\{ \begin{array}{c} v \in {\rm H}^1_a(\Omega)\,,\,\, v_{\surd\surd\surd} \in {\rm L}^2(\Omega)\,,\, (\nabla v)_{\surd} \in ({\rm L}^2(\Omega))^2\,, \\ ({\rm d}^2 v)^{\surd} \in ({\rm L}^2(\Omega))^4 \end{array} \right\}.$$

Norms and semi-norms:

$$\|v\|_{0,\,a}^2 \;=\; \int_\Omega y\,|v|^2 \;\mathrm{d}x\,\mathrm{d}y$$

$$|v|_{1,\,a}^2 \;=\; \int_\Omega \left(\frac{1}{y}\,|v|^2 \;+\; y\,|\nabla v|^2\right)\,\mathrm{d}x\,\mathrm{d}y\,, \qquad \|v\|_{1,\,a}^2 \;=\; \|v\|_{0,\,a}^2 \;+\; |v|_{1,\,a}^2$$

$$|v|_{2,\,a}^2 \;=\; \int_\Omega \left(\frac{1}{y^3}\,|v|^2 \;+\; \frac{1}{y}\,|\nabla v|^2 \;+\; y\,|\mathrm{d}^2 v|^2\right)\,\mathrm{d}x\,\mathrm{d}y\,,$$

$$\|v\|_{2,\,a}^2 \;=\; \|v\|_{1,\,a}^2 \;+\; |v|_{2,\,a}^2$$

The condition $\quad u \;=\; 0 \qquad$ on Γ_0
is incorporated inside the choice of the axi-space $\mathrm{H}^1_a(\Omega)$.

Sobolev space that takes into account the Dirichlet boundary condition

$$V \;=\; \{v \in \mathrm{H}^1_a(\Omega)\,,\;\; \gamma v = 0 \text{ on } \Gamma_{\mathrm{D}}\}\,.$$

Variational formulation: $\begin{cases} u \in V \\ a(u\,,\,v) \;=< b\,,\,v >\,,\;\forall\, v \in V\,. \end{cases}$

We observe that $\quad a(v\,,\,v) \;=\; |v|^2_{1,\,a}\,, \quad \forall\, v \in \mathrm{H}^1_a(\Omega)\,,$

The existence and uniqueness of the solution of problem is (relatively !) easy according to the so-called Lax-Milgram-Vishik's lemma.

See the article of B. Mercier and G. Raugel !

Very simple, but fundamental remark

Consider $v(x\,,\,y) \;=\; \sqrt{y}\,(a\,x \,+\, b\,y \,+\, c)\,, \qquad (x\,,\,y) \in K \in \mathcal{T}^2\,,$

Then we have $\sqrt{y}\;\nabla v(x\,,\,y) \;=\; \Big(a\,y\,,\; \dfrac{1}{2}\,(a\,x \,+\, 3b\,y \,+\, c)\Big)\,.$

P_1 : the space of polynomials of total degree less or equal to 1

We have $v_{\sqrt{}} \in P_1 \;\Longrightarrow\; (\nabla v)^{\sqrt{}} \in (P_1)^2\,.$

A two-dimensional conforming mesh $\mathcal{T}$

- $\mathcal{T}^0$ set of vertices
- $\mathcal{T}^1$ set of edges
- $\mathcal{T}^2$ set of triangular elements.

Linear space $P_1^{\surd} = \{v, v_{\surd} \in P_1\}$.

Degrees of freedom $< \widetilde{\delta}_S \,, v >$ for v regular, $S \in \mathcal{T}^0 : < \widetilde{\delta}_S \,, v > = v_{\surd}(S)$

Proposition 1. Unisolvance property.
$K \in \mathcal{T}^2$ be a triangle of the mesh $\mathcal{T}$,
Σ the set of linear forms $< \widetilde{\delta}_S \,, \bullet >$, $S \in \mathcal{T}^0 \cap \partial K$
$P_1^{\surd}$ defined above.
Then the triple $(K \,, \Sigma \,, P_1^{\surd})$ is unisolvant.

Proposition 2. Conformity of the axi-finite element
The finite element $(K \,, \Sigma \,, P_1^{\surd})$ is conforming in space $\mathcal{C}^0(\overline{\Omega})$.

Proposition 3. Conformity in the axi-space $\mathrm{H}_a^1(\Omega)$.
The discrete space $\mathrm{H}_{\mathcal{T}}^{\surd}$ is included in the axi-space $\mathrm{H}_a^1(\Omega)$:

$$\mathrm{H}_{\mathcal{T}}^{\surd} \subset \mathrm{H}_a^1(\Omega) \,.$$

$\Omega =]0,1[^2$, $\Gamma_D = \emptyset$

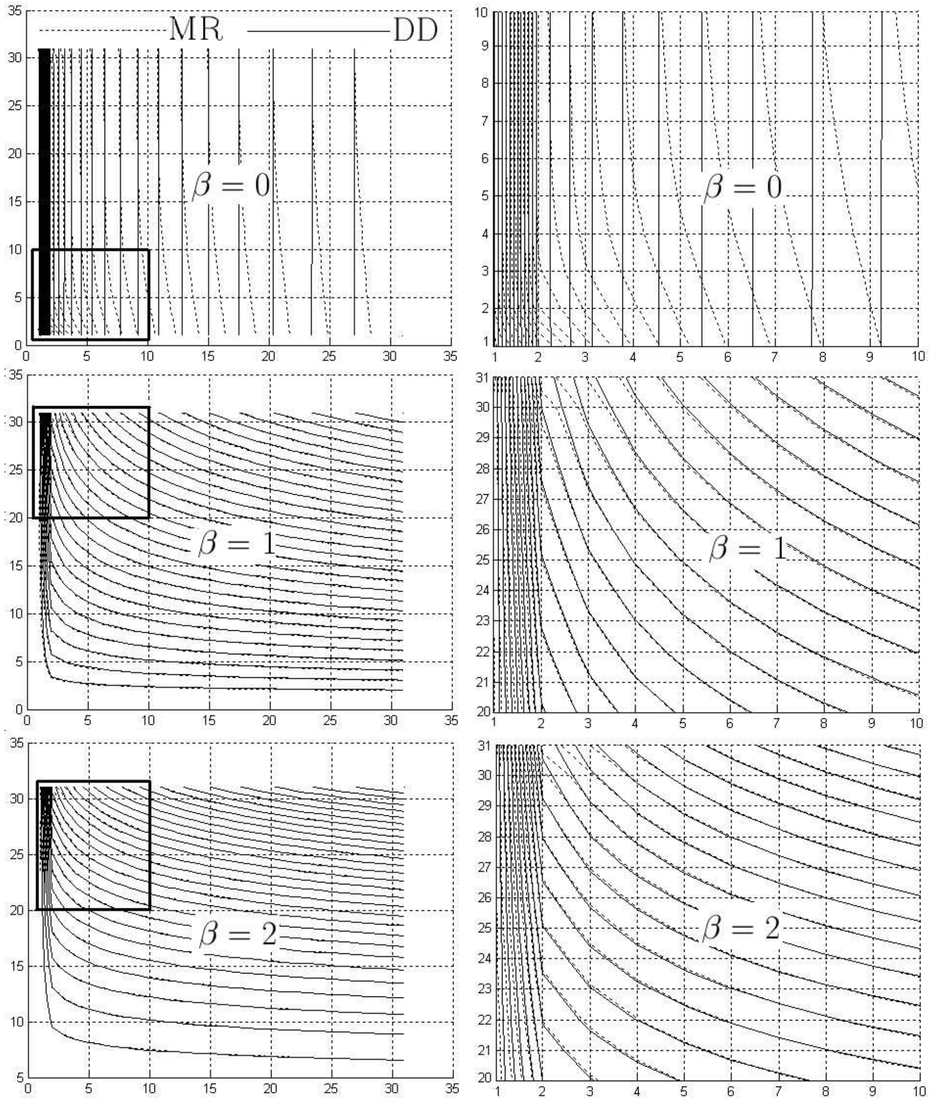
Parameters $\alpha > 0,\ \beta > 0,$

Right hand side: $f\,(y,x) \equiv y^\alpha \left[\left(\alpha^2 - 1\right) \frac{x^\beta}{y^2} + \beta(\beta-1)x^{\beta-2} \right]$

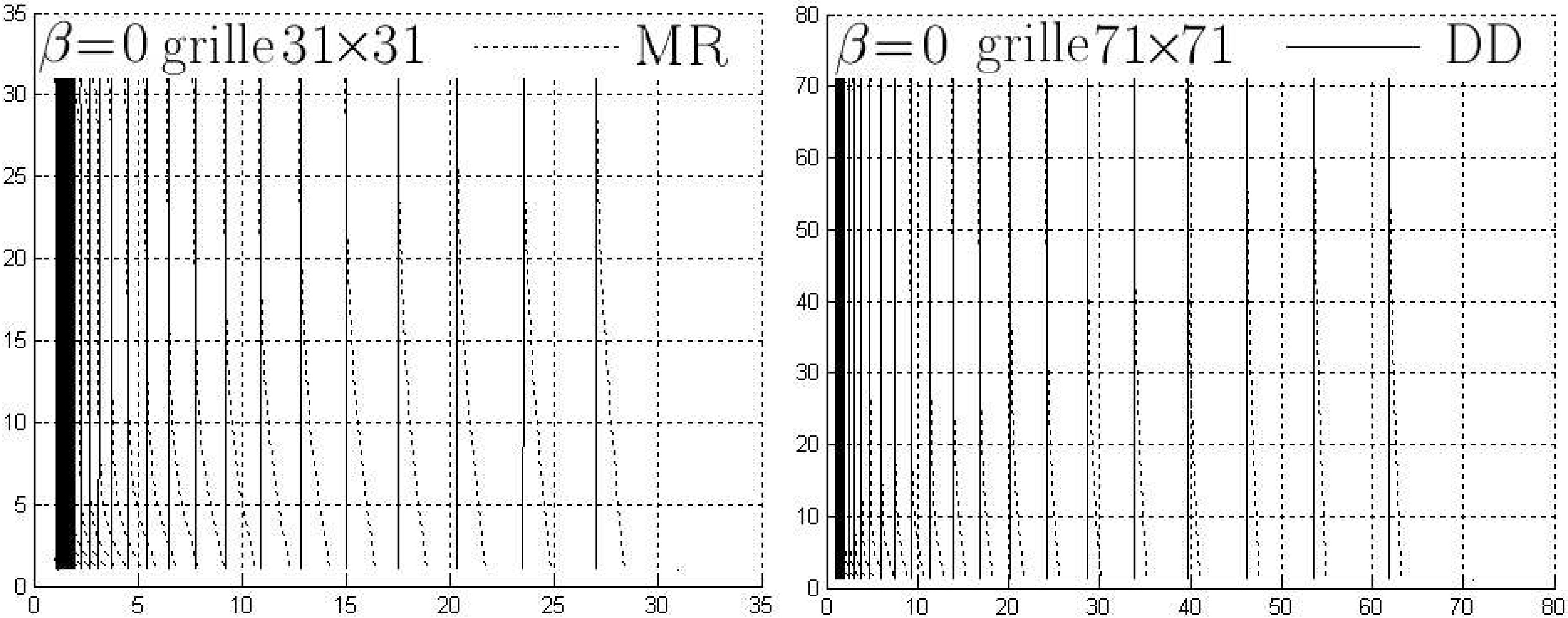
Neumann datum:
$g(x,\,y) = \alpha$ if $y = 1$, $-\beta y^\alpha x^{\beta-1}$ if $x = 0$, βy^α if $x = 1$.

Solution: $u(x,\,y) \equiv y^\alpha x^\beta$.

Comparison between
the present method (DD)
the use of classical P_1 finite elements (MR)



MR
DD
$\beta = 0$
$\beta = 0$
$\beta = 1$
$\beta = 1$
$\beta = 2$
$\beta = 2$


β=0 grille31×31
MR
β=0 grille71×71
DD

Numerical study of the convergence properties

Test cases :

$\alpha = 1/4$, $\alpha = 1/3$, $\alpha = 2/3$

$\beta = 0$, $\beta = 1$, $\beta = 2$

Three norms: $\|v\|_{0,\,a}$ $|v|_{1,\,a}$ $\|v\|_{\ell^\infty}$

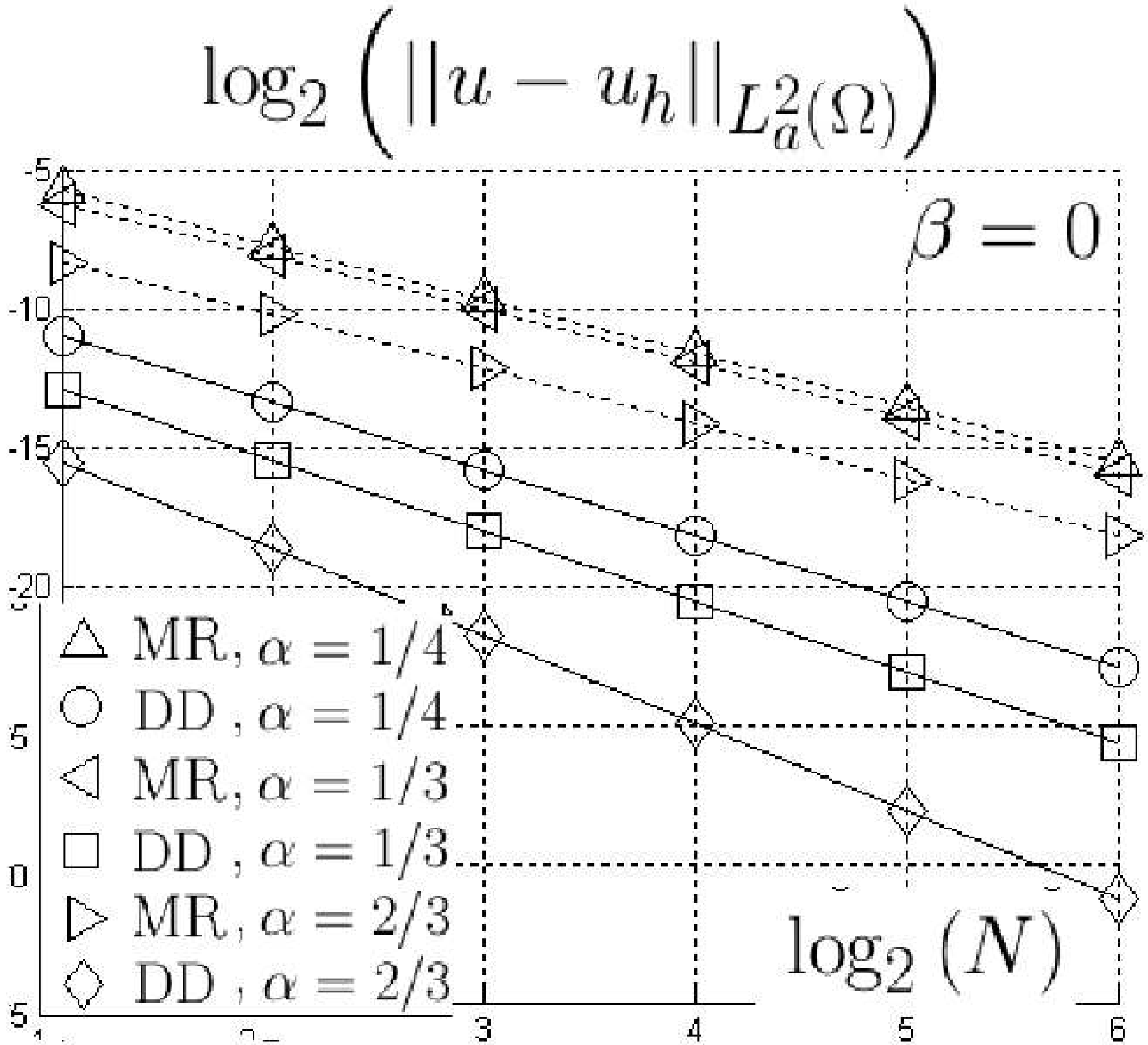
Order of convergence easy (?) to see.

Example : $\beta = 0$ and $\alpha = 2/3$:
our axi-finite element has a rate of convergence $\simeq 3$ for the $\|\bullet\|_{0,\,a}$ norm.

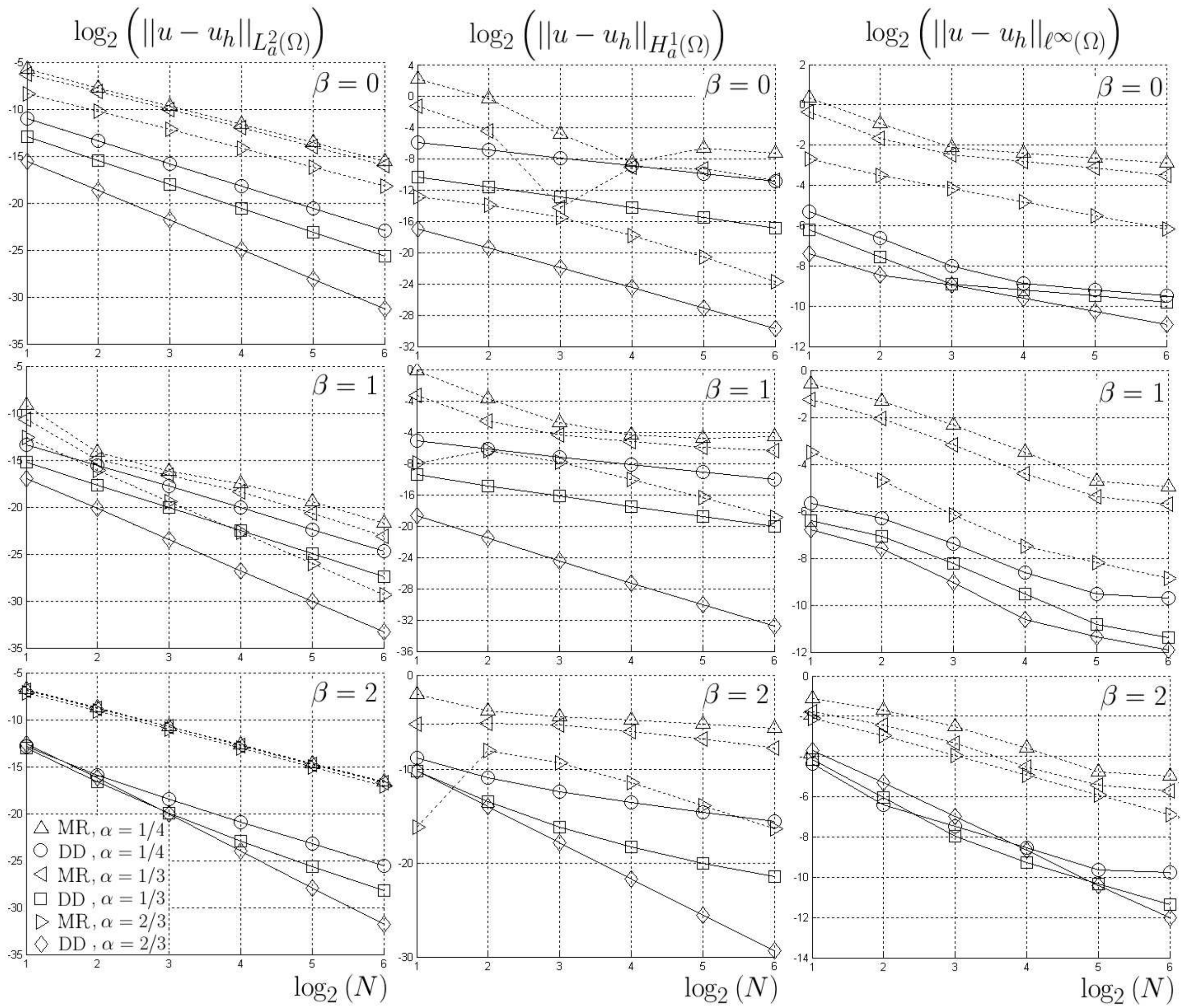
Synthesis of these experiments:

same order of convergence than with the classical approach

errors much more smaller!



$\log_2 \left(||u - u_h||_{L^2_a(\Omega)} \right)$
$\beta = 0$
MR, $\alpha = 1/4$
DD , $\alpha = 1/4$
MR, $\alpha = 1/3$
DD , $\alpha = 1/3$
MR, $\alpha = 2/3$
DD , $\alpha = 2/3$
$\log_2 (N)$
-5
-10
-15
-20
3
4
5
6



$\log_2\left(||u-u_h||_{L^2_a(\Omega)}\right)$
$\log_2\left(||u-u_h||_{H^1_a(\Omega)}\right)$
$\log_2\left(||u-u_h||_{\ell^\infty(\Omega)}\right)$
$\beta = 0$
$\beta = 1$
$\beta = 2$
△ MR, $\alpha = 1/4$
○ DD, $\alpha = 1/4$
◁ MR, $\alpha = 1/3$
□ DD, $\alpha = 1/3$
▷ MR, $\alpha = 2/3$
◇ DD, $\alpha = 2/3$
$\log_2(N)$

Discrete space for the approximation of the variational problem:

$$V_{\mathcal{T}} = \mathrm{H}^{\surd}_{\mathcal{T}} \cap V\,.$$

Discrete variational formulation: $\begin{cases} u_{\mathcal{T}} \in V_{\mathcal{T}} \\ a(u_{\mathcal{T}}\,,\,v) = < b\,,\,v >\,,\ \forall\, v \in V_{\mathcal{T}}\,. \end{cases}$

Estimate the error $\|u - u_{\mathcal{T}}\,\|_{1,\,a}$

Study the interpolation error $\|u - \Pi_{\mathcal{T}} u\,\|_{1,\,a}$

What is the interpolate $\Pi_{\mathcal{T}} u$??

Proposition 4. Lack of regularity.
Hypothesis: $u \in \mathrm{H}^2_a(\Omega)$.
Then $u_{\surd}$ belongs to the space $\mathrm{H}^1(\Omega)$ and $\|u_{\surd}\|_{1,\,\Omega} \leq C\,\|u\|_{2,\,a}$

Introduce $v \equiv u\sqrt{\cdot}$. Small calculus: $\nabla v = -\frac{1}{2y\sqrt{y}}\, u \,\nabla y + \frac{1}{\sqrt{y}}\, \nabla u\,.$

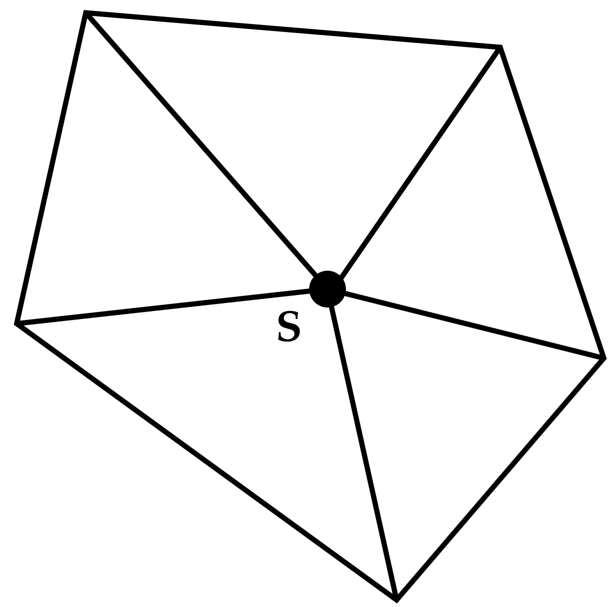
Then $\int_\Omega |v|^2 \,\mathrm{d}x\,\mathrm{d}y \le \int_\Omega \frac{1}{y}\, |u|^2 \,\mathrm{d}x\,\mathrm{d}y \le C\, \|u\|^2_{2,\,a}$

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y \le 2 \int_\Omega \Big(\frac{1}{4y^3}\, |u|^2 + \frac{1}{y}\, |\nabla u|^2 \Big) \,\mathrm{d}x\,\mathrm{d}y \le C\, \|u\|^2_{2,\,a}\,. \qquad \square$$

Derive (formally !) two times:

$$\mathrm{d}^2 v = \frac{3}{4\, y^2 \sqrt{y}}\, u \,\nabla y \bullet \nabla y - \frac{1}{y\sqrt{y}}\, \nabla u \bullet \nabla y + \frac{1}{\sqrt{y}}\, \mathrm{d}^2 u$$

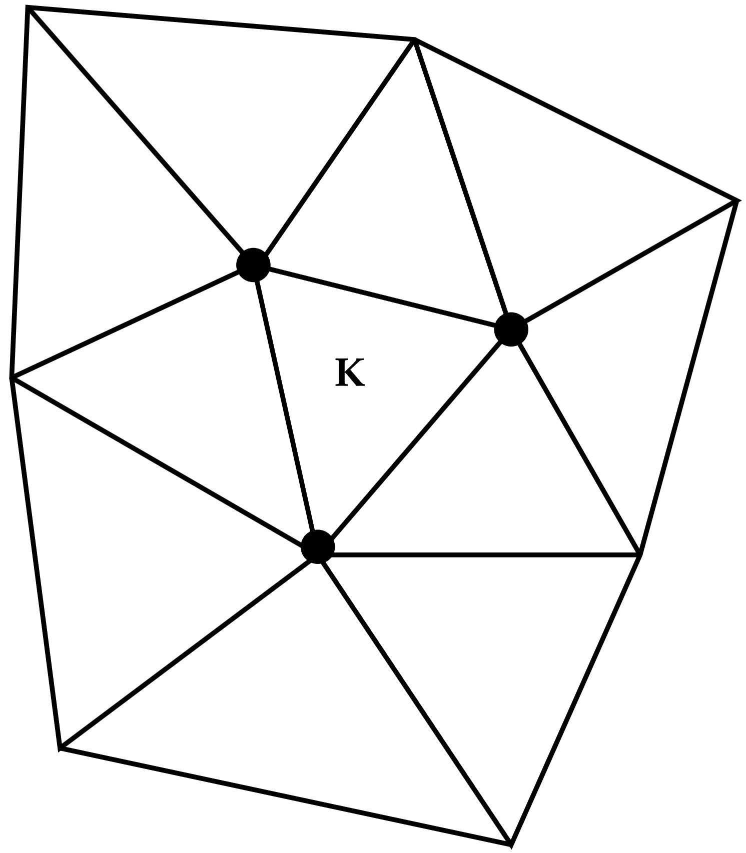
Even if u is regular, v has no reason to be continuous.



Vicinity Ξ_S of the vertex $S \in \mathcal{T}^0$.

Degree of freedom $$< \delta_S^{\mathcal{C}} \,,\, v > = \frac{1}{|\Xi_S|} \int_{\Xi_S} v(x) \, \mathrm{d}x \, \mathrm{d}y \,, \quad S \in \mathcal{T}^0$$

Clément's interpolation: $$\Pi^{\mathcal{C}} v \,=\, \sum_{S \in \mathcal{T}^0} < \delta_S^{\mathcal{C}} \,,\, v > \, \varphi_S \,.$$



Vicinity Z_K for a given triangle $K \in \mathcal{T}^2$.

$$|v - \Pi^{\mathcal{C}} v|_{0,\,K} \;\leq\; C\, h_{\mathcal{T}}\, |v|_{1,\,Z_K}\,, \qquad |v - \Pi^{\mathcal{C}} v|_{1,\,K} \;\leq\; C\, |v|_{1,\,Z_K}\,,$$

$$|v - \Pi^{\mathcal{C}} v|_{1,\,K} \;\leq\; C\, h_{\mathcal{T}}\, |v|_{2,\,Z_K}\,.$$

Interpolate Πu by conjugation: $\qquad \Pi u \,=\, \big(\Pi^{\mathcal{C}} u_{\surd}\big)^{\surd}$

id est $\qquad \Pi u(x,\, y) \,=\, \sqrt{y}\, \big(\Pi^{\mathcal{C}} v\big)(x,\, y)\,, \qquad (x,\, y) \in K \in \mathcal{T}^2$

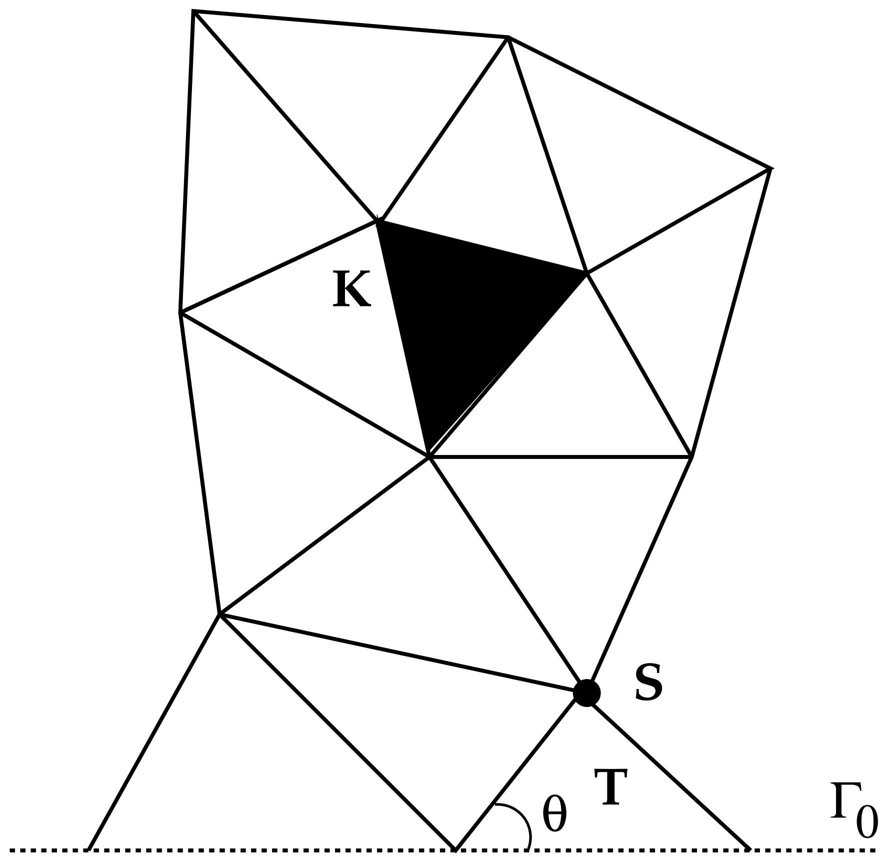
Theorem 1. An interpolation result.
Relatively strong hypotheses concerning the mesh $\mathcal{T}$
Let $u \in \mathrm{H}^2_a(\Omega)$ and Πu defined above.
Then we have $\qquad \| u - \Pi u \|_{1,\, a} \,\leq\, C\, h_{\mathcal{T}}\, \| u \|_{2,\, a}\,.$

$$\begin{aligned}
\int_\Omega \frac{1}{y}\, |u - \Pi u|^2 \,\mathrm{d}x\,\mathrm{d}y \,&=\, \int_\Omega \frac{1}{y}\, |u - \sqrt{y}\,\Pi^{\mathcal{C}} v|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&=\, \int_\Omega |v - \Pi^{\mathcal{C}} v|^2 \,\mathrm{d}x\,\mathrm{d}y \,=\, \| v - \Pi^{\mathcal{C}} v \|^2_{0,\, \Omega} \\
&\leq\, C\, h^2_{\mathcal{T}}\, |v|^2_{1,\, \Omega} \\
&\leq\, C\, h^2_{\mathcal{T}}\, \| u \|^2_{2,\, a}
\end{aligned}$$

$$\nabla\Big(\sqrt{y}\;(v-\Pi^{\mathcal{C}}v)\Big) \;=\; \frac{1}{2\sqrt{y}}\,(v-\Pi^{\mathcal{C}}v)\nabla y \;+\; \sqrt{y}\;\nabla\big(v-\Pi^{\mathcal{C}}v\big)\,.$$

$$\int_{\Omega} y\,|\nabla\big(u-\Pi u\big)|^2\,\mathrm{d}x\,\mathrm{d}y \;\le$$
$$\le\; \int_{\Omega}|v-\Pi^{\mathcal{C}}v|^2\,\mathrm{d}x\,\mathrm{d}y \;+\; 2\int_{\Omega} y^2\,|\nabla\big(v-\Pi^{\mathcal{C}}v\big)\,|^2\,\mathrm{d}x\,\mathrm{d}y$$

$$\Omega_+ \;=\; \{K\in\mathcal{T}^2\,,\,\mathrm{dist}\,(Z_K\,,\,\Gamma_0)>0\} \qquad\qquad \Omega_- \;=\; \Omega\setminus\Omega_+\,.$$



Triangle element K that belongs to the sub-domain Ω_+.

Theorem 2. First order approximation
relatively strong hypotheses concerning the mesh $\mathcal{T}$
u solution of the continuous problem: $u \in \mathrm{H}^2_a(\Omega)$,
Then we have $\|u - u_{\mathcal{T}}\|_{1,\,a} \,\leq\, C\, h_{\mathcal{T}}\, \|u\|_{2,\,a}\,.$

Proof: classical with Cea's lemma!

"Axi-finite element"

Interpolation properties founded of the underlying axi-Sobolev space

First numerical tests: good convergence properties

Numerical analysis based on Mercier-Raugel contribution (1982)

See also Gmati (1992), Bernardi *et al.* (1999)

May be all the material presented here is well known ?!