Vorticity-velocity-pressure and stream function-vorticity formulations for the Stokes problem

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Résumé
Nous étudions le problème de Stokes pour les fluides incompressibles en deux et trois dimensions, sur des domaines bornés à frontière régulière. Nous utilisons pour cela une formulation tourbillon-vitesse-pression et introduisons un nouvel espace de Hilbert pour le tourbillon. Nous développons une formulation mixte abstraite qui donne un cadre variationnel précis et conduit à un problème de Stokes bien posé faisant intervenir une nouvelle condition limite en vitesse-tourbillon. Dans le cas particulier de domaines bidimensionnels simplement connexes avec des conditions limites homogènes, nous décrivons complètement le lien avec la formulation classique en fonction courant-tourbillon et nous montrons que la formulation tourbillon-vitesse-pression est une extension mathématique naturelle de celle-ci.
Abstract
We study the Stokes problem of incompressible fluid dynamics in two and

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three-dimension spaces, for general bounded domains with smooth boundary. We use the vorticity-velocity-pressure formulation and introduce a new Hilbert space for the vorticity. We develop an abstract mixed formulation that gives a precise variational frame and conducts to a well-posed Stokes problem involving a new velocity-vorticity boundary condition. In the particular case of simply connected bidimensional domains with homogeneous boundary conditions, the link with the classical stream function-vorticity formulation is completely described, and we show that the vorticity-velocity-pressure formulation is a natural mathematical extension of the previous one.

Keywords : Fluid mechanics, Stokes equation, mixed formulations, inf-sup condition, vector field decomposition.

AMS (MOS) classification : 35Q30 - 46E35 - 76D03

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1 Introduction

∘ Let Ω be a bounded connected domain of $\mathbb{R}^N$ $(N = 2 \text{ or } 3)$ with a boundary $\partial\Omega = \Gamma$. To fix ideas, we will suppose that Γ is Lipschitz continuous but when it will be necessary to increase the regularity of the boundary, it will be quoted in the text. The Stokes problem modelizes the stationary equilibrium of an incompressible viscous fluid when the velocity u is sufficiently small to neglect the nonlinear terms (see *e.g.* Landau-Lifchitz [LL71]). From a mathematical point of view, this problem is the first step in order to consider the nonlinear Navier-Stokes equations of incompressible fluids, as proposed *e.g.* by Lions [Lio69], Temam [Tem77] or Girault-Raviart [GR86]. The Stokes problem can be classically written with primal formulation involving velocity u and pressure p:

$$\left\{ \begin{array}{cccc} -\nu\Delta u + \nabla p & = & f & \text{in} \quad \Omega \\ \text{div}\, u & = & 0 & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \Gamma \,, \end{array} \right. \tag{1}$$

where $\nu > 0$ is the kinematic viscosity and f the datum of external forces.

∘ Our motivation comes from the numerical simulations in computational fluid dynamics. The Marker And Cell ("MAC") method was proposed by Harlow and Welch [HW65] and is also known as the C-grid of Arakawa [Ara66]. It contains staggered grids relative to velocity and pressure and is still very popular when used in industrial computer softwares as Flow3d of Harper, Hirt and Sicilian [HHS83] or Phoenics developed by Patankar and Spalding [PS72]. This discretization is founded on the use of a cartesian mesh : velocity is defined with the help of fluxes on the faces of the mesh and pressure is supposed to be constant in each cell (Figure 1), with an analogous finite difference method for Maxwell equations [Yee66] : we refer to this methodology with the acronym HaWAY, for Harlow, Welch, Arakawa, Yee. Our objective is to generalize these degrees of freedom to arbitrary meshes that respect the usual topological constraints associated with finite elements (see *e.g.* Ciarlet [Cia78]) and in particular to triangles (Figure 2) or tetrahedra.

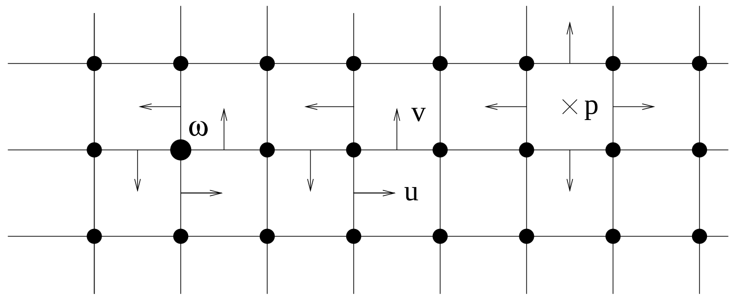

FIG. 1 – *The HaWAY scheme on a cartesian mesh.*

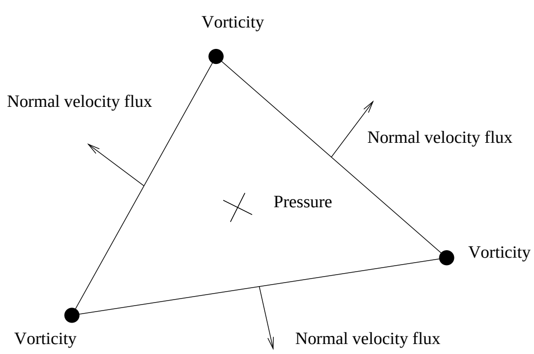

FIG. 2 – *The HaWAY scheme on a triangular mesh.*

Some years ago, Nicolaïdes [Nic89] has proposed a new interpretation of the HaWAY method with the help of dual finite volumes for triangular meshes. An analysis of the HaWAY scheme as a numerical quadrature for finite elements has also been proposed by Girault and Lopez [GL96].

∘ From the numerical point of view, this HaWAY discretization can be seen as the search of an approximation of velocity field conforming in the $H(\mathrm{div}, \Omega)$ Sobolev space with the help of the Raviart-Thomas [RT77] (when $N = 2$) and Nédélec [Néd80] (when $N = 3$) finite element of degree one. The approximation of pressure field in space $L^2(\Omega)$ is associated with discontinuous finite elements of degree zero. This vision, also adopted by Nicolaïdes, is a variational crime for the Stokes problem (1), where velocity classically belongs to finite dimensional linear spaces that are included in the Sobolev space $H^1(\Omega)$ (see *e.g.* Adams [Ada75]).

∘ In this paper, we recall the variational formulation which was previously proposed in ([Dub92], [Dub95]) involving the three fields of vorticity, velocity and pressure. A particularity of this formulation is that boundary conditions can be considered in a very general way. Previous works of Beghe, Conca, Murat and Pironneau [BCMP87] and Girault [Gir88] appear as particular cases of what we obtain. Finally, boundary condition in (1) can in our sense be seen as a mixed Dirichlet-Neumann boundary condition. The basic idea of our formulation is the same as the one used in stream function-vorticity formulation (Glowinski [Glo73], Ciarlet-Raviart [CR74], Girault [Gir76]) : we introduce the vorticity as a new unknown. But, the latter use the fact that a solenoidal vector field u (satisfying div $u = 0$) can *a priori* be represented as the curl of some stream function ψ: $u = \text{curl}\ \psi$. For the complete generality of the approach, we have here chosen to do not represent the solenoidal velocity field u with a stream function ψ for multiple reasons. First, any representation of the type $u = \text{curl}\ \psi$ precludes flows with sinks and sources (Foias-Temam [FT78]). Moreover, this representation is in the numerical practice restricted to two-dimensional domains even if Roux, Dupuy and one of the authors ([Rou84], [DD86], [Dub90]) have done first attempts in three-dimensional domains with Nédélec's vectorial finite elements [Néd80] conforming in space $H(\text{curl}, \Omega)$. Let us notice also a recent paper of Amara, Barucq and Duloué [ABD99], developing also a tridimensional stream function and vorticity formulation.

∘ The scope of this work is then the following. In Section 2, we recall the formulation involving the three fields vorticity, velocity and pressure. In Section 3, we study the two-dimensional case, which was already intensively analyzed by Glowinski [Glo73], Ciarlet-Raviart [CR74], Glowinski-Pironneau [GP79], Bernardi, Girault and Maday [BGM92], Ruas [Rua95a] among others. We then define an appropriate functional space in Section 4. In Section 5, we develop an abstract approach and exhibit the technical inf-sup hypotheses that are sufficient to satisfy. Then we prove that this triple formulation conducts to a mathematically well-posed problem with continuous dependence on the data. We give in Section 6 the main result of this article that expresses the conditions under which the Stokes problem in vorticity-velocity-pressure formulation is well-posed. These conditions are completely nontrivial for a general tridimensional domain Ω that is bounded, connected, non simply connected and with a non-connected boundary. For proving it, we have to generalize the representation theorem for vector fields that is summarized in Bendali, Dominguez and Gallic [BDG85]. A particular emphasis is given on the boundary condition for the tangential velocity. Finally, the last section deals again with the two-dimensional case and with the link between the stream function-vorticity formulation and the vorticity-velocity-pressure one. It then allows to enlarge the frame where our formulation is well-posed.

2 Vorticity-velocity-pressure formulation

In the following, all notation and formulae are supposed to be correct when Ω is a two- or a three-dimensional domain, and N will stand for the dimension.

2.1 Notation and functional spaces

∘ We shall consider the following spaces (see for example [Ada75]) : we denote $\mathcal{D}(\Omega)$ the space of all indefinitely differentiable functions from Ω to $\mathbb{R}$ with compact support, $\mathcal{D}'(\Omega)$ the space of distributions which is the dual space of $\mathcal{D}(\Omega)$ and $L^2(\Omega)$ the space of all classes of functions which are square integrable. Space $L^2_0(\Omega)$ is composed of functions in $L^2(\Omega)$ whose mean value is zero. Space $H^1(\Omega)$ consists of functions $\varphi \in L^2(\Omega)$ for which all partial derivatives $\dfrac{\partial \varphi}{\partial x_i}$ (in the distribution sense) belong to the space $L^2(\Omega)$:

$$H^1(\Omega) = \left\{ \varphi \in L^2(\Omega) \ , \ \forall i \in \{1 \ldots N\} \ , \ \frac{\partial \varphi}{\partial x_i} \in L^2(\Omega) \right\} \quad .$$

Symbols $\| \bullet \|_{1,\Omega}$ (respectively $|\bullet|_{1,\Omega}$) denote usual norms (respectively semi-norms) in Sobolev space $H^1(\Omega)$. In a similar way, we define space $H^2(\Omega)$ as the space of functions of $H^1(\Omega)$ for which the first partial derivatives belong to $H^1(\Omega)$. The associated norms and semi-norms are respectively noted $\| \ . \ \|_{2,\Omega}$ and $| \ . \ |_{2,\Omega}$. The Sobolev space $H^1_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in the sense of the norm $\| \bullet \|_{1,\Omega}$. In the sequel, $(\bullet, \bullet)_0$ and $\| \bullet \|_{0,\Omega}$ denotes respectively the standard inner product and the norm in $L^2(\Omega)$ and $\langle \bullet, \bullet \rangle_{-1,1}$ the duality product between $H^1_0(\Omega)$ and its topological dual space $H^{-1}(\Omega)$. Finally, γ shall denote the trace operator from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$ or from $H^2(\Omega)$ onto $H^{3/2}(\Gamma)$.

∘ **Space $H^1_0(\Omega; \Gamma_1)$, $\Gamma_1 \subset \Gamma$.**
For any subset Γ_1 of the boundary Γ, we define the space $H^1_0(\Omega; \Gamma_1)$ composed of functions of $H^1(\Omega)$ whose trace is zero on Γ_1.

$$H^1_0(\Omega; \Gamma_1) = \left\{ \varphi \in H^1(\Omega), \quad \begin{array}{ll} \gamma\varphi = 0 \text{ on } \Gamma_1 & \text{if meas } (\Gamma_1) \neq 0 \\ (\varphi, 1)_0 = 0 & \text{if meas } (\Gamma_1) = 0 \end{array} \right\} \quad .$$

Notice that $H^1_0(\Omega; \Gamma) = H^1_0(\Omega)$ and $H^1_0(\Omega; \emptyset) = H^1(\Omega) \cap L^2_0(\Omega)$. We shall note Γ_1^c the complementary of Γ_1. Then by definition, traces of functions in

$H_0^1(\Omega;\Gamma_1)$ belong to space $H_{00}^{1/2}(\Gamma_1^c)$ (see Lions-Magenes [LM68]):

$$H_{00}^{1/2}(\Gamma_1^c) = \left\{ \gamma\varphi \ , \ \varphi \in H^1(\Omega) \ \text{ such that } \ \gamma\varphi = 0 \text{ on } \Gamma_1 \right\} \quad .$$

We have $H_{00}^{1/2}(\Gamma) = H^{1/2}(\Gamma)$. Finally, for any space $H_{00}^{1/2}(\Gamma_1)$, $(H_{00}^{1/2}(\Gamma_1))'$ will denote its topological dual space and we can remark that $(H_{00}^{1/2}(\Gamma))' = H^{-1/2}(\Gamma)$.

∘ **Space $H(\text{div},\Omega)$.**
First, let us recall that, for all vector field v in $\mathbb{R}^N$, div v is defined by:

$$\text{div } v = \sum_{i=1}^{N} \frac{\partial v_i}{\partial x_i} \quad .$$

Following Duvaut-Lions [DL72], we then define $H(\text{div},\Omega)$ the space of all vector fields that belong to space $(L^2(\Omega))^N$ and whose divergence is in $L^2(\Omega)$. We have classically:

$$H(\text{div},\Omega) = \left\{ v \in (L^2(\Omega))^N \ , \ \text{div } v \in L^2(\Omega) \right\} \quad , \tag{2}$$

which is a Hilbert space equipped with the norm:

$$\| v \|_{\text{div},\Omega} = \left(\left(\sum_{j=1}^{N} \| v_j \|_{0,\Omega}^2 \right) + \|\text{div } v\|_{0,\Omega}^2 \right)^{1/2} . \tag{3}$$

∘ **Normal trace in $H(\text{div},\Omega)$.**
Now, let us consider any subset Γ_1 of Γ whose measure is non zero. If φ belongs to $H_0^1(\Omega;\Gamma_1^c)$, its trace $\gamma\varphi$ belongs to $H_{00}^{1/2}(\Gamma_1)$. Following [Dub02] or Fernandes and Gilardi [FG97]), the normal trace on Γ_1, denoted by $\widetilde{\gamma_{\Gamma_1}\bullet} v$ is a linear form acting on functions that are zero on the complementary of Γ_1 in Γ.

$$\begin{array}{rcl} \widetilde{\gamma_{\Gamma_1}\bullet} : H(\text{div},\Omega) & \longrightarrow & (H_{00}^{1/2}(\Gamma_1))' \\ v & \longmapsto & \widetilde{\gamma_{\Gamma_1}\bullet} v \quad , \end{array}$$

which is defined, for all $\varphi \in H_0^1(\Omega;\Gamma_1^c)$ and $v \in H(\text{div},\Omega)$, by:

$$\langle \widetilde{\gamma_{\Gamma_1}\bullet} v, \gamma\varphi \rangle_{(H_{00}^{1/2}(\Gamma_1))', H_{00}^{1/2}(\Gamma_1)} = (v, \nabla\varphi)_0 \ + \ (\text{div } v, \varphi)_0 \quad .$$

As they coincide on regular functions, in all the sequel, the normal trace $\widetilde{\gamma_{\Gamma_1}\bullet} v$ will be shortly denoted by $v \bullet n_{|\Gamma_1}$. Finally, we define the following space:

Definition 1 *We note* $H_0(\text{div},\Omega) = \left\{ v \in H(\text{div},\Omega) \ , \ v \bullet n_{|\Gamma} = 0 \right\} \quad .$

∘ **Space** $H(\text{curl}, \Omega)$**.**
We recall that if v is a vectorial field defined on $\Omega \subset \mathbb{R}^3$, then curl v is also a vectorial field, defined by:

$$\text{curl } v = \begin{pmatrix} \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3} \\ \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2} \end{pmatrix} \quad . \tag{4}$$

Then, we define:

$$H(\text{curl}, \Omega) = \left\{ v \in (L^2(\Omega))^3 \ , \ \text{curl } v \in \left(L^2(\Omega)\right)^3 \right\} \quad .$$

When $\Omega \subset \mathbb{R}^2$ and φ is a scalar field defined on Ω, then curl φ is the vectorial field defined by:

$$\text{curl } \varphi = \begin{pmatrix} \dfrac{\partial \varphi}{\partial x_2} \\ -\dfrac{\partial \varphi}{\partial x_1} \end{pmatrix} \quad . \tag{5}$$

Here again, we can define:

$$H(\text{curl}, \Omega) = \left\{ \varphi \in L^2(\Omega) \ , \ \text{curl } \varphi \in \left(L^2(\Omega)\right)^2 \right\} \quad .$$

To be compatible with $N = 2$ or $N = 3$ dimensions, we set:

$$H(\text{curl}, \Omega) = \left\{ \varphi \in (L^2(\Omega))^{2N-3} \ , \ \text{curl } \varphi \in \left(L^2(\Omega)\right)^N \right\}. \tag{6}$$

which is equipped with the norm:

$$\| \varphi \|_{\text{curl},\Omega} = \left(\sum_{j=1}^{2N-3} \| \varphi_j \|^2_{0,\Omega} \ + \sum_{j=1}^{N} \| (\text{curl } \varphi)_j \|^2_{0,\Omega} \right)^{1/2} . \tag{7}$$

Remark 2.1 *Be aware that* $H(\text{curl}, \Omega)$ *is equal to* $H^1(\Omega)$ *in two dimensions and different from* $(H^1(\Omega))^3$ *in three.*

If Ω is contained in $\mathbb{R}^2$ and v is a vectorial field defined on Ω, the following scalar field, still denoted by curl v is obtained from the previous definition (4) by taking the last component of v equal to zero:

$$\text{curl } v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \ , \quad \text{when } \Omega \subset \mathbb{R}^2 \text{ and } v : \Omega \longrightarrow \mathbb{R}^2 \quad . \tag{8}$$

∘ **Tangential trace in $H(\mathrm{curl}, \Omega)$.**
Following Levillain [Lev91] or [Dub02], we define the space of **tangential vector functions** that are zero on the component Γ_1^c of the boundary (n is the outer normal to the boundary):

$$TH_{00}^{1/2}(\Gamma_1) = \left\{ \gamma\xi \; , \; \xi \in (H^1(\Omega))^N \; , \; \gamma\xi \bullet n \; \equiv \; 0 \text{ on } \Gamma \; , \; \gamma\xi \times n = 0 \text{ on } \Gamma_1^c \right\} \quad .$$

The tangential part of any vector $\gamma\xi$ is: $\xi_t \equiv \gamma\xi - (\gamma\xi \bullet n)\, n$. Then, for elements of space $TH_{00}^{1/2}(\Gamma_1)$, we have: $\xi_t = \gamma\xi$. In tridimensional domains, there exists a tangential trace from $H(\mathrm{curl}, \Omega)$ to $\left(TH_{00}^{1/2}(\Gamma_1)\right)'$, where $\left(TH_{00}^{1/2}(\Gamma_1)\right)'$ denotes the topological dual space of $TH_{00}^{1/2}(\Gamma_1)$,

$$\begin{aligned} \widetilde{\gamma_{\Gamma_1} \times} : H(\mathrm{curl}, \Omega) \;\; &\longrightarrow \;\; (TH_{00}^{1/2}(\Gamma_1))' \\ \varphi \;\; &\longmapsto \;\; \widetilde{\gamma_{\Gamma_1} \times} \varphi \quad , \end{aligned}$$

which is defined in the following way. Let ξ_t be in $TH_{00}^{1/2}(\Gamma_1)$, then we set:

$$\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \xi_t \rangle_{\left(TH_{00}^{1/2}(\Gamma_1)\right)', \; TH_{00}^{1/2}(\Gamma_1)} = (\varphi, \mathrm{curl}\; \xi)_0 - (\mathrm{curl}\; \varphi, \xi)_0 \quad .$$

As they coincide on regular functions, in all the sequel the tangential trace $\widetilde{\gamma_{\Gamma_1} \times} \varphi$ will be shortly denoted by $\varphi \times n_{|\Gamma_1}$.

2.2 Vorticity-velocity-pressure formulation

∘ We suppose that boundary Γ of domain Ω is decomposed with the help of **two** independent partitions:

$$\Gamma = \overline{\Gamma_m} \cup \overline{\Gamma_p} \;\; \text{with} \;\; \Gamma_m \cap \Gamma_p = \emptyset \quad , \tag{9}$$

$$\Gamma = \overline{\Gamma_\theta} \cup \overline{\Gamma_t} \;\; \text{with} \;\; \Gamma_\theta \cap \Gamma_t = \emptyset \quad . \tag{10}$$

We suppose that different types of data are given on each part of Γ: normal velocity g_0 on Γ_m , constraint Π_0 on Γ_p , tangential vorticity θ_0 on Γ_θ and tangential velocity σ_0 on Γ_t . In [Dub92] and [Dub02], it was proposed to write the Stokes problem by means of a vorticity-velocity-pressure formulation. We introduce the vorticity ω:

$$\omega = \mathrm{curl}\; u \tag{11}$$

and the Stokes problem reads formally:

$$\mathrm{curl}\; \omega - \delta \nabla \mathrm{div}\; u + \nabla p = f \text{ in } \Omega \tag{12}$$

$$\mathrm{div}\ u = 0 \text{ in } \Omega \quad , \tag{13}$$

with the very general boundary conditions:

$$u{\bullet}n = g_0 \text{ on } \Gamma_m \tag{14}$$

$$p - \delta \mathrm{div}\ u = \Pi_0 \text{ on } \Gamma_p \tag{15}$$

$$\omega \times n = \theta_0 \text{ on } \Gamma_\theta \tag{16}$$

$$n \times u \times n = \sigma_0 \text{ on } \Gamma_t \quad , \tag{17}$$

with ω given in (11) and the kinematic viscosity taken equal to 1. In the previous formulae, the constant δ is equal to 1. But, as we consider u divergence free, it is possible to forget div u in the formulation. So, the constant δ can be taken equal to 0. Then, in the following, δ will be either 0 or 1.

∘ We have already established (see [Dub02] for three-dimensional case and [Sal99] for two-dimensional case) that *modulo* some technical hypotheses recalled in Section 6.3, problem (11-17) is well-posed for the triplet (ω, u, p) in a particular case and under the restrictive hypothesis:

$$\Gamma_\theta = \Gamma_m \ \text{ and } \ \Gamma_t = \Gamma_p \quad . \tag{18}$$

In the sequel, we will **restrict** first to the case of homogeneous Dirichlet boundary conditions:

$$g_0 = 0 \text{ on } \Gamma_m \quad , \tag{19}$$

$$\theta_0 = 0 \text{ on } \Gamma_\theta \quad , \tag{20}$$

and, second, to a **particular decomposition of the boundary** Γ:

$$\Gamma_m \ \equiv \ \Gamma \quad \text{and} \quad \Gamma_p \ \equiv \ \emptyset \quad . \tag{21}$$

∘ We have precedently proposed (see [Dub92]) to formulate problem (11-17) in Sobolev spaces with the help of velocity vector space $H(\mathrm{div}, \Omega)$ such that $v{\bullet}n$ is zero on $\Gamma_m = \Gamma$. More precisely, we set:

$$X = \big\{ v \in H(\mathrm{div}, \Omega) \ , \ v{\bullet}n_{|\Gamma} = 0 \big\} \ = H_0(\mathrm{div}, \Omega) \quad . \tag{22}$$

We have proceeded in an analogous way with vorticity by setting:

$$\widetilde{W} = \Big\{ \varphi \in H(\mathrm{curl}, \Omega) \ , \ \varphi \times n_{|\Gamma_\theta} = 0 \Big\} . \tag{23}$$

Finally, meas (Γ_p) being zero, the space for the pressure is:

$$Y = L_0^2(\Omega) \quad . \tag{24}$$

Remark 2.2 *In relation (23), $\varphi \times n_{|\Gamma_\theta} = 0$ means rigorously that $\varphi \times n$ is zero in dual space $(TH_{00}^{1/2}(\Gamma_\theta))'$, analogously with the normal trace in $H(\mathrm{div}, \Omega)$.*

3 Classical bidimensional case

3.1 Stream function-vorticity formulation

○ In this section, we suppose that Ω is a bounded connected and simply connected domain of $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega \equiv \Gamma$. These hypotheses allow to consider the classical stream function-vorticity formulation of the Stokes problem. We will point out the formal link between the two formulations.

○ We choose a set of boundary conditions that consists in giving all the components of the velocity field on the entire boundary :

$$u = 0 \quad \text{on} \;\; \Gamma \quad . \tag{25}$$

With notation introduced in (11-17), boundary condition (25) corresponds to "Dirichlet-Neumann" boundary conditions in vorticity-velocity-pressure formulation :

$$\begin{array}{lllllll} \Gamma_m & = & \Gamma & \qquad g_0 & \equiv & 0 & , \\ \Gamma_t & = & \Gamma & \qquad \sigma_0 & \equiv & 0 & . \end{array}$$

The unknown velocity field u belongs to space X introduced in relation (22) and satisfies also incompressibility relation (13). Then, taking into account hypotheses done on domain Ω (see *e.g.* Girault and Raviart [GR86]), there exists a stream function ψ that belongs to space $H_0^1(\Omega)$ in such a way that u is represented as the curl of the scalar field ψ :

$$u = \text{curl}\ \psi \equiv \left(\frac{\partial \psi}{\partial x_2} , \;\; - \frac{\partial \psi}{\partial x_1} \right)^{\rm t} , \quad \psi \in H_0^1(\Omega) \quad . \tag{26}$$

Then, it is possible to write equations (11) and (12) under the form :

$$\omega + \Delta \psi = 0 \ \text{in} \ \Omega \quad , \tag{27}$$

$$- \Delta \omega = \text{curl}\ f \ \text{in} \ \Omega \quad . \tag{28}$$

Taking into account representation (26), boundary conditions for stream function are :

$$\psi = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial n} = 0 \quad \text{on} \ \Gamma \quad , \tag{29}$$

where $\dfrac{\partial \psi}{\partial n}$ is the normal derivative along Γ. These equations are the Stokes problem in stream function-vorticity formulation which was well studied (Glowinski-Pironneau [GP79], [GR86]).

We have just seen formally that the vorticity-velocity-pressure problem corresponds to the stream function-vorticity problem when we restrict to bidimensional case and particular boundary conditions. We had also observed in an earlier work ([Sal99], [DSS02b]) that this correspondence is still valid after discretization with low degree finite elements.

∘ Consider now the problem (27)-(28) under a variational form with the following Hilbert space introduced by Bernardi, Girault and Maday [BGM92]:

$$M(\Omega) = \big\{ \varphi \in L^2(\Omega) \,,\; \Delta\varphi \in H^{-1}(\Omega) \big\} \quad , \tag{30}$$

where $H^{-1}(\Omega)$ is the topological dual space of $H_0^1(\Omega)$ with the associated norm:

$$H^{-1}(\Omega) \ni \theta \;\longmapsto\; \| \, \theta \, \|_{-1,\Omega} = \sup_{v \,\in\, H_0^1(\Omega)} \frac{\langle \theta, v \rangle_{-1\,,\,1}}{\| \, \nabla v \, \|_{0,\Omega}} \quad . \tag{31}$$

Consequently, the norm on space $M(\Omega)$ is defined by the relation:

$$M(\Omega) \ni \varphi \;\longmapsto\; \| \, \varphi \, \|_{M} = \Big(\| \, \varphi \, \|_{0,\Omega}^2 + \| \, \Delta\varphi \, \|_{-1,\Omega}^2 \Big)^{1/2} \quad . \tag{32}$$

If f is given in space $(L^2(\Omega))^2$, the variational formulation of problem (27), (28), (29) is the following:

$$\psi \in H_0^1(\Omega) \quad , \; \omega \in M(\Omega) \quad , \tag{33}$$

$$(\omega, \varphi)_0 \; + \; \langle \Delta\varphi, \psi \rangle_{-1\,,\,1} = 0 \quad , \quad \forall \varphi \in M(\Omega) \tag{34}$$

$$\langle \Delta\omega, \zeta \rangle_{-1\,,\,1} = -(f, \mathrm{curl}\; \zeta)_0 \quad , \quad \forall \zeta \in H_0^1(\Omega) \tag{35}$$

and we have the following result due to [BGM92].

Theorem 3.1 The stream function-vorticity problem is well-posed.
If Ω is a bounded connected and simply connected domain of $\mathbb{R}^2$ with a Lipschitz boundary Γ and if datum f belongs to space $(L^2(\Omega))^2$, the variational problem (33), (34), (35) admits a unique solution $(\psi, \omega) \in H_0^1(\Omega) \times M(\Omega)$ that depends continuously on the datum f :

$$\exists \; C \; > \; 0 \,, \quad \| \, \nabla\psi \, \|_{0,\Omega} + \| \, \omega \, \|_{M} \;\; \leq \; C \;\; \| \, f \, \|_{0,\Omega} \,. \tag{36}$$

The proof of this theorem is based on a general result derived by Brezzi [Bre74] (see also Babuška [Bab71]).

Theorem 3.2 Mixed formulation.
Let Z and M be two Hilbert spaces, $M \times M \ni (\omega, \varphi) \longmapsto a(\omega, \varphi) \in \mathbb{R}$ and

$Z \times M \ni (\zeta,\varphi) \longmapsto b(\zeta,\varphi) \in \mathbb{R}$ *two continuous bilinear forms such that* $b(\bullet,\bullet)$ *satisfies the so-called inf-sup condition:*

$$\exists\, \beta \;>\; 0 \;,\; \inf_{\zeta\in Z} \sup_{\varphi\in M} \frac{b(\zeta,\varphi)}{\parallel \zeta \parallel_{Z} \parallel \varphi \parallel_{M}} \;\geq\; \beta \quad , \tag{37}$$

and bilinear form $a(\bullet,\bullet)$ *is elliptic on the right kernel* K *of* $b(\bullet,\bullet)$ *:*

$$K = \{\varphi \in M \;,\; \forall \zeta \in Z \;,\; b(\zeta,\varphi) = 0\} \tag{38}$$

$$\exists\, \alpha \;>\; 0 \;,\; \forall\, \varphi \in K \;,\; a(\varphi,\varphi) \;\geq\; \alpha \parallel \varphi \parallel_{M} \quad . \tag{39}$$

Then, for each pair $(\rho,\sigma) \in M' \times Z'$*, the mixed variational problem:*

$$\psi \in Z \quad , \quad \omega \in M \tag{40}$$

$$a(\omega,\varphi) \;+\; b(\psi,\varphi) = \langle \rho,\varphi \rangle_{M',M} \quad , \quad \forall\, \varphi \in M \tag{41}$$

$$b(\zeta,\omega) = \langle \sigma,\zeta \rangle_{Z',Z} \quad , \quad \forall \zeta \in Z \quad , \tag{42}$$

has a unique solution $(\psi,\omega) \in Z \times M$ *that continuously depends on datum* (ρ,σ) *:*

$$\exists\, C \;>\; 0 \;,\; \parallel \psi \parallel_{Z} + \parallel \omega \parallel_{M} \;\leq\; C \;\big(\parallel \rho \parallel_{M'} + \parallel \sigma \parallel_{Z'}\big) \quad . \tag{43}$$

<u>Proof of Theorem 3.1</u>

The present proof is a variant of the one proposed in [BGM92]. We just give it for completeness of our study.

- With the notation of Theorem 3.2, we make the following choice:

$$\begin{array}{ll} Z = H^1_0(\Omega) & M = M(\Omega) \\ a(\omega,\varphi) = (\omega,\varphi)_0 \;\;, & \omega \in M(\Omega) \;,\; \varphi \in M(\Omega) \\ b(\zeta,\varphi) = \langle \Delta\varphi,\zeta \rangle_{-1,1} \;\;, & \zeta \in H^1_0(\Omega) \;,\; \varphi \in M(\Omega) \quad . \end{array}$$

- The proof of the inf-sup condition (37) is elementary thanks to the introduction of the Poincaré constant P:

$$\parallel \zeta \parallel_{0,\Omega} \;\leq\; P \parallel \nabla\zeta \parallel_{0,\Omega} \quad , \quad \forall \zeta \in H^1_0(\Omega) \quad .$$

First, we remark that, if φ belongs to $H^1_0(\Omega)$, φ belongs also to $M(\Omega)$ and verifies:

$$\|\varphi\|_{M} \leq \sqrt{1+P^2}\|\nabla\varphi\|_{0,\Omega}. \tag{44}$$

Indeed, for $\varphi \in H_0^1(\Omega) \subset M(\Omega)$ and $\zeta \in H_0^1(\Omega)$, $\langle -\Delta\varphi, \zeta\rangle_{-1,1} = (\nabla\varphi, \nabla\zeta)_0$, then $\|\Delta\varphi\|_{-1,\Omega} \leq \|\nabla\varphi\|_{0,\Omega}$ which proves (44). Then, for a fixed ζ in $H_0^1(\Omega)$, we have (take $\varphi = -\zeta$ and use (44)) :

$$\sup_{\varphi\in M(\Omega)} \frac{\langle \Delta\varphi, \zeta\rangle_{-1,1}}{\|\nabla\zeta\|_{0,\Omega}\|\varphi\|_M} \geq \frac{\langle \Delta\zeta, \zeta\rangle_{-1,1}}{\|\nabla\zeta\|_{0,\Omega}\|\zeta\|_M} \geq \frac{\|\nabla\zeta\|_{0,\Omega}^2}{\sqrt{1+P^2}\|\nabla\zeta\|_{0,\Omega}^2}.$$

The inf-sup condition with $\beta = \dfrac{1}{\sqrt{1+P^2}}$ is proved.

- The kernel K, defined in (38), can be evaluated and we have :

$$K = \left\{\varphi \in L^2(\Omega)\ ,\ \Delta\varphi = 0 \text{ in } H^{-1}(\Omega)\right\} \quad .$$

Then, the L^2 scalar product is clearly elliptic (with $\alpha = 1$) on space K relatively to the norm (32) in space $M(\Omega)$. ■

∘ First we have proposed in Section 2.2 to search the vorticity for the (ω, u, p) formulation, in a subspace of $H(\text{curl}, \Omega)$, which is equal to $H^1(\Omega)$ in two dimensions (see (23)). But it is now understood ([BGM92]) that the (ψ, ω) problem, which can be seen as a particular case of the (ω, u, p) formulation, is well-posed when the vorticity is searched in space $M(\Omega)$ and not in space $H^1(\Omega)$. As $M(\Omega)$ is different from $H^1(\Omega) = H(\text{curl}, \Omega)$ in the two-dimensional case, we *a priori* have to change the space $\widetilde{W}$ (see (23)) where we look for the vorticity in the (ω, u, p) formulation in order to obtain a well-posed problem. The adequate space for the vorticity will be introduced further.

Remark 3.3 *We refer to our previous studies for cases where (ω, u, p) is well-posed with ω in a subspace of $H(\text{curl}, \Omega)$ [Dub02], [Sal99], [DSS02b], and for difficulties associated with the discretization of the space $M(\Omega)$ in the (ψ, ω) formulation to [GP79], [DSS00], [DSS02a] and [ASS02].*

3.2 Properties of space $M(\Omega)$

In this section we shall give some properties of the space $M(\Omega)$ and a density result useful for the last section of this paper.

∘ Let us recall the definition of the space $M(\Omega)$:

$$M(\Omega) = \left\{\varphi \in L^2(\Omega)\ ,\ \Delta\varphi \in H^{-1}(\Omega)\right\} \quad ,$$

and the associated norm of an element φ of $M(\Omega)$:

$$\| \varphi \|_M = \left(\| \varphi \|_{0,\Omega}^2 + \| \Delta\varphi \|_{-1,\Omega}^2\right)^{1/2} \quad .$$

Lemma 3.4 *Trace in space $M(\Omega)$.*
Let Ω be a simply connected open bounded domain in $\mathbb{R}^2$, with a Lipschitz boundary Γ. Then, there exists a trace operator, still denoted by γ, which is a continuous application from $M(\Omega)$ in $\big(H^{1/2}(\Gamma)\big)' = H^{-1/2}(\Gamma)$. Consequently, there exists a strictly positive constant C such that, for all φ in $M(\Omega)$, we have :

$$\| \gamma\varphi \|_{-1/2,\Gamma} \ \leq \ C \ \| \varphi \|_{M} \quad . \tag{45}$$

Proof
• Let us remark that for all g in $H^{1/2}(\Gamma)$, there exists ξ in $H^2(\Omega) \cap H_0^1(\Omega)$ such that the normal derivative $\dfrac{\partial \xi}{\partial n}$ is equal to g in space $H^{1/2}(\Gamma)$. Then, for all φ in $M(\Omega)$, expression $(\varphi, \Delta\xi)_0 - \langle \Delta\varphi, \xi \rangle_{-1,1}$ is well defined and we can set :

$$\langle \gamma\varphi, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = (\varphi, \Delta\xi)_0 - \langle \Delta\varphi, \xi \rangle_{-1,1} \quad . \tag{46}$$

Let us begin to remark that, by construction, we have :

$$|\langle \gamma\varphi, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}| \ \leq \ 2 \ \| \varphi \|_{M} \ \ \| \xi \|_{2,\Omega} \quad , \tag{47}$$

which proves that (46) defines a continuous operator on $H^2(\Omega)$.
• Now, we shall show that $\langle \gamma\varphi, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$ is effectively only function of g. We observe that, for all $\delta \in \mathcal{D}(\Omega)$, we have :

$$\langle \Delta\varphi, \delta \rangle_{-1,1} = \langle \Delta\varphi, \delta \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = (\varphi, \Delta\delta)_0 \quad ,$$

and then $\langle \gamma\varphi, \dfrac{\partial \delta}{\partial n} \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 0$ for all $\delta \in \mathcal{D}(\Omega)$. Now, using continuity (47) and density of $\mathcal{D}(\Omega)$ in $H_0^2(\Omega)$, we deduce that :

$$\langle \gamma\varphi, \frac{\partial \delta}{\partial n} \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = 0 \quad , \quad \forall \delta \in H_0^2(\Omega) \quad .$$

Finally, if ξ and η are two functions of $H^2(\Omega) \cap H_0^1(\Omega)$, such that $\dfrac{\partial \xi}{\partial n} = g = \dfrac{\partial \eta}{\partial n}$ on Γ, the difference $\delta = \xi - \eta$ belongs to $H_0^2(\Omega)$ and we have :

$$\langle \gamma\varphi, \frac{\partial \xi}{\partial n} \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \langle \gamma\varphi, \frac{\partial \eta}{\partial n} \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \langle \gamma\varphi, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \quad ,$$

which proves that $\langle \gamma\varphi, g \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$ only depends on the function g of $H^{1/2}(\Gamma)$.
• In a last step, we shall show that $\gamma\varphi$ is continuous on $H^{1/2}(\Gamma)$. Using

again the continuity property (47), and as $\langle \gamma\varphi, g\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}$ only depends on g, we deduce that :

$$|\langle \gamma\varphi, g\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}| \;\leq\; 2 \parallel \varphi \parallel_M \inf_{\zeta\in H^2(\Omega)\,\cap\, H^1_0(\Omega),\; \frac{\partial\zeta}{\partial n}=g} \parallel \zeta \parallel_{2,\Omega} \quad .$$

Then, thanks to the trace theorem [LM68], there exists a positive constant C independent of g such that :

$$|\langle \gamma\varphi, g\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}| \;\leq\; C \parallel \varphi \parallel_M \parallel g \parallel_{1/2,\Gamma} .$$

The previous inequality is valid for all φ in $M(\Omega)$ and for all g in $H^{1/2}(\Gamma)$, which proves that $\gamma\varphi$ is a continuous operator from $M(\Omega)$ in $H^{-1/2}(\Gamma)$, and leads to the announced inequality (45). ■

∘ We introduce the space of harmonic functions of $L^2(\Omega)$:

$$\mathcal{H}(\Omega) = \left\{\varphi \in L^2(\Omega)\ ,\ \Delta\varphi = 0 \in \mathcal{D}'(\Omega)\right\} \quad ,$$

and we have the following decomposition of space $M(\Omega)$:

Lemma 3.5 *Decomposition of space $M(\Omega)$.*
We have :

$$M(\Omega) = H^1_0(\Omega) \oplus \mathcal{H}(\Omega) \quad .$$

Proof
We split any element φ of $M(\Omega)$ into two parts : $\varphi = \varphi^0 + \varphi^\Delta$. On the one hand, the component φ^0 is uniquely defined in space $H^1_0(\Omega)$, since $\Delta\varphi$ belongs to $H^{-1}(\Omega)$, as the solution of the Dirichlet problem :

$$\left\{ \begin{array}{lcll} \Delta\varphi^0 & = & \Delta\varphi & \text{in } \Omega \\ \gamma\varphi^0 & = & 0 & \text{on } \gamma \end{array} \right. .$$

On the other hand, we set : $\varphi^\Delta = \varphi - \varphi^0$. Then, function φ^Δ belongs to $L^2(\Omega)$ as φ and φ^0. Moreover, by construction, we have : $\Delta\varphi^\Delta = 0$ in Ω. Then, φ^Δ belongs to $\mathcal{H}(\Omega)$. Let us remark that $\gamma\varphi^\Delta = \gamma\varphi$ in space $H^{-1/2}(\Gamma)$ (see Lemma 3.4). ■

Lemma 3.6 *Scalar product in $M(\Omega)$ (see also [Rua95b]).*
Let φ and ξ be two elements of space $M(\Omega)$. Using the previous decomposition (see Lemma 3.5), we can write : $\varphi = \varphi^0 + \varphi^\Delta$ and $\xi = \xi^0 + \xi^\Delta$. Then, the scalar product in $M(\Omega)$ associated with norm (32) can be written as :

$$(\varphi, \xi)_M = (\varphi, \xi)_0 \;+\; (\nabla\varphi^0, \nabla\xi^0)_0 \quad . \tag{48}$$

Proof
The scalar product in $M(\Omega)$ can be expressed by :

$$(\varphi,\xi)_{_M} = \frac{1}{2}\left(\| \varphi+\xi \|^2_{_M} - \| \varphi \|^2_{_M} - \| \xi \|^2_{_M}\right) \quad .$$

For any function ψ of $M(\Omega)$, we have: $\| \psi \|^2_{_M} = \| \psi \|^2_{0,\Omega} + \| \Delta\psi \|^2_{-1,\Omega}$. Now, introducing the decomposition given in Lemma 3.5, as $\Delta\psi^{\Delta}$ is zero, we obtain :

$$\| \psi \|^2_{_M} = \| \psi \|^2_{0,\Omega} + \| \Delta\psi^0 \|^2_{-1,\Omega} \quad ,$$

for all $\psi \in M(\Omega)$. Moreover, we have :

$$\| \Delta\psi^0 \|_{-1,\Omega} = \sup_{\zeta\in H^1_0(\Omega)} \frac{\langle \Delta\psi^0, \zeta\rangle_{-1,1}}{\| \nabla\zeta \|_{0,\Omega}} = \sup_{\zeta\in H^1_0(\Omega)} \frac{-(\nabla\psi^0, \nabla\zeta)_{_0}}{\| \nabla\zeta \|_{0,\Omega}} = \| \nabla\psi^0 \|_{0,\Omega}$$

as ψ^0 belongs to $H^1_0(\Omega)$. Then, we obtain for the $M(\Omega)$-scalar product :

$$\begin{aligned}(\varphi,\xi)_{_M} &= \frac{1}{2}\left(\| \varphi+\xi \|^2_{_0} - \| \varphi \|^2_{_0} - \| \xi \|^2_{_0}\right) \\ &+ \frac{1}{2}\left(\| \nabla\varphi^0 + \nabla\xi^0 \|^2_{0,\Omega} - \| \nabla\varphi^0 \|^2_{0,\Omega} - \| \nabla\xi^0 \|^2_{0,\Omega}\right) \quad .\end{aligned}$$

In the above expression, the first block gives the standard $L^2(\Omega)$-scalar product between φ and ξ, and the second the $L^2(\Omega)$-scalar product between $\nabla\varphi^0$ and $\nabla\xi^0$, which achieves the proof. ■

Proposition 3.7 *Density of $H^1(\Omega)$ in $M(\Omega)$.*
Let Ω be a simply connected open bounded domain in $\mathbb{R}^2$, with a Lipschitz boundary Γ. Space $H^1(\Omega)$ is dense in space $M(\Omega)$ for the norm $\| \bullet \|_{_M}$.

Proof
- This proof is close to the one of Theorem 2.4 in [GR86], and is based on the following property : a subspace $\mathcal{S}$ of a Hilbert space M is dense in M if and only if every element of M' that vanishes on $\mathcal{S}$ also vanishes on M.
- Let $\widehat{\varphi}$ belong to $(M(\Omega))'$. As it is a Hilbert space, the Riesz theorem proves that there exists a function, denoted by φ, in $M(\Omega)$ such that :

$$\langle \widehat{\varphi}, \xi\rangle_{(M(\Omega))', M(\Omega)} = (\varphi,\xi)_{_M} \quad , \quad \forall\, \xi \in M(\Omega) \quad .$$

Using the decomposition introduced in Lemma 3.5 and the expression of the $M(\Omega)$-scalar product derived in (48), we have for all $\xi \in M(\Omega)$:

$$(\varphi,\xi)_{_M} = (\varphi,\xi)_{_0} + (\nabla\varphi^0, \nabla\xi^0)_{_0} \quad .$$

We suppose now that $\widehat{\varphi}$ vanishes on $H^1(\Omega)$. Then, we obtain:

$$(\varphi, \xi)_0 \ + \ (\nabla\varphi^0, \nabla\xi^0)_0 = 0 \ , \ \forall \xi \in H^1(\Omega) \quad .$$

In this relation, we have used the splitting of ξ considered as an element of $M(\Omega)$. Moreover, as φ^0 belongs to $H_0^1(\Omega)$, we have also:

$$\begin{aligned} (\nabla\varphi^0, \nabla\xi^0)_0 &= -\langle \Delta\xi^0, \varphi^0 \rangle_{-1,1} \\ &= -\langle \Delta\xi, \varphi^0 \rangle_{-1,1} \quad \text{as } \Delta\xi^0 = \Delta\xi \\ &= (\nabla\varphi^0, \nabla\xi)_0 \quad \text{as } \xi \in H^1(\Omega) \quad . \end{aligned}$$

Then, we have:

$$(\varphi, \xi)_0 \ + \ (\nabla\varphi^0, \nabla\xi)_0 = 0 \ , \ \forall \xi \in H^1(\Omega) \quad .$$

This equality implies that in the distribution sense:

$$\varphi - \text{div } \nabla\varphi^0 = 0 \text{ in } \mathcal{D}'(\Omega) \quad .$$

Then, φ^0 is solution in $H_0^1(\Omega)$ of the following problem:

$$\left\{ \begin{array}{rcccl} \text{div } \nabla\varphi^0 & = & \Delta\varphi^0 & = & \varphi \quad \text{in } \Omega \\ & & \gamma\varphi^0 & = & 0 \quad \text{on } \Gamma \quad . \end{array} \right.$$

And we can observe that $\Delta\varphi^0$, which is equal to φ, belongs to $L^2(\Omega)$.

• As $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, let $(\varphi_k)_{k\geq 1}$ be a sequence of $\mathcal{D}(\Omega)$ that tends to φ^0 in $H_0^1(\Omega)$. Then, we have the following relations:

$$\begin{aligned} \varphi_k &\stackrel{k\longrightarrow\infty}{\longrightarrow} \varphi^0 \text{ in } H_0^1(\Omega) \quad , \\ \Delta\varphi_k &\stackrel{k\longrightarrow\infty}{\longrightarrow} \Delta\varphi^0 = \varphi \text{ in } \mathcal{D}'(\Omega). \end{aligned}$$

Then, we have:

$$\langle \Delta\varphi_k, \psi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \stackrel{k\longrightarrow\infty}{\longrightarrow} \langle \Delta\varphi^0, \psi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \text{ for all } \psi \in \mathcal{D}(\Omega) \quad ,$$

which can be rewritten, as $\Delta\varphi_k$ and $\Delta\varphi^0$ are both in $L^2(\Omega)$:

$$(\Delta\varphi_k, \psi)_0 \stackrel{k\longrightarrow\infty}{\longrightarrow} (\Delta\varphi^0, \psi)_0 \text{ for all } \psi \in \mathcal{D}(\Omega) \quad .$$

Finally, as $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$ and if ψ belongs to $L^2(\Omega)$, $(\Delta\varphi^0, \psi)_0$ is the limit, when k tends to infinity, of $(\Delta\varphi_k, \psi)_0$.

• Let us now prove that, for any arbitrary element ξ of $M(\Omega)$, we have:

$$\langle \widehat{\varphi}, \xi \rangle_{(M(\Omega))', M(\Omega)} = (\varphi, \xi)_M = 0 \quad .$$

Using the expression of the scalar product (48) and, as we have seen above, we obtain:

$$\begin{aligned}
(\varphi,\xi)_M &= (\varphi,\xi)_0 + (\nabla\varphi^0,\nabla\xi^0)_0 \\
&= (\varphi,\xi)_0 - \langle\Delta\xi^0,\varphi^0\rangle_{-1,1} \\
&= (\varphi,\xi)_0 - \langle\Delta\xi,\varphi^0\rangle_{-1,1} \quad \text{as } \Delta\xi^0 = \Delta\xi \\
&= (\Delta\varphi^0,\xi)_0 - \langle\Delta\xi,\varphi^0\rangle_{-1,1} \quad \text{as } \Delta\varphi^0 = \varphi \\
&= \lim_{k\longrightarrow\infty} (\Delta\varphi_k,\xi)_0 - \lim_{k\longrightarrow\infty} \langle\Delta\xi,\varphi_k\rangle_{-1,1} \\
&= \lim_{k\longrightarrow\infty} \left(\langle\xi,\Delta\varphi_k\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} - \langle\Delta\xi,\varphi_k\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \right),
\end{aligned}$$

which leads to the result: $(\varphi,\xi)_M = 0$ for all ξ in $M(\Omega)$. ■

4 New functional space for vorticity

In this section, we shall define the new space W where we search for the vorticity as announced above. Instead of being a subspace of $H(\text{curl},\Omega)$, W will be a subspace of a new functional space $H(\text{curl},\text{div}*,\Omega)$ that we introduce in this section. We will define a weak rotational operator acting on functions in $H(\text{curl},\text{div}*,\Omega)$ and a "co-curl" operator taking its values in the space of velocities. We will need to define a tangential trace on this space to impose boundary conditions on the vorticity in the formulation.

4.1 Functional space and weak rotational operator

○ We have introduced above the space $H_0(\text{div},\Omega)$ as:

$$H_0(\text{div},\Omega) = \left\{ v \in H(\text{div},\Omega)\ ,\ v\bullet n_{|\Gamma} = 0 \right\} \quad .$$

We first recall how to deal with the dual space $(H_0(\text{div},\Omega))'$. As space $(H_0(\text{div},\Omega))'$ is a subspace of $(\mathcal{D}'(\Omega))^N$, if we consider a linear form T of $(\mathcal{D}'(\Omega))^N$, T belongs to space $(H_0(\text{div},\Omega))'$ if and only if T is continuous for the $H(\text{div},\Omega)$-topology, *ie* if there exists a constant $C > 0$ such that:

$$\forall v \in (\mathcal{D}(\Omega))^N,\ |\langle T,v\rangle_{(\mathcal{D}'(\Omega))^N,(\mathcal{D}(\Omega))^N}| \leq C \left(\|\, v \,\|^2_{0,\Omega} + \|\, \text{div}\ v \,\|^2_{0,\Omega} \right)^{1/2} .$$

○ For $\varphi \in (L^2(\Omega))^{2N-3}$, the distribution curl φ is well defined in $(\mathcal{D}'(\Omega))^N$ and is given by:

$$\langle \text{curl}\ \varphi, v\rangle_{(\mathcal{D}'(\Omega))^N,(\mathcal{D}(\Omega))^N} = (\varphi, \text{curl}\ v)_0 \quad , \quad \forall v \in (\mathcal{D}(\Omega))^N$$

We restrict ourselves to fields $\varphi \in (L^2(\Omega))^{2N-3}$ such that the distribution curl φ is **continuous** for the $H(\text{div},\Omega)$-norm. It thus gives a mathematical sense to the duality product $\langle \text{curl}\ \omega, v\rangle$ when v is a vector field that belongs to space $H(\text{div},\Omega)$, as suggested to us by Amara [Ama97].

Definition 2 *Functional space for vorticity.*
We set $H(\text{curl},\text{div}*,\Omega)$ *the following space:*

$$H(\text{curl},\text{div}*,\Omega) = \left\{ \begin{array}{l} \varphi \in (L^2(\Omega))^N \ , \ \exists\ C \ > \ 0 \ , \ \forall v \in (\mathcal{D}(\Omega))^N \ , \\ |(\varphi, \text{curl}\ v)_0| \leq C \left(\| \ v \ \|_{0,\Omega}^2 + \| \ \text{div}\ v \ \|_{0,\Omega}^2 \right)^{1/2} \end{array} \right\} \quad (49)$$

∘ We denote by $\langle \bullet, \bullet \rangle_{\text{div}*,\text{div}}$ the duality product between $H_0(\text{div},\Omega)$ and its dual. Then, the norm of an element of $(H_0(\text{div},\Omega))'$ is defined as follows:

$$(H_0(\text{div},\Omega))' \ni \zeta \longmapsto \| \ \zeta \ \|_{\text{div}*,\Omega} = \sup_{v \in H_0(\text{div},\Omega)} \frac{\langle \zeta, v \rangle_{\text{div}*,\text{div}}}{\| \ v \ \|_{\text{div},\Omega}} \quad . \qquad (50)$$

Proposition and definition 4.1 *Weak rotational operator.*
∘ *For functions* φ *in* $H(\text{curl},\text{div}*,\Omega)$, *the application denoted by* $R_* \varphi$ *and defined by:*

$$(\mathcal{D}(\Omega))^N \ni v \longmapsto \ \langle R_* \varphi, v \rangle_{\text{div}*,\text{div}} \ = \ (\varphi, \text{curl}\ v)_0, \qquad (51)$$

is continuous from space $(\mathcal{D}(\Omega))^N$ *in* $\mathbb{R}$ *for the* $H(\text{div},\Omega)$ *-topology.*
∘ *Then, for* $\varphi \in H(\text{curl},\text{div}*,\Omega)$, *the application* $R_*\varphi$ *is uniquely extended by continuity to space* $H_0(\text{div},\Omega)$ *and the application:*

$$R_* : H(\text{curl},\text{div}*,\Omega) \ni \varphi \longmapsto R_*\varphi \in (H_0(\text{div},\Omega))'$$

is thus well defined.
∘ *The application:*

$$H(\text{curl},\text{div}*,\Omega) \ni \varphi \longmapsto \| \ \varphi \ \|_{\text{curl},\text{div}*,\Omega} = \left(\| \ \varphi \ \|_{0,\Omega}^2 + \| \ R_*\varphi \ \|_{\text{div}*,\Omega}^2 \right)^{1/2} \qquad (52)$$

is a norm on the space $H(\text{curl},\text{div}*,\Omega)$. *Moreover, for this norm and the associated scalar product,* $H(\text{curl},\text{div}*,\Omega)$ *is a Hilbert space.*

Proof

Thanks to the density of $(\mathcal{D}(\Omega))^N$ in $H_0(\text{div},\Omega)$, we can use the extension theorem for uniformly continuous functions due to the continuity properties (49). ■

∘ The next proposition shows that $H(\text{curl}, \text{div}*, \Omega)$ is not empty as the space $H(\text{curl}, \Omega)$ is contained in it, and that R_* is the natural extension of the curl operator as they both coincide on regular functions.

Proposition 4.2
When φ belongs to $H(\text{curl}, \Omega)$, then $R_*\varphi$ is equal to curl φ *in $(H_0(\text{div}, \Omega))'$ and we have:*

$$\langle R_*\varphi, v\rangle_{\text{div}*,\text{div}} = (\text{curl }\varphi, v)_0 \ , \ \varphi \in H(\text{curl}, \Omega) \ , \ v \in H_0(\text{div}, \Omega). \quad (53)$$

Proof
For all $\varphi \in H(\text{curl}, \Omega)$, φ belongs to $(L^2(\Omega))^{2N-3}$ and curl φ to $(L^2(\Omega))^N$. The application: $H_0(\text{div}, \Omega) \ni v \longmapsto (\text{curl }\varphi, v)_0 \in \mathbb{R}$ is linear and continuous for the $H(\text{div}, \Omega)-$topology, so belongs to $(H_0(\text{div}, \Omega))'$. Moreover, for all $v \in (\mathcal{D}(\Omega))^N$:

$$\begin{aligned}(\text{curl }\varphi, v)_0 &= (\varphi, \text{curl } v)_0 \quad \text{by integrating by parts} \\ &= \langle R_*\varphi, v\rangle_{\text{div}*,\text{div}} \text{ by Definition (51).}\end{aligned} \quad (54)$$

¿From which we deduce that $R_*\varphi = \text{curl }\varphi$ in $(\mathcal{D}'(\Omega))^N$ and since $(\mathcal{D}(\Omega))^N$ is dense in $H_0(\text{div}, \Omega)$, $R_*\varphi = \text{curl }\varphi$ in $(H_0(\text{div}, \Omega))'$, which leads to (53). ■

Remark 4.3 *Notice that we can define $\langle R_*\varphi, v\rangle_{\text{div}*,\text{div}}$ for a function v in $H_0(\text{div}, \Omega)$, but that we do not know how to define $\langle R_*\varphi, v\rangle_{\text{div}*,\text{div}}$ for a generic function v in $H(\text{div}, \Omega)$.*

As $R_*\varphi$ and curl φ coincide on regular functions, in all the sequel, we will drop the notation $R_*\varphi$ for those of curl φ, $\varphi \in H(\text{curl}, \text{div}*, \Omega)$.

4.2 Properties of space $H(\text{curl}, \text{div}*, \Omega)$

In this section, we prove the basic properties of the new space introduced above.

Proposition 4.4 *Density of $H(\text{curl}, \Omega)$ in $H(\text{curl}, \text{div}*, \Omega)$.*
The space $H(\text{curl}, \Omega)$ is dense in $H(\text{curl}, \text{div}*, \Omega)$ for the norm $\| \bullet \|_{\text{curl},\text{div}*,\Omega}$.

Proof
- Here again, this proof is based on the following property: a subspace $\mathcal{S}$ of a Hilbert space M is dense in M if and only if every element of M' that vanishes on $\mathcal{S}$ also vanishes on M.
- Let $\widehat{\varphi}$ belong to $(H(\text{curl}, \text{div}*, \Omega))'$. As space $H(\text{curl}, \text{div}*, \Omega)$ is a Hilbert

space (see Proposition 4.1), the Riesz theorem proves that there exists a function, denoted by φ, in $H(\mathrm{curl}, \mathrm{div}*, \Omega)$ such that:

$$\begin{aligned}\langle \widehat{\varphi}, \xi \rangle_{(H(\mathrm{curl},\mathrm{div}*,\Omega))', H(\mathrm{curl},\mathrm{div}*,\Omega)} &= (\varphi, \xi)_{\mathrm{curl},\mathrm{div}*,\Omega}\ ,\ \forall\ \xi \in H(\mathrm{curl}, \mathrm{div}*, \Omega) \\ &= (\varphi, \xi)_0 + (\mathrm{curl}\ \varphi, \mathrm{curl}\ \xi)_{(H_0(\mathrm{div},\Omega))'}\end{aligned}$$

By applying again the Riesz theorem for curl $\varphi \in (H_0(\mathrm{div}, \Omega))'$, we can find an element of $H_0(\mathrm{div}, \Omega)$, denoted by $\rho\varphi$ such that:

$$(\varphi, \xi)_{\mathrm{curl},\mathrm{div}*,\Omega} = (\varphi, \xi)_0 + \langle \mathrm{curl}\ \xi, \rho\varphi \rangle_{\mathrm{div}*,\mathrm{div}} \quad .$$

We suppose now that $\widehat{\varphi}$ vanishes on $H(\mathrm{curl}, \Omega)$ *ie*:

$$(\varphi, \xi)_{\mathrm{curl},\mathrm{div}*,\Omega} = 0\ ,\ \forall \xi \in H(\mathrm{curl}, \Omega) \quad .$$

Using Proposition 4.2, as $\rho\varphi$ belongs to $H_0(\mathrm{div}, \Omega)$, we obtain:

$$(\varphi, \xi)_0 + (\mathrm{curl}\ \xi, \rho\varphi)_0 = 0\ ,\ \forall \xi \in H(\mathrm{curl}, \Omega) \quad .$$

Let us now introduce $\widetilde{\varphi}$ and $\widetilde{\rho\varphi}$ the extensions to $\mathbb{R}^N$, by zero outside Ω, of functions φ and $\rho\varphi$. Let us remark that $\widetilde{\rho\varphi}$ belongs to $H(\mathrm{div}, \mathbb{R}^N)$ as $\rho\varphi \bullet n_{|\Gamma} = 0$. Moreover, let us notice that, for all function $\widetilde{\xi} \in (\mathcal{D}(\mathbb{R}^N))^{2N-3}$, its restriction on Ω, say ξ, belongs to $H(\mathrm{curl}, \Omega)$. Then, the above formula leads to the following relations:

$$\begin{aligned}0 &= (\varphi, \xi)_0 + (\mathrm{curl}\ \xi, \rho\varphi)_0 \\ &= (\widetilde{\varphi}, \xi)_0 + (\mathrm{curl}\ \xi, \widetilde{\rho\varphi})_0 \\ &= (\widetilde{\varphi}, \widetilde{\xi})_{L^2(\mathbb{R}^N)} + (\mathrm{curl}\ \widetilde{\xi}, \widetilde{\rho\varphi})_{L^2(\mathbb{R}^N)} \quad \forall \widetilde{\xi} \in (\mathcal{D}(\mathbb{R}^N))^{2N-3} \quad .\end{aligned}$$

This equality implies that in the distributions sense:

$$\widetilde{\varphi} + \mathrm{curl}\ \widetilde{\rho\varphi} = 0 \text{ in } (\mathcal{D}'(\mathbb{R}^N))^{2N-3}.$$

Thus, $\widetilde{\rho\varphi}$ belongs to $H(\mathrm{curl}, \mathbb{R}^N)$ since $\widetilde{\varphi}$ belongs to $(L^2(\mathbb{R}^N))^{2N-3}$. As $\widetilde{\rho\varphi}$ belongs to $H(\mathrm{div}, \mathbb{R}^N)$, we deduce that $\widetilde{\rho\varphi}$ belongs to $H(\mathrm{curl}, \mathbb{R}^N) \cap H(\mathrm{div}, \mathbb{R}^N) = (H^1(\mathbb{R}^N))^N$ (see Weber [Web80] or [GR86]). Moreover, $\widetilde{\rho\varphi}$ being identically zero outside Ω, we deduce that $\rho\varphi$ belongs to $(H_0^1(\Omega))^N$ (see [GR86]).

• As $(\mathcal{D}(\Omega))^N$ is dense in $(H_0^1(\Omega))^N$, let $(\psi_k)_{k\geq 1}$ be a sequence of $(\mathcal{D}(\Omega))^N$ that tends to $\rho\varphi$ in $(H_0^1(\Omega))^N$. Then, we have the following relations:

$$\begin{aligned}\psi_k &\overset{k\longrightarrow\infty}{\longrightarrow} \rho\varphi \text{ in } H_0(\mathrm{div}, \Omega) \quad , \\ \mathrm{curl}\ \psi_k &\overset{k\longrightarrow\infty}{\longrightarrow} \mathrm{curl}\ \rho\varphi = -\varphi \text{ in } (L^2(\Omega))^{2N-3} \quad ,\end{aligned}$$

• Let us now prove that, for any arbitrary element ξ of $H(\text{curl}, \text{div}*, \Omega)$, we have:

$$\langle \widehat{\varphi}, \xi \rangle_{(H(\text{curl},\text{div}*,\Omega))', H(\text{curl},\text{div}*,\Omega)} = (\varphi, \xi)_{\text{curl},\text{div}*,\Omega} = 0 \quad .$$

We have seen that $\rho\varphi$ belongs to $(H_0^1(\Omega))^N$, which is contained in $H_0(\text{div}, \Omega)$. Using the above convergences, we obtain:

$$\begin{aligned}(\varphi, \xi)_{\text{curl},\text{div}*,\Omega} &= (\varphi, \xi)_0 + \langle \text{curl}\ \xi, \rho\varphi \rangle_{\text{div}*,\text{div}} \\ &= \lim_{k \longrightarrow \infty} [(-\text{curl}\ \psi_k, \xi)_0 + \langle \text{curl}\ \xi, \psi_k \rangle_{\text{div}*,\text{div}}].\end{aligned}$$

As ψ_k belongs to $(\mathcal{D}(\Omega))^N$, using (51), we have:

$$\langle \text{curl}\ \xi, \psi_k \rangle_{\text{div}*,\text{div}} = (\xi, \text{curl}\ \psi_k)_0.$$

Then we obtain: $(\varphi, \xi)_{\text{curl},\text{div}*,\Omega} = 0$ for any $\xi \in H(\text{curl}, \text{div}*, \Omega)$, which finishes to prove that $H(\text{curl}, \Omega)$ is dense in $H(\text{curl}, \text{div}*, \Omega)$. ■

○ Because of the previous density property, the scope of this paragraph is to define an extension of the tangential trace, naturally defined in $H(\text{curl}, \Omega)$ (see Section 2.1), for functions of $H(\text{curl}, \text{div}*, \Omega)$. Let Γ_1 be an arbitrary subset of the boundary Γ, and n the outer normal along Γ, let us recall that we have defined the following space of tangential boundary vector functions that are different from zero on Γ_1:

$$TH_{00}^{1/2}(\Gamma_1) = \left\{ \gamma\xi \ , \ \xi \in (H^1(\Omega))^N \ , \ \gamma\xi \bullet n \ \equiv \ 0 \text{ on } \Gamma \ , \ \gamma\xi \times n = 0 \text{ on } \Gamma_1^c \right\} \quad .$$

Let us observe that an element ξ of $(H^1(\Omega))^N$ such that: $\xi \bullet n = 0$ on Γ, belongs also to space $H_0(\text{div}, \Omega)$. Then, we can define the following tangential trace.

Proposition 4.5 *Tangential trace operator on* $H(\text{curl}, \text{div}*, \Omega)$.
Let Ω *be an open bounded domain in* $\mathbb{R}^N$ *whose boundary* Γ *is such that* $\Gamma_1 \subset \Gamma$. *There exists a continuous application* $\widetilde{\gamma_{\Gamma_1} \times}$ *from* $H(\text{curl}, \text{div}*, \Omega)$ *in* $\left(TH_{00}^{1/2}(\Gamma_1) \right)'$*:*

$$\begin{aligned}\widetilde{\gamma_{\Gamma_1} \times} : H(\text{curl}, \text{div}*, \Omega) &\longrightarrow \left(TH_{00}^{1/2}(\Gamma_1) \right)' \\ \varphi &\longmapsto \widetilde{\gamma_{\Gamma_1} \times}\varphi \quad ,\end{aligned}$$

Let $\gamma\xi$ *be in* $TH_{00}^{1/2}(\Gamma_1)$*, associated with* $\xi \in (H^1(\Omega))^N \ \cap \ H_0(\text{div}, \Omega)$*. Then, the normal component of* $\gamma\xi$ *is reduced to zero and the previous trace operator is defined by:*

$$\langle \widetilde{\gamma_{\Gamma_1} \times}\varphi, \gamma\xi \rangle = (\varphi, \text{curl}\ \xi)_0 \ - \ \langle \text{curl}\ \varphi, \xi \rangle_{\text{div}*,\text{div}} \quad . \tag{55}$$

Proof

• Let $\varphi \in H(\text{curl}, \text{div}*, \Omega)$ and $\xi \in (H^1(\Omega))^N \cap H_0(\text{div}, \Omega)$. Then we remark that: $\langle \text{curl } \varphi, \xi \rangle_{\text{div}*,\text{div}}$ is well defined. In a first step, we shall show that $\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\xi \rangle$ is effectively only function of the trace of ξ. Let us begin to remark that, by construction, we have:

$$|\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\xi \rangle| \ \leq \ 2 \ \| \varphi \|_{\text{curl},\text{div}*,\Omega} \| \xi \|_{1,\Omega}, \qquad (56)$$

which proves that (55) define a continuous operator on $(H^1(\Omega))^N$. Then, for all $\delta \in (\mathcal{D}(\Omega))^N$, we have thanks to Definition (51):

$$\langle \text{curl } \varphi, \delta \rangle_{\text{div}*,\text{div}} = (\varphi, \text{curl } \delta)_0 \quad .$$

Then $\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \delta \rangle = 0$ for all $\delta \in (\mathcal{D}(\Omega))^N$ (*cf* (55)). And, using continuity (56) and density of $(\mathcal{D}(\Omega))^N$ in $(H_0^1(\Omega))^N$, we deduce that:

$$\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\delta \rangle = 0 \ \ , \ \ \forall \delta \in (H_0^1(\Omega))^N \quad .$$

Finally, if ξ and η are two functions of $(H^1(\Omega))^N \cap H_0(\text{div}, \Omega)$, such that $\gamma\xi = \gamma\eta$ on Γ_1 and $\gamma\xi = \gamma\eta = 0$ on Γ_1^c, then the difference $\delta = \xi - \eta$ belongs to $(H_0^1(\Omega))^N$ and we have:

$$\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\xi \rangle = \langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\eta \rangle \ \ ,$$

which proves that $\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\xi \rangle$ only depends on the trace $\gamma\xi$ of ξ on Γ_1.

• In a second step, we shall show that $\widetilde{\gamma_{\Gamma_1} \times} \varphi$ is a continuous function on $TH_{00}^{1/2}(\Gamma_1)$. Using again the continuity property (56), and as $\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\xi \rangle$ only depends on ξ, we deduce that:

$$|\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\xi \rangle| \ \leq \ 2 \, \| \varphi \|_{\text{curl},\text{div}*,\Omega} \inf_{\zeta \in (H^1(\Omega))^3, \ \gamma\zeta = \gamma\xi} \| \zeta \|_{1,\Omega} \quad .$$

Then, thanks to the trace theorem [LM68], there exists a positive constant C independent of ξ such that:

$$|\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\xi \rangle| \ \leq \ C \, \| \varphi \|_{\text{curl},\text{div}*,\Omega} \| \gamma\xi \|_{1/2,\Gamma} \quad .$$

The previous inequality remains valid for all φ in $H(\text{curl}, \text{div}*, \Omega)$ and ξ in $(H^1(\Omega))^N \cap H_0(\text{div}, \Omega)$, such that $\gamma\xi$ belongs to $TH_{00}^{1/2}(\Gamma_1)$, which proves that $\widetilde{\gamma_{\Gamma_1} \times}$ is a continuous operator from $H(\text{curl}, \text{div}*, \Omega)$ in $(TH_{00}^{1/2}(\Gamma_1))'$. ■

Proposition 4.6 *Case of regular functions.*
Let Ω be an open bounded domain in $\mathbb{R}^N$ whose boundary Γ is such that $\Gamma_1 \subset \Gamma$. If function φ belongs to $H(\text{curl}, \Omega)$, $\widetilde{\gamma_{\Gamma_1} \times}$ is equal to the "tangential trace" of φ (see Section 2.1):

$$\widetilde{\gamma_{\Gamma_1} \times} \varphi = \varphi \times n_{|\Gamma_1} \ \text{ if } \varphi \in H(\text{curl}, \Omega). \qquad (57)$$

Proof

Let us notice that if φ belongs to $H(\text{curl}, \Omega)$, we can rewrite, thanks to Proposition 4.2, the duality product $\langle \text{curl } \varphi, \xi \rangle_{\text{div}*,\text{div}}$:

$$\langle \text{curl } \varphi, \xi \rangle_{\text{div}*,\text{div}} \quad = \quad (\text{curl } \varphi, \xi)_0 \quad , \quad \text{for all } \ \xi \in (H^1(\Omega))^N \ \cap \ H_0(\text{div}, \Omega) \quad .$$

Then, with the help of Green's formula and by definition of the tangential trace in $H(\text{curl}, \Omega)$, we have for all φ in $H(\text{curl}, \Omega)$:

$$\langle \widetilde{\gamma_{\Gamma_1} \times} \varphi, \gamma\xi \rangle \quad = \quad \langle \varphi \times n_{|\Gamma_1}, \gamma\xi \rangle \quad .$$

which achieves the proof. ■

Thanks to this last proposition, as the traces coincide on regular functions, in all the sequel, we will shortly denote $\widetilde{\gamma_{\Gamma_1} \times} \varphi$ by $\varphi \times n_{|\Gamma_1}$, for all φ in $H(\text{curl}, \text{div}*, \Omega)$.

4.3 Definition of the vorticity space

Let Γ_θ and Γ_t be a partition of the boundary : $\Gamma = \overline{\Gamma_\theta} \cup \overline{\Gamma_t}$ with $\Gamma_\theta \cap \Gamma_t = \emptyset$. We can now define the new vectorial space where we shall search for the vorticity. This space is composed of functions of $H(\text{curl}, \text{div}*, \Omega)$ introduced previously, whose tangential trace is null on the part Γ_θ of Γ. We set :

$$W = \left\{ \varphi \in H(\text{curl}, \text{div}*, \Omega) \ , \ \varphi \times n_{|\Gamma_\theta} = 0 \text{ in } \left(TH_{00}^{1/2}(\Gamma_\theta) \right)' \right\} \quad . \tag{58}$$

This definition allows to introduce the following curl operator $R : W \longrightarrow X'$ for functions of W. Let us define l as the canonical injection from W to $H(\text{curl}, \text{div}*, \Omega)$. In Definition 4.1, we have introduced a curl operator from $H(\text{curl}, \text{div}*, \Omega)$ to $(H_0(\text{div}, \Omega))' = X'$, named R_* in this case. Then, we define $R = R_*.l$:

$$\begin{array}{lccccc} R_* & : & H(\text{curl}, \text{div}*, \Omega) & \xrightarrow{R_*} & X' \\ & & \uparrow l & & \downarrow \\ R & : & W & \longrightarrow & X'. \end{array} \tag{59}$$

The expression $\langle R\varphi, v \rangle_{X',X}$ is now well defined for all $\varphi \in W$, $v \in X$.
The norm in W is naturally defined as follows :

$$\begin{aligned} \| \varphi \|^2_{\text{curl},\text{div}*,\Omega} \quad &= \quad \| \varphi \|^2_{0,\Omega} + \| R\varphi \|^2_{\text{div}*,\Omega} \\ &= \quad \| \varphi \|^2_{0,\Omega} + \left(\sup_{v \in H_0(\text{div}, \Omega)} \frac{\langle R\varphi, v \rangle_{\text{div}*,\text{div}}}{\| v \|_{\text{div},\Omega}} \right)^2 . \end{aligned}$$

The following lemma will be useful in the sequel.

Lemma 4.7 *There exists a constant* $C > 0$ *such that, for all function* φ *of* $(H^1(\Omega))^{2N-3} \cap W$*, we have:*

$$\| \varphi \|_W = \| \varphi \|_{\mathrm{curl},\mathrm{div}*,\Omega} \leq C \, \| \varphi \|_{1,\Omega} .$$

Proof
First, let us recall that:

$$\| \varphi \|^2_{\mathrm{curl},\mathrm{div}*,\Omega} = \| \varphi \|^2_{0,\Omega} + \left(\sup_{v \in H_0(\mathrm{div},\Omega)} \frac{\langle R\varphi, v \rangle_{\mathrm{div}*,\mathrm{div}}}{\| v \|_{\mathrm{div},\Omega}} \right)^2 .$$

As space $(H^1(\Omega))^{2N-3}$ is contained in $H(\mathrm{curl},\Omega)$, Proposition 4.2 shows that $R_*\varphi$ is equal to curl φ in $(H_0(\mathrm{div},\Omega))'$. Then, thanks to Cauchy-Schwarz inequality, we deduce that:

$$\begin{aligned} \| \varphi \|^2_{\mathrm{curl},\mathrm{div}*,\Omega} &\leq \| \varphi \|^2_{0,\Omega} + \sup_{v \in H_0(\mathrm{div},\Omega)} \frac{\| \mathrm{curl}\, \varphi \|^2_{0,\Omega} \| v \|^2_{0,\Omega}}{\| v \|^2_{\mathrm{div},\Omega}} \\ &\leq \| \varphi \|^2_{0,\Omega} + \| \mathrm{curl}\, \varphi \|^2_{0,\Omega} \\ &\leq C \, \| \varphi \|^2_{1,\Omega} \end{aligned}$$

which gives the announced result. ■

5 Abstract result

In this section, we propose an abstract three-fields formulation which is a general way to consider the Stokes problem in (ω, u, p) formulation. To keep some flexibility for the interpretation of the variational formulation, we introduce a mass operator J to represent in a rigorous way the formal equality "$\omega = \mathrm{curl}\, u$" which is not obvious to write when u does not belong to $H(\mathrm{curl},\Omega)$. We can state the major result of this section, *ie* the necessary hypotheses to obtain a well-posed problem from the abstract formulation.

Theorem 5.1 *Triple mixed variational formulation.*
Preliminaries
Let W*,* X *and* Y *be three Hilbert spaces, with their respective scalar products* $(\bullet,\bullet)_W$*,* $(\bullet,\bullet)_X$ *and* $(\bullet,\bullet)_Y$*, and respective norms* $\| \bullet \|_W$*,* $\| \bullet \|_X$ *and* $\| \bullet \|_Y$*. We suppose that there exists two continuous mappings* $R : W \longrightarrow X'$ *and* $D : X \longrightarrow Y'$*. We define the polar space of* Ker$D$*:*

$$(\mathrm{Ker}D)^0 = \left\{ \xi \in X' \, , \, \langle \xi, v \rangle_{X',X} = 0 \, , \, \forall \, v \in \mathrm{Ker}D \right\} , \qquad (60)$$

and the subspace V of W :

$$\begin{aligned} V &= \Big\{\varphi \in W \ , \ \langle R\varphi, v\rangle_{X',X} = 0 \ , \ \forall\ v \in \mathrm{Ker}D\Big\} \quad , \\ &= \big\{\varphi \in W \ , \ R\varphi \in (\mathrm{Ker}D)^0\big\} \quad . \end{aligned} \tag{61}$$

We introduce the canonical injection $i : \mathrm{Ker}D \longrightarrow X$*, a continuous operator* $J : W \longrightarrow W'$ *and* r *the Riesz isomorphism from* Y' *to* Y*. Moreover we introduce* $D' : Y \longrightarrow X'$ *and* $R' : X \longrightarrow W'$ *dual operators of* D *and* R *respectively :*

$$\langle D'\zeta, x\rangle_{X',X} = \langle \zeta, Dx\rangle_{Y,Y'} \ , \quad \forall \zeta \in Y \ , \ \forall x \in X \quad ,$$

$$\langle R'\eta, \varphi\rangle_{W',W} = \langle \eta, R\varphi\rangle_{X,X'} \ , \quad \forall \eta \in X \ , \ \forall \varphi \in W \quad ,$$

and a real parameter δ*.*

Hypotheses
Assume that :

- $\exists\ a > 0 \ , \ \inf\limits_{\substack{q\in Y\\ q\neq 0}} \sup\limits_{\substack{v\in X\\ v\neq 0}} \dfrac{\langle q, Dv\rangle_{Y,Y'}}{\| v \|_X \| q \|_Y} \ \geq\ a \quad ;$ (62)
- $\exists\ b > 0 \ , \ \inf\limits_{\substack{v\in \mathrm{Ker}D\\ v\neq 0}} \sup\limits_{\substack{\varphi\in W\\ \varphi\neq 0}} \dfrac{\langle v, R\varphi\rangle_{X,X'}}{\| v \|_X \| \varphi \|_W} \ \geq\ b \quad ;$ (63)
- J is elliptic on V :
  $\exists\ c > 0 \ , \ \langle J\varphi, \varphi\rangle_{W',W} \ \geq\ c \ \| \varphi \|_W^2$ for all $\varphi \in V$. (64)

Conclusion
Then, for any $\sigma = (\lambda, \mu, \nu)$ *in* $W' \times X' \times Y'$*, the problem : find* $\xi = (\omega, u, p)$ *in* $W \times X \times Y$ *such that for all* $\eta = (\varphi, v, q)$ *in* $W \times X \times Y$ *:*

$$\left\{ \begin{array}{lcll} \langle J\omega, \varphi\rangle_{W',W} - \langle R'u, \varphi\rangle_{W',W} & = & \langle \lambda, \varphi\rangle_{W',W} & \forall \varphi \in W \\ \langle R\omega, v\rangle_{X',X} - \langle D'(p - \delta r D u), v\rangle_{X',X} & = & \langle \mu, v\rangle_{X',X} & \forall v \in X \\ \langle Du, q\rangle_{Y',Y} & = & \langle \nu, q\rangle_{Y',Y} & \forall q \in Y, \end{array} \right.$$

has a unique solution $(\omega(\sigma), u(\sigma), p(\sigma)) \in W \times X \times Y$ *which continuously depends on data* $\sigma = (\lambda, \mu, \nu)$ i.e *there exists* $C > 0$ *such that, for all* $\sigma \in W' \times X' \times Y'$ *:*

$$\| \omega(\sigma) \|_W + \| u(\sigma) \|_X + \| p(\sigma) \|_Y \ \leq\ C \ \| \sigma \|_{W'\times X'\times Y'} \quad . \tag{65}$$

Let us begin by rewriting hypotheses (62) and (63).

Proposition 5.2 *Interpretation of hypotheses (62)-(63).*

$$\circ \quad D \in \mathrm{Isom}((\mathrm{Ker}D)^{\perp}, Y') \quad , \tag{66}$$

$$\circ \quad D' \in \mathrm{Isom}(Y, (\mathrm{Ker}D)^{0}) \quad , \tag{67}$$

where $(\mathrm{Ker}D)^{\perp} = \{x \in X \ , \ (v,x)_{X} = 0 \ , \ \forall v \in \mathrm{Ker}D\}$.

$$\circ \quad R'i = R'|_{\mathrm{Ker}D} \in \mathrm{Isom}(\mathrm{Ker}D, V^{0}) \quad , \tag{68}$$

$$\circ \quad i'R \in \mathrm{Isom}(V^{\perp}, (\mathrm{Ker}D)') \quad , \tag{69}$$

where we have set: $V^{0} = \Big\{\xi \in W' \ , \ \langle \xi, \varphi \rangle_{W',W} = 0 \ , \ \forall \varphi \in V\Big\}$ *, and:* $V^{\perp} = \{w \in W \ , \ (w, \varphi)_{W} = 0 \ , \ \forall \varphi \in V\}$ *. Finally,* $(\mathrm{Ker}D)'$ *is the dual space of* $\mathrm{Ker}D$*.*

To prove this proposition, we need the classical result of Ladyzenskaya-Babuška-Brezzi which is proved in [LU68], [Bab71] or [Bre74].

Theorem 5.3 *Ladyzenskaya-Babuška-Brezzi theorem.*
Let T *and* M *be two Hilbert spaces whose respective norms are denoted by* $\| \bullet \|_{T}$ *and* $\| \bullet \|_{M}$*, scalar products by* $(\bullet,\bullet)_{T}$ *and* $(\bullet,\bullet)_{M}$ *and dual spaces by* T' *and* M'*. Let* $T \times M \ni (t,\mu) \longmapsto b(t,\mu) \in \mathbb{R}$ *be a bilinear continuous form,* $B : T \longrightarrow M'$ *a linear operator and* $B' : M \longrightarrow T'$ *its dual defined by:*

$$\forall t \in T \ , \ \forall \mu \in M \ , \ \langle Bt, \mu \rangle_{M',M} = \langle t, B'\mu \rangle_{T,T'} = b(t,\mu)$$

We define K *the left kernel of* $b(\bullet,\bullet)$*, its polar* K^{0} *and its orthogonal* $K^{\perp}$ *:*

- $\circ \ K = \{t \in T \ , \ b(t,\mu) = 0 \ , \ \forall \mu \in M\} \quad ;$
- $\circ \ K^{0} = \Big\{\theta \in T' \ , \ \langle \theta, t \rangle_{T',T} = 0 \ , \ \forall t \in K\Big\} \quad ;$
- $\circ \ K^{\perp} = \{\theta \in T \ , \ (\theta, t)_{T} = 0 \ , \ \forall t \in K\} \quad .$

The three following conditions are equivalent:

(i) $\quad \exists \, \beta \ > \ 0 \ , \ \inf\limits_{\substack{\mu \in M \\ \mu \neq 0}} \ \sup\limits_{\substack{t \in T \\ t \neq 0}} \dfrac{b(t,\mu)}{\| \, t \, \|_{T} \| \, \mu \, \|_{M}} \ \geq \ \beta \quad ;$

(ii) $\quad B'$ *is an isomorphism from* M *onto* K^{0} *and:*
$\quad \exists \, \beta \ > \ 0 \ , \ \forall \mu \in M \ , \ \ \| \, B'\mu \, \|_{T'} \ \geq \ \beta \, \| \, \mu \, \|_{M} \quad ;$

(iii) *B is an isomorphism from* $K^\perp$ *onto* M' *and:*

$$\exists\, \beta > 0\ ,\ \forall t \in K^\perp\ ,\quad \| Bt \|_{M'} \ \geq\ \beta \| t \|_T \quad .$$

The constant β *is the same in the three inequalities.*

Proof of Proposition 5.2

• We first apply Theorem 5.3 with $T = X$ and $M = Y$. Then, we set: $b(v,q) = \langle Dv, q\rangle_{Y',Y}$. Therefore, we have $B = D$, $B' = D'$ and:

$$K = \Big\{ v \in X\ ,\ \langle Dv, q\rangle_{Y',Y} = 0\ ,\ \forall q \in Y \Big\} = \mathrm{Ker}D \quad .$$

We conclude that:

$$B \in \mathrm{Isom}(K^\perp, M') \quad \Leftrightarrow \quad D \in \mathrm{Isom}((\mathrm{Ker}D)^\perp, Y') \quad ,$$

$$B' \in \mathrm{Isom}(M, K^0) \quad \Leftrightarrow \quad D' \in \mathrm{Isom}(Y, (\mathrm{Ker}D)^0) \quad .$$

• We apply again Theorem 5.3 with $T = W$ and $M = \mathrm{Ker}D$. Here, we set: $b(v,\varphi) = \langle iv, R\varphi\rangle_{X,X'}$. Then, we have:

$$\langle iv, R\varphi\rangle_{X,X'} = b(v,\varphi) = \langle v, i'R\varphi\rangle_{\mathrm{Ker}D,(\mathrm{Ker}D)'} = \langle \varphi, R'iv\rangle_{W,W'}$$

which leads to: $B = i'R$, $B' = R'i = R'\,|_{\mathrm{Ker}D}$. As:

$$\begin{aligned} K &= \Big\{ \varphi \in W\ ,\ \langle v, R\varphi\rangle_{X,X'} = 0\ ,\ \forall v \in \mathrm{Ker}D \Big\} \\ &= \big\{ \varphi \in W\ ,\ R\varphi \in (\mathrm{Ker}D)^0 \big\} = V \quad , \end{aligned}$$

we obtain:

$$B \in \mathrm{Isom}(K^\perp, M') \quad \Leftrightarrow \quad i'R \in \mathrm{Isom}(V^\perp, (\mathrm{Ker}D)') \quad ,$$

$$B' \in \mathrm{Isom}(M, K^0) \quad \Leftrightarrow \quad R'\,|_{\mathrm{Ker}D} \in \mathrm{Isom}(\mathrm{Ker}D, V^0) \quad .$$

■

Proof of Theorem 5.1

We introduce the operator $A : W \times X \times Y \longrightarrow W' \times X' \times Y'$ by the following matrix:

$$A = \begin{pmatrix} J & -R' & 0 \\ R & \delta D' r D & -D' \\ 0 & D & 0 \end{pmatrix}$$

We just have to prove that the continuous operator A is bijective due to the Banach isomorphism theorem.

- **The operator A is injective**.

Let $\xi = (\omega, u, p) \in W \times X \times Y$ be the solution of $A\xi = 0$:

$$J\omega - R'u = 0 \text{ in } W' \quad , \tag{70}$$
$$R\omega - D'(p - \delta r Du) = 0 \text{ in } X' \quad , \tag{71}$$
$$Du = 0 \text{ in } Y' \quad . \tag{72}$$

$*$ Let us apply equation (71) on any vector $v \in \text{Ker}D$. As:

$$\langle D'(p - \delta r Du), v \rangle_{X',X} = \langle p - \delta r Du, \underbrace{Dv}_{=0} \rangle_{Y,Y'} = 0 \quad ,$$

we have, for all v in $\text{Ker}D$:

$$\langle R\omega, v \rangle_{X',X} = 0 \quad ,$$

thus $R\omega$ belongs to $(\text{Ker}D)^0$ *ie* ω belongs to V.
$*$ Equation (72) $Du = 0$, means $u \in \text{Ker}D$.
$*$ If we test equation (70) with the particular vector $\varphi = \omega$, we obtain:

$$\langle J\omega, \omega \rangle_{W',W} - \langle R'u, \omega \rangle_{W',W} = 0 \quad .$$

As ω belongs to V and as u belongs to $\text{Ker}D$, we have:

$$\langle R'u, \omega \rangle_{W',W} = \langle u, R\omega \rangle_{X,X'} = 0 \quad .$$

So, $\langle J\omega, \omega \rangle_{W',W} = 0$ and hypothesis (64) leads to $\omega \equiv 0$.
$*$ Taking into account that ω and Du are zero, equation (71) becomes $D'p = 0$ in Y'. As D' is an isomorphism from Y onto $(\text{Ker}D)^0$ (hypothesis (67)), p is zero.
$*$ Finally, equation (70) is reduced to $R'u = 0$ in W'. As u belongs to $\text{Ker}D$ and as $R'|_{\text{Ker}D}$ is an isomorphism from $\text{Ker}D$ onto V^0, see (68), we obtain $u = 0$.

Therefore, we have proved that if $\sigma = (0, 0, 0)$, then $\xi = (0, 0, 0)$. It means that A is an injective operator.

- **The operator A is surjective.**

Let $\sigma = (\lambda, \mu, \nu) \in W' \times X' \times Y'$ and $\xi = (\omega, u, p) \in W \times X \times Y$ be the solution of $A\xi = \sigma$:

$$J\omega - R'u = \lambda \text{ in } W' \quad , \tag{73}$$
$$R\omega - D'(p - \delta r Du) = \mu \text{ in } X' \quad , \tag{74}$$
$$Du = \nu \text{ in } Y' \quad . \tag{75}$$

$*$ Let us split $\omega \in W$ and $u \in X$ into two orthogonal components:

$$\omega = \omega_1 + \omega_2 \;\in\; V \oplus V^{\perp} = W \quad ,$$

$$u = u_1 + u_2 \;\in\; \text{Ker } D \oplus (\text{Ker } D)^{\perp} = X \quad .$$

$*$ Let us apply equation (74) on any vector v in Ker D. As:

$$\langle D'(p - \delta r D u), v\rangle_{X',X} = \langle p - \delta r D u, \underbrace{Dv}_{=0}\rangle_{Y,Y'} = 0 \quad ,$$

we obtain for all v in Ker$D$:

$$\begin{aligned} \langle R\omega, v\rangle_{X',X} = \langle i'R\omega, iv\rangle_{(\text{Ker}D)',\text{Ker}D} &= \langle \mu, v\rangle_{X',X} \quad , \\ \underbrace{\langle i'R\omega_1, iv\rangle_{(\text{Ker}D)',\text{Ker}D}}_{=0 \text{ as } v \,\in\, \text{Ker}D \;,\; \omega_1 \in V} + \langle i'R\omega_2, iv\rangle_{(\text{Ker}D)',\text{Ker}D} &= \langle \mu, v\rangle_{X',X} \quad , \end{aligned} \tag{76}$$

ie $i'R\omega_2 = \mu$ in $(\text{Ker}D)'$ because $\mu \in X' \subset (\text{Ker}D)'$. Taking into account that $i'R$ is an isomorphism from $V^{\perp}$ onto $(\text{Ker } D)'$, *cf* (69), there exists a unique ω_2 in $V^{\perp}$ such that $i'R\omega_2 = \mu$ in $(\text{Ker } D)'$.
$*$ From equation (75), we deduce that $Du_2 = \nu$ in Y' and from the isomorphism (66), we obtain that there exists a unique u_2 in $(\text{Ker } D)^{\perp}$.
$*$ Using both decompositions, equation (73) can be rewritten as follows:

$$J\omega_1 = \lambda - J\omega_2 + R'u_1 + R'u_2 \text{ in } W' \quad . \tag{77}$$

We first test this equation with some function φ in V. As $u_1 \in$ Ker D:

$$\langle R'u_1, \varphi\rangle_{W',W} = \langle u_1, R\varphi\rangle_{X,X'} = 0 \;,\; \forall \varphi \in V \quad ,$$

equation (77) implies:

$$\langle J\omega_1, \varphi\rangle_{W',W} = \langle \lambda - J\omega_2 + R'u_2, \varphi\rangle_{W',W} \;,\; \forall \varphi \in V \quad , \tag{78}$$

where ω_2 and u_2 are given by previous steps. If we note $\eta \equiv \lambda - J\omega_2 + R'u_2$, we have to find $\omega_1 \in V$ such that:

$$\langle J\omega_1, \varphi\rangle_{W',W} = \langle \eta, \varphi\rangle_{W',W} \;,\;\; \forall \varphi \in V \quad . \tag{79}$$

Applying the Lax-Milgram's lemma [LM54] (hypothesis (64) is the ellipticity on V), there exists a unique ω_1 in V solution of (79).
$*$ We report that last result in equation (77) and we obtain:

$$R'u_1 = -\lambda + J\omega - R'u_2 \quad . \tag{80}$$

As $\langle R'u_1, \varphi \rangle_{W',W} = 0$ for all u_1 in Ker D and all φ in V, we have constructed ω_1 such that $-\lambda + J\omega - R'u_2$ belongs to V^0 (see (78)). Then equation (80) can be considered in the Hilbert space V^0. As $R'|_{\text{Ker } D}$ is an isomorphism from Ker$D$ onto V^0 (hypothesis (68)), there exists a unique $u_1 \in$ Ker$D$ satisfying (80).

$*$ Finally, equation (74) can be written: $-D'p = \mu - R\omega - D'\delta r D u \equiv \zeta$. We have constructed ω_2 such that functional ζ belongs to (Ker $D)^0$ and is also independent of ω_1 (see (76)). Finally, D' is an isomorphism from Y onto (Ker $D)^0$ (hypothesis (67)) and we deduce that there exists a unique $p \in Y$ such that $D'p = \zeta$. ■

Remark 5.4 *Previous proof gives an algorithm for obtaining all different fields:*

- ○ *First $\omega_2 \in V^\perp$ and $u_2 \in$ (Ker $D)^\perp$ are obtained independently.*
- ○ *Then ω_2 and u_2 lead to $\omega_1 \in V$.*
- ○ *Vorticity field ω and component u_2 of u allow to obtain $u_1 \in$ Ker D.*
- ○ *Finally ω and u give the pressure $p \in Y$.*

6 Application to the Stokes problem

The aim of this section is to apply the abstract result (Theorem 5.1) to the Stokes problem. As we look for the velocity in space $X = H_0(\text{div}, \Omega)$, curl u can only be defined in the distribution sense. So equation $\omega =$ curl u should be verified in a weak sense. First, we give a sense to curl u for u in $H_0(\text{div}, \Omega)$ by duality. For doing this, we define $R : W \subset H(\text{curl}, \text{div}*, \Omega) \longrightarrow X'$ with the help of the curl operator, introduced previously, from $H(\text{curl}, \text{div}*, \Omega)$ to $(H_0(\text{div}, \Omega))' = X'$ (see (59)). Then the curl will be the dual operator $R' : X \longrightarrow W'$ that gives a sense to curl u in W' for $u \in X$. Second, we introduce an abstract mass operator $J : W \longrightarrow W'$ and the equation $\omega =$ curl u becomes in a weak general version in W':

$$J\omega = R'u \text{ in } W' \ , \ u \in X \ , \ \omega \in W.$$

The precise choice of the operator J will be discussed further.

○ Let us recall that Γ_θ and Γ_t is a partition of the boundary Γ of the domain Ω such that $\Gamma = \overline{\Gamma_\theta} \cup \overline{\Gamma_t}$ with $\Gamma_\theta \cap \Gamma_t = \emptyset$. We have set the following spaces:

$$W = \left\{ \varphi \in H(\text{curl}, \text{div}*, \Omega) \ , \ \varphi \times n_{|\Gamma_\theta} = 0 \right\} \ ,$$
$$X = H_0(\text{div}, \Omega) \ , \ Y = L^2_0(\Omega).$$

∘ We now introduce the divergence operator D. Defining r the Riesz isomorphism from $L^2(\Omega)$ onto $(L^2(\Omega))'$, and s the canonical injection of X in $H(\mathrm{div},\Omega)$, as div is a continuous operator from $H(\mathrm{div},\Omega)$ to $L^2(\Omega)$, we first set :

$$D_0 \quad : \quad H(\mathrm{div},\Omega) \ni v \quad \longmapsto \quad D_0 v = \mathrm{div}\ v \in L^2(\Omega).$$

Space Y is equal to $L^2_0(\Omega)$ and is a subspace of $L^2(\Omega)$. So we have $(L^2(\Omega))' \subset Y'$. If we note t the corresponding canonical injection from $(L^2(\Omega))'$ to Y', abstract operator D is then equal to $t.r.D_0.s$:

$$\begin{array}{cccccc} D_0 & : & H(\mathrm{div},\Omega) & \longrightarrow & L^2(\Omega) \\ & & \uparrow & & \downarrow r \\ & & \mid s & & (L^2(\Omega))' \\ & & \mid & & \downarrow t \\ D & : & X & \longrightarrow & Y' \end{array}$$

And finally $\langle Dv, q\rangle_{Y',Y}$ is well defined for all $v \in X$ and $q \in Y$.

∘ With the previous notation, operator R is defined in (59), the Stokes problem (11-17) becomes :

$$\left\{ \begin{array}{l} \text{find } \omega \in W \ , \quad u \in X \ , \quad p \in Y \quad \text{such that :} \\ J\omega = R'u \quad \text{in } W' \quad , \\ R\omega - D'(p - \delta r D u) = f \quad \text{in } X' \quad , \\ Du = 0 \quad \text{in } Y', \end{array} \right. \tag{81}$$

with δ a constant either equal to 0 or 1.

The variational formulation of the previous problem is obtained by making first equation of (81) acting on $\varphi \in W$, second equation of (81) on $v \in X$ and last equation on $q \in Y$. It gives :

$$\left\{ \begin{array}{ll} \text{find } (\omega, u, p) \in W \times X \times Y \text{ such that ,} & \text{for all } (\lambda,\mu,\nu) \in W' \times X' \times Y' : \\ \langle J\omega, \varphi\rangle_{W',W} - \langle R'u, \varphi\rangle_{W',W} & = \langle \lambda, \varphi\rangle_{W',W} \ , \ \forall\ \varphi \in W \\ \langle R\omega, v\rangle_{X',X} - \langle D'(p - \delta r D u), v\rangle_{X',X} & = \langle \mu, v\rangle_{X',X} \ , \ \forall\ v \in X \\ \langle Du, q\rangle_{Y',Y} & = \langle \nu, q\rangle_{Y',Y} \ , \ \forall\ q \in Y. \end{array} \right.$$

Problem (81) is obtained from the previous one by taking :

$$\lambda = 0 \quad , \quad \mu = f \quad , \quad \nu = 0 \quad .$$

6.1 Co-Curl operator

In this subsection, we introduce a new operator called a "co-curl" $\rho : W \longrightarrow X$ that allows to easily manipulate the curl of functions of W as functions in space $H_0(\text{div}, \Omega)$, whose normal trace is null on the boundary. The co-curl is an useful tool to finally define the mass operator J introduced above. We will see that a remarkable property of the co-curl operator ρ is that $\text{div}(\rho\varphi)$ is well defined for $\varphi \in W$ but it is not null in general!

∘ Let us take φ in W. By definition, $R\varphi$, defined in (59), belongs to X'. As X is a Hilbert space, by applying Riesz theorem to $R\varphi$, for all $\varphi \in W$, there exists a unique $\rho\varphi \in X$ such that for all $v \in X$:

$$\langle R\varphi, v\rangle_{X',X} \;=\; (\rho\varphi, v)_{\text{div}} = (\rho\varphi, v)_0 + (\text{div }\rho\varphi, \text{div } v)_0 \quad . \qquad (82)$$

Moreover, for all φ in W, we have: $\| R\varphi \|_{X'} = \| \rho\varphi \|_{\text{div},\Omega}$. Let us notice that the norm in W is defined as follows: $\| \varphi \|_W^2 = \| \varphi \|_{0,\Omega}^2 + \| R\varphi \|_{X'}^2$ and, using (82), we obtain:

$$\begin{aligned} \| \varphi \|_W^2 &= \| \varphi \|_{0,\Omega}^2 + \| \rho\varphi \|_{\text{div},\Omega}^2 \\ \| \varphi \|_W^2 &= \| \varphi \|_{0,\Omega}^2 + \| \rho\varphi \|_{0,\Omega}^2 + \| \text{div }\rho\varphi \|_{0,\Omega}^2 \,. \end{aligned}$$

We list the properties implied by the introduction of the co-curl operator.

Lemma 6.1 *Let us recall that we have introduced in relation (61) the following kernel:*

$$V = \Big\{ \varphi \in W \;,\; \langle R\varphi, v\rangle_{X',X} = 0 \;,\; \forall\, v \in \text{Ker}D \Big\} \quad .$$

Then, the kernel V can be characterized with the help of the co-curl ρ by:

$$V = \{ \varphi \in W \;,\; \rho\varphi \in (\text{Ker}D)^\perp \} \quad ,$$

where

$$(\text{Ker}D)^\perp = \{ v \in X \;,\; (v, w)_{\text{div}} = 0 \;,\; \forall\, w \in \text{Ker}D \} \,.$$

<u>Proof</u>
The proof is straightforward as, for $\varphi \in V$ and for all $v \in \text{Ker}D$, we have:

$$\langle R\varphi, v\rangle_{X',X} = 0 = (\rho\varphi, v)_{\text{div}} \quad ,$$

which means that $\rho\varphi$ belongs to $(\text{Ker}D)^\perp$. ■

Definition 3 *Leray projection operator and K operator.*
Consider $v \in X = H_0(\text{div}, \Omega)$ and the orthogonal decomposition:

$$v = v_1 + v_2 \in \text{Ker }D \oplus (\text{Ker }D)^\perp \quad .$$

Then, we define the two following operators:

$$L : X \ni v \longmapsto Lv = v_1 \in \text{Ker } D \quad ,$$
$$K : X \ni v \longmapsto Kv = v_2 \in (\text{Ker } D)^{\perp} \quad .$$

Remark that L is the Leray projection operator (see Leray [Ler34]).

Taking $\varphi \in W$ and using the above decomposition for $\rho\varphi \in X$ (see (82)), we obtain:

$$\rho\varphi = L\rho\varphi + K\rho\varphi \ \in \ \text{Ker } D \oplus (\text{Ker } D)^{\perp} \quad .$$

So, for all $\varphi \in W$, we have:

$$\| \varphi \|_W^2 = \| \varphi \|_{0,\Omega}^2 + \| \rho\varphi \|_{0,\Omega}^2 + \| \text{div } \rho\varphi \|_{0,\Omega}^2 \quad ,$$

or else:

$$\| \varphi \|_W^2 = \| \varphi \|_{0,\Omega}^2 + \| L\rho\varphi \|_{0,\Omega}^2 + \| K\rho\varphi \|_{0,\Omega}^2 + \| \text{div } K\rho\varphi \|_{0,\Omega}^2 \,. \qquad (83)$$

Lemma 6.2 *For all function $\varphi \in (\mathcal{D}(\Omega))^{2N-3}$, we have:*

$$\text{curl } \varphi = \rho\varphi = L\rho\varphi \quad \text{and} \quad K\rho\varphi = 0 \quad .$$

Proof

First, as φ belongs to $(\mathcal{D}(\Omega))^{2N-3}$, curl φ belongs to $H_0(\text{div}, \Omega)$, as $\rho\varphi$. Moreover, as $(\mathcal{D}(\Omega))^{2N-3}$ is contained in $H(\text{curl}, \Omega)$, from Proposition 4.2, we deduce that for all $v \in H_0(\text{div}, \Omega)$:

$$\begin{aligned} \langle \text{curl } \varphi, v \rangle_{\text{div}*,\text{div}} &= (\text{curl } \varphi, v)_0 = (\text{curl } \varphi, v)_{\text{div}} \\ &= (\rho\varphi, v)_{\text{div}} \text{ by (82).} \end{aligned}$$

So, curl φ is equal to $\rho\varphi$ in $H_0(\text{div}, \Omega) = X$. Moreover, as div curl $\varphi \equiv 0$, $\rho\varphi$ belongs to Ker D. Then, $\rho\varphi = L\rho\varphi$ and $K\rho\varphi = 0$. ■

Remark 6.3 *For any sufficiently regular function φ, equal to zero on the boundary of Ω, we have seen that: $\rho\varphi =$ curl φ in $H_0(\text{div}, \Omega)$.*
As div curl $\varphi \equiv 0$*, we obtain $K\rho\varphi = 0$. But a function φ of W is not so regular and only its co-curl is defined in $H_0(\text{div}, \Omega)$. Then, the component $K\rho\varphi$ is the non divergence free part of the (co-)curl $\rho\varphi$ of φ! (see Section* 6.4*, proposition 6.18).*

Lemma 6.4 *Let us recall the definition of the kernel V :*

$$V = \Big\{ \varphi \in W \ , \ \langle R\varphi, v \rangle_{X',X} = 0 \ , \ \forall \ v \in \mathrm{Ker} D \Big\} \quad .$$

We can now characterize this kernel with the help of the Leray projection operator :

$$V = \{ \varphi \in W \ , \ L\rho\varphi \equiv 0 \} \quad .$$

Proof

The proof is straightforward as, for all $\varphi \in W$, we have :

$$\rho\varphi = L\rho\varphi + K\rho\varphi \in \mathrm{Ker}\ D \oplus (\mathrm{Ker}\ D)^{\perp} \quad .$$

By Lemma 6.1, we know that $\varphi \in V$ implies $\rho\varphi \in (\mathrm{Ker}\ D)^{\perp}$. So $L\rho\varphi \equiv 0$. ■

6.2 Mass operator

With the help of the co-curl operator ρ, we can define the mass operator J. Our first idea to define J is to consider the Riesz isomorphism from W to W' :

$$\langle J\omega, \varphi \rangle_{W',W} = \langle J_1\omega, \varphi \rangle_{W',W} \ \equiv \ (\omega, \varphi)_W \quad ,$$

where we recall that :

$$(\omega, \varphi)_W = (\omega, \varphi)_0 + (L\rho\omega, L\rho\varphi)_0 + (K\rho\omega, K\rho\varphi)_0 + (\mathrm{div}\ K\rho\omega, \mathrm{div}\ K\rho\varphi)_0 \ .$$

Second, we consider a second functional that verify the hypothesis of ellipticity (64) on V which is : $\langle J\omega, \varphi \rangle_{W',W} = \langle J_2\omega, \varphi \rangle_{W',W}$, with :

$$\langle J_2\omega, \varphi \rangle_{W',W} = (\omega, \varphi)_0 + (K\rho\omega, K\rho\varphi)_0 + (\mathrm{div}\ K\rho\omega, \mathrm{div}\ K\rho\varphi)_0 \quad .$$

Third, we will use the L^2-scalar product :

$$\langle J\omega, \varphi \rangle_{W',W} = \langle J_3\omega, \varphi \rangle_{W',W} = (\omega, \varphi)_0 \quad ,$$

that is *a priori* not relevant for our formulation because of the hypothesis of ellipticity (64) on V.

6.3 Vector field representation

In this section, we shall use theorems of vector field representation proven in different references. The first ones in [ABDG98] are given for homogeneous conditions on all the boundary, supposed Lipschitz or $\mathcal{C}^{1,1}$ (see also [BDG85]).

The second reference [Dub02], which is needed here, is a vector field representation with mixed boundary conditions and this theorem needs a strong hypothesis : the boundary Γ of Ω is of $\mathcal{C}^2-$class. It has to be improved in the future, in order that this paper has a better use in practice.

So, let us assume now that the boundary Γ of Ω is of $\mathcal{C}^2-$class (when this last and strong hypothesis can be released, it will be quoted in the text). We suppose that Γ is split into two subsets Γ_1 and Γ_2 that compose a partition :

$$\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \ \text{ with } \ \Gamma_1 \cap \Gamma_2 = \emptyset \quad .$$

We introduce some functional spaces :

$$H_0^1(\Omega;\Gamma_1,\Gamma_2) = \left\{\varphi \in (H^1(\Omega))^{2N-3} \ , \ \gamma\varphi{\bullet}n = 0 \text{ on } \Gamma_1 \ , \ \gamma\varphi \times n = 0 \text{ on } \Gamma_2\right\} \quad ,$$

$$M^0(\Omega;\Gamma_1,\Gamma_2) = \left\{ \begin{array}{l} \varphi \in (L^2(\Omega))^N \ , \ \text{div } \varphi = 0 \ , \ \text{curl } \varphi = 0 \\ \varphi{\bullet}n_{|\Gamma_1} = 0 \text{ in } (H_{00}^{1/2}(\Gamma_1)^N)' \ , \ \varphi \times n_{|\Gamma_2} = 0 \text{ in } (TH_{00}^{1/2}(\Gamma_2))' \end{array} \right\}$$

$$M^1(\Omega;\Gamma_1,\Gamma_2) = \left\{ \begin{array}{l} \varphi \in (H^1(\Omega))^N \ , \ \text{div } \varphi = 0 \ , \ \text{curl } \varphi = 0 \\ \gamma\varphi{\bullet}n = 0 \text{ on } \Gamma_1 \ , \ \gamma\varphi \times n = 0 \text{ on } \Gamma_2 \end{array} \right\} \quad ,$$

Then $\Pi^1_{\Gamma_1,\Gamma_2}$ will be the orthogonal projector from $(L^2(\Omega))^N$ onto $M^1(\Omega;\Gamma_1,\Gamma_2)$ and $\Pi^0_{\Gamma_1,\Gamma_2}$ the projector from $(L^2(\Omega))^N$ onto $M^0(\Omega;\Gamma_1,\Gamma_2)$. Let us begin by a first result :

Lemma 6.5 *We suppose that Γ is a $\mathcal{C}^2-$class regular boundary. Then, for all $\varphi \in H_0^1(\Omega;\Gamma_1,\Gamma_2)$, we have :*

$$\| \varphi \|_{1,\Omega} \ \leq \ C \ \left(\| \Pi^1_{\Gamma_1\Gamma_2}\varphi \|^2_{0,\Omega} + \| \text{ div } \varphi \|^2_{0,\Omega} + \| \text{ curl } \varphi \|^2_{0,\Omega}\right)^{1/2} \quad .$$

Proof is derived in [BDG85] when $\Gamma_1 = \Gamma$ or $\Gamma_1 = \emptyset$, and in [Dub02] in a more general case. Then, we have the two following theorems :

Theorem 6.6 *Space $M^0(\Omega;\Gamma_1,\Gamma_2)$ is finite dimensional.*
Let Ω be an open, bounded, connected domain with a Lipschitz boundary. If we can choose smooth cuts Σ_j, $j = 1,\ldots,M$ in order that the interior of Ω, obtained by removing the cuts from Ω is simply connected, then space $M^0(\Omega;\Gamma_1,\Gamma_2)$ is finite dimensional.

The proof is derived in [FG97].

Theorem 6.7 *Representation of vector fields.*
Assume that Ω *verifies hypotheses of Lemma 6.5 and let* (Γ_1, Γ_2) *be a partition of the boundary* Γ*. Let* $u \in (L^2(\Omega))^N$ *be a vector field. Then there exists two potentials* φ *and* ψ *satisfying the condition :*

$$\left\{ \begin{array}{l} \varphi \in H^1_0(\Omega; \Gamma_1) \quad , \\ \psi \in H^1_0(\Omega; \Gamma_1, \Gamma_2) \quad , \end{array} \right.$$

and such that u has the following orthogonal decomposition in space $(L^2(\Omega))^N$ *:*

$$u = \nabla\varphi + \text{curl } \psi + \Pi^0_{\Gamma_2,\Gamma_1} \; u \quad .$$

Moreover, if we impose the supplementary following conditions to vector potential ψ *when* $N = 3$ *:*

$$\text{div } \psi = 0 \;\; \text{in} \;\; \Omega \quad , \quad \Pi^1_{\Gamma_1,\Gamma_2}\psi = 0 \quad ,$$

they are uniquely and continuously defined :

$$\exists \; C \; > \; 0 \;\; , \quad \| \, \varphi \, \|_{1,\Omega} \;\; \leq \;\; C \;\; \| \, u \, \|_{0,\Omega} \quad , \quad \| \, \psi \, \|_{1,\Omega} \;\; \leq \;\; C \;\; \| \, u \, \|_{0,\Omega} \quad .$$

From this theorem, whose proof can be found in [FG97] and Dubois [Dub02], we deduce the following lemma :

Lemma 6.8 *Representation of space* Ker$D$.
○ *If we suppose that* Γ *is a* $\mathcal{C}^2-$*class regular boundary, then all functions in* Ker$D$ *can be orthogonally split as follows :*

$$v = \text{curl } \chi + \zeta \quad ,$$

with $\chi \in H^1_0(\Omega; \emptyset, \Gamma)$ *and* $\zeta = \Pi^0_{\Gamma,\emptyset} \; v \in M^0(\Omega; \Gamma, \emptyset)$.
○ *If* Ω *is a connected and simply connected open bounded domain of* $\mathbb{R}^N$ *with a* $\mathcal{C}^{1,1}$ *boundary, then all functions in* Ker$D$ *can be written as follows :*

$$v = \text{curl } \chi \quad ,$$

with $\chi \in H^1_0(\Omega; \emptyset, \Gamma)$.

Proof

• As v belongs to X, it belongs to $(L^2(\Omega))^N$ and we can apply the Theorem 6.7 with $\Gamma_1 = \emptyset$ and $\Gamma_2 = \Gamma$:

$$v = \nabla\varphi + \text{curl }\psi + \zeta \quad ,$$

where $\varphi,\ \psi,\ \zeta$ are uniquely defined, respectively in the following spaces :

$$\begin{cases} \varphi \in H^1_0(\Omega;\emptyset) = \{\varphi \in H^1(\Omega)\ ,\ (\varphi,1)_0 = 0\} \quad , \\ \psi \in H^1_0(\Omega;\emptyset,\Gamma) \quad , \\ \zeta = \Pi^0_{\Gamma,\emptyset}\ v \ \in\ M^0(\Omega;\Gamma,\emptyset) \quad . \end{cases}$$

• Due to the orthogonal decomposition, scalar φ is defined as the variational solution of :

$$\begin{cases} \varphi \in H^1_0(\Omega;\emptyset) \\ (\nabla\varphi,\nabla\eta)_0 = (v,\nabla\eta)_0 \quad , \quad \forall\eta \in H^1_0(\Omega;\emptyset) \quad . \end{cases}$$

Using the Green formula, we obtain, for all η in $H^1_0(\Omega;\emptyset)$:

$$(v,\nabla\eta)_0 = - \underbrace{(\text{div } v,\eta)_0}_{= 0 \text{ as } v \in \text{Ker}D} + \langle v{\bullet}n,\eta\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \quad ,$$

Moreover $\langle v{\bullet}n,\eta\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}$ is also zero : as v is in X, $v{\bullet}n_{|\Gamma}$ belongs to $H^{-1/2}(\Gamma)$ and is zero on the whole boundary Γ. Then, scalar φ verifies :

$$\begin{cases} \varphi \in H^1_0(\Omega;\emptyset) \\ (\nabla\varphi,\nabla\eta)_0 = 0 \quad , \quad \forall\eta \in H^1_0(\Omega;\emptyset) \quad , \end{cases}$$

which means that $\nabla\varphi$ is equal to zero. Finally, the decomposition of vector v in $X \cap \text{Ker}D$ is reduced to : $v = \text{curl }\chi\ +\ \zeta$, with $\chi \in H^1_0(\Omega;\emptyset,\Gamma)$ and $\zeta = \Pi^0_{\Gamma,\emptyset}\ v \in M^0(\Omega;\Gamma,\emptyset)$.

• When Ω is connected and simply connected with a $\mathcal{C}^{1,1}$ boundary, $M^0(\Omega;\Gamma,\emptyset)$ is reduced to zero (see *e.g* [ABDG98]) and the decomposition of v is reduced to : $v = \text{curl }\chi,\ \chi \in H^1_0(\Omega;\emptyset,\Gamma)$. ∎

Corollary 6.9

$$X = \text{Ker } D \oplus (\text{Ker } D)^\perp \quad ,$$

with :

$$\text{Ker } D = \left\{\text{curl }\chi + \zeta\ ,\ \chi \in H^1_0(\Omega;\emptyset,\Gamma)\ ,\ \zeta \in M^0(\Omega;\Gamma,\emptyset)\right\} \quad ,$$

and :

$$(\text{Ker } D)^\perp = \left\{\nabla\varphi\ ,\ \varphi \in H^1_0(\Omega;\emptyset)\ ,\ \Delta\varphi \in L^2(\Omega)\ ,\ \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma\right\} \quad .$$

Proof

The characterization of Ker D is an application of the Lemma 6.8. Let v be a vector of X. As we have done previously, we know that v can be split as:

$$v = \nabla\varphi + \underbrace{\text{curl } \psi + \zeta}_{\in \text{ Ker}D} \quad ,$$

where φ is uniquely defined in $H_0^1(\Omega;\emptyset)$ and such that:

$$(\nabla\varphi, \nabla\eta)_0 = -(\text{div } v, \eta)_0 \quad , \quad \forall \eta \in H_0^1(\Omega;\emptyset) \quad .$$

Integrating by parts, we deduce that previous function φ is the weak solution of the following laplacian problem:

$$\left\{ \begin{array}{rcl} \Delta\varphi & = & \text{div } v \quad \text{in } \Omega \quad , \\ \dfrac{\partial\varphi}{\partial n} & = & 0 \quad \text{on } \Gamma \quad . \end{array} \right.$$

So, $(\text{Ker } D)^\perp$ is the space of functions:

$$\left\{ \nabla\varphi \ , \ \varphi \in H_0^1(\Omega;\emptyset) \ , \ \Delta\varphi \in L^2(\Omega) \ , \ \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma \right\} \quad ,$$

where the normal trace $\dfrac{\partial\varphi}{\partial n}$ has to be considered in $H^{-1/2}(\Gamma)$. ■

6.4 Theoretical study of generalized Stokes problems

We apply in this section the theoretical result proposed in Section 5. We detail the proof of the following theorem.

Theorem 6.10 *Well-posedness of a generalization of the Stokes problem.*
∘ *Let Ω be an open bounded connected domain of $\mathbb{R}^N$ with a boundary denoted by Γ. Let $(\Gamma_t, \Gamma_\theta)$ be a partition of Γ. We consider the spaces:*
$W = \left\{ \varphi \in H(\text{curl}, \text{div}*, \Omega) \ , \ \varphi \times n_{|\Gamma_\theta} = 0 \right\}$, $X = H_0(\text{div}, \Omega)$, $Y = L_0^2(\Omega)$; *and the operators introduced in the previous section acting on these spaces:* $R : W \longrightarrow X'$; $D : X \longrightarrow Y'$; $J : W \longrightarrow W'$. *Functional associated with operator J is either equal to:*

$$\langle J_1\omega, \varphi\rangle_{W',W} = (\omega, \varphi)_W \quad ,$$

where:

$$(\omega, \varphi)_W = (\omega, \varphi)_0 + (L\rho\omega, L\rho\varphi)_0 + (K\rho\omega, K\rho\varphi)_0 + (\text{div } K\rho\omega, \text{div } K\rho\varphi)_0 \ ,$$

or :

$$\langle J_2\omega, \varphi\rangle_{W',W} = (\omega, \varphi)_0 + (K\rho\omega, K\rho\varphi)_0 + (\text{div } K\rho\omega, \text{div } K\rho\varphi)_0,$$

with operators K and L given in Definition 3. Finally, we denote by r the Riesz isomorphism from Y' to Y and by δ a constant either equal to 0 or 1.
∘ *We assume one of the two following hypotheses on Ω :*
(i) Ω is a connected and simply connected open bounded domain of $\mathbb{R}^N$ with a $\mathcal{C}^{1,1}$ boundary,
(ii) Ω is an open bounded connected domain of $\mathbb{R}^N$ with a $\mathcal{C}^2$-class boundary and there exists some analytical subset Γ_0 of Γ such that :

$$\text{meas }(\Gamma_0) \neq 0 \quad \text{and} \quad \Gamma_0 \subset \Gamma_t \quad . \tag{84}$$

∘ *Then, the following problem :*

$$\begin{cases} \text{find } (\omega, u, p) \in W \times X \times Y \text{ such that :} \\ \langle J\omega, \varphi\rangle_{W',W} - \langle R'u, \varphi\rangle_{W',W} & = \langle\lambda, \varphi\rangle_{W',W} \ , \quad \forall\ \varphi \in W \\ \langle R\omega, v\rangle_{X',X} - \langle D'(p - \delta r D u), v\rangle_{X',X} & = \langle\mu, v\rangle_{X',X} \ , \quad \forall\ v \in X \\ \langle Du, q\rangle_{Y',Y} & = \langle\nu, q\rangle_{Y',Y} \ , \quad \forall\ q \in Y \end{cases}$$

is well-posed : there exists $C > 0$ such that, for all $(\lambda, \mu, \nu) \in W' \times X' \times Y'$:

$$\|\ \omega\ \|_W + \|\ u\ \|_X + \|\ p\ \|_Y \ \leq\ C\ \ \|\ \lambda\ \|_{W'} + \|\ \mu\ \|_{X'} + \|\ \nu\ \|_{Y'} \ .$$

Remark 6.11
∘ *As the normal velocity is null along the whole boundary Γ, hypothesis (84) implies that the velocity is completely known at least on an analytical part of the boundary.*
∘ *In [Dub02] and [Sal99], a different case is studied : it is assumed that $\Gamma_m = \Gamma_\theta$ and that no singularity exists,* ie *$M^0(\Omega; \Gamma_m, \Gamma_p) = \{0\}$. But the condition $\Gamma_m = \Gamma$ is not necessary.*
∘ *Notice that when $\Omega \subset \mathbb{R}^2$ is connected and simply connected, this theorem allows the classical stream function-vorticity formulation to treat enlarged boundary conditions on the tangential velocity.*

We shall need the four next lemmas to prove the Theorem 6.10.

Lemma 6.12 *Ellipticity (64).*
In the frame of Theorem 6.10, the operators J_1 and J_2 are elliptic on space $V = \{\varphi \in W\ ,\ L\rho\varphi = 0\}$.

Proof

For all $\varphi \in V$, we have:

$$\begin{aligned} \| \varphi \|_W^2 &= \| \varphi \|_{0,\Omega}^2 + \| K\rho\varphi \|_{0,\Omega}^2 + \| \operatorname{div} K\rho\varphi \|_{0,\Omega}^2 \\ &= \langle J_1\varphi, \varphi \rangle_{W',W} = \langle J_2\varphi, \varphi \rangle_{W',W} \quad , \end{aligned}$$

and the result is obvious. ■

Lemma 6.13 *Inf-sup condition for velocity-pressure (62).*
There exists a strictly positive constant a such that:

$$\inf_{\substack{q \in Y \\ q \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{\langle q, Dv \rangle_{Y,Y'}}{\| v \|_X \| q \|_Y} \geq a \quad .$$

Proof

The proof is not repeated here as it is a very classical result (see *e.g* [RT77], [DSS01] among others). ■

Lemma 6.14 *Inf-sup condition for velocity-vorticity (63).*
We assume that there exists some analytical subset Γ_0 of Γ such that:

$$\text{meas } (\Gamma_0) \neq 0 \text{ and } \Gamma_0 \subset \Gamma_t \quad .$$

Then, there exists a strictly positive constant b such that:

$$\inf_{\substack{v \in \text{Ker} D \\ v \neq 0}} \sup_{\substack{\varphi \in W \\ \varphi \neq 0}} \frac{\langle R\varphi, v \rangle_{X',X}}{\| v \|_X \| \varphi \|_W} \geq b \quad .$$

Proof

We prove the inequality by contradiction. We suppose that there exists a sequence $(v_k)_{k \in \mathbb{N}}$ of elements of Ker D such that, for all integer k, we have: $\| v_k \|_{\text{div},\Omega} = \| v_k \|_{0,\Omega} = 1$, and:

$$\forall \varphi \in W \ , \ \langle R\varphi, v_k \rangle_{X',X} \leq \frac{1}{k} \| \varphi \|_W \quad . \tag{85}$$

• The field v_k belongs to $(L^2(\Omega))^N \cap \text{Ker } D$ for all $k \in \mathbb{N}$. So, using Theorem 6.7 and exactly the same argument as in Lemma 6.8 with $\Gamma_1 = \Gamma_t$ and $\Gamma_2 = \Gamma_\theta$, v_k can be split as follows: $v_k = \text{curl } \psi_k + \xi_k$, with $\psi_k \in H_0^1(\Omega; \Gamma_t, \Gamma_\theta)$ and $\xi_k = \Pi^0_{\Gamma_\theta,\Gamma_t} v_k \in M^0(\Omega; \Gamma_\theta, \Gamma_t)$.

• Second, let us show now that curl ψ_k tends to zero in $(L^2(\Omega))^N$. As ψ_k belongs to $(H^1(\Omega))^{2N-3}$ which is contained in $H(\text{curl}, \Omega)$, using Proposition

4.2, we deduce that ψ_k belongs to $H(\text{curl}, \text{div}*, \Omega)$. Moreover, by definition of $H^1_0(\Omega; \Gamma_\theta, \Gamma_t)$, $\psi_k \times n_{|\Gamma_\theta} = 0$ (see (57)), which means that ψ_k belongs to space W, so $R\psi_k = \text{curl}\ \psi_k$ is well defined. And we can take $\varphi = \psi_k$ in (85). Then, thanks to the orthogonality of the decomposition, we obtain, for all $k \in \mathbb{N}$:

$$\begin{aligned} \langle R\psi_k, v_k \rangle_{X',X} &= (\text{curl}\ \psi_k, v_k)_0 \ = \| \text{curl}\ \psi_k \|^2_{0,\Omega} \\ &\leq \frac{1}{k} \| \psi_k \|_W \\ &\leq \frac{1}{k} \| \psi_k \|_{1,\Omega} \ \text{(by Lemma 4.7)} \end{aligned}$$

Now, using Lemma 6.5 and unicity conditions for ψ_k given in Theorem 6.7, we have $\Pi^1_{\Gamma_t,\Gamma_\theta} \psi_k = 0$ and div $\psi_k = 0$, in three dimensions. In two dimensions, we use a generalized Poincaré inequality as meas $(\Gamma_t) \neq 0$. In both cases, we deduce from the previous inequality that :

$$\| \text{curl}\ \psi_k \|^2_{0,\Omega} \ \leq \ \frac{C}{k} \| \text{curl}\ \psi_k \|_{0,\Omega} \quad ,$$

and after simplification :

$$\| \text{curl}\ \psi_k \|_{0,\Omega} \ \leq \ \frac{C}{k} \quad ,$$

which means that curl ψ_k goes to zero in $(L^2(\Omega))^N$ as k goes to infinity.

• The sequence $(v_k)_{k\in\mathbb{N}}$ is bounded in $(L^2(\Omega))^N$. We have seen above that v_k can be split into $v_k = \text{curl}\ \psi_k + \xi_k$, and curl ψ_k tends strongly to zero in $(L^2(\Omega))^N$. Then the sequence $(\xi_k)_{k\in\mathbb{N}}$ is also bounded in $(L^2(\Omega))^N$. Moreover, space $M^0(\Omega; \Gamma_\theta, \Gamma_t)$ is finite dimensional (see Theorem 6.6), so there exists a sub-sequence, still denoted by ξ_k and $\xi \in M^0(\Omega; \Gamma_\theta, \Gamma_t)$ such that ξ_k strongly converges to ξ. Finally, we deduce from these results that v_k converges towards ξ in $(L^2(\Omega))^N$. Moreover, as div $v_k = 0$ for all $k \in \mathbb{N}$, div v_k converges towards $0 = \text{div}\ \xi$, as ξ belongs to $M^0(\Omega; \Gamma_\theta, \Gamma_t)$. Then, v_k converges towards ξ in $H(\text{div}, \Omega)$. And, by continuity of the normal trace on $H(\text{div}, \Omega)$, we deduce that $\xi{\bullet}n_{|\Gamma} = \lim_{k\longrightarrow\infty} v_k{\bullet}n_{|\Gamma} = 0$ in $H^{-1/2}(\Gamma)$.

• As Ω is bounded, we can cover $\overline{\Omega}$ with a finite number of open balls B_l, for $l = 0$ to $l = L$, such that :

$$\overline{\Omega} \subset \cup_{l=1}^L B_l \ \text{ and } \ B_l \cap B_{l+1} \neq \emptyset \ \text{ for all } \ l \in \{0\ ,\ L-1\} \quad .$$

$*$ Let us choose B_0 the ball containing the analytical subset $\Gamma_0 \subset \Gamma_t$ (hypothesis (84)). As B_0 is simply connected and as curl $\xi = 0$, we deduce that ξ

can be written as $\nabla\mu_0$ in B_0 with $\mu_0 \in H^1(B_0)$ (see [BDG85] for example). Moreover, $\xi \times n_{|\Gamma} = 0$ on Γ_t. So we obtain :

$$\nabla\mu_0 \times n_{|\Gamma_t} = \mathrm{curl}_\Gamma(\gamma\mu_0) = 0 \text{ on } \Gamma_t \quad .$$

Then, we have : $\nabla_\Gamma(\gamma\mu_0) = n \times \mathrm{curl}_\Gamma(\gamma\mu_0) = 0$ on Γ_t. We deduce that $\gamma\mu_0$ is constant on Γ_t thus on Γ_0, and we finally can choose the constant equal to zero. Then μ_0 is solution of the following problem :

$$\begin{cases} \Delta\mu_0 &= \mathrm{div}\,\xi &= 0 & \text{in } B_0 \quad , \\ \dfrac{\partial\mu_0}{\partial n} &= \xi\bullet n_{|\Gamma} &= 0 & \text{on } \Gamma_0 \subset \Gamma \quad , \\ \mu_0 &= 0 & & \text{on } \Gamma_0 \subset \Gamma_t \quad . \end{cases}$$

So μ_0 verifies a Cauchy problem on Γ_0 which is supposed analytical (84). Then, $\mu_0 \equiv 0$ on B_0 (see Landis [Lan59]).
$*$ Consider now B_1 intersecting B_0. On B_1 which is simply connected, we have : $\xi = \nabla\mu_1$. Let us introduce an analytical line Γ_1 contained in the intersection of B_0 and B_1. Then, exactly as above, μ_1 verifies :

$$\begin{cases} \Delta\mu_1 &= 0 \text{ in } B_1 \quad , \\ \dfrac{\partial\mu_1}{\partial n} &= 0 \text{ on } \Gamma_1 \quad , \\ \mu_1 &= 0 \text{ on } \Gamma_1 \quad . \end{cases}$$

As Γ_1 is analytical, μ_1 is identically zero on B_1.
$*$ The same argument, applied on other balls, leads to $\xi \equiv 0$ on Ω, which is impossible : as $v_k \overset{k\longrightarrow\infty}{\longrightarrow} \xi$ in $L^2(\Omega)$, $\| \xi \|_{0,\Omega} = 1$ thanks to $\| v_k \|_{0,\Omega} = 1$. ∎

Lemma 6.15 *Inf-sup condition (63) when Ω is a connected and simply connected open bounded domain of $\mathbb{R}^N$.*
We assume that Ω is a connected and a simply connected open bounded domain in $\mathbb{R}^N$ whose boundary Γ is supposed of class $\mathcal{C}^{1,1}$. Then, there exists a strictly positive constant b such that :

$$\inf_{\substack{v\in \mathrm{Ker}D \\ v\neq 0}} \sup_{\substack{\varphi\in W \\ \varphi\neq 0}} \frac{\langle R\varphi, v\rangle_{X',X}}{\| v \|_X \| \varphi \|_W} \;\geq\; b \quad .$$

Proof

We are in the particular case of a connected and simply connected open bounded domain of $\mathbb{R}^N$ with a $\mathcal{C}^{1,1}$ boundary, we can use the second part of Lemma 6.8 : there exists $\chi \in H^1_0(\Omega; \emptyset, \Gamma)$ such that $v = \mathrm{curl}\chi$ and satisfying

$\|\chi\|_{1,\Omega} \leq C_0 \|v\|_{0,\Omega}$. With Proposition 4.2, χ belongs to $H(\mathrm{curl}, \mathrm{div}*, \Omega)$ and $R\,\chi$ is equal to curl χ in $(H_0(\mathrm{div},\Omega))'$. So, for all v in Ker D, we have:

$$\begin{aligned}\sup_{\substack{\varphi\in W\\ \varphi\neq 0}} \frac{\langle R\varphi, v\rangle_{X',X}}{\| v \|_X \| \varphi \|_W} &\geq \frac{(\mathrm{curl}\ \chi, v)_0}{\| v \|_X \| \chi \|_W} = \frac{\| v \|_{0,\Omega}^2}{\| v \|_X \| \chi \|_W} = \frac{\| v \|_{0,\Omega}^2}{\| v \|_{0,\Omega} \| \chi \|_W} \\ &\geq \frac{\| v \|_{0,\Omega}}{C \| \chi \|_{1,\Omega}} \quad \text{with Lemma 4.7} \\ &\geq \frac{1}{CC_0}\ .\end{aligned}$$

Then, the desired inf-sup condition is proved. ■

Proof of Theorem 6.10

We prove that hypotheses of the abstract Theorem 5.1 from the previous section are satisfied. The proof is divided into three steps: Lemma 6.12 is ellipticity (64), Lemmas 6.13 and 6.14 are inf-sup conditions (62) and (63). The particular case of a connected and simply connected domain is proved with the help of Lemma 6.15. ■

6.5 Towards a new boundary condition

We now interpret the variational solution of the problem studied in Theorem 6.10 for different choices of the mass operator J. The first choice J_1 conducts to an elliptic problem which is not the Stokes problem, while the second one J_2 can be re-interpreted as the Stokes system of partial differential equations, but with a non classical boundary condition.

Proposition 6.16 *Mass operator J_1 associated with the W-scalar product. We set:*

$$\langle J_1\omega, \varphi\rangle_{W',W} = (\omega,\varphi)_0 + (L\rho\omega, L\rho\varphi)_0 + (K\rho\omega, K\rho\varphi)_{\mathrm{div}} \equiv (\omega,\varphi)_W.$$

Operators K and L are given in Definition 3.
Under hypotheses of Theorem 6.10, taking $\lambda = 0$, $\mu = f \in (L^2(\Omega))^N$ and $\nu = 0$, the solution $(\omega, u, p) \in W \times X \times Y$ of the problem:

$$\left\{\begin{array}{lcll} \langle J_1\omega, \varphi\rangle_{W',W} - \langle R'u, \varphi\rangle_{W',W} & = & 0 & \forall\, \varphi \in W \quad , \\ \langle R\omega, v\rangle_{X',X} - \langle D'(p - \delta r D u), v\rangle_{X',X} & = & (f, v)_0 & \forall\, v \in X \quad , \\ \langle Du, q\rangle_{Y',Y} & = & 0 & \forall\, q \in Y \quad . \end{array}\right. \qquad (86)$$

is such that:

$$\omega + \mathrm{curl}\ L\rho\omega = \mathrm{curl}\ u \text{ in } (\mathcal{D}'(\Omega))^{2N-3} \quad .$$

In other words, the operator J_1, associated with the natural scalar product in W, does not satisfy the first equation of the Stokes problem, which is: $\omega = \text{curl } u$ *in* $(\mathcal{D}'(\Omega))^{2N-3}$.

<u>Proof</u>

Let us consider the first equation of problem (86). We take φ in $(\mathcal{D}(\Omega))^{2N-3}$ and rewrite the duality product. Thanks to (53), we obtain:

$$\langle R'u, \varphi \rangle_{W',W} = \langle R\varphi, u \rangle_{X',X} = (\text{curl } \varphi, u)_0 \quad .$$

Using Lemma 6.2, as φ belongs to $(\mathcal{D}(\Omega))^{2N-3}$, we have $K\rho\varphi = 0$ and $L\rho\varphi = \text{curl } \varphi$. So we obtain for all $\varphi \in (\mathcal{D}(\Omega))^{2N-3}$:

$$(\omega, \varphi)_0 + (L\rho\omega, \text{curl } \varphi)_0 = (u, \text{curl } \varphi)_0 \quad .$$

Then, the first equation of problem (86) leads to: $\omega + \text{curl } L\rho\omega = \text{curl } u$ in $(\mathcal{D}'(\Omega))^{2N-3}$. ■

∘ The previous proposition shows that the first natural choice J_1 as the functional J is not the good one. Let us now examine the second one.

Proposition 6.17 *Boundary mass operator J_2 that guarantees ellipticity.*
We set:

$$\langle J_2\omega, \varphi \rangle_{W',W} = (\omega, \varphi)_0 + (K\rho\omega, K\rho\varphi)_{\text{div}} \quad .$$

The operator K is given in Definition 3.
Under hypotheses of Theorem 6.10, taking $\lambda = 0$, $\mu = f \in (L^2(\Omega))^N$ and $\nu = 0$, the solution $(\omega, u, p) \in W \times X \times Y$ of the problem:

$$\left\{ \begin{array}{lcll} \langle J_2\omega, \varphi \rangle_{W',W} - \langle R'u, \varphi \rangle_{W',W} & = & 0 & \forall\, \varphi \in W \quad , \\ \langle R\omega, v \rangle_{X',X} - \langle D'(p - \delta r D u), v \rangle_{X',X} & = & (f, v)_0 & \forall\, v \in X \quad , \\ \langle Du, q \rangle_{Y',Y} & = & 0 & \forall\, q \in Y \quad . \end{array} \right. \tag{87}$$

is such that:

$$\left\{ \begin{array}{lcll} \omega & = & \text{curl } u & \text{in } (\mathcal{D}'(\Omega))^N \quad , \\ \text{curl } \omega + \nabla p & = & f & \text{in } (\mathcal{D}'(\Omega))^N \quad , \\ \text{div } u & = & 0 & \text{in } \mathcal{D}'(\Omega) \quad , \end{array} \right.$$

These are the partial differential equations associated with the Stokes problem.

<u>Proof</u>

- Let us consider the first equation of problem (87). Exactly as in the previous proof, when we take φ in $(\mathcal{D}(\Omega))^{2N-3}$, we obtain $K\rho\varphi = 0$ with Lemma 6.2. Then, we can rewrite the duality product, thanks to (53), as:

$$\langle R'u, \varphi\rangle_{W',W} = \langle R\varphi, u\rangle_{X',X} = (\text{curl } \varphi, u)_0 \quad ,$$

which leads to:

$$(\omega, \varphi)_0 = (\text{curl } \varphi, u)_0 = \langle \text{curl } u, \varphi\rangle_{(\mathcal{D}'(\Omega))^{2N-3},(\mathcal{D}(\Omega))^{2N-3}},$$

which means: $\omega = \text{curl } u$ in $(\mathcal{D}'(\Omega))^{2N-3}$. It is exactly the first equation of the Stokes problem.

- We consider now the last equation of problem (87): $Du = 0$ in Y'. As D is nothing else than the divergence operator, we have $Du = \text{div } u$. Then, solution u of problem (87) is divergence free, which is the third equation of the Stokes problem.
- Finally, let us consider the second equation of problem (87). As $Du = 0$, choosing virtual fields v in space $(\mathcal{D}(\Omega))^N$, it becomes:

$$\langle R\omega, v\rangle_{X',X} \; - \; (p, \text{div } v)_0 = (f, v)_0 \quad , \quad \forall v \in (\mathcal{D}(\Omega))^N \quad .$$

Using Definition (51), for all v in $(\mathcal{D}(\Omega))^N$, we have:

$$\langle R\omega, v\rangle_{X',X} = (\omega, \text{curl } v)_0 = \langle \text{curl } \omega, v\rangle_{(\mathcal{D}'(\Omega))^N,(\mathcal{D}(\Omega))^N}.$$

These two equations lead to: $\text{curl } \omega \; + \; \nabla p = f$ in $(\mathcal{D}'(\Omega))^N$. It is the second equation of the Stokes problem. ■

○ According to Lemma 6.12, the term $(K\rho\omega, K\rho\varphi)_{\text{div}}$ is, in some sense, the minimal one to obtain ellipticity of the functional J on the kernel V, without more explicit conditions on the domain Ω. Moreover, this complementary term appears on the boundary only (it is zero for regular functions, see Lemma 6.2) and it is associated with a non classical boundary condition, as it is developed in the next proposition.

Proposition 6.18 *A new boundary condition for the Stokes operator.*
Under hypotheses of Theorem 6.10, the solution $(\omega, u, p) \in W \times X \times Y$ of the problem (87) studied in proposition 6.17 is such that the velocity u belongs to $H(\text{curl}, \Omega)$. Moreover, it formally satisfies the following boundary conditions of

○ *non penetrability, $u \bullet n = 0$ on Γ,*
○ *given tangential vorticity on a part Γ_θ of the boundary, $\omega \times n = 0$ on Γ_θ,*
○ *and a new coupled condition between tangential velocity and vorticity:*

$$u \times n_{|\Gamma_t} = \text{curl}_\Gamma \; \gamma\chi \; \textit{on } \Gamma_t, \tag{88}$$

where the scalar function χ is associated with the vorticity ω through the following relation:

$$\left\{ \begin{array}{rcl} \Delta\chi & = & \text{div } (\rho\omega) \quad \text{in } \Omega \quad , \\ \dfrac{\partial\chi}{\partial n} & = & 0 \quad \text{on } \Gamma \quad , \end{array} \right. \tag{89}$$

in which $\rho\omega$ is the Riesz representant of $R\omega$ in space X.

Remark 6.19 *The last boundary condition appears mathematically but is not contained in the mechanical model for which we have: $u \times n_{|\Gamma_t} = 0$. We do not have a simple physical interpretation of this boundary condition.*

Proof of Proposition 6.18

- First, let us recall that we have seen in the previous proposition that curl $u = \omega$. As ω belongs to $(L^2(\Omega))^{2N-3}$, function u belongs to $H(\text{curl}, \Omega)$.
- Moreover, the natural Dirichlet condition on normal velocity is a consequence of the choice of space X. Then we have:

$$u \bullet n = 0 \quad \text{on} \quad \Gamma \quad .$$

In a similar manner, the choice of space W leads to:

$$\omega \times n = 0 \quad \text{on} \quad \Gamma_\theta \quad .$$

Finally, the only difficult point deals with the study of the tangential component of the velocity on Γ_t. The demonstration will be done in three dimensions, but it is analogous in two dimensions.

- We consider again the first equation of problem (87), and we choose φ in $(H^1(\Omega))^3$ such that its tangential trace $\gamma\varphi \times n$ is zero on Γ_θ and its normal trace $\gamma\varphi \bullet n$ is zero on the whole boundary Γ. As $\Gamma_t = \Gamma_\theta^c$, its trace $\gamma\varphi$ belongs to $TH_{00}^{1/2}(\Gamma_t)$, by definition of this space. Moreover, for regular functions, $\varphi \times n_{|\Gamma_\theta}$ is equal to $\gamma\varphi \times n$ on Γ_θ, so φ belongs to W. As u belongs to $H(\text{curl}, \Omega)$, tangential trace of u is well defined in $(TH_{00}^{1/2}(\Gamma_t))'$. Then, using again (53) and integrating by parts, we obtain:

$$\langle R\varphi, u \rangle_{X',X} = (u, \text{curl } \varphi)_0 = (\varphi, \text{curl } u)_0 \; + \; \langle \gamma\varphi, n \times u \times n \rangle_{TH_{00}^{1/2}(\Gamma_t),(TH_{00}^{1/2}(\Gamma_t))'}$$

Introducing this relation in the first equation of problem (87) and taking into account the fact that $\omega = \text{curl } u$, we deduce that:

$$\langle \gamma\varphi, n \times u \times n \rangle_{TH_{00}^{1/2}(\Gamma_t),(TH_{00}^{1/2}(\Gamma_t))'} = (K\rho\omega, K\rho\varphi)_{\text{div}} \quad , \tag{90}$$

for all φ in space $(H^1(\Omega))^3$ such that $\gamma\varphi$ is in $TH^{1/2}_{00}(\Gamma_t)$. Then, the tangential trace of u, which is formally written: $n \times u \times n$, is zero in space $(TH^{1/2}_{00}(\Gamma_t))'$, which is still formally written $n \times u \times n = 0$ on Γ_t, if the expression $(K\rho\omega, K\rho\varphi)_{\text{div}}$ is zero for all φ in space $(H^1(\Omega))^3$. But, in general, it is not the case. More precisely, using the orthogonal decomposition of X (see Corollary 6.9), we have:

$$(K\rho\omega, K\rho\varphi)_{\text{div}} = (K\rho\omega, \rho\varphi)_{\text{div}} = \langle R\varphi, K\rho\omega\rangle_{X',X} \quad ,$$

by definition of the Riesz operator ρ. As φ belongs to space $(H^1(\Omega))^3$, using (53), we deduce that:

$$(K\rho\omega, K\rho\varphi)_{\text{div}} = (K\rho\omega, \text{curl } \varphi)_0 \quad .$$

Let us now observe that, by definition, $K\rho\omega$ belongs to $(\text{Ker } D)^\perp$ which is contained in space $\{\nabla\chi\ ,\ \chi \in H^1_0(\Omega; \emptyset)\}$ (see Corollary 6.9). So, there exists a function χ in $H^1_0(\Omega; \emptyset)$ such that: $K\rho\omega = \nabla\chi$. Then, we have:

$$(K\rho\omega, K\rho\varphi)_{\text{div}} = (\text{curl } \varphi, \nabla\chi)_0 \quad ,$$

which is equal to a boundary term: $\langle \gamma\varphi, \gamma(\nabla\chi) \times n\rangle_{TH^{1/2}_{00}(\Gamma_t),(TH^{1/2}_{00}(\Gamma_t))'}$ (see [Dub90]). Introducing this relation in (90), we deduce that, formally:

$$\langle \gamma\varphi, n \times u \times n\rangle_{TH^{1/2}_{00}(\Gamma_t),(TH^{1/2}_{00}(\Gamma_t))'} = \langle \gamma\varphi, \gamma(\nabla\chi) \times n\rangle_{TH^{1/2}_{00}(\Gamma_t),(TH^{1/2}_{00}(\Gamma_t))'}$$

for all φ in $(H^1(\Omega))^3$, null on Γ_θ and such that $\gamma\varphi\bullet n = 0$ on the whole boundary Γ. Then, we obtain:

$$n \times u \times n_{|\Gamma_t} = \gamma(\nabla\chi) \times n \;\text{ on }\; \Gamma_t \quad ,$$

where χ is associated with the non divergence free part of the co-curl of the vorticity ω. Finally, the equality $\gamma(\nabla\chi) \times n = \text{curl}_\Gamma\, \gamma\chi$, which is the surfacic curl operator (see [CB68]), leads to the expected result. ■

∘ Let us remark that the boundary condition on the tangential velocity (88-89) is not classical. To recover the boundary condition of given tangential velocity (17), it is sufficient for the term $(K\rho\omega, K\rho\varphi)_{\text{div}}$ to be identically zero. It would be the case if the mass operator J is equal to the $(L^2(\Omega))^{2N-3}$-norm *ie* J_3, and if this operator is elliptic on V. We develop this point in the following theorem; a (ω, u, p) formulation compatible with the classical Stokes problem is therefore a direct consequence of Theorem 6.10 and of the previous remarks.

Theorem 6.20 *Well-posedness of the Stokes problem.*
∘ *Let Ω be a bounded connected domain of $\mathbb{R}^N$, ($N = 2$ or 3) with a boundary denoted by Γ. Let $(\Gamma_t, \Gamma_\theta)$ be a partition of Γ. We consider the spaces:*

$W=\left\{\varphi\in H(\text{curl},\text{div}*,\Omega)\ ,\ \varphi\times n_{|\Gamma_\theta}=0\right\}$, $X=H_0(\text{div},\Omega)$, $Y=L^2_0(\Omega)$, *the operators introduced in the previous section acting on these spaces:* $R: W\longrightarrow X'$ *;* $D: X\longrightarrow Y'$ *;* $J: W\longrightarrow W'$*. We denote by* r *the Riesz isomorphism from* Y' *to* Y *and by* δ *a constant either equal to* 0 *or* 1.

∘ *We consider a functional associated with the operator* J_3 *chosen equal to the* $(L^2(\Omega))^{2N-3}$*-scalar product:*

$$\langle J_3\omega,\varphi\rangle_{W',W}=(\omega,\varphi)_0 \quad ,$$

and we assume that J_3 *is elliptic on the kernel* V*, which is:*

$$V=\left\{\varphi\in W\ ,\ \langle R\varphi,v\rangle_{X',X}=0\ ,\ \forall\ v\in \text{Ker}D\right\} \quad .$$

∘ *We assume one of the two following hypotheses on* Ω*:*
(i) Ω *is a connected and simply connected open bounded domain of* $\mathrm{I\!R}^N$ *with a* $\mathcal{C}^{1,1}$ *boundary,*
(ii) Ω *is an open bounded connected domain of* $\mathrm{I\!R}^N$ *with a* $\mathcal{C}^2$*-class boundary and there exists some analytical subset* Γ_0 *of* Γ *such that:*

$$\text{meas}\ (\Gamma_0)\neq 0 \quad\text{and}\quad \Gamma_0\subset\Gamma_t \quad .$$

∘ *Then, the following problem:*

$$\left\{\begin{array}{lcll}
\text{find } (\omega,u,p)\in W\times X\times Y \text{ such that :} & & & \\
\langle J_3\omega,\varphi\rangle_{W',W}-\langle R'u,\varphi\rangle_{W',W} & = & \langle\lambda,\varphi\rangle_{W',W}\ , & \forall\ \varphi\in W\\
\langle R\omega,v\rangle_{X',X}-\langle D'(p-\delta rDu),v\rangle_{X',X} & = & \langle\mu,v\rangle_{X',X}\ , & \forall\ v\in X\\
\langle Du,q\rangle_{Y',Y} & = & \langle\nu,q\rangle_{Y',Y}\ , & \forall\ q\in Y
\end{array}\right.$$

is well-posed and is exactly the following Stokes problem.

$$\left\{\begin{array}{rcll}
\omega-\text{curl}\ u & = & \lambda & \text{in }\Omega,\\
\text{curl}\ \omega-\delta\nabla\text{div}\ u+\nabla p & = & \mu & \text{in }\Omega,\\
\text{div}\ u & = & \nu & \text{in }\Omega,\\
u\bullet n & = & 0 & \text{on }\Gamma,\\
n\times u\times n & = & 0 & \text{on }\Gamma_t,\\
\omega\times n & = & 0 & \text{on }\Gamma_\theta.
\end{array}\right.$$

<u>Proof</u>
For the well-posedness, we prove that hypotheses of the abstract Theorem 5.1 are satisfied. Ellipticity (64) is here assumed, Lemmas 6.13 and 6.14 are inf-sup conditions (62) and (63). The particular case of the connected and simply connected domain is proved by Lemma 6.15.

As J_3 is equal to the $(L^2(\Omega))^{2N-3}$-scalar product, the re-interpretation of the first equation of the variational formulation is obvious as done in proposition 6.17. The last point is the boundary condition on tangential velocity. Refer to the proof of proposition 6.18 for the details. Taking φ in space $(H^1(\Omega))^{2N-3}$ such that $\gamma\varphi$ is in $TH_{00}^{1/2}(\Gamma_t)$, using (53) and integrating by parts, we obtain:

$$\langle R\varphi, u\rangle_{X',X} = (u, \text{curl } \varphi)_0 = (\varphi, \text{curl } u)_0 \ + \ \langle \gamma\varphi, n \times u \times n\rangle_{TH_{00}^{1/2}(\Gamma_t),(TH_{00}^{1/2}(\Gamma_t))'}.$$

Introducing this relation in the first equation of problem (87) and taking into account the fact that $\omega = \text{curl } u$, we deduce that:

$$\langle \gamma\varphi, n \times u \times n\rangle_{TH_{00}^{1/2}(\Gamma_t),(TH_{00}^{1/2}(\Gamma_t))'} = 0,$$

for all φ in space $(H^1(\Omega))^{2N-3}$ such that $\gamma\varphi$ is in $TH_{00}^{1/2}(\Gamma_t)$. Then, the tangential trace of u, which is $n \times u \times n$, is zero in space $(TH_{00}^{1/2}(\Gamma_t))'$, which is formally written: $n \times u \times n = 0$ on Γ_t. ■

The problem of the ellipticity of J_3 in the general case is still open, except in a slightly more general case than the stream function-vorticity formulation, as we shall see in the next section.

7 The bi-dimensional case revisited

In this section, we consider the particular case of a two-dimensional simply connected domain Ω whose boundary is supposed to be of class $\mathcal{C}^{1,1}$. The boundary Γ is split into Γ_θ and Γ_t. In a first step, we show that the mass operator J_3, associated with the L^2- scalar product, is elliptic on V in this configuration, and we though obtain an extension of the frame of the classical (ψ, ω) formulation. In a second step, in the case $\Gamma_t \equiv \Gamma$ we prove the complete equivalence between the vorticity-velocity-pressure formulation and the classical (ψ, ω) one.

7.1 A well-posed formulation of the (ω, u, p) Stokes problem in the bidimensional case

∘ In the vorticity-velocity-pressure formulation in two-dimensional domains, the space W for the vorticity is $\left\{\varphi \in H(\text{curl}, \text{div}*, \Omega)\ ,\ \varphi \times n_{|\Gamma_\theta} = 0\right\}$. Let us also recall that the curl operator is defined in (59). First, let us compare spaces for vorticity: W, $H(\text{curl}, \text{div}*, \Omega)$ and $M(\Omega) = \{\varphi \in L^2(\Omega),$ $\Delta\varphi \in H^{-1}(\Omega)\}$.

Lemma 7.1 *Space $H(\mathrm{curl}, \mathrm{div}*, \Omega)$ (and then W) is imbedded in $M(\Omega)$ and, for all φ in $H(\mathrm{curl}, \mathrm{div}*, \Omega)$, we have:*

$$\forall \chi \in H_0^1(\Omega) \ , \ \langle -\Delta\varphi, \chi \rangle_{-1,1} = \langle \mathit{curl}\ \varphi, \mathrm{curl}\ \chi \rangle_{X',X} \quad . \tag{91}$$

Moreover, this imbedding is continuous:

$$\forall \varphi \in H(\mathrm{curl}, \mathrm{div}*, \Omega) \ , \ \ \| \varphi \|_M \ \ \leq \ \ \| \varphi \|_{\mathrm{curl},\mathrm{div}*,\Omega} \quad . \tag{92}$$

<u>Proof</u>
Let us take a function $\varphi \in H(\mathrm{curl}, \mathrm{div}*, \Omega)$. On the one hand, $\langle \mathrm{curl}\ \varphi, v \rangle_{X',X}$ is defined for all $v \in X$. On the other hand, for all $\chi \in \mathcal{D}(\Omega)$, we have:

$$\begin{aligned} -\langle \Delta\varphi, \chi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \ &= \ \langle \varphi, \mathrm{curl}\ \mathrm{curl}\ \chi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\ &= \ (\varphi, \mathrm{curl}\ \mathrm{curl}\ \chi)_0 \text{ as } \varphi \in L^2(\Omega) \quad . \end{aligned}$$

Let us remark that curl χ belongs to X, because the tangential derivative of χ is zero along Γ. Then, by definition of the curl operator (see (51)), we obtain:

$$\begin{aligned} |\langle \Delta\varphi, \chi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| \ &= \ |\langle \mathrm{curl}\ \varphi, \mathrm{curl}\ \chi \rangle_{X',X}| \\ &\leq \ \| \mathrm{curl}\ \varphi \|_{X'} \underbrace{\| \mathrm{curl}\ \chi \|_X}_{=\|\nabla\chi\|_{0,\Omega}} \quad . \end{aligned}$$

This relation proves that $\Delta\varphi$ is a linear and continuous form on $H_0^1(\Omega)$, and then belongs to $H^{-1}(\Omega)$. So φ belongs to $M(\Omega)$. Moreover, the above inequality leads to:

$$\| \Delta\varphi \|_{-1,\Omega} = \sup_{\chi \in H_0^1(\Omega)} \frac{\langle \Delta\varphi, \chi \rangle_{-1,1}}{\| \nabla\chi \|_{0,\Omega}} \ \leq \ \| \mathrm{curl}\ \varphi \|_{X'} \quad .$$

Then, by definition of the two norms, we have:

$$\| \varphi \|_M^2 = \| \varphi \|_{0,\Omega}^2 + \| \Delta\varphi \|_{-1,\Omega}^2 \ \ \leq \ \ \| \varphi \|_{0,\Omega}^2 + \| \mathrm{curl}\ \varphi \|_{X'}^2 = \| \varphi \|_{\mathrm{curl},\mathrm{div}*,\Omega}^2$$

which gives the announced result. ■

Proposition 7.2 *Comparison of the vorticity spaces.*
Let Ω be a connected and simply connected open bounded domain in $\mathbb{R}^2$ whose boundary Γ is supposed to be of class $\mathcal{C}^{1,1}$. Then, space $H(\mathrm{curl}, \mathrm{div}*, \Omega)$, defined in (49), is equal to space $M(\Omega)$ defined in (30) and the norms of these two spaces are equivalent.

Proof

• Due to Lemma 7.1, the only point to prove is that space $M(\Omega)$ is continuously imbedded in $H(\text{curl}, \text{div}*, \Omega)$. First, it means that we have to show that any function φ of $M(\Omega)$ has a weak curl which belongs to X', with $X = H_0(\text{div}, \Omega)$. Using the density of space $H^1(\Omega)$ in space $M(\Omega)$ (see Proposition 3.7) we first assume that φ belongs to $H^1(\Omega)$.

• Let us consider a function v in $(\mathcal{D}(\Omega))^2$, which is contained in X. Using the decomposition recalled in Theorem 6.7, we can split v : $v = \nabla\chi + \text{curl}\ \psi$. There is no special function as Ω is simply connected (see *e.g* [GR86]). Moreover, function χ is the unique solution in $H^1(\Omega)/\mathbb{R}$ of the homogeneous Neumann problem :

$$\left\{ \begin{array}{rcll} \Delta\chi & = & \text{div}\ v & \text{in}\ \Omega \\ \dfrac{\partial\chi}{\partial n} & = & 0 & \text{on}\ \Gamma \end{array} \right. .$$

Using regularity results (see [ADN59]), as div v belongs to $L^2(\Omega)$, we deduce that χ belongs to $H^2(\Omega)$ and there exists $C > 0$ such that :

$$\| \chi \|_{2,\Omega} \leq C \ \| \text{div}\ v \|_{0,\Omega} \quad . \tag{93}$$

In a similar way, function ψ is the unique solution in $H^1_0(\Omega)$ of the homogeneous Dirichlet problem :

$$\left\{ \begin{array}{rcll} \Delta\psi & = & -\text{curl}\ v & \text{in}\ \Omega \\ \psi & = & 0 & \text{on}\ \Gamma \end{array} \right. .$$

In a variational form, the partial differential equation becomes :

$$(\text{curl}\ \psi, \text{curl}\ \zeta)_0 = (\text{curl}\ v, \zeta)_0 = (v, \text{curl}\ \zeta)_0 \quad , \quad \text{for all}\ \zeta \in H^1_0(\Omega) \quad .$$

Then, there exists $C > 0$ such that :

$$\| \nabla\psi \|_{0,\Omega} = \| \text{curl}\ \psi \|_{0,\Omega} \leq C \ \| v \|_{0,\Omega} \quad . \tag{94}$$

• Now, let us calculate $\langle \text{curl}\ \varphi, v \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$ for all φ in $H^1(\Omega)$ and v in $(\mathcal{D}(\Omega))^2$. Using the previous decomposition, we have :

$$\begin{array}{rcl} \langle \text{curl}\ \varphi, v \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} & = & (\text{curl}\ \varphi, v)_0 \\ & = & (\text{curl}\ \varphi, \nabla\chi)_0 \ + \ (\text{curl}\ \varphi, \text{curl}\ \psi)_0 \quad . \end{array}$$

On the one hand, as φ belongs to $H^1(\Omega)$ and ψ to $H^1_0(\Omega)$, we obtain :

$$(\text{curl}\ \varphi, \text{curl}\ \psi)_0 = -\langle \Delta\varphi, \psi \rangle_{-1,1} \quad .$$

On the other hand, as χ belongs to $H^2(\Omega)$, its tangential derivative $\dfrac{\partial \chi}{\partial t}$ is in $H^{1/2}(\Gamma)$. Moreover, the trace $\gamma\varphi$ belongs to $H^{-1/2}(\Gamma)$ because φ is in $M(\Omega)$. Then, we have:

$$(\text{curl } \varphi, \nabla\chi)_0 = \langle \gamma\varphi, \frac{\partial \chi}{\partial t} \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \quad .$$

Finally, using (93) and (94), and the trace continuity from $M(\Omega)$ to $H^{-1/2}(\Gamma)$ (see (45)), and from $H^2(\Omega)$ to $H^{3/2}(\Gamma)$, we obtain:

$$\begin{aligned}
|\langle \text{curl } \varphi, v \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}| &= |\ \langle \gamma\varphi, \frac{\partial \chi}{\partial t} \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} - \langle \Delta\varphi, \psi \rangle_{-1,1}\ | \\
&\leq \|\ \gamma\varphi\ \|_{-1/2,\Gamma} \ \|\ \gamma\chi\ \|_{3/2,\Gamma} + \|\ \Delta\varphi\ \|_{-1,\Omega} \ \|\ \psi\ \|_{1,\Omega} \\
&\leq C\ \|\ \gamma\varphi\ \|_{-1/2,\Gamma} \ \|\ \chi\ \|_{2,\Omega} + \|\ \Delta\varphi\ \|_{-1,\Omega} \ \|\ \psi\ \|_{1,\Omega} \\
&\leq C\ \big(\|\ \varphi\ \|_{M} \ \|\ \text{div } v\ \|_{0,\Omega} + \|\ \varphi\ \|_{M} \ \|\ v\ \|_{0,\Omega}\big) \\
&\leq C\ \|\ \varphi\ \|_{M} \ \|\ v\ \|_{\text{div},\Omega} \quad .
\end{aligned}$$

This inequality proves that curl φ defines a linear form on $(\mathcal{D}(\Omega))^2$, which is continuous for the $H(\text{div}, \Omega)$-topology: curl φ belongs to X'. It means φ belongs to $H(\text{curl}, \text{div}*, \Omega)$ for all φ of $H^1(\Omega)$.

• Then, observing that, in the above inequality, the continuity constant depends on the $M(\Omega)$-norm, we deduce that any function φ of $M(\Omega)$ has a weak curl which belongs to X', by density of space $H^1(\Omega)$ in space $M(\Omega)$ (see Proposition 3.7).

• Finally, let us compare the two norms. Using the density of $(\mathcal{D}(\Omega))^2$ in X, the above inequality leads to:

$$\|\ \text{curl } \varphi\ \|_{X'} = \sup_{v \in X} \frac{\langle \text{curl } \varphi, v \rangle_{X',X}}{\|\ v\ \|_{\text{div},\Omega}} = \sup_{v \in (\mathcal{D}(\Omega))^2} \frac{\langle \text{curl } \varphi, v \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}}{\|\ v\ \|_{\text{div},\Omega}} \leq C\ \|\ \varphi\ \|_{M}$$

for all φ of $H^1(\Omega)$ and then, by density, for all φ in $M(\Omega)$. Then, the definitions (52) and (32) of the two norms lead obviously to:

$$\forall \varphi \in H(\text{curl}, \text{div}*, \Omega) = M(\Omega),$$
$$\|\ \varphi\ \|_{\text{curl},\text{div}*,\Omega} = \sqrt{\|\varphi\|_0^2 + \|\text{curl } \varphi\|_{X'}^2} \leq C\ \|\ \varphi\ \|_{M} \ .$$

■

∘ We have already introduced the space $\mathcal{H}(\Omega)$ of harmonic functions of $L^2(\Omega)$. Then we obtain the following characterization of space V given by: $V = \{\varphi \in W\ ,\ L\rho\varphi = 0\}$ (see Lemma 6.4).

Lemma 7.3 *Let Ω be a connected and simply connected open bounded domain in $\mathbb{R}^2$ whose boundary Γ is supposed of class $\mathcal{C}^{1,1}$. Then, space V is imbedded in $\mathcal{H}(\Omega)$. Moreover, if $\Gamma_t \equiv \Gamma$ (*ie *$\Gamma_\theta = \emptyset$), spaces V and $\mathcal{H}(\Omega)$ are equal.*

Proof

- First, let us remark that, for all ζ in $H_0^1(\Omega)$, curl ζ belongs to X and is divergence free. Conversely, as Ω is simply connected, for all v in Ker D, the subspace of divergence free functions of X, there exists ζ in $H_0^1(\Omega)$ such that $v = \text{curl } \zeta$.
- We have seen in Lemma 7.1 that space W is imbedded in $M(\Omega)$. So, let φ be an element of W, and ζ be in $H_0^1(\Omega)$. We recall equality (91):

$$\langle -\Delta\varphi, \zeta \rangle_{-1,1} = \langle R\varphi, \text{curl } \zeta \rangle_{X',X} \quad .$$

Then, using the definition of the co-curl operator (see (82)), we have:

$$\langle R\varphi, \text{curl } \zeta \rangle_{X',X} = (\rho\varphi, \text{curl } \zeta)_{\text{div}} = (\rho\varphi, \text{curl } \zeta)_0 \quad ,$$

as curl ζ is divergence free. Moreover, if we introduce the Leray operator L (see Definition 3), which is the projector from X to Ker D, we obtain: $(\rho\varphi, \text{curl } \zeta)_0 = (L\rho\varphi, \text{curl } \zeta)_0$, as curl ζ belongs to Ker D. Finally, for all ζ in $H_0^1(\Omega)$ and all φ in W, we have:

$$\langle -\Delta\varphi, \zeta \rangle_{-1,1} = \langle R\varphi, \text{curl } \zeta \rangle_{X',X} = (L\rho\varphi, \text{curl } \zeta)_0 \quad . \tag{95}$$

In a similar way, for any element v of Ker D, we obtain:

$$\langle R\varphi, v \rangle_{X',X} = (L\rho\varphi, v)_0 \quad , \quad \forall\ v\ \in \text{Ker } D \quad , \quad \forall\ \varphi\ \in W \quad . \tag{96}$$

- Relation (95) shows that if φ belongs to V, $L\rho\varphi$ is zero and then $\langle -\Delta\varphi, \zeta \rangle_{-1,1} = 0$ for all ζ in $H_0^1(\Omega)$, which means that $\Delta\varphi = 0$: φ belongs to $\mathcal{H}(\Omega)$. So space V is always imbedded in $\mathcal{H}(\Omega)$.
- Conversely, if Γ_θ is empty, any element of $\mathcal{H}(\Omega)$ belongs to $M(\Omega)$ so to W, as $M(\Omega) = W = H(\text{curl}, \text{div}*, \Omega)$ (Proposition 7.2). Moreover, if φ is harmonic, we obtain with (95): $(L\rho\varphi, \text{curl } \zeta)_0 = 0$ for all ζ in $H_0^1(\Omega)$. For all v in Ker D, there exists ζ in $H_0^1(\Omega)$ such that $v = \text{curl } \zeta$. So we have: $(L\rho\varphi, v)_0 = 0$, for all v in Ker D. As $L\rho\varphi$ also belongs to Ker D, we deduce that $L\rho\varphi = 0$ and φ belongs to V. ■

∘ A first important consequence of the previous results is to enlarge the case in which the velocity-vorticity-pressure formulation is well-posed. This is done in the following theorem.

Theorem 7.4 *Well-posedness of the Stokes problem in the bidimensional case.*

○ *Let Ω be an open bounded connected and simply connected domain with its boundary Γ assumed to be $\mathcal{C}^{1,1}$. Let $(\Gamma_t, \Gamma_\theta)$ be a partition of Γ. We consider the following functional spaces:* $W = \left\{ \varphi \in H(\text{curl}, \text{div}*, \Omega) \ , \ \varphi \times n_{|\Gamma_\theta} = 0 \right\}$, $X = H_0(\text{div}, \Omega)$, $Y = L^2_0(\Omega)$, *and the operators acting on these spaces:* $R : W \longrightarrow X'$; $D : X \longrightarrow Y'$; $J_3 : W \longrightarrow W'$, *where the functional associated with operator* J_3 *is the* $L^2(\Omega)$*-scalar product:*
$\langle J_3\omega, \varphi \rangle_{W',W} = (\omega, \varphi)_0 \quad \forall \varphi \in W$.
We denote by r the Riesz isomorphism from Y' to Y and by δ a constant either equal to 0 or 1.
○ *Then, for all f in $(L^2(\Omega))^2$, the following problem:*

$$\begin{cases} \text{find } (\omega, u, p) \in W \times X \times Y \text{ such that :} & & \\ \langle J_3\omega, \varphi \rangle_{W',W} - \langle R'u, \varphi \rangle_{W',W} & = \ 0 \ , & \forall \ \varphi \in W \\ \langle R\omega, v \rangle_{X',X} - \langle D'(p - \delta r D u), v \rangle_{X',X} & = \ (f, v)_0 \ , & \forall \ v \in X \\ \langle Du, q \rangle_{Y',Y} & = \ 0 \ , & \forall \ q \in Y \end{cases}$$

is well-posed and is exactly the following Stokes problem:

$$\begin{cases} \omega - \text{curl } u & = & 0 & \text{in } \Omega, \\ \text{curl } \omega - \delta \nabla \text{div } u + \nabla p & = & f & \text{in } \Omega, \\ \text{div } u & = & 0 & \text{in } \Omega, \\ u \bullet n & = & 0 & \text{on } \Gamma, \\ u \times n & = & 0 & \text{on } \Gamma_t, \\ \omega & = & 0 & \text{on } \Gamma_\theta. \end{cases}$$

Proof

Thanks to Theorem 6.20, the only point to prove is the W-ellipticity of the $L^2(\Omega)$-norm on $V = \left\{ \varphi \in W \ , \ \langle R\varphi, v \rangle_{X',X} = 0, \ \forall v \in \text{Ker} D \right\}$.
We have proved in Lemma 7.1 that $W \subset M(\Omega)$. By the way, due to the equivalence of the norms of $H(\text{curl}, \text{div}*, \Omega)$ and $M(\Omega)$ (Proposition 7.2), there exists a strictly positive constant C such that:

$$\| \varphi \|_M \ \geq \ C \ \| \varphi \|_{\text{curl,div}*,\Omega} \ = \ C \| \varphi \|_W, \quad \forall \varphi \in W.$$

Moreover, as space V is imbedded in $\mathcal{H}(\Omega)$ (Lemma 7.3), we have:

$$\| \varphi \|_M = \| \varphi \|_0, \quad \forall \varphi \in V,$$

The two above relations lead to:

$$\| \varphi \|_0 \ \geq \ C \ \| \varphi \|_W, \quad \forall \varphi \in V.$$

which is the announced result. ■

7.2 Link with stream function-vorticity formulation

Here again, we restrict ourselves to the two-dimensional case when Ω is connected and simply connected. Then, Γ is connected and every divergence free function can be written as a curl of some stream function. Let us recall that we have seen in Section 2 that stream function-vorticity and vorticity-velocity-pressure formulations have a formal link when boundary conditions are reduced to $u = 0$ on Γ, which means $\Gamma_m = \Gamma_t = \Gamma$. We want here to precise the mathematical link between solutions of both formulations.

∘ The spaces associated with the vorticity-velocity-pressure formulation are: $X = H_0(\text{div}, \Omega)$, $W = H(\text{curl}, \text{div}*, \Omega)$, and $Y = L_0^2(\Omega)$. We note $(\theta, u, p) \in W \times X \times Y$ the solution of the well-posed formulation given in Theorem 7.4. We also introduce the solution (ψ, ω) in $H_0^1(\Omega) \times M(\Omega)$ of the stream function-vorticity Stokes problem. Let us recall that we have proved that the spaces for vorticity, W and $M(\Omega)$, are equal in this case (see Proposition 7.2). Then, the natural questions are to find the link between $\omega \in M(\Omega)$ and $\theta \in W$, on one hand, and on the other hand, the link between $u \in X$ and curl ψ which belongs naturally to $H_0(\text{div}, \Omega)$. The answers are given in the following theorem:

Theorem 7.5 *Let Ω be a connected and simply connected open bounded domain in $\mathbb{R}^2$ whose boundary Γ is supposed to be of class $\mathcal{C}^{1,1}$. Let f belong to $(L^2(\Omega))^2$. Let us recall that the solutions of the stream function-vorticity formulation are ω and ψ:*

$$\left\{ \begin{array}{lll} \omega \in M(\Omega) \text{ and } \psi \in H_0^1(\Omega) & & \\ (\omega, \varphi)_0 + \langle \Delta\varphi, \psi \rangle_{-1,1} & = 0 \;, & \forall \varphi \in M(\Omega) \;, \\ -\langle \Delta\omega, \zeta \rangle_{-1,1} & = (f, \text{curl } \zeta)_0 \;, & \forall \zeta \in H_0^1(\Omega) \;, \end{array} \right. \tag{97}$$

while the solutions of the vorticity-velocity-pressure formulation are θ, u and p such that:

$$\left\{ \begin{array}{lll} \theta \in W = M(\Omega), u \in X = H_0(\text{div}, \Omega) & \text{and} \quad p \in Y = L_0^2(\Omega) & \\ (\theta, \varphi)_0 - \langle u, \mathit{curl}\ \varphi \rangle_{X,X'} & = \quad 0 \;, & \forall \varphi \in W, \\ \langle \mathit{curl}\ \theta, v \rangle_{X',X} - \langle D'(p - \delta r D u), v \rangle_{X',X} & = \quad (f, v)_0 \;, & \forall\, v \in X \\ \langle D u, q \rangle_{Y',Y} & = \quad 0 \;, & \forall q \in Y \;. \end{array} \right. \tag{98}$$

Then, the solutions of these two formulations of the Stokes problem are equivalent in the following sense:

- ∘ *The vorticities ω and θ are equal.*

◦ *The velocities* curl ψ *and* u *are equal.*

Proof

• Taking v in Ker D, the second equation of (98) becomes:

$$\langle \text{curl } \theta, v\rangle_{X',X} = (f, v)_0 \quad , \quad \forall\, v \ \in \text{Ker } D \quad .$$

Then, using (96), we obtain:

$$\langle \text{curl } \theta, v\rangle_{X',X} = (L\rho\theta, v)_0 = (f, v)_0 \quad , \quad \forall\, v \ \in \text{Ker } D \quad .$$

Let us examine now the second equation of (97). Equality (95) leads to:

$$\langle -\Delta\omega, \zeta\rangle_{-1,1} = (L\rho\omega, \text{curl } \zeta)_0 = (f, \text{curl } \zeta)_0 \quad ,$$

for all ζ in $H^1_0(\Omega)$. As any element v of Ker D is a curl, we obtain:

$$(L\rho\omega, v)_0 = (f, v)_0 \quad , \quad \forall\, v \ \in \text{Ker } D \quad .$$

Finally, we obtain:

$$(L\rho\theta, v)_0 = (f, v)_0 = (L\rho\omega, v)_0 \quad , \quad \forall\, v \ \in \text{Ker } D \quad .$$

As $L\rho\theta$ and $L\rho\omega$ belong to Ker D, we deduce that: $L\rho\theta = L\rho\omega$ in Ker D. Then $\omega - \theta$ belongs to space $V = \{\varphi \in W \ , \ L\rho\varphi = 0\}$ (see Lemma 6.4).

• Let us now prove that θ and ω are equal. Then, as the velocity u is divergence free (see the third equation of (98)), the use of relation (96) in the first equation of (98) leads to:

$$(\theta, \varphi)_0 = (u, L\rho\varphi)_0 \quad , \quad \forall \varphi \in W \quad .$$

In a same manner, using equality (95) in the first equation of the stream function-vorticity formulation (97), we obtain:

$$(\omega, \varphi)_0 = (L\rho\varphi, \text{curl } \psi)_0 \quad , \quad \forall \varphi \in W \quad .$$

Then, subtracting the two above equations, we find:

$$(\theta, \varphi)_0 - (\omega, \varphi)_0 = (u, L\rho\varphi)_0 - (L\rho\varphi, \text{curl } \psi)_0 \quad , \quad \forall \varphi \in W \quad ,$$

or else:

$$(u - \text{curl } \psi, L\rho\varphi)_0 = (\theta - \omega, \varphi)_0 \quad , \quad \forall \varphi \in W \quad . \tag{99}$$

If we choose φ in V, we have: $L\rho\varphi = 0$ and the above equality gives:

$$(\theta - \omega, \varphi)_0 = 0 \quad , \quad \forall \varphi \in V \quad .$$

As $\omega \ - \ \theta$ belongs to space V, we can choose $\varphi = \omega \ - \ \theta$ and we obtain: $\omega = \theta$.

• To finish, we study the difference $u - \text{curl } \psi$. Using equality (99) and the previous result $\omega = \theta$, we deduce:

$$(u - \text{curl } \psi, L\rho\varphi)_0 = 0 \quad , \quad \forall \varphi \in W \quad .$$

Taking φ in $\mathcal{D}(\Omega)$, we have $L\rho\varphi = \text{curl } \varphi$ (see Lemma 6.2). Then, we obtain:

$$(u - \text{curl } \psi, \text{curl } \varphi)_0 = 0 \quad , \quad \forall \varphi \in \mathcal{D}(\Omega) \quad .$$

As u belongs to Ker D, there exists ζ in $H^1_0(\Omega)$ such that $u = \text{curl } \zeta$ and we have:

$$(\text{curl } (\psi - \zeta), \text{curl } \varphi)_0 = 0 \quad , \quad \forall \varphi \in \mathcal{D}(\Omega) \quad .$$

Then $\psi = \zeta$ and u is equal to curl ψ, by density of $\mathcal{D}(\Omega)$ in $H^1_0(\Omega)$. ■

8 Conclusion

The vorticity-velocity-pressure formulation for the Stokes problem of incompressible fluids mechanics, proposed in [Dub92], does not give satisfying numerical results for classical Dirichlet boundary condition on velocity as numerically established in [Sal99] and [DSS02b] because the vorticity is searched in $H(\text{curl}, \Omega)$, which is not the appropriate functional space. We have shown in this article that the appropriate space $M(\Omega)$ for the stream function-vorticity formulation ([BGM92]) can be extended into a new functional space $H(\text{curl}, \text{div}*, \Omega)$ that we have defined. Then, we have proposed to extend the vorticity-velocity-pressure formulation with this new vectorial space for the vorticity. A difficulty that arises is to deal with a general "mass operator" J introduced in Section 6.2 in order to write in a variational way the equation $\omega = \text{curl } u$. The results, that we have proved, are summarized below.

○ For a bounded connected domain Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$, with $u \bullet n = 0$ on Γ, we choose a mass operator J defined as $J = J_2$ in Section 6.2. It leads to a well-posed variational problem if Ω is connected and simply connected or if there exists some analytical subset Γ_0 of Γ_t ($\Gamma_t \subset \Gamma$) such that: meas $(\Gamma_0) \ \neq \ 0$. Moreover, the interpretation of the vorticity-velocity-pressure formulation gives the partial differential equations of the Stokes problem, with a **new** boundary condition on Γ_t for the tangential velocity. This new boundary condition is detailed in the following and is defined with non classical objects. First, for ω in the space of vorticity $W \subset H(\text{curl}, \text{div}*, \Omega)$ (defined at relations (58), the co-curl operator $\rho\omega$ (see Section 6.1) is the Riesz representant of the weak rotational operator in the space $X = H_0(\mathrm{div}, \Omega)$ (see Section 4.1). Second, function χ is the variational solution of the following problem:

$$\left\{ \begin{array}{rcl} \Delta\chi & = & \mathrm{div}\ (\rho\omega) \quad \text{in } \Omega \quad , \\ \dfrac{\partial \chi}{\partial n} & = & 0 \quad \text{on } \Gamma \quad , \end{array} \right.$$

When $\omega \in H(\mathrm{curl}, \Omega)$, $\rho\omega = \mathrm{curl}\ \omega$ in a sufficiently weak sense and $\chi \equiv 0$. Nevertheless, function χ is not null for a general vorticity field. Then, on the subset Γ_t of Γ, the new boundary condition take the algebraic form:

$$n \times u \times n = \mathrm{curl}_\Gamma(\gamma\chi).$$

The mechanical interpretation of this condition should be improved in the future.

∘ For a bounded connected domain Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$, with $u \bullet n = 0$ on Γ, if the $(L^2(\Omega))^{2N-3}$ scalar product is elliptic on the kernel V defined at Lemma 6.1 and if Ω is connected and simply connected or if there exists some analytical subset Γ_0 of Γ_t such that: meas $(\Gamma_0) \neq 0$, the vorticity-velocity-pressure formulation is well-posed and is exactly the Stokes problem, with the classical boundary conditions: $u \bullet n = 0$ on Γ; $\omega \times n = 0$ on Γ_θ and $n \times u \times n = 0$ on Γ_t.

∘ In the particular case where Ω is a connected, simply connected, open bounded domain of $\mathbb{R}^2$, we have proved that the $L^2(\Omega)$-scalar product is elliptic on the kernel V: the vorticity-velocity-pressure formulation is well-posed and is exactly the Stokes problem, with the classical boundary conditions. Finally, if $\Gamma_t = \Gamma$, our formulation and the classical stream function-vorticity one give exactly the same fields of vorticity and velocity.

∘ The first next step is to reduce the hypothesis on the regularity of the boundary for the decomposition of vector fields, following the ideas of [ABDG98] and [Dub02]. Then, the second one is to study the discretization strategies in order to extend the HaWAY method to triangles. A third direction is to establish the link between our vorticity-velocity-pressure formulation and the three-dimensional stream function-vorticity one which was proposed by Amara and *al* [AB99], [ABD99].

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Références

[AB99] M. Amara and C. Bernardi. Convergence of a finite element discretization of the Navier-Stokes equations in vorticity and stream function formulation. *M2AN*, 33(5):1033–1056, 1999.

[ABD99] M. Amara, H. Barucq, and M. Duloué. Une formulation mixte convergente pour le système de Stokes tridimensionnel. *C. R. Acad. Sci. Paris, Ser I*, 328:935–938, 1999.

[ABDG98] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional nonsmooth domains. *Mathematical Methods in the Applied Sciences*, 21(9):823–864, 1998.

[Ada75] R.A. Adams. *Sobolev spaces.* Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[ADN59] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions in elliptic partial differential equations satisfying general boundary conditions I. *Comm. Pure and Appl. Math.*, 12:–, 1959.

[Ama97] M. Amara. Personal communication, June 1997.

[Ara66] A. Arakawa. Computational design for long-term numerical integration of the equations of fluid motion. *J. of Computational Physics*, 1:119–143, 1966.

[ASS02] T. Abboud, M. Salaün, and S. Salmon. Integral method for the Stokes problem. *C. R. Acad. Sc. Paris, Ser I*, 334:71–76, 2002.

[Bab71] I. Babuška. Error bounds for finite element method. *Numer. Math.*, 16:322–333, 1971.

[BCMP87] C. Bègue, C. Conca, F. Murat, and O. Pironneau. A nouveau sur les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression. *C. R. Acad. Sc. Paris, Ser I*, 304:23–28, 1987.

[BDG85] A. Bendali, J.M. Dominguez, and S. Gallic. A variational approach for the vector potential formulation of the Stokes and Navier-Stokes problems arising in three dimensional domains. *J. Math. Anal. Appl.*, 107:537–560, 1985.

[BGM92] C. Bernardi, V. Girault, and Y. Maday. Mixed spectral element approximation of the Navier-Stokes equations in stream function and vorticity formulation. *IMA Journal Numer. Anal.*, 12:565–608, 1992.

[Bre74] F. Brezzi. On the existence, uniqueness and approximation of saddle point problems arising from Lagrange multiplyers. *R.A.I.R.O. R2*, pages 129–151, 1974.

[CB68] Y. Choquet-Bruhat. *Géométrie différentielle et systèmes extérieurs*. Dunod, 1968. Paris.

[Cia78] P.G. Ciarlet. *The finite element method for elliptic problems*, volume 4 of *Studies in Mathematics and its applications*. North-Holland, 1978.

[CR74] P. Ciarlet and P.A. Raviart. A mixed finite element method for the biharmonic equation. *Mathematical Aspects of Finite Elements in Partial Differential Equations*, pages 125–145, 1974.

[DD86] F. Dubois and J.M. Dupuy. A three dimensional vector potential formulation for solving transonic flow with mixed finite elements. In 6^{th} *International Symposium on Finite elements methods in flow problems*, Antibes, France, 1986.

[DL72] G. Duvaut and J-L. Lions. *Les inéquations em mécanique et en physique*. Dunod, Paris, 1972.

[DSS00] F. Dubois, M. Salaün, and S. Salmon. Harmoniques discrètes pour le problème de Stokes. *C.R. Acad. Sci. Paris, Ser I*, 331:827–832, 2000.

[DSS01] F. Dubois, M. Salaün, and S. Salmon. Vorticity-velocity-pressure and vorticity stream function formulations for the Stokes problem. *Report IAT/CNAM 353/01*, 2001.

[DSS02a] F. Dubois, M. Salaün, and S. Salmon. Discrete harmonics for stream function-vorticity Stokes problem. *Numer. Math.*, 92(4):711–742, 2002.

[DSS02b] F. Dubois, M. Salaün, and S. Salmon. First vorticity-velocity-pressure numerical scheme for the Stokes problem. *Report IAT/CNAM 356/02, to appear in Computer Methods in Applied Mechanics and Engineering*, 2002.

[Dub90] F. Dubois. Discrete vector potential representation of a divergence-free vector field in three-dimensional domains : numerical analysis of a model problem. *SIAM J. of Numerical Analysis*, 27(5):1103–1141, 1990.

[Dub92] F. Dubois. Une formulation tourbillon-vitesse-pression pour le problème de Stokes. *C. R. Acad. Sc. Paris, Ser I*, 314:277–280, 1992.

[Dub95] F. Dubois. Vorticity-velocity-pressure formulation for the Stokes problem. Venise (Italie), 1995. Ninth International Conference on Finite Elements in Fluids.

[Dub02] F. Dubois. Vorticity-velocity-pressure formulation for the Stokes problem. *Math. Meth. in the Applied Sciences*, 25:1091–1119, 2002.

[FG97] P. Fernandes and G. Gilardi. Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. *Mathematical Models and Methods in Applied Sciences*, 7(7):957–991, 1997.

[FT78] C. Foias and R. Temam. Remarques sur les équations de Navier-Stokes stationnaires et les phénomèmes successifs de bifurcation. *Ann. Scuola Normale Sup. Pisa*, 5(1):29–63, 1978.

[Gir76] V. Girault. A combined finite element and Marker and Cell method for solving Navier-Stokes equations. *Numerische Mathematik*, 26(183):39–59, 1976.

[Gir88] V. Girault. Incompressible finite element methods for Navier-Stokes equations with nonstandard boundary conditions in $\mathbb{R}^3$. *Mathematics of Computation*, 51(183):55–74, 1988.

[GL96] V. Girault and H. Lopez. Finite element error estimates for the MAC scheme. *IMA Journal Numer. Anal.*, 16:347–379, 1996.

[Glo73] R. Glowinski. Approximation externes par éléments finis de Lagrange d'ordre un et deux du problème de Dirichlet pour l'opérateur biharmonique et méthode itérative de résolution des problèmes approchés. *Topics in Numerical Analysis*, pages 123–171, 1973.

[GP79] R. Glowinski and O. Pironneau. Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem. *SIAM*, 21(2):167–211, 1979.

[GR86] V. Girault and P-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[HHS83] R.P. Harper, C.W. Hirt, and J.M. Sicilian. Flow2d : a computer program for transient, two-dimensional flow analysis. *Flow Science Inc.*, FSI-83-00-01, 1983.

[HW65] F. H. Harlow and J. E. Welch. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *Physics of Fluids*, 8:2182–2189, 1965.

[Lan59] E.M. Landis. Sur certaines propriétés qualitatives dans la théorie des équations elliptiques et paraboliques. *Uspehi Mat. Nauk.*, 14:21–85, 1959.

[Ler34] J. Leray. Sur les mouvements d'un liquide visqueux remplissant l'espace. *Acta Math.*, 63:193–248, 1934.

[Lev91] F. Levillain. Couplage éléments finis-équations intégrales pour la résolution des équations de maxwell en milieu hétérogènes. Phd thesis, Ecole Polytechnique, 1991.

[Lio69] J. L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1969.

[LL71] L. Landau and E. Lifchitz. *Mécanique des fluides*, volume VI of *Physique théorique*. Editions Mir, 1971.

[LM54] P. Lax and N. Milgram. Parabolic equations, in contributions to the theory of partial differential equations. In *Annals of mathematical studies*, 1954.

[LM68] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications.Vol. I.* Dunod, Paris, 1968.

[LU68] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and quasilinear elliptic equations*. Academic Press, New York, 1968.

[Néd80] J.C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. *Numerische Mathematik*, 35:315–341, 1980.

[Nic89] R.A. Nicolaides. Direct discretization of planar div-curl problems. ICASE Report 89-76, NASA, 1989.

[PS72] S.W. Patankar and D.B. Spalding. A calculation procedure for heat, mass and momentum transfer in three-dimensional parabolic flows. *Int. J. Heat Mass Transfer*, 15:1787–1806, 1972.

[Rou84] F.X. Roux. Méthode d'éléments finis mixtes et hybrides. Seminar, Paris 6 University, 1984.

[RT77] P.-A. Raviart and J.-M. Thomas. A mixed finite element method for 2-nd order elliptic problems. *Lectures Notes in Mathematics*, 306:292–315, 1977.

[Rua95a] V. Ruas. Approximating harmonic bases for a decoupled solution of viscous flow problems in $\psi - \omega$ formulation. *Z. Angew. Math. Mech.*, 75(5):407–408, 1995.

[Rua95b] V. Ruas. On the velocity-vorticity formulation of the Stokes system in two- and three-dimension spaces. *Mechanics Research Communications*, 22(5):511–517, 1995.

[Sal99] S. Salmon. *Développement numérique de la formulation tourbillon-vitesse-pression pour le problème de Stokes*. PhD thesis, Université Paris VI, December 1999.

[Tem77] R. Temam. *Theory and numerical analysis of the Navier-Stokes equations*. North Holland, Amsterdam, 1977.

[Web80] C. Weber. A local compactness theorem for Maxwell's equations. *Math. Meth. in the Appl. Sci.*, 2:12–25, 1980.

[Yee66] K.S. Yee. Numerical solution of initial boundary value problems in isotropic media. *I.E.E.E. Trans. antennas propagation*, 1966.