From algebraic topology to software reliability in Electromagnetism: The case of Smith normal forms

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Abstract. A complex of linear operators A_i between rungs of a scale of linear spaces V_i, satisfying $A_{i+1} o A_i = 0$, is in "Smith normal form" when bases in each V_i are chosen in such a way that the matrix representative of each A_i reduces to a unit matrix bloc flanked with three zero blocks. Reducing to Smith form is a useful tool in homology, where the A_i's stand for the boundary operators of all dimensions, for it allows one to read off topological invariants such as Betti numbers, etc.

The aim of this introductory talk is to explain the relevance of all this to computational electromagnetics: Smith forms are used, there, to build special finite element bases as required when computational domains have non-trivial topology (presence of "loops" and/or "holes" in the 3D case). They also, we shall suggest, provide a way to check on mesh-generators, thus improving software reliability.

In more detail: Modern codes in electromagnetics use tens of millions of mesh elements, whose spatial arrangement is described by so-called "incidence matrices", the above A_i's. Generating meshes of such size is a complex process, which mixes automatic procedures with users' interventions, hence the unintended introduction of "defects", such as (unintended) holes and loops. As the size of numerical models increases, such mesh unreliablility is becoming a major industrial problem. Algebraic analysis of the complex of incidence matrices, by extracting the topological information they encode, can reveal these defects. [The relevance of algebraic topology, there, is not so surprising if one thinks, by analogy, of what happened in crystallography, where "defects" of a different kind (but not so different, after all) are associated with topological properties.]