Un estimateur *a posteriori* asymptotiquement exacte

Pour une méthode de Volumes Finis et la méthode E.F. P1-conforme.

R. LUCE*, B. WOHLMUTH°

*Laboratoire de Mathématiques Appliqués. Université de PAU.

°Institute of Applied Analysis and Numerical Simulations. University of Stuttgart

(*A local a posteriori error estimator based on equilibrated fluxes*. A paraître dans SINUM)

Problème elliptique du second ordre

Ω est un ouvert borné de $\mathbb{R}^2$

$$\begin{cases} div\ \mathbf{u} + f = 0 \text{ dans } \Omega & \textit{relation fondamentale } (\text{"}\textit{équilibre}\text{"}, \text{"}\textit{continuité}\text{"} \ldots \\ \mathbf{u} = A\ \mathbf{grad}\ p \text{ dans } \Omega & \textit{loi constitutive } (\textit{Darcy}) \\ + \text{ condition aux limites} & \end{cases}$$

$$\begin{cases} -div(A\ \mathbf{grad}\ p) = f \\ + \text{ condition aux limites} \end{cases}$$

Conditions aux limites: Dirichlet homogène

Volumes finis

E.F. (P1-conforme)

Formulation faible
(type Petrov-Galerkin)

Trouver $p \in H_0^1(\Omega)$ et $\mathbf{u} \in H(div, \Omega)$

$$\begin{cases} \int_\Omega q \, div \, \mathbf{u} \, dx = -\int_\Omega f \, q \, dx \quad \forall q \in L^2(\Omega) \\ \int_\Omega \mathbf{u} . \mathbf{v} \, dx = \int_\Omega A \, \mathbf{grad} \, p . \mathbf{v} \, dx \quad \forall \mathbf{v} \in \left(L^2(\Omega)\right)^2 \end{cases}$$

Formulation variationnelle

Trouver $p \in H_0^1(\Omega)$)

$$\int_\Omega A \, \mathbf{grad} \, p \, \mathbf{grad} \, \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \phi \in H_0^1(\Omega)$$

Problèmes approchés

V.F Trouver $p_h \in P_h \subset H_0^1(\Omega)$ et $\mathbf{u}_h \in U_h \subset H(div, \Omega)$

$$\begin{cases} \int_\Omega q_h \, div \, \mathbf{u}_h \, dx = -\int_\Omega f \, q_h \, dx & \forall q_h \in Q_h \subset L^2(\Omega) \\ \int_\Omega \mathbf{u}_h \, . \, \mathbf{v}_h \, dx = \int_\Omega A \, \mathbf{grad} \, p_h \, . \, \mathbf{v}_h \, dx & \forall \mathbf{v}_h \in V_h \subset (L^2(\Omega)^2) \end{cases}$$

E.F Trouver $p_h \in P_h \subset H_0^1(\Omega)$

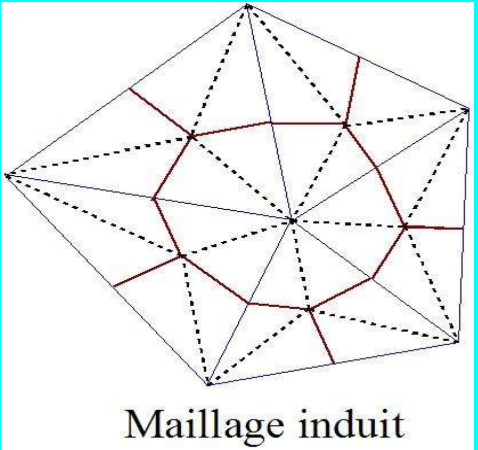
$$\int_\Omega A \, \mathbf{grad} \, p_h \, \mathbf{grad} \, \phi_h dx = \int_\Omega f \phi_h dx \quad \forall \phi_h \in P_h$$

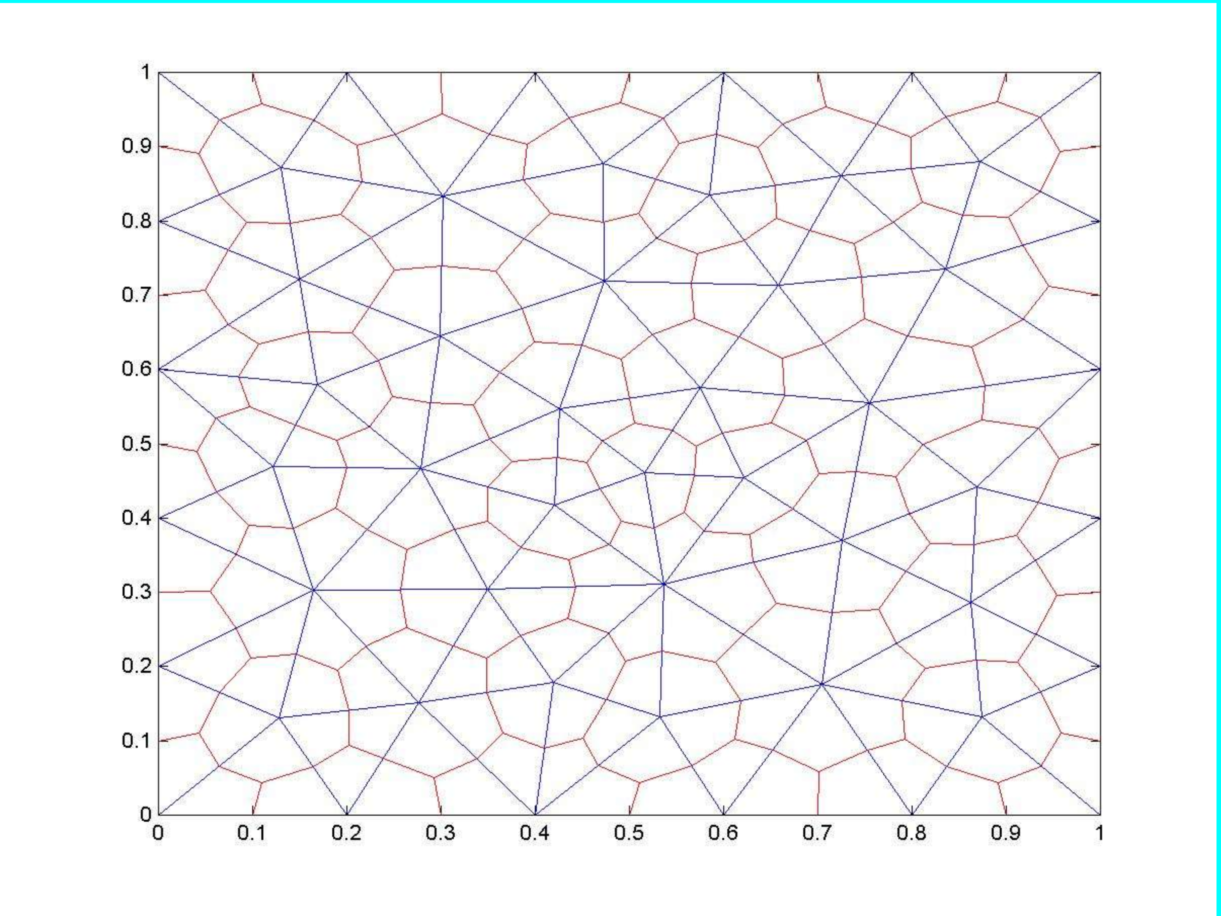
Maillages

$\kappa_h(K)$

$\tau_h(T)$

$S_h(t)$





$\varepsilon_h(\hat{e})$ arêtes rouges

Les espaces de discrétisation

$$\overline{\Omega} = \bigcup_{T \in \tau_h} T \qquad \overline{\Omega} = \bigcup_{T \in \kappa_h} K$$

$$P_h = \left\{ p_h \in H_0^1(\Omega);\, p_h\big|_T \in P_1(T),\, T \in \tau_h \right\},$$

$$Q_h = \left\{ q_h \in L^2(\Omega);\, q_h\big|_T \in P_0(K),\, K \in \kappa_h \right\}$$

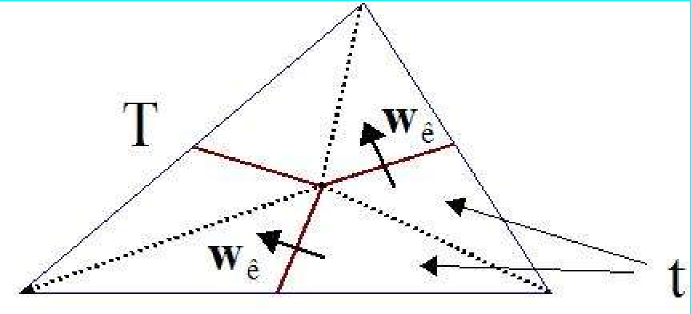
$$dim(P_h) = dim(Q_h)$$

$$RT_h = \left\{ \mathbf{u} \in H(div;\Omega), \mathbf{u}\big|_t \in RT_0(t), t \in S_h \right\}$$

$$U_h = \left\{ \mathbf{u}_h \in RT_h ; \mathrm{div}\, \mathbf{u}_h \in Q_h \right\}$$

$$D(K) = \left\{ \mathbf{u} \in H(div, K); div\, \mathbf{u} \in P_0(K) \; et \; \mathbf{u}\big|_t \in RT_0(t), t \in S_h \right\}$$

$D_0(K) = \left\{ \mathbf{u} \in D(K); \mathbf{u}\,.\,\mathbf{n} = 0 \right\}$ est un sous espace de dim $= 1$



$$\mathbf{w}_K = \beta\, rot \Psi_K$$

avec Ψ_K une fonction de base P_1 sur S_h

$$V_h = vect(\mathbf{w}_{\hat{e}}, \hat{e} \in \boldsymbol{\varepsilon}_h) \oplus vect(\mathbf{w}_K ; K \in \kappa_h)$$

$$dim(U_h) = dim(V_h)$$

Schéma volumes finis

(1) $\displaystyle\int_{\partial K} \mathbf{u}_h \,.\, \mathbf{n}\, d\sigma = -\int_K f\, dx \quad \forall K \in \kappa_h$

(2) $\displaystyle\int_\Omega \mathbf{u}_h \,.\, \mathbf{w}_{\hat{e}}\, dx - \int_\Omega A\,\mathbf{grad}\, p_h \,.\, \mathbf{w}_{\hat{e}}\, dx = 0 \;\; \forall \hat{e} \in \boldsymbol{\varepsilon}_h$

(3) $\displaystyle\int_\Omega \mathbf{u}_h \,.\, \mathbf{w}_K\, dx - \int_\Omega A\,\mathbf{grad}\, p_h \,.\, \mathbf{w}_K\, dx = 0 \;\; \forall K \in \kappa_h$

Intégration approchée

$$\int_\Omega \mathbf{u}_h \,.\, \mathbf{w}_{\hat{e}}\, dx \approx Aire(\mathrm{supp}\,\mathbf{w}_e)\, \mathbf{u}_h\big|_{\hat{e}} \,.\, \mathbf{w}_{\hat{e}}$$

$$(2) \to (2)_h \qquad \mathbf{u_h} \,.\, \mathbf{n}_{\hat{e}} = A\,\mathbf{grad}\, p_h \,.\, \mathbf{n}_{\hat{e}}$$

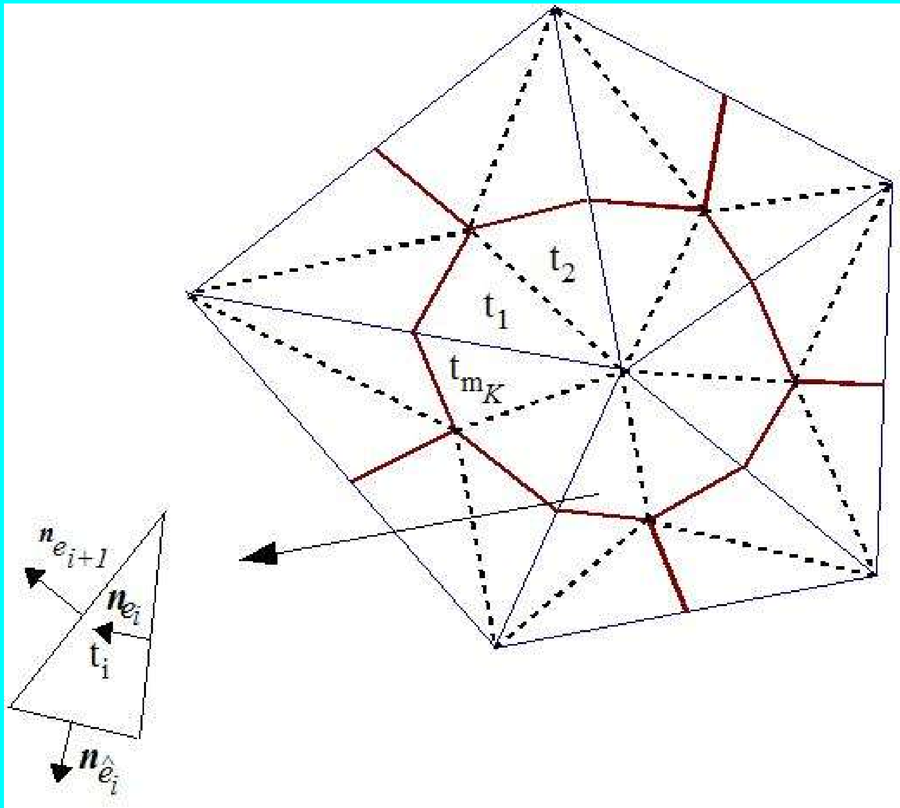
En reportant $(2)_h$ dans (1), on obtient un schéma volumes finis qui fournit une solution p_h dans H^1.

Un post traitement local donne $\mathbf{u}_h$ dans H(div,$\boldsymbol{\Omega}$) vérifiant

$$div\, \mathbf{u}_h + f_h = 0 \quad \text{dans } \boldsymbol{\Omega}$$

$$\phi_{\hat{e}_i} = \mathbf{u}_h . \mathbf{n}_{\hat{e}_i} = A\, \mathbf{grad}\, p_h . \mathbf{n}_{\hat{e}_i} ,$$

$$\phi_{e_i} = \mathbf{u}_h . \mathbf{n}_{e_i}$$



$$div\, \mathbf{u}_h = f_h \Rightarrow \phi_{e_{i+1}} - \phi_{e_i} = |t_i| f_h - \phi_{\hat{e}_i} = b_i$$

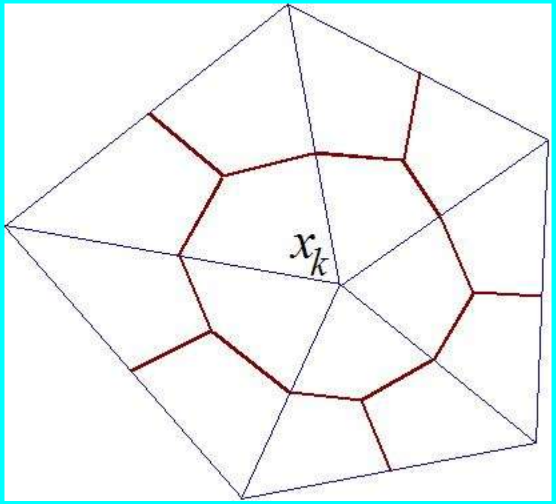
$$(3) \Rightarrow \sum_{i=1}^{m_K} \alpha_i \phi_{e_i} = g_K$$

$$\begin{pmatrix} 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & \ddots & \ddots & & \\ & & 0 & 1 & -1 \\ \alpha_1 & \dots & & \dots & \alpha_{m_K} \end{pmatrix} \begin{pmatrix} \phi_{e_1} \\ \vdots \\ \\ \\ \phi_{e_k} \end{pmatrix} = \begin{pmatrix} b_i \\ \\ \\ b_{m_K-1} \\ g_K \end{pmatrix}$$

$$\Pi_Q v\big|_K := \frac{1}{|K|}\int_K v\,dx \qquad v \in L^2(\Omega)$$

$$\Pi_w v\big|_t := \frac{1}{|t|}\int_t v\,dx \qquad v \in L^2(\Omega)$$

$$I_Q v\big|_K := v(x_K) \qquad v \in C(\Omega)$$



$$P_Q v\big|_K = \frac{1}{|K|}\int_\Omega v\phi_{x_K}\,dx \qquad v \in L^2(\Omega)$$

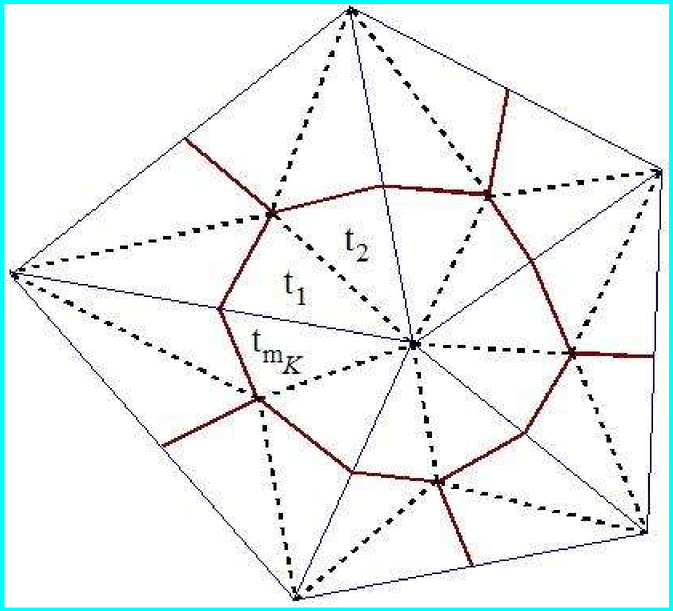
$$\left\|v - P_Q v\right\|_{0;K} \leq C h_K |v|_{1,\omega_K} \qquad v \in H^1(\Omega)$$

$$|||\mathbf{v}|||_{0,\Omega}^2 = \int_\Omega A^{-1}\mathbf{v}.\mathbf{v}\,dx$$

Estimateur *a posteriori*

$$\eta_K^2 := \left\| (\mathbf{u}_h - A\,\mathbf{grad}\, p_h) \right\|_{0;K}^2 = \sum_{t_K \in K \cap S_h} \left\| (\mathbf{u}_h - A\,\mathbf{grad}\, p_h) \right\|_{0;t_K}^2$$

$$\eta^2 = \sum_K \eta_K^2$$



Volumes finis

$$div\mathbf{u}_h = -\Pi_Q f$$

$$\|e_h\| = \left(\|u - u_h\|_{0,\Omega}^2 + \|A\,\mathbf{grad}(p - p_h)\|_{0,\Omega}^2 \right)^{1/2}$$

$$\|e_h\|^2 = \eta^2 + 2\int_\Omega \left(f - \Pi_Q f \right)\left(p - p_h \right) dx$$

$$\|e_h\| \leq \eta + C\xi$$

$$\xi^2 = \sum_{K \in \kappa_h} \frac{h_K^2}{a_K} \|f - \Pi_Q f\|_{0;K}^2 + \sum_{t \in S_h} \frac{h_t^2}{a_t} \|f - \Pi_W f\|_{0;t}^2$$

P_1- conforme

$$div\mathbf{u}_h = -P_Q f$$

$$\|e_h\| = \|A\,\mathbf{grad}(p - p_h)\|_{0,\Omega}$$

$$\|e_h\|^2 \leq \eta \|e_h\| + \int_\Omega \left(f - P_Q f \right)\left(p - p_h \right) dx$$

$$\|e_h\| \leq \eta + C\xi$$

$$\xi^2 = \sum_{K \in \kappa_h} \frac{h_K^2}{a_K} \|f - P_Q f\|_{0;K}^2 + \sum_{t \in S_h} \frac{h_t^2}{a_t} \|f - \Pi_W f\|_{0;t}^2$$

ERREUR GLOBALE

$$u_h^p = \frac{A\,\mathbf{grad}\,p_h + u_h}{2}$$

- $\forall \varepsilon > 0$

$$\frac{1}{4}(1-\varepsilon)\eta^2 - C(1+\frac{1}{\varepsilon})\xi^2 \leq \left\| A\,\mathbf{grad}\,p - u_h^p \right\|_0^2 \leq \frac{1}{4}(1+\varepsilon)\eta^2 - C(1+\frac{1}{\varepsilon})\xi^2$$

$$(1-\varepsilon)\eta^2 - C(1+\frac{1}{\varepsilon})\xi^2 \leq \left\| u - u_h \right\|_{0,\Omega}^2 + \left\| A\,\mathbf{grad}(p - p_h) \right\|_{0,\Omega}^2 \leq (1+\varepsilon)\eta^2 + C(1+\frac{1}{\varepsilon})\xi^2$$

- $f \in H^s(\Omega) \quad (0 < s \leq 1)$

$$1 - O(h^s) \leq \frac{4\left\| A\,\mathbf{grad}\,p - u_h^p \right\|_{0;\Omega}^2}{\eta^2} \leq 1 + O(h^s)$$

$$1 - O(h^s) \leq \frac{\left\| u - u_h \right\|_{0;\Omega}^2 + \left\| A\,\mathbf{grad}(p - p_h) \right\|_{0;\Omega}^2}{\eta^2} \leq 1 + O(h^s)$$

Estimateurs locaux

Equivalence avec les sauts des flux discrets

$$c\sum_{e\in\varepsilon_K}\frac{h_e}{a_e}\left\|\left[A\,\mathbf{grad}\,p_h.n_e\right]\right\|_{0;e}^2\leq\eta_K^2\leq C\sum_{e\in\varepsilon_K}\frac{h_e}{a_e}\left\|\left[A\,\mathbf{grad}\,p_h.n_e\right]\right\|_{0;e}^2$$

$$c\eta_K^2\leq\frac{h_K^2}{a_K}\left\|\Pi_Q f\right\|_{0;K}^2+\sum_{e\in\varepsilon_K}\frac{h_e}{a_e}\left\|\left[A\,\mathbf{grad}\,p_h.n_e\right]\right\|_{0;e}^2\leq C\eta_K^2$$ V.F

E.F $$c\eta_K^2\leq\frac{h_K^2}{a_K}\left\|P_Q f\right\|_{0;K}^2+\sum_{e\in\varepsilon_K}\frac{h_e}{a_e}\left\|\left[A\,\mathbf{grad}\,p_h.n_e\right]\right\|_{0;e}^2\leq C\eta_K^2$$

On montre que

$$\eta_K^2\equiv\sum_{e\in\varepsilon_K}\frac{h_e}{a_e}\left(\left\|\left[A\,\mathbf{grad}\,p_h.n_e\right]\right\|_{0;e}^2+\left\|\left\{\mathbf{u}_h-A\,\mathbf{grad}\,p_h\right\}.n_e\right\|_{0;e}^2\right)$$

$$\sum_{e\in\varepsilon_K}\frac{h_e}{a_e}\left\|\left\{\mathbf{u}_h-A\,\mathbf{grad}\,p_h\right\}.n_e\right\|_{0;e}^2\leq C\sum_{e\in\varepsilon_K}\frac{h_e}{a_e}\left\|\left[A\,\mathbf{grad}\,p_h.n_e\right]\right\|_{0;e}^2$$

V.F. $$\eta_K \leq C\left(\|e_h\|_K + \frac{h_K^2}{a_K} \|f - \Pi_Q f\|_{0;K}^2 \right)$$

E.F. $$\eta_K \leq C\left(\|e_h\|_K + \frac{h_K^2}{a_K} \|f - P_Q f\|_{0;K}^2 \right)$$

Quel type d'estimateur a-t-on?

TESTS NUMERIQUES

$$e_p = \left\| A\,\mathbf{grad}(p - p_h) \right\|_{0,\Omega}^2 \qquad \sigma_p = \frac{\sqrt{e_p}}{\eta}$$

$$e_u = \left\| A\,\mathbf{grad}\,p - u_h^p \right\|_{0,\Omega}^2 \qquad \sigma_p = \frac{2\sqrt{e_u}}{\eta}$$

$$e_{up} = \left\| u - u_h \right\|_{0,\Omega}^2 + \left\| A\,\mathbf{grad}(p - p_h) \right\|_{0,\Omega}^2 \qquad \sigma_{up} = \frac{\sqrt{e_{up}}}{\eta}$$

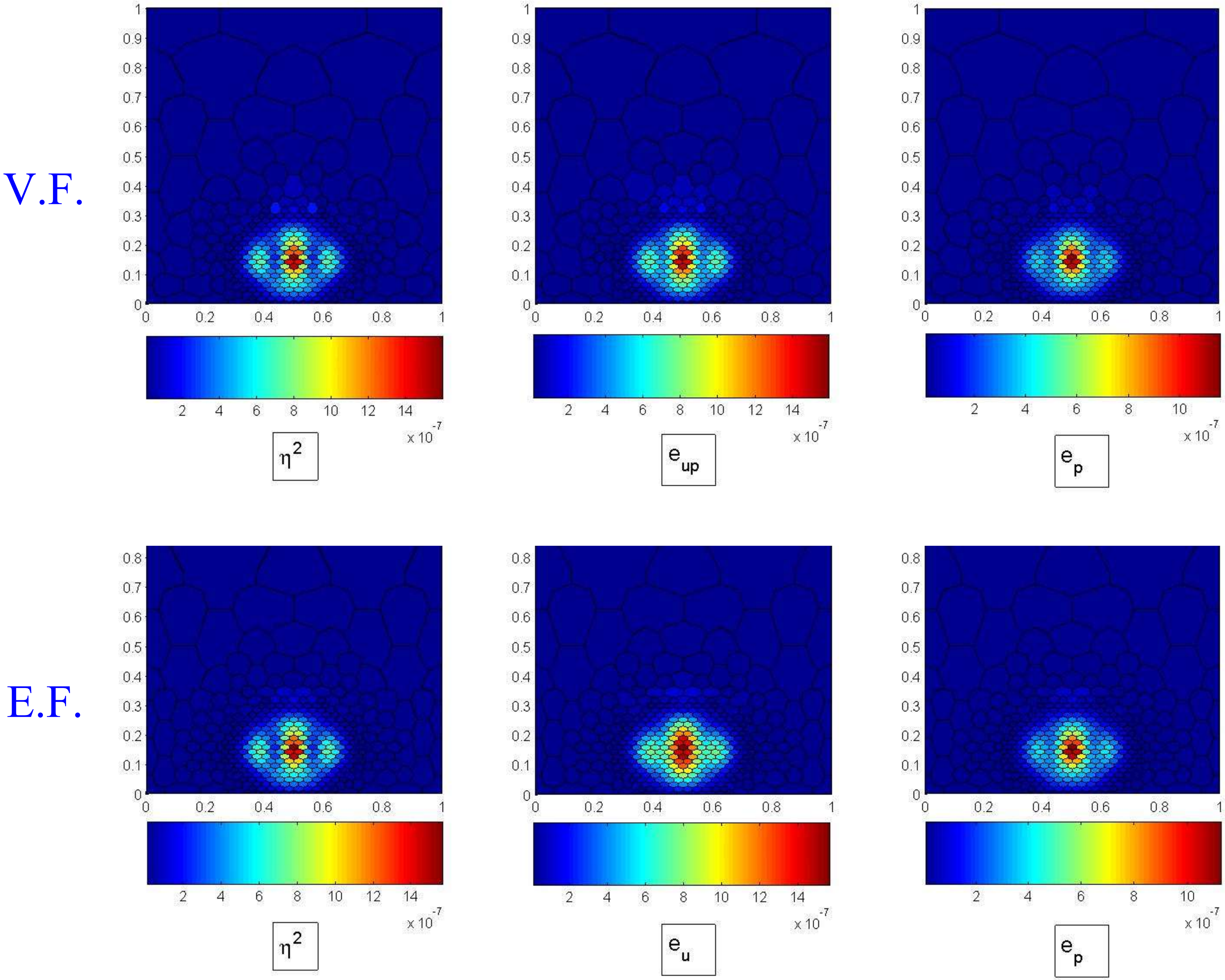
$$p(x,y) = x(x-1)y(y-1)\,\mathbf{exp}(-100(x-\frac{1}{2})^2 - 100(y-\frac{117}{1000})^2)$$

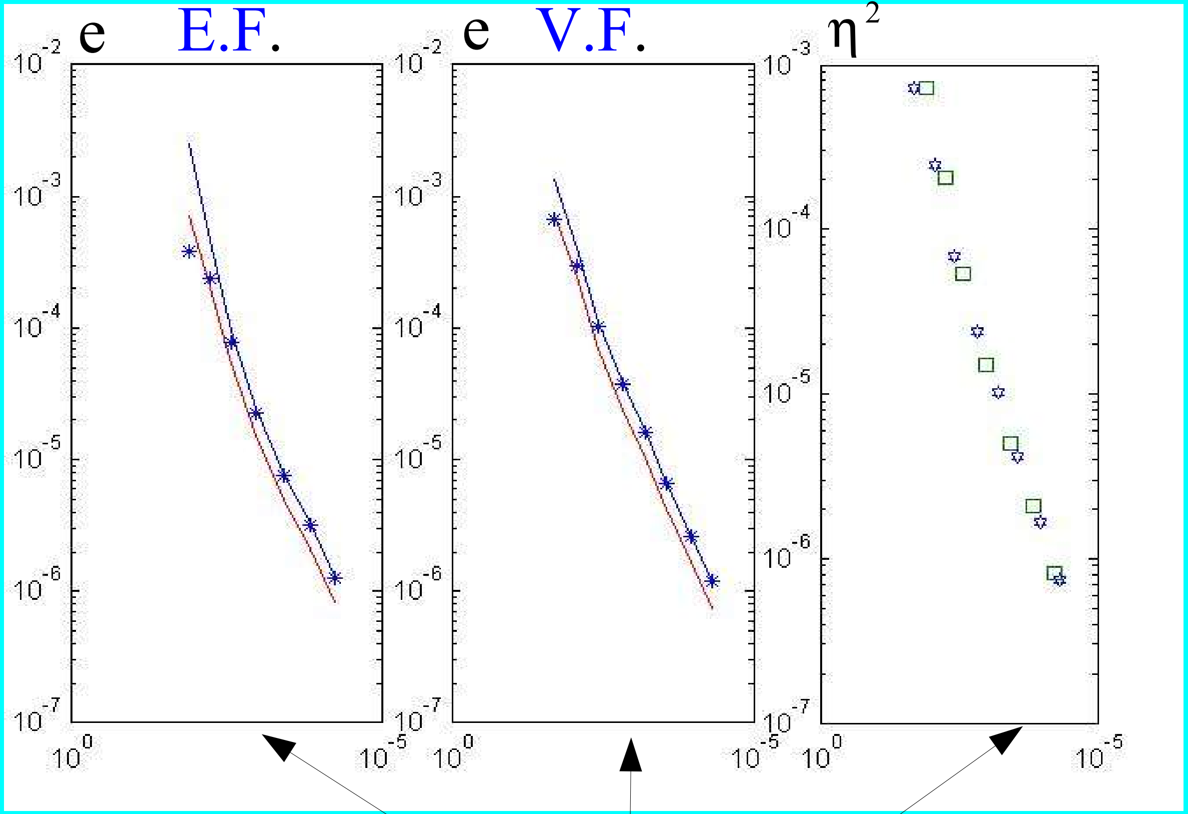
V.F.

Nodes	η^2	e_p	σ_p	e_u	σ_u	e_{up}	σ_{up}
5	4.6415E-4	4.2285E-3	3.018	4.2757E-3	6.070	8.7835E-3	4.350
13	7.8068E-4	3.9305E-3	2.244	3.3364E-3	4.135	7.0631E-3	3.008
22	3.5421E-4	2.0710E-3	2.418	2.1427E-3	4.919	4.4626E-3	3.550
51	6.7632E-4	7.2815E-4	1.038	4.9787E-4	1.716	1.3339E-3	1.404
119	2.9918E-4	2.4466E-4	0.904	1.2578E-4	1.297	4.0115E-4	1.158
273	1.0153E-4	6.9201E-5	0.826	2.9736E-5	1.082	1.1024E-4	1.042
690	3.7415E-5	2.3979E-5	0.801	9.7250E-6	1.020	3.8158E-5	1.010
1627	1.6235E-5	1.0292E-5	0.796	4.1622E-6	1.013	1.6442E-5	1.006
3744	6.5097E-6	4.1858E-6	0.802	1.6412E-6	1.004	6.5372E-6	1.002
9195	2.5983E-6	1.6673E-6	0.801	6.5218E-7	1.002	2.6035E-6	1.001
20976	1.1810E-6	7.4773E-7	0.796	2.9577E-7	1.001	1.1820E-6	1.000

E.F.

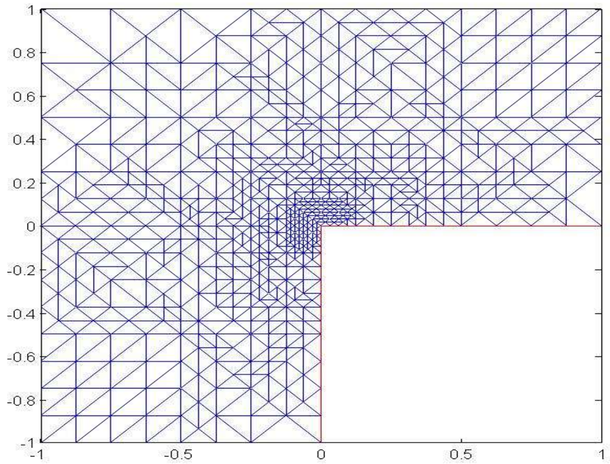
Nodes	η^2	e_p	σ_p	e_u	σ_u	e_{up}	σ_{up}
5	9.1557e-2	2.2687e-1	1.574	1.5916e-1	2.637	3.6409e-1	1.994
13	9.5695e-3	2.9197e-2	1.747	2.4084e-2	3.173	5.2953e-2	2.352
35	1.1260e-3	3.4171e-3	1.742	2.8726e-3	3.194	6.3082e-3	2.367
83	3.8608e-4	7.1832e-4	1.364	6.1726e-4	2.529	1.4276e-3	1.923
178	2.3465e-4	2.0749e-4	0.940	1.1744e-4	1.415	3.5221e-4	1.225
387	7.6880e-5	5.4286e-5	0.840	2.3628e-5	1.109	8.5696e-5	1.056
945	2.2791e-5	1.5119e-5	0.814	6.0886e-6	1.034	2.3573e-5	1.017
2703	7.6171e-6	4.9707e-6	0.807	1.9542e-6	1.013	7.7170e-6	1.007
7175	3.1784e-6	2.0624e-6	0.805	8.1224e-7	1.011	3.2137e-6	1.006
17299	1.2704e-6	8.1788e-7	0.802	3.2002e-7	1.004	1.2752e-6	1.002

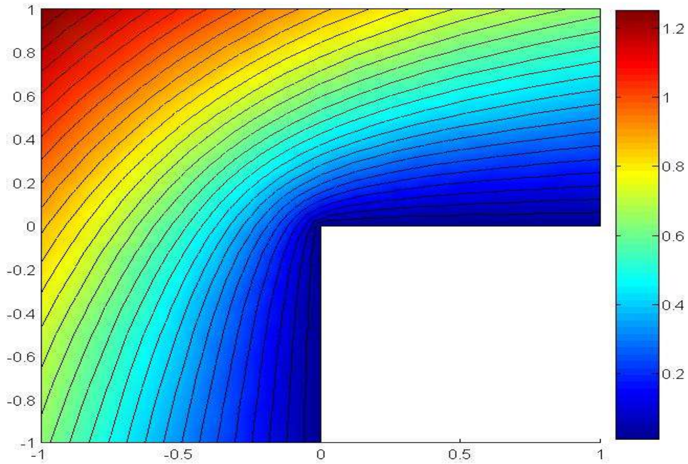

V.F.
E.F.
η2
eup
ep
eu
x 10-7



$$-\log\left(\frac{1}{n}\right)$$

$$p(r,\theta) = r^{2/3}\cos(2/3\theta)$$



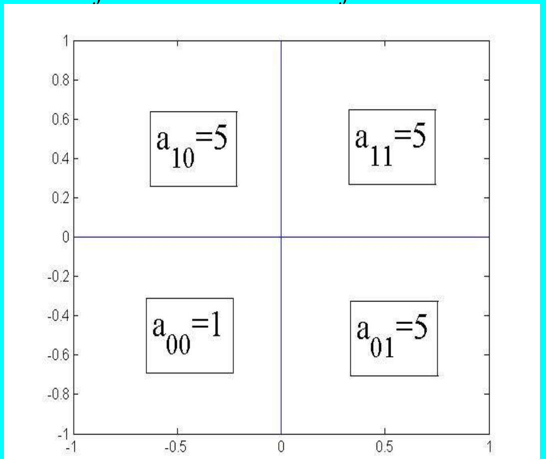


Nodes	η^2	e_p	σ_p	e_u	σ_u	e_{up}	σ_{up}
11	3.3138E-1	1.3396E-1	0.636	8.6231E-2	1.020	3.3815E-1	1.010
22	1.4867E-1	6.4454E-2	0.658	3.9337E-2	1.029	1.5301E-1	1.015
52	6.3707E-2	2.8323E-2	0.667	1.6173E-2	1.008	6.4200E-2	1.004
95	3.1662E-2	1.5084E-2	0.690	8.1340E-3	1.014	3.2099E-2	1.007
237	1.2372E-2	5.8879E-3	0.690	3.1081E-3	1.002	1.2402E-2	1.001
594	5.2163E-3	2.5088E-3	0.694	1.3062E-3	1.001	5.2206E-3	1.000
1406	2.1786E-3	1.0757E-3	0.703	5.4559E-4	1.001	2.1805E-3	1.000
3437	8.9261E-4	4.4444E-4	0.706	2.2321E-4	1.000	8.9273E-4	1.000
8706	3.6039E-4	1.8037E-4	0.708	9.0183E-5	1.000	3.6056E-4	1.000
21751	1.4688E-4	7.4038E-5	0.710	3.6748E-5	1.000	1.4693E-4	1.000

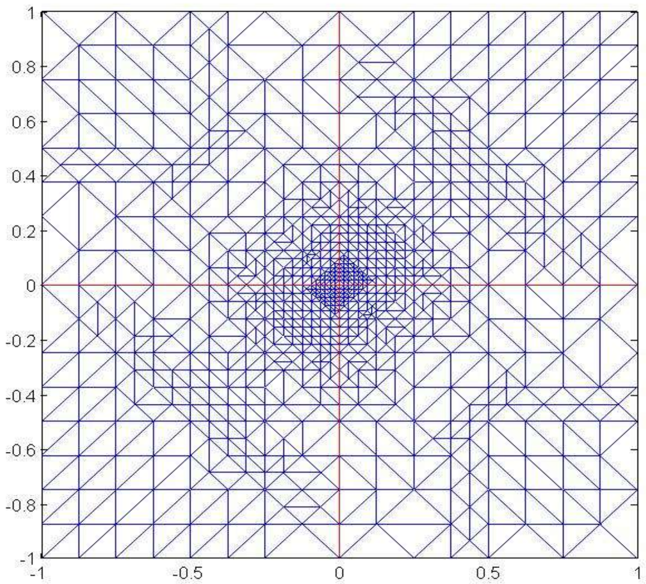
$$p(r,\theta) = r^{\alpha}(\beta_{ij}\sin(\alpha\theta) + \gamma_{ij}\cos(\alpha\theta))$$

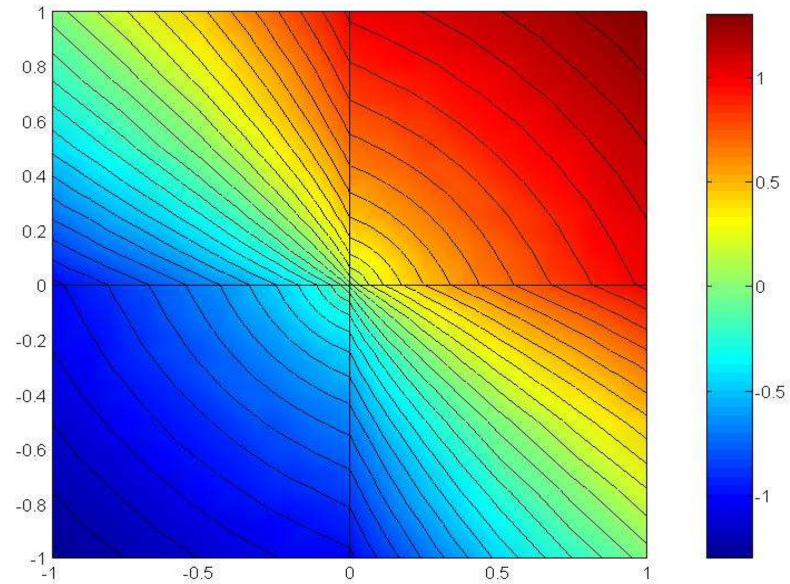
$$A = a_{ij} I$$

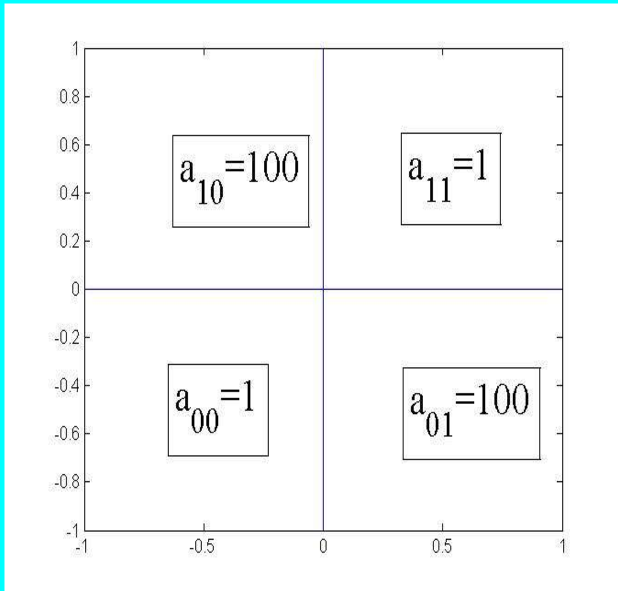
$$\alpha \approx 0.535$$

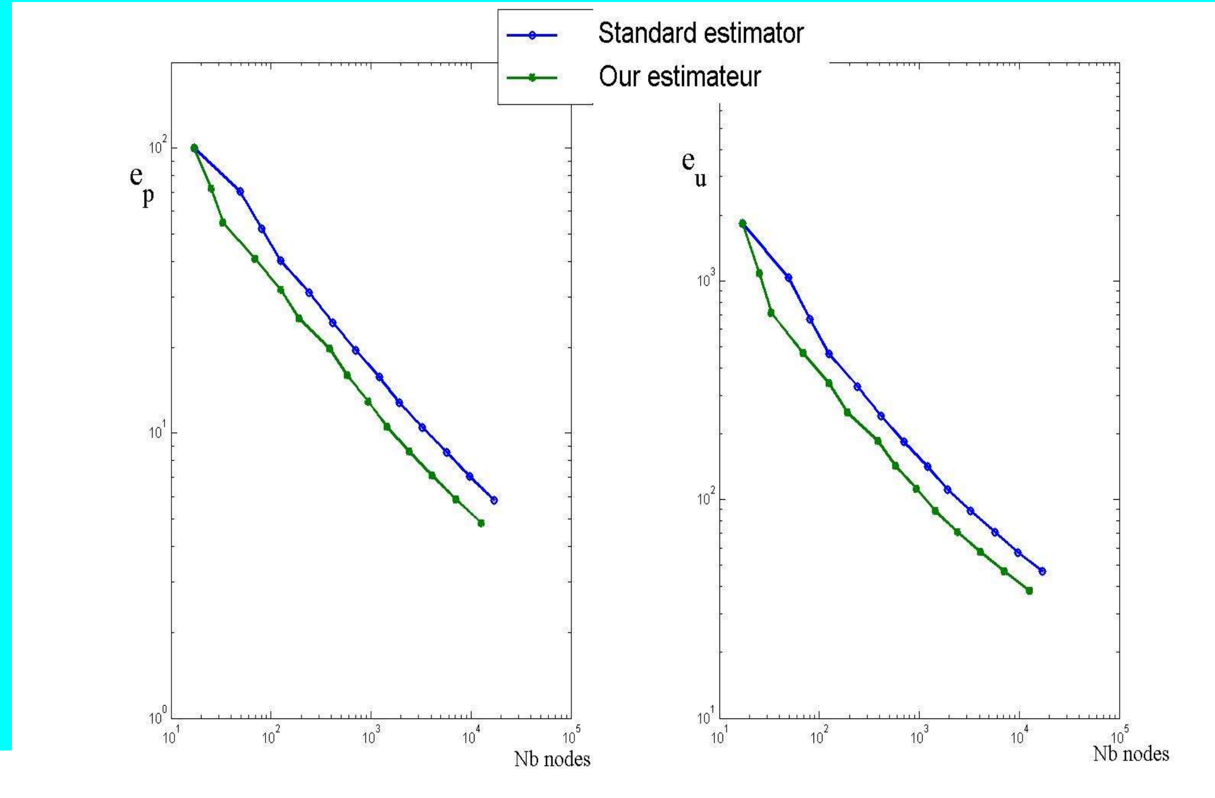


Nodes	η^2	e_p	σ_p	e_u	σ_u	e_{up}	σ_{up}
17	3.7972E+0	1.5019E+0	0.629	1.0015E+0	1.027	3.9016E+0	1.014
49	1.7419E+0	7.2512E-1	0.645	4.3943E-1	1.005	1.7498E+0	1.002
77	8.9621E-1	3.9538E-1	0.664	2.2680E-1	1.006	9.0170E-1	1.003
151	4.5087E-1	2.0979E-1	0.682	1.1502E-1	1.010	4.5547E-1	1.005
333	2.2049E-1	1.0224E-1	0.681	5.5336E-2	1.002	2.2092E-1	1.001
767	1.0569E-1	4.9834E-2	0.687	2.6579E-2	1.003	1.0600E-1	1.002
1775	4.9744E-2	2.3458E-2	0.687	1.2541E-2	1.004	4.9955E-2	1.002
4151	2.3677E-2	1.1120E-2	0.685	5.9297E-3	1.001	2.3698E-2	1.000
9621	1.1194E-2	5.2559E-3	0.685	2.7995E-3	1.000	1.1196E-2	1.000

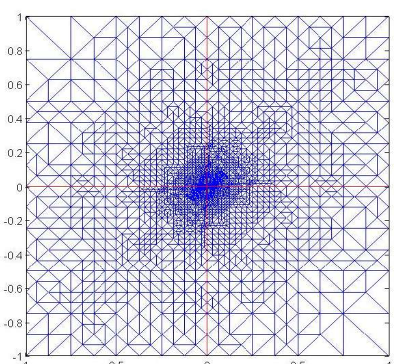


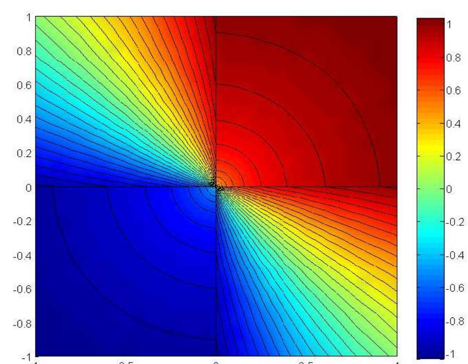






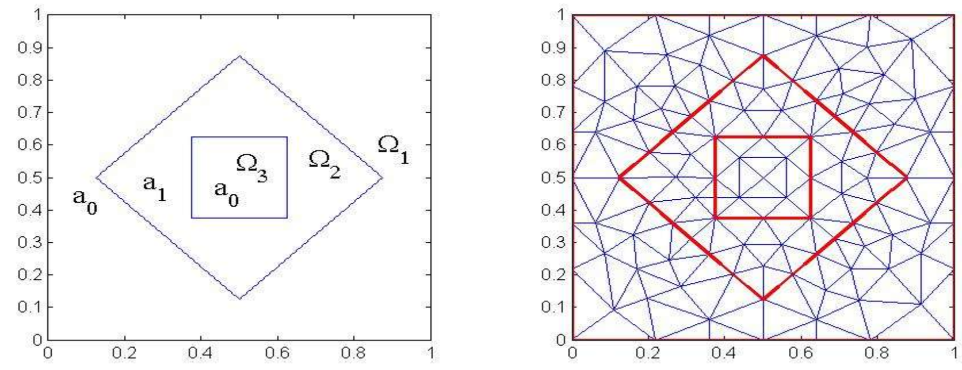
$\alpha \approx 0.127$



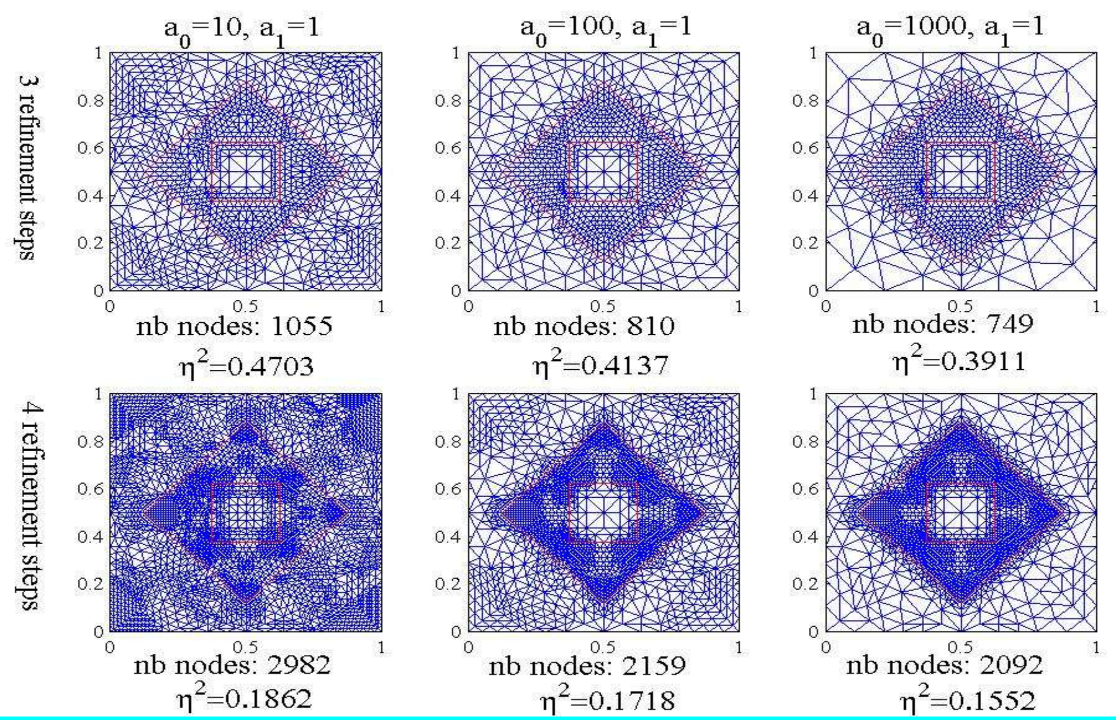


$$f = 100$$

$$A = a_i I$$



a0
a1
a0
Ω3
Ω2
Ω1



a0=10, a1=1
a0=100, a1=1
a0=1000, a1=1
3 refinement steps
nb nodes: 1055
η2=0.4703
nb nodes: 810
η2=0.4137
nb nodes: 749
η2=0.3911
4 refinement steps
nb nodes: 2982
η2=0.1862
nb nodes: 2159
η2=0.1718
nb nodes: 2092
η2=0.1552