A THREE-DIMENSIONAL VECTOR POTENTIAL FORMULATION FOR SOLVING TRANSONIC FLOWS WITH MIXED FINITE ELEMENTS

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We present in this paper a three dimensional extension of the work done by AMARA. We firstly recall the physical model. The iterative numerical algorithm of resolution introduces an elliptic problem which a variational formulation involving a vector potential is given in section 2. Furthermore we describe its approximation using the NEDELEC's element conforming in H (curl). Finally preliminary numerical experiment is presented in a subsonic case.

1. Potential transonic flow.

We study an inviscid fluid in stationary irrotational and isentropic evolution. With those assumptions it is known [1] that only the two physical fields density ρ and velocity $\vec{u}$ are necessary to its description. The corresponding partial differential equations are

mass conservation	(1.1)	$\mathrm{div}\ (\rho\ \vec{u}) = 0$
no vorticity	(1.2)	$\mathrm{curl}\ \vec{u} = 0$

furthermore the Bernoulli theorem can be written :

$$\rho = \{1 + \frac{\gamma - 1}{2} M_\infty^2 (1 - |\vec{u}|^2)\}^{1/\gamma-1} \tag{1.3}$$

where M_∞ is the upstream Mach number and γ the specific heat ratio. This problem is non-linear, and the type of the equations is mixed : elliptic in the subsonic zone and hyperbolic in the supersonic one. Moreover an entropy condition has to be added to select the physical shock waves [2,3] .

The domain Ω is discretized with a triangulation T_h of finite elements (we used tetrahedrons ans prisms) and the density ρ is taken to be constant in each element. The set of equations (1.1), (1.2), (1.3) is solved numerically with a fixed point technique and an artificial density [4,5] .

2. Formulation of the linear problem.

In all this section we investigate the linear problem for the volocity ; so the density ρ is a fixed positive function in $L^\infty(\Omega)$.

$$(P) \quad \begin{cases} (2.1) & \mathrm{div}\ (\rho\ \vec{u}) = 0 & \Omega \\ (2.2) & \mathrm{curl}\ \vec{u} = 0 & \Omega \\ (2.3) & \rho\ \vec{u}\ .\ \vec{n} = g & \Gamma \end{cases}$$

where $\vec{n}$ is the exterior normal of the boundary Γ. Since we restrict our study to the case where both Ω and Γ are connected domains, the equation (2.1) gives the existence of a vector potential ψ [6] such that

$$\rho\ \vec{u} - \mathrm{curl}\ \psi = 0 \qquad \Omega \tag{2.4}$$

The condition (2.3) is then equivalent to

$$\mathrm{div}_\Gamma\ (\psi \times \vec{n}) = g \tag{2.5}$$

where div_Γ is the Laplace-Beltrami divergence operator [7]. Note that the conditions (2.4), (2.5) do not insure the uniqueness of a potential ψ ; it is therefore necessary to add gauge conditions [8,9] like

$$(2.6) \qquad \operatorname{div} \psi = 0 \qquad \Omega$$

$$(2.7) \qquad \operatorname{curl}_\Gamma (\psi \times \vec{n}) = 0 \qquad \Gamma$$

• Let us introduce now some functional spaces :

$$H(\text{curl}) = \{ \chi \in L^2(\Omega),\ \text{curl}\ \chi \in L^2(\Omega) \}$$

$$H^0(\text{curl}) = \{ \chi \in H(\text{curl}),\ \chi \times \vec{n} = 0 \text{ on } \Gamma \}$$

$$V = \{ \chi \in H^0(\text{curl}),\ \operatorname{div} \chi = 0 \}$$

$$U = \{ u \in H(\operatorname{div}),\ \operatorname{div} u = 0,\ u \cdot \vec{n} = 0 \text{ on } \Gamma \}$$

we know that there is some vector field $\alpha \in H^1(\Omega)$ satisfying (2.8) $\operatorname{div}_\Gamma(\alpha \times \vec{n}) = g$ and the problem (P) is equivalent to the variational problem

$$(Q) \quad \begin{cases} \text{find } \phi \in V \\ (2.10)\ \forall \chi \in V,\ \displaystyle\int_\Omega \frac{1}{\rho} \text{curl}\ \phi \cdot \text{curl}\ \chi\, dx = - \int_\Omega \frac{1}{\rho} \text{curl}\ \alpha \cdot \text{curl}\ \chi\, dx . \end{cases}$$

which admits a unique solution since Poincaré inequality holds in V [6] .

3. Conformial element in H (curl).

As the continuous problem (Q) is well posed in a subspace of H (curl), a natural idea is to discretize this vector space. NEDELEC [10] proposed the following finite element (K, P_K, Σ_K) [fig.1] : K is a tetrahedron,

$$P_K = \{ \varphi,\ \exists\ \alpha, \beta \in \mathbb{R}^3 ,\ \forall x \in K ,\ \varphi(x) = \alpha + \beta \times x \}$$

$$\Sigma_K = \{ \varphi \cdot \tau ,\ \tau \text{ is an edge of } K \} .$$

Let us introduce also the finite dimensional spaces :

$$X_h = \{ \varphi_h \in H(\text{curl}),\ \forall K \in \mathcal{T}_h ,\ \varphi_{h|_K} \in P_K \}$$

$$X_h^0 = \{ \varphi_h \in X_h ,\ \varphi_h \times \vec{n} = 0 \text{ on } \Gamma \}$$

$$U_h = \{ \text{curl}\ \varphi_h ,\ \varphi_h \in X_h^0 \}$$

$$H^1_{0,h} = \{ v_h \in H^1_0,\ v_h \text{ polynomial of degree 1 in each } K \in \mathcal{T}_h \}$$

This was extended to prisms.

• We now focus on the boundary condition (2.8). In the continuous case the existence of α satisfying (2.8) is given by a Laplace-Beltrami problem on the manifold Γ. We discretize the boundary condition (2.8) in X_h by the relations

$$(3.1) \qquad \sum_{\tau \in T} \alpha_h \cdot \tau = g_h(T) \qquad \forall T \text{ triangle of } \Gamma$$

where $g_h(T)$ is constant. We just need a procedure to compute explicitly such an α_h : the edges of the triangulation lying on Γ allow to define a graph g [11] in the set of all the vertices of Γ. Let A be a tree in this graph g [fig.2] . We choose the space $W_h(\Gamma)$ containing α_h as follow :

(3.2) $\qquad W_h(\Gamma) = \{ \varphi \in X_h \ , \ \varphi \cdot \tau = 0 \ \text{ if } \tau \in \Gamma \cap A \ \text{ or } \tau \notin \Gamma \}$

and if $\int_\Gamma g_h \, d\gamma = 0$ then (3.1) admits a unique solution $\alpha_h \in W_h(\Gamma)$.

- Assume now that α_h is computed, we solve numerically the problem

$$(Q_h) \quad \begin{cases} \text{find } \phi_h \in V_h \\ \forall \chi_h \in V_h \ , \ \int_\Omega \frac{1}{\rho} \operatorname{curl}\phi_h \cdot \operatorname{curl}\chi_h \, dx = - \int_\Omega \frac{1}{\rho} \operatorname{curl}\alpha_h \cdot \operatorname{curl}\chi_h \, dx \end{cases}$$

we have to <u>choose</u> the space V_h, $V_h \subset X_h^0$, in order to adapt the gauge condition (2.6) in a finite dimensional space. The constraint on V_h is to have :

(3.3) $\qquad$ the application curl : $V_h \to U_h$ is one to one

(like in the continuous case for curl : $V \to U$). Since the kernel of curl : $X_h^0 \to U_h$ is not null, V_h must be a <u>strict</u> subspace of X_h^0. Then if V_h is any subspace of X_h^0 satisfying (3.3), the solution ϕ (resp ϕ_h) of (Q) (resp (Q_h)) are such that the error $\| \operatorname{curl} \phi - \operatorname{curl} \phi_h \| L^2$ vanishes like h. Note that the fields ϕ and ϕ_h have no physical interpretation, furthermore it does not exist a priori any simple mathematical relation between them. NEDELEC in [12] proposed to use

$$\tilde{V}_h = \{ \varphi_h \in X_h^0 \ , \ \int_\Omega \varphi_h \cdot \nabla\theta_h \, dx = 0 \qquad \forall \theta_h \in H^1_{0,h} \}$$

A basis of $\tilde{V}_h$ is non local in space, and difficult to compute. For this reason we choose V_h in a purely algebrical way, following [13] and including references in an other context.

The space X_h^0 is generated by the edges of $\mathcal{T}_h$ which are <u>not</u> lying in Γ. Consider the graph g' of the edges coupling two vertices of $\mathcal{T}_h$ which are not on Γ (g' is a strict subset of all the edges of X_h^0), and A' a tree in this graph. Furthermore let τ' be an edge joining up one vertex of Γ and one in the interior of Ω. Then V_h, defined by

$$V_h = \{ \varphi_h \in X_h^0 \ , \ \varphi_h \cdot \tau = 0 \ \text{ if } \tau \in A' \text{ or } \tau = \tau' \}$$

satisfies the condition (3.3) [fig.3] .

4. Implementation.

The method described in paragraph 3 is beeing developped on the CRAY 1-S of CCVR. First results have been obtained in a subsonic case and compared with a 2D computation to validate the code [14] : from the channel of GAMM 79 workshop

[15] a 3D mesh using prisms has been generated [fig.4] (71 × 20 × 4 elements, 6879 unknowns). The linear problem is solved by a preconditioning conjugate gradient algorithm. Several preconditioning matrices have been tested. A good and cheap choise seems to be the SSOR method. The figure [fig.5] shows isomach curves on the top and the bottom of the channel for both 2D and 3D computations.

• Références.

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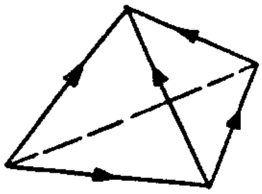

Figure 1

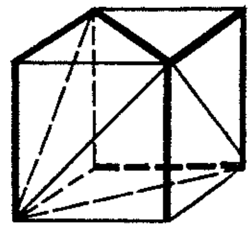

Figure 2 Example of a boundary tree if the domain is a cube divided in five tetrahedrons

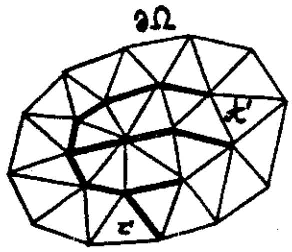

Figure 3

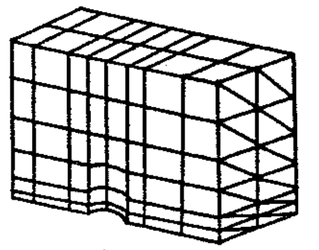

Figure 4

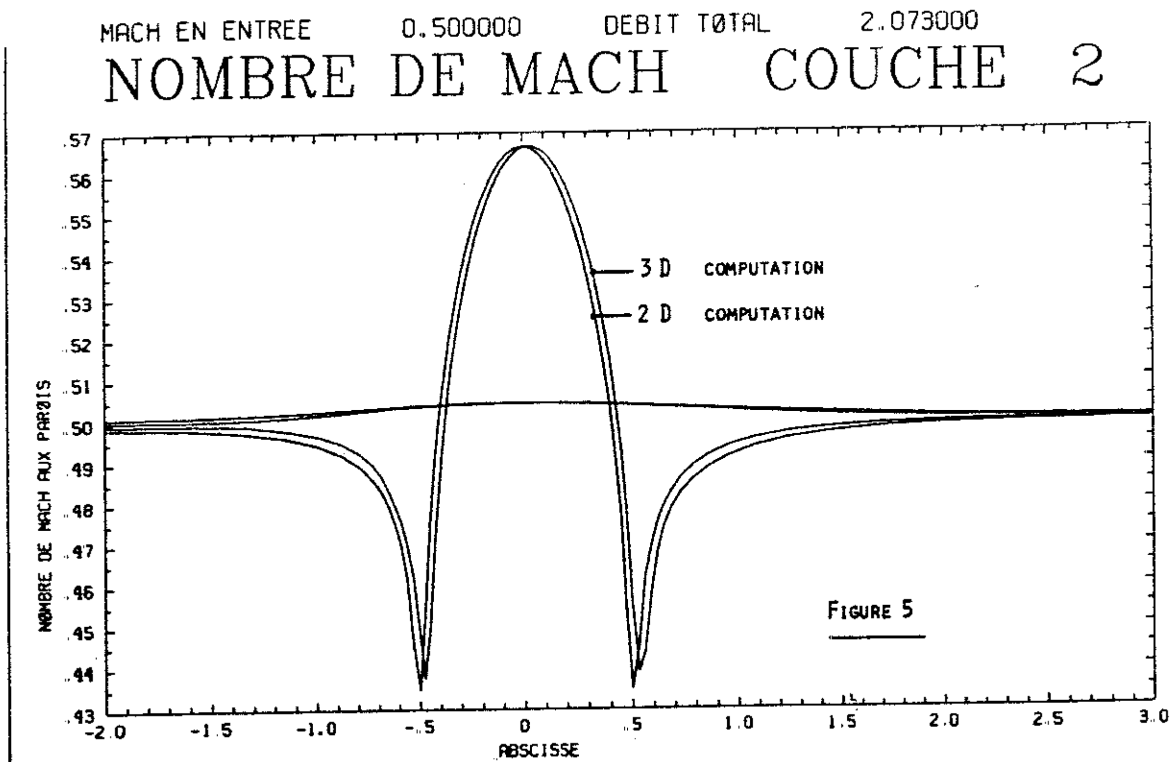

FIGURE 5