Properties of Multipoint Flux Approximations

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Outline

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Reservoir flow equations

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- The medium is strongly heterogeneous.
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- Here, we study control volume formulations for an elliptic model equation on quadrilateral grids.
- This guarantees local conservation, important for the hyperbolic part.

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Model equation

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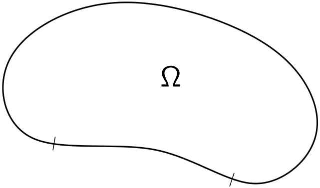

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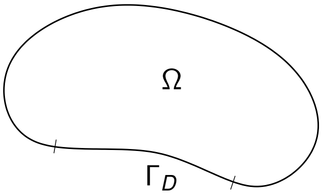

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\boldsymbol{q} \cdot \boldsymbol{n} &= q_N && \text{on } \Gamma_N
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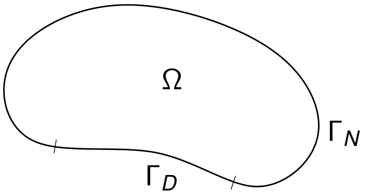

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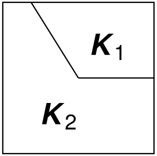

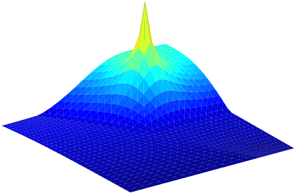

Monotonicity

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$G(\boldsymbol{x}, \xi) \geq 0$ implies that the operator $\mathcal{T}$, defined by

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We must show that $\mathcal{T}$ is monotone for all Ω with homogeneous Dirichlet boundary conditions on $\partial\Omega$.

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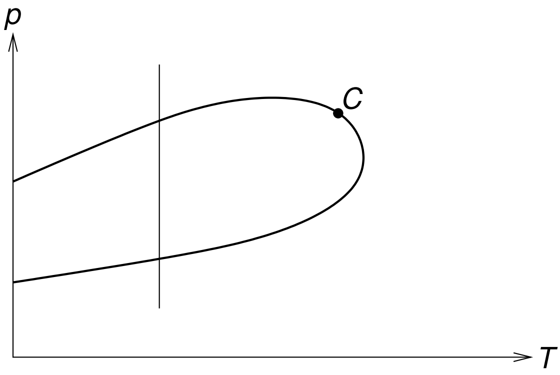

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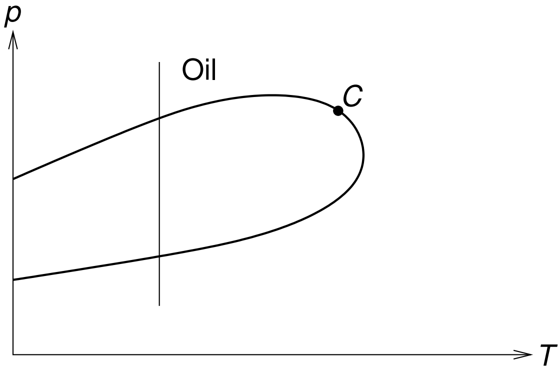

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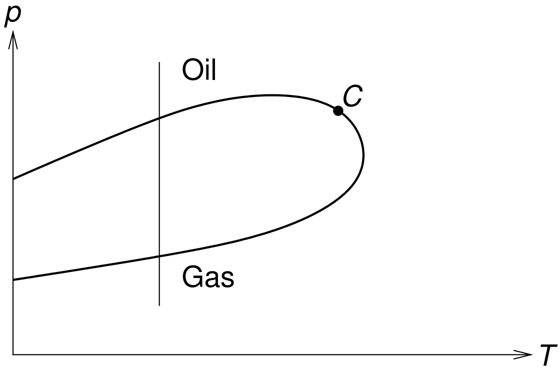

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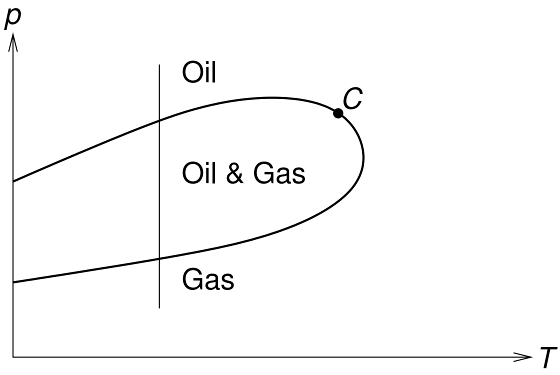

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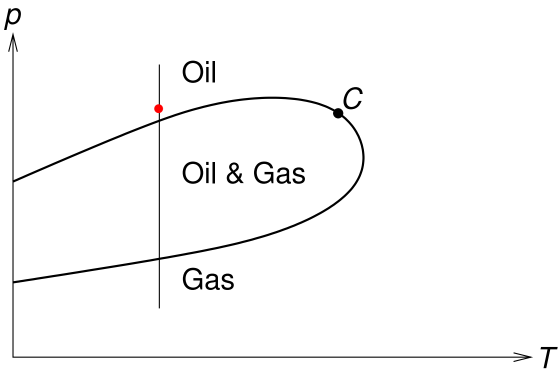

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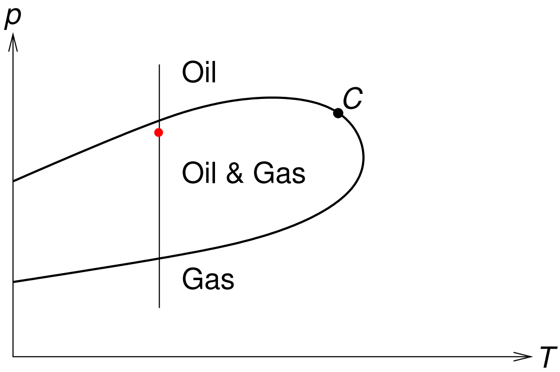

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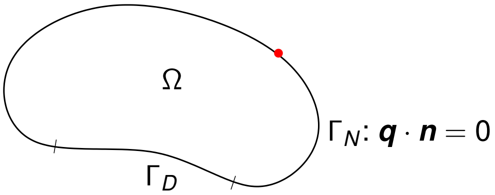
- Maximum principle $\Rightarrow$ Extrema lie on the boundary.
- E. Hopf (1952): If there is an extremum on the boundary, then $\boldsymbol{q} \cdot \boldsymbol{n} \neq 0$.
- Hence, extrema cannot occur on no-flow boundaries.



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- Method: Generalize harmonic averaging to 2D and 3D by requiring continuity in flux and (weak) continuity in potential.

Control-volume formulation

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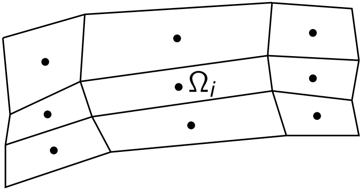

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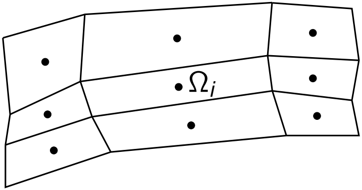

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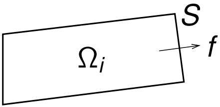

$$f = \int_S \boldsymbol{q} \cdot \boldsymbol{n} \, d\sigma$$

O-method

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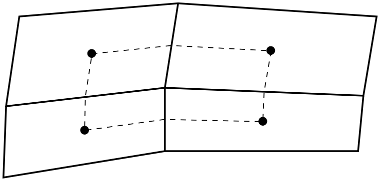
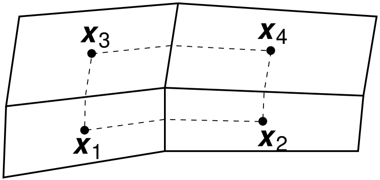


Cells with common corner

O-method



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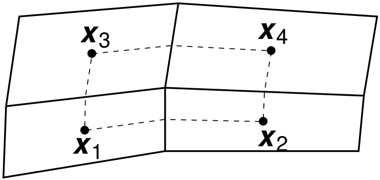

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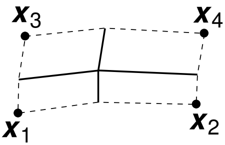

Interaction volume

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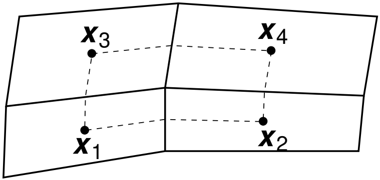

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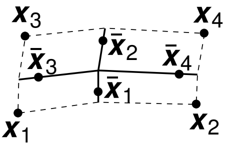

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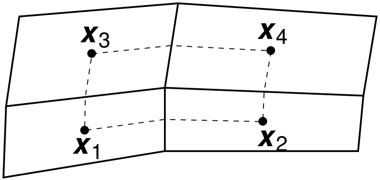

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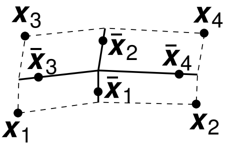

Interaction volume

- Determine the flux through the half edges from the interaction of linear potentials in the four cells.

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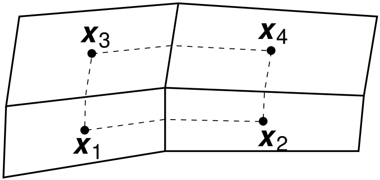

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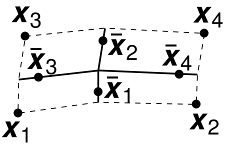

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- Require continuous potential at $\bar{\boldsymbol{x}}_i$ and continuous flux through the half edges.

Flux equations in an interaction region

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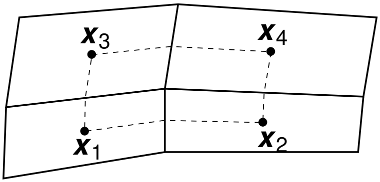

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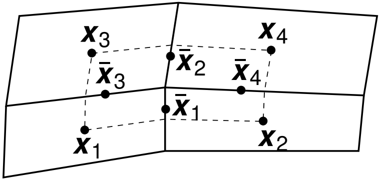

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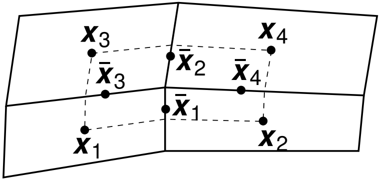

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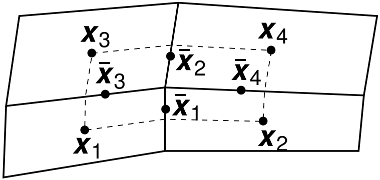

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$\Rightarrow$ Local explicit expression for the half-edge fluxes

Flux expression

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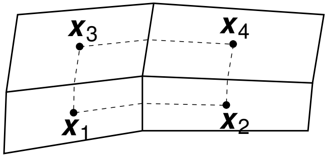

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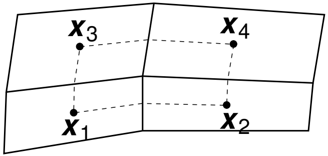
$$f_i = \sum_{j=1}^{4} t_{i,j} u_j \quad \text{where} \quad \sum_{j=1}^{4} t_{i,j} = 0$$

Flux expression



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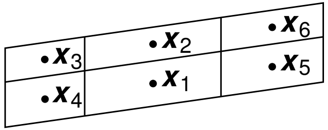

Flux stencil

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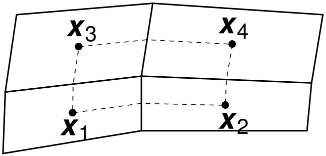

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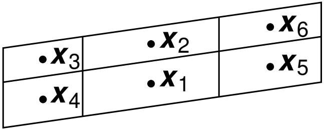

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Multipoint flux approximation (MPFA)

Stencils

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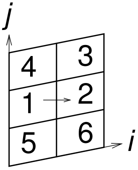
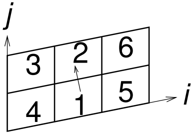


i-flux stencil



j-flux stencil

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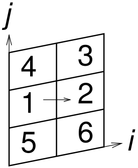

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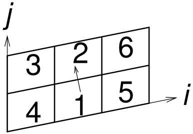

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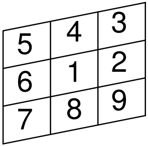

Cell stencil

3D stencil

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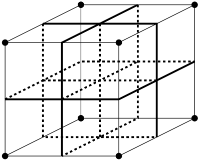

Interaction volume

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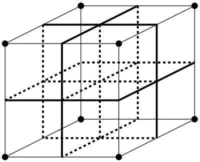

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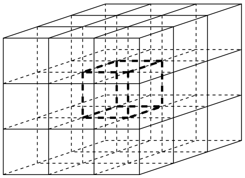

Cell stencil

Polygonal and triangular grids

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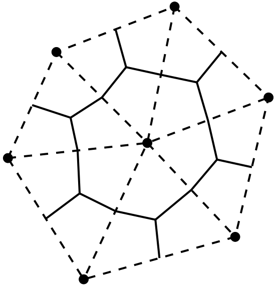
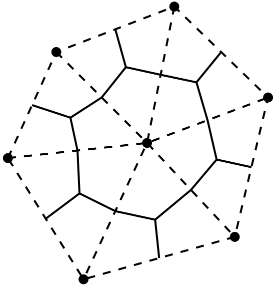
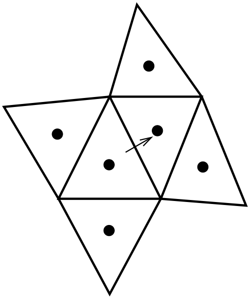


Cell stencil in polygonal grid

Polygonal and triangular grids



Cell stencil in polygonal grid



Flux stencil in triangular grid

MPFA O-method

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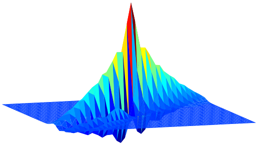
- For non-parallelogram quadrilaterals with strong irregularity, convergence may be lost.

MPFA O-method

- For non-parallelogram quadrilaterals with strong irregularity, convergence may be lost.
- For high skewness combined with strong aspect or anisotropy ratio, oscillating solutions may occur.



Anisotropy ratio 1 : 1000
$\theta = 30^\circ$
Square grid

Challenges

Challenges

- Are there MPFA-methods with a larger domain of validity for convergence and monotonicity?

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- Are there methods which behave less oscillatory when monotonicity cannot be assured?

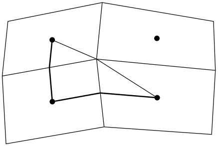
Challenges

- Are there MPFA-methods with a larger domain of validity for convergence and monotonicity?
- Are there methods which behave less oscillatory when monotonicity cannot be assured?
- Does such a new method have disadvantages?

Outline

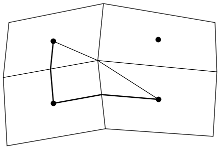
L-method

L-method



L-shaped coupling

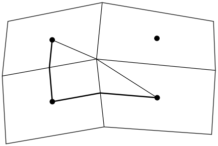
L-method



L-shaped coupling

Inside the “triangle”:

L-method

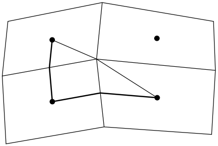


L-shaped coupling

Inside the “triangle”:

- Linear potential in each cell

L-method

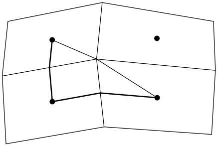


L-shaped coupling

Inside the “triangle”:

- Linear potential in each cell
- Full potential continuity

L-method

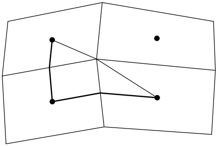


L-shaped coupling

Inside the “triangle”:

- Linear potential in each cell
- Full potential continuity
- Flux continuity

L-method

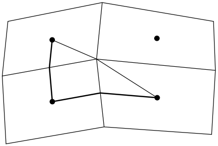


L-shaped coupling

Inside the “triangle”:

- Linear potential in each cell
- Full potential continuity
- Flux continuity
- $3 \cdot 2 = 6$ deg. of freedom

L-method



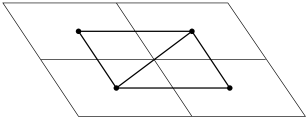
L-shaped coupling

Inside the “triangle”:

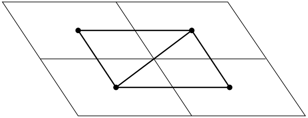
- Linear potential in each cell
- Full potential continuity
- Flux continuity
- $3 \cdot 2 = 6$ deg. of freedom
- $2 \cdot 3 = 6$ conditions

Interaction region

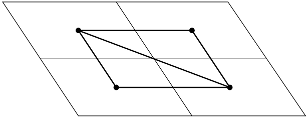
Interaction region



Interaction region



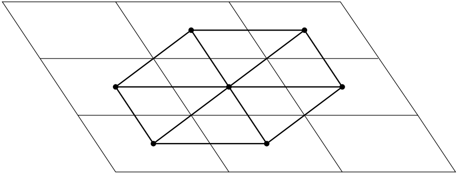
Short diagonal



Long diagonal

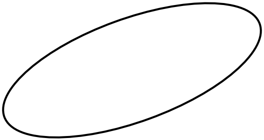
7-point cell stencil

7-point cell stencil



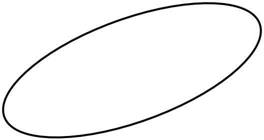
Cell stencil

7-point cell stencil



Permeability ellipse
$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{K}^{-1}\boldsymbol{x} = 1$

7-point cell stencil



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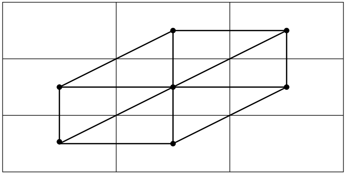
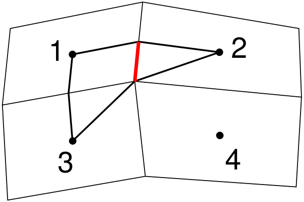


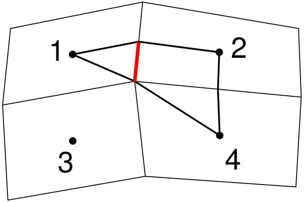
Cell stencil

General case

General case

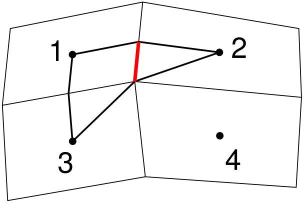


Triangle 1
Transmissibilities: t_j^1



Triangle 2
Transmissibilities: t_j^2

General case



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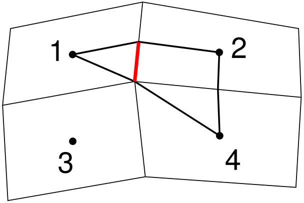


Triangle 2
Transmissibilities: t_j^2

If $\left|t_1^1\right| < \left|t_2^2\right|$, triangle 1 is chosen, else triangle 2 is chosen.

Top edge

Top edge

+	−
+	

+	−
	+

Top edge

+	−
+	

+	−
	+

- For moderate skewness, the choice will be between these cases, and it is natural to choose the case to the left.

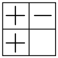
Top edge

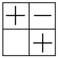
+	−
+	

+	−
	+

- For moderate skewness, the choice will be between these cases, and it is natural to choose the case to the left.
- For homogeneous medium and uniform grid, the criterion always chooses the “natural” seven-point stencil.

Top edge





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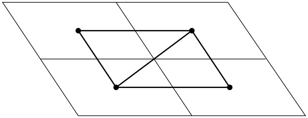
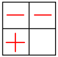
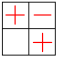


Top edge





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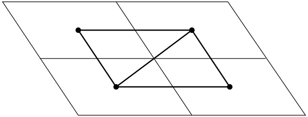

3D L-stencils

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In 3 dimensions there are 4 L-stencils with 4 cells.

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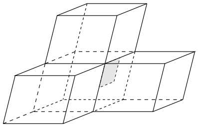

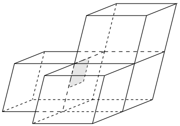

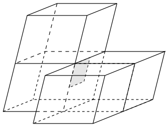

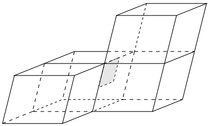

3D flux stencils

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In 3 dimensions the flux stencil contains 6 till 10 cells, and the cell stencil has 13 till 19 cells.

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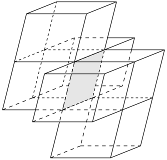
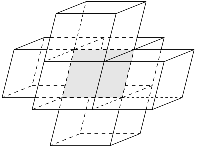
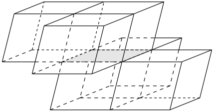


i direction



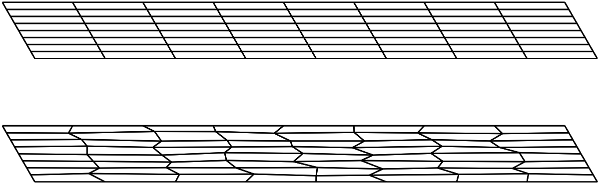
j direction



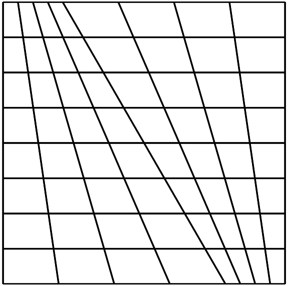
k direction

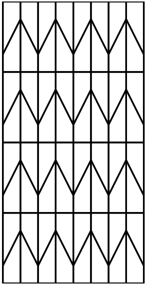
Outline

Test grids



Test grids

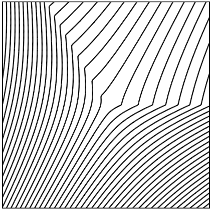




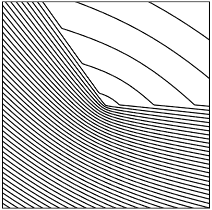
Test cases, streamlines

Smooth solution:

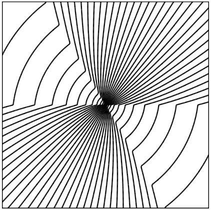
Nonsmooth solutions:



$u \in H^{2.29}$



$u \in H^{1.79}$



$u \in H^{1.24}$

Comparisons

Comparisons

Compare convergence behavior of L-, O(0)- and O(0.5)-method.

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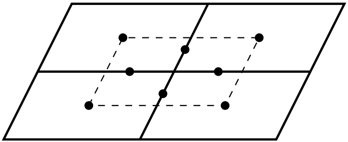
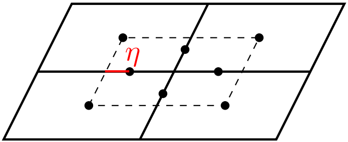


O(η)-method

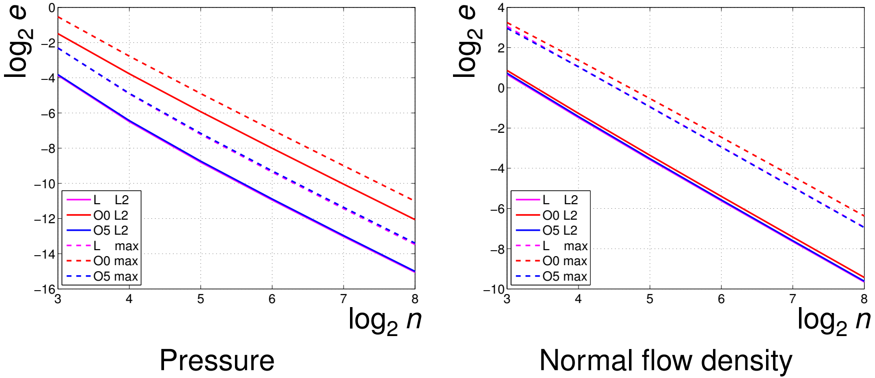
Comparisons

Compare convergence behavior of L-, O(0)- and O(0.5)-method.

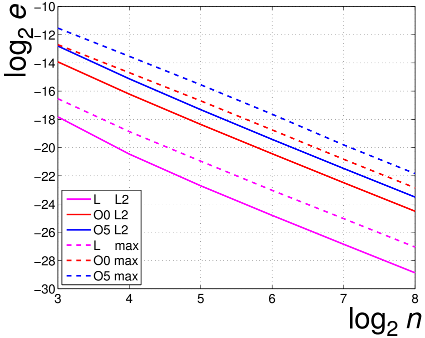


O(η)-method

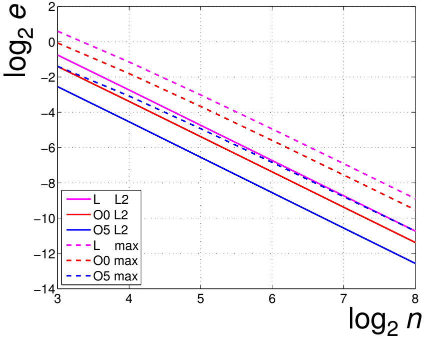
Parallelogram grid, aspect ratio 1, angle 30 $^\circ$



Parallelogram grid, aspect ratio 0.01, angle 30 °

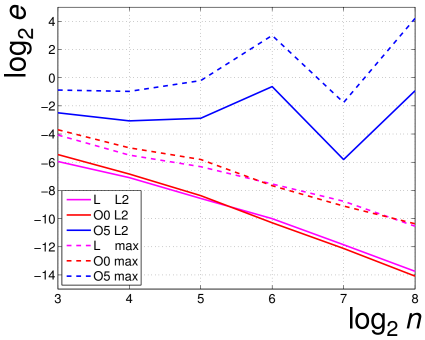


Pressure

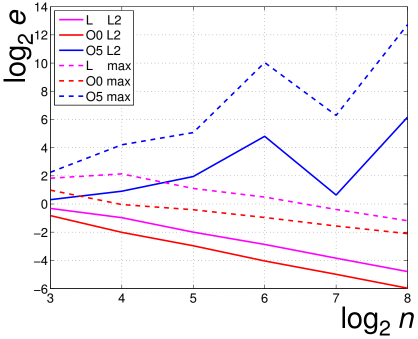


Normal flow density

Perturbed parallelogram grid, aspect ratio 0.1, angle 30 ◦



Pressure



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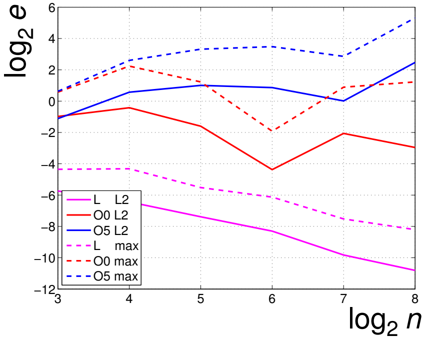
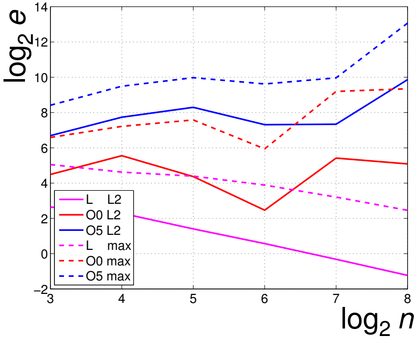
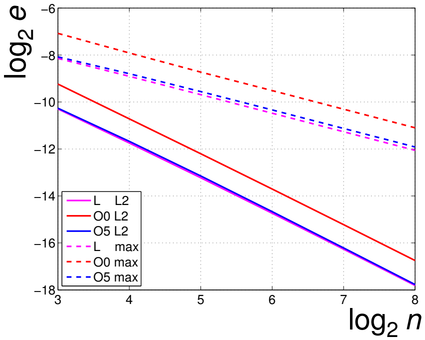


Pressure

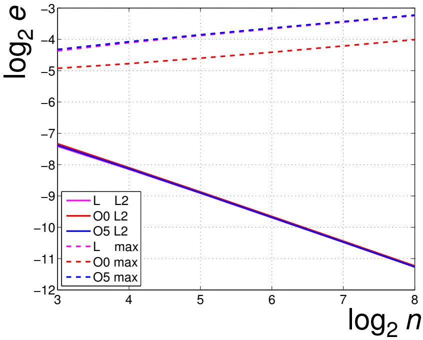


Normal flow density

Flow around a corner, $u \in H^{1.79}$



Pressure



Normal flow density

Convergence tests

Convergence tests

Tested L^2 and L^∞ convergence for

- solutions $u \in H^{1+\alpha}$, $\alpha > 0$,

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- solutions $u \in H^{1+\alpha}$, $\alpha > 0$,
- smooth and rough grids,
- grid aspect ratios between 10^{-2} and 10^2,

Convergence

Convergence

On rough quadrilateral grids, the simulation tests indicate that if $u \in H^{1+\alpha}$, $\alpha > 0$, then

$$\|u_h - u\|_{L^2} \sim h^{\min\{2,2\alpha\}}$$
$$\|u_h - u\|_{L^\infty} \sim h^{\min\{2,\alpha\}}$$
$$\|(\boldsymbol{q}_h - \boldsymbol{q}) \cdot \boldsymbol{n}\|_{L^2} \sim h^{\min\{1,\alpha\}}$$

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On smooth quadrilateral grids, stronger flow density bounds apply:

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Convergence

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$$\|(\boldsymbol{q}_h - \boldsymbol{q}) \cdot \boldsymbol{n}\|_{L^\infty} \sim h^{\min\{2,\alpha-1\}}$$

These rates apply to “moderate” aspect ratios.

Outline

Discrete monotonicity

Discrete monotonicity

Solution of differential equation with homogeneous Dirichlet boundary conditions

$$u = \mathcal{T} q,$$

where the operator $\mathcal{T}$ is a monotone operator.

Discrete monotonicity

Solution of differential equation with homogeneous Dirichlet boundary conditions

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Associated discrete system

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{q}.$$

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Solution of differential equation with homogeneous Dirichlet boundary conditions

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The matrix $\boldsymbol{A}^{-1}$ is monotone if

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Solution of differential equation with homogeneous Dirichlet boundary conditions

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The matrix $\boldsymbol{A}^{-1}$ is monotone if

$$\boldsymbol{A}^{-1} \geq \boldsymbol{O}.$$

Then

$$\boldsymbol{q} \geq \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{u} \geq \boldsymbol{0}.$$

Discrete maximum principle

Discrete maximum principle

Natural discrete analogue of the maximum principle:

$$\boldsymbol{A}^{-1} \geq \boldsymbol{O}$$

for all subgrids with homogeneous Dirichlet boundary conditions.

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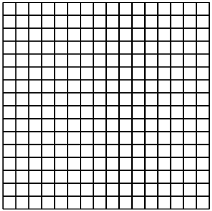

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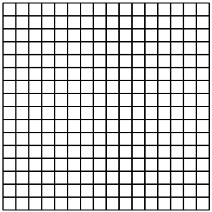
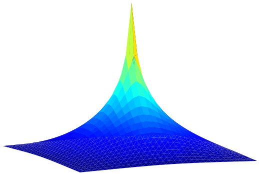
for all subgrids with homogeneous Dirichlet boundary conditions.
The associated scheme is then called monotone.

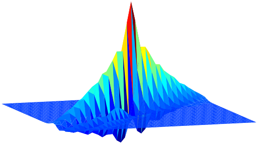




$$\boldsymbol{A}^{-1} \geq \boldsymbol{O}$$

Nonmonotone examples

Nonmonotone examples



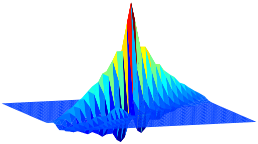
Anisotropy ratio 1 : 1000

$\theta = \pi/6$

$\eta = 0$

$\boldsymbol{A}^{-1} \not\geq \boldsymbol{O}$

Nonmonotone examples



Anisotropy ratio 1 : 1000
$\theta = \pi/6$
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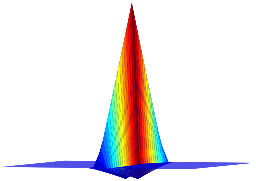

Anisotropy ratio 1 : 10000
$\theta = 0$
$\eta = 0.5$
$\boldsymbol{A}^{-1} \not\geq \boldsymbol{O}$

Outline

Monotone matrices

Monotone matrices

Conditions for $\boldsymbol{A}^{-1} \geq \boldsymbol{O}$

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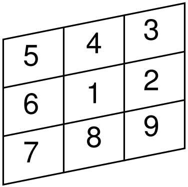

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then the splitting $\boldsymbol{A} = \boldsymbol{B} - \boldsymbol{C}$ is weakly regular. It follows:

$$\boldsymbol{A}^{-1} \geq \boldsymbol{O} \qquad \Leftrightarrow \qquad \rho(\boldsymbol{B}^{-1}\boldsymbol{C}) < 1$$

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- Generalization of M-matrix theory.

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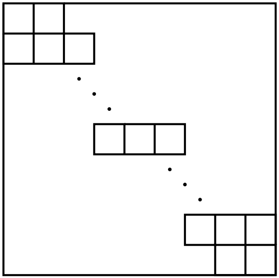
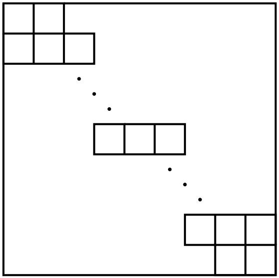
- These conditions are only sufficient.

Monotonicity criteria



A block-tridiagonal

Monotonicity criteria



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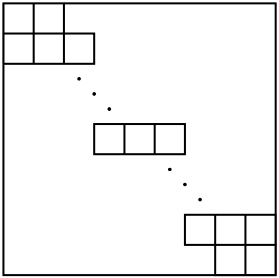

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- $\boldsymbol{A} = \boldsymbol{B} - \boldsymbol{C}$,
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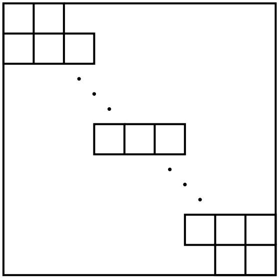

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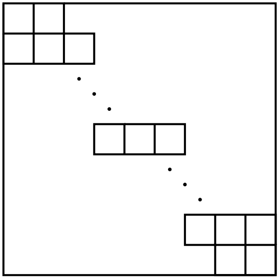

$\boldsymbol{A}$ block-tridiagonal

- $\boldsymbol{A} = \boldsymbol{B} - \boldsymbol{C}$,
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- Different orderings yield different conditions.

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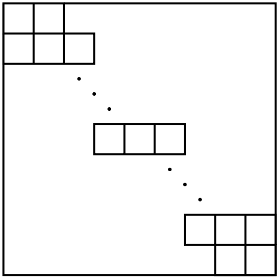

$\boldsymbol{A}$ block-tridiagonal

- $\boldsymbol{A} = \boldsymbol{B} - \boldsymbol{C}$,
- $\boldsymbol{B} =$ diagonal blocks,
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- Different orderings yield different conditions.
- Use rowwise or columnwise orderings.

Rowwise ordering

$$
\begin{aligned}
m_1^{i,j} &> 0\\
m_2^{i,j} &< 0\\
m_6^{i,j} &< 0\\
m_4^{i,j} &< 0\\
m_8^{i,j} &< 0\\
m_1^{i,j} + m_2^{i,j} + m_6^{i,j} &> 0\\
m_2^{i,j} m_4^{i,j-1} - m_3^{i,j-1} m_1^{i,j} &> 0\\
m_6^{i,j} m_4^{i,j-1} - m_5^{i,j-1} m_1^{i,j} &> 0\\
m_2^{i,j} m_8^{i,j+1} - m_9^{i,j+1} m_1^{i,j} &> 0\\
m_6^{i,j} m_8^{i,j+1} - m_7^{i,j+1} m_1^{i,j} &> 0
\end{aligned}
$$

Columnwise ordering

$$
\begin{aligned}
m_1^{i,j} &> 0 \\
m_2^{i,j} &< 0 \\
m_4^{i,j} &< 0 \\
m_6^{i,j} &< 0 \\
m_8^{i,j} &< 0 \\
m_1^{i,j} + m_4^{i,j} + m_8^{i,j} &> 0 \\
m_4^{i,j} m_2^{i-1,j} - m_3^{i-1,j} m_1^{i,j} &> 0 \\
m_4^{i,j} m_6^{i+1,j} - m_5^{i+1,j} m_1^{i,j} &> 0 \\
m_8^{i,j} m_2^{i-1,j} - m_9^{i-1,j} m_1^{i,j} &> 0 \\
m_8^{i,j} m_6^{i+1,j} - m_7^{i+1,j} m_1^{i,j} &> 0
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- Agreement with numerical observations.

Homogeneous medium, uniform grid

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$$\begin{aligned} m_1 &> 0 \\ \max\{m_2, m_4\} &< 0 \\ m_1 + 2\max\{m_2, m_4\} &> 0 \\ m_2 m_4 - \max\{m_3, m_5\} \cdot m_1 &> 0 \end{aligned}$$

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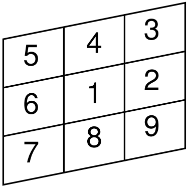

Monotonicity

Monotonicity

Homogeneous medium
Uniform grid

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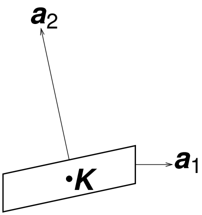
Homogeneous medium
Uniform grid



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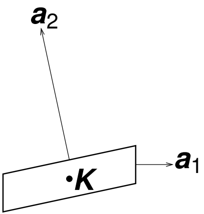

Define

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} = \frac{1}{V} \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}^{\mathrm{T}} \boldsymbol{K} \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$$

Monotonicity

Homogeneous medium
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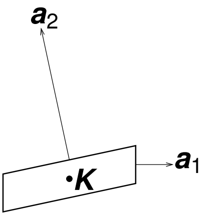

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Ellipticity implies

$$c \leq \sqrt{ab}$$

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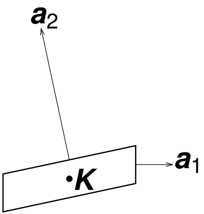

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Ellipticity implies

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Monotonicity & conservation & exact solution for uniform flow imply

$$c \leq \min\{a, b\}.$$

Monotonicity

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Assume $a \leq b$

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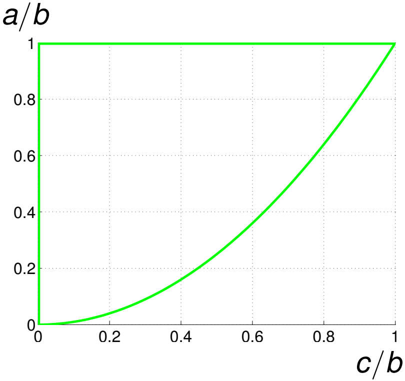

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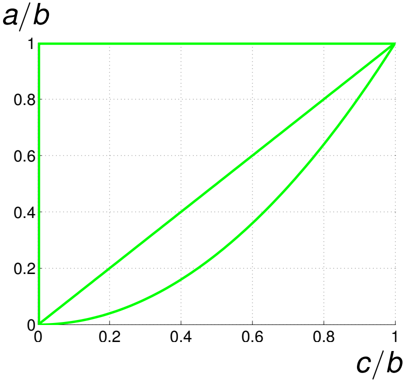

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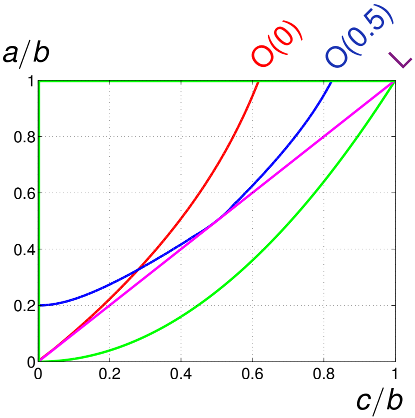

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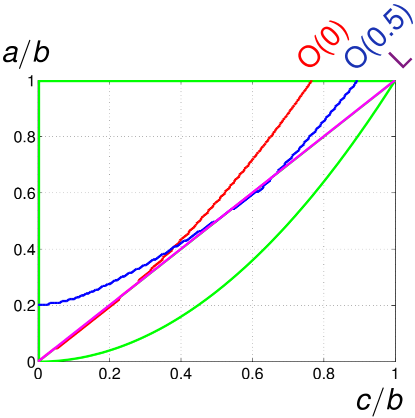

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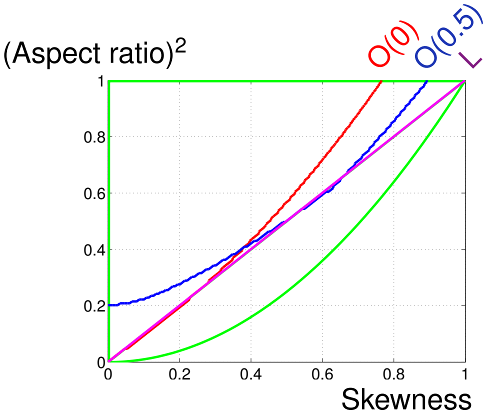

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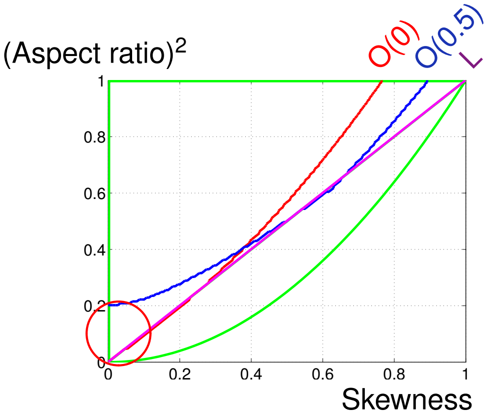

Layered media and uniform grids. O(0)-method.

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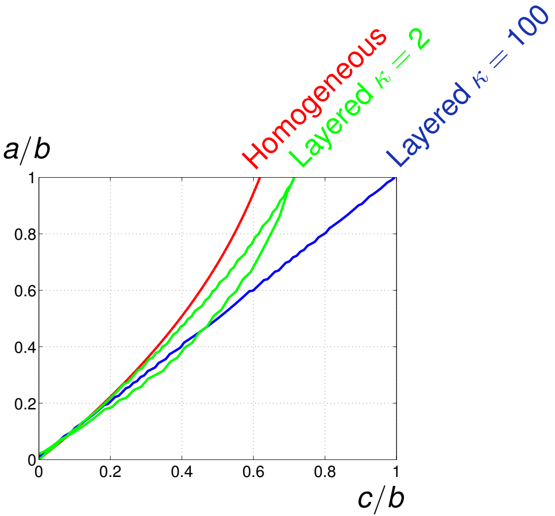
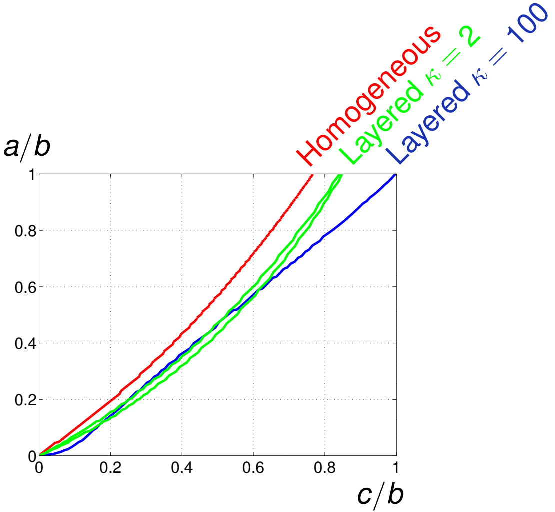


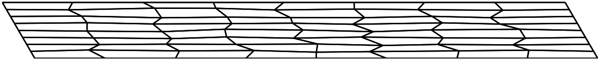
Layered media and uniform grids. O(0)-method.



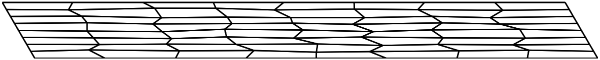
Outline

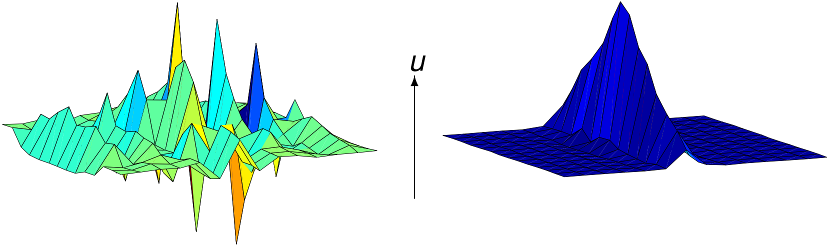
Oscillations

Oscillations



Oscillations



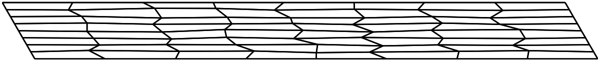


O(0)-method

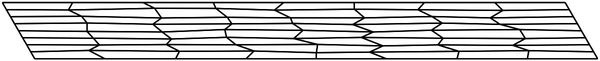
L-method

Oscillations

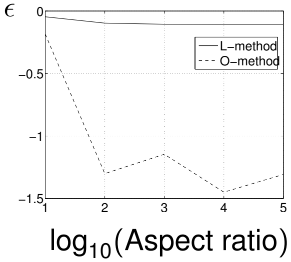
Oscillations



Oscillations



$$\epsilon = \min_j \left\{ \frac{\min_i \left[\boldsymbol{A}^{-1}\right]_{i,j}}{\max_i \left[\boldsymbol{A}^{-1}\right]_{i,j}} \right\}$$



A case with no-flow boundary

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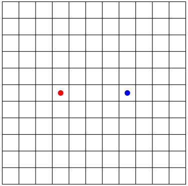
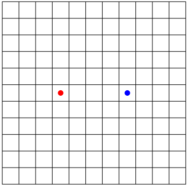


- pressure 0
- pressure 1

A case with no-flow boundary



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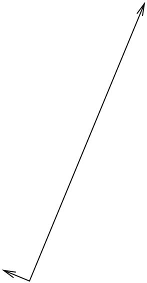
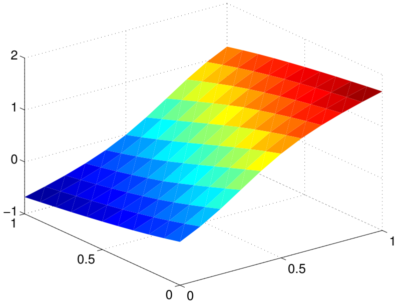


Anisotropy 1:1000
Angle 67.5 °

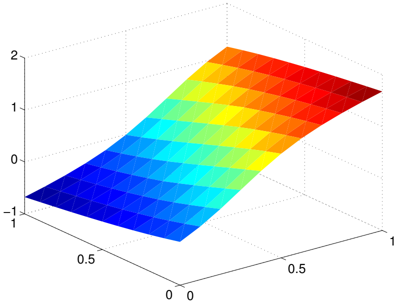
11×11 grid

11 × 11 grid



L-method

11 × 11 grid



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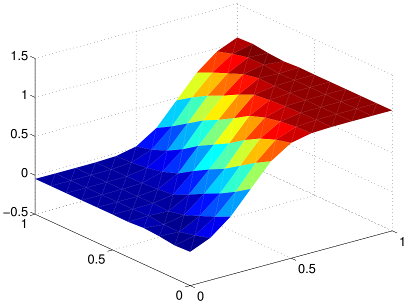
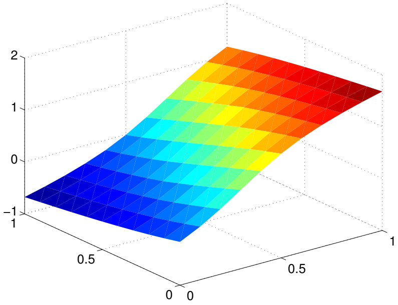
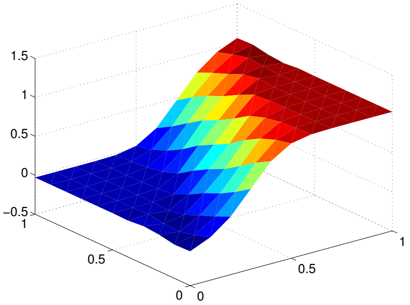


O(0)-method

11 × 11 grid

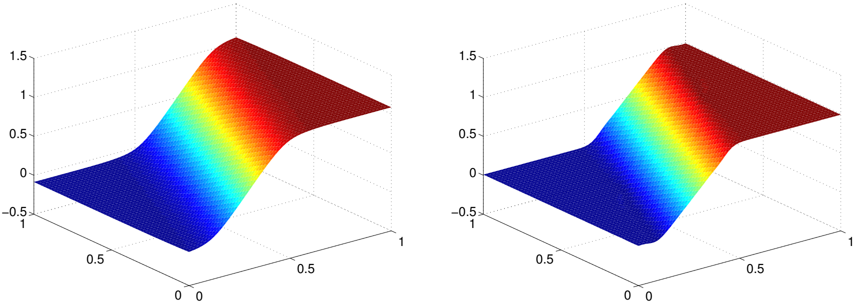


L-method



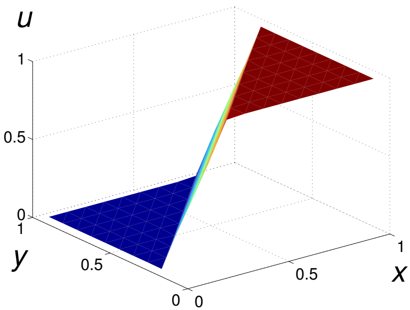
O(0.5)-method

55 $\times$ 55 grid

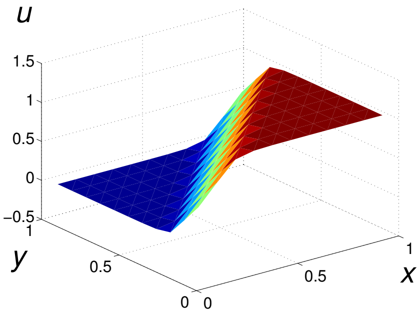


L-method　　　　O(0)-method

11 × 11 grid, Angle 45°



L-method



O(0)-method

No-flow boundary extrema

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- Continuous maximum principle ensures no extrema on no-flow boundaries (Hopf, 1952).

No-flow boundary extrema

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- No discrete no-flow boundary extrema when the matrix of coefficients is a diagonally dominant M-matrix.

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Monotonicity and absence of boundary extrema

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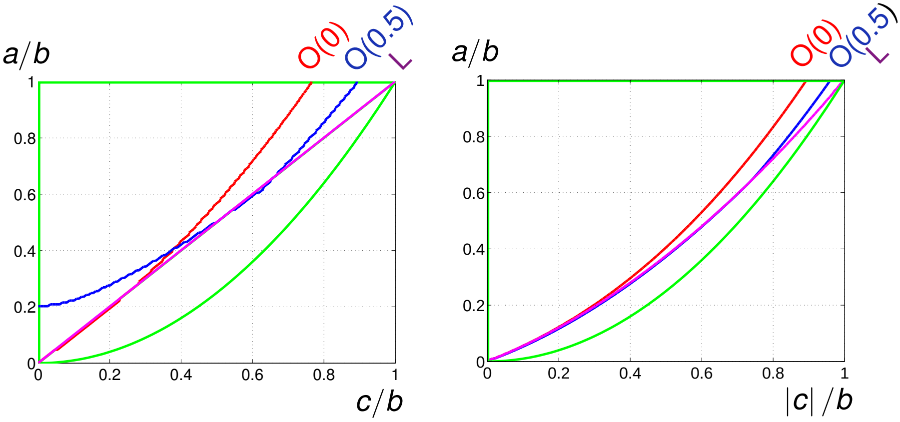
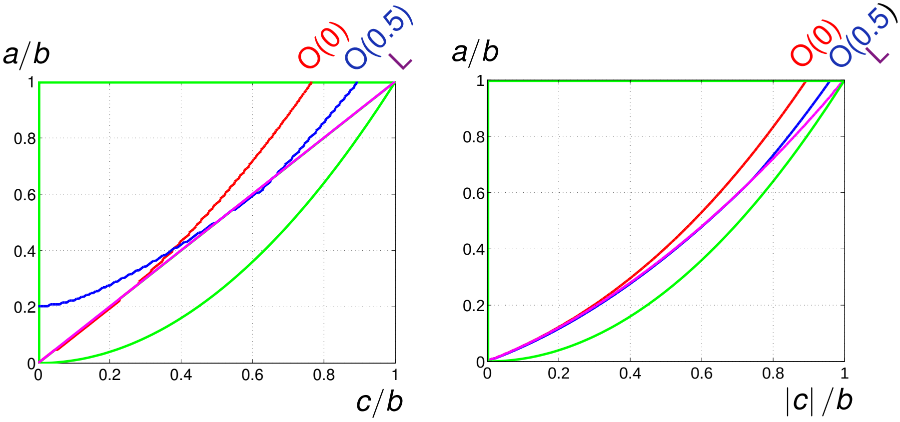


Monotonicity

No boundary extrema

Monotonicity and absence of boundary extrema



Monotonicity | No boundary extrema

This indicates that monotone methods never yield solutions with discrete extrema on no-flow boundaries.

Outline

Nonmatching grids

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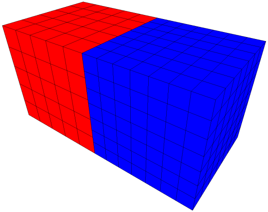

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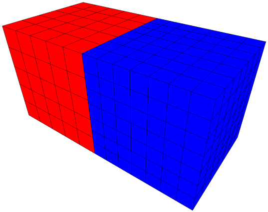
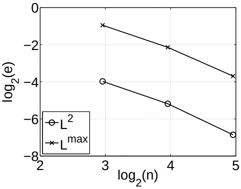
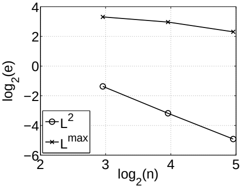


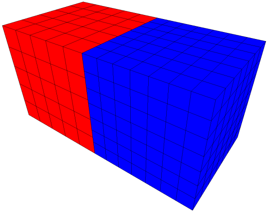


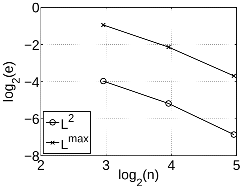
Pressure



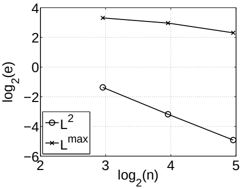
Normale flow density

Nonmatching grids





Pressure

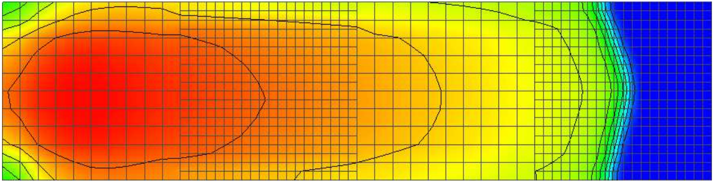


Normale flow density

L^2 convergence order: 1.7 for pressure and flow density

Two-phase flow

Two-phase flow



Saturation contours

Summary

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- Good convergence and monotonicity properties can be shown for chosen stencils.
- The solutions are exact for linear pressure fields.
- Monotonicity conditions are generally more restrictive than L^2 convergence conditions.
- The L-method may also be used for nonmatching grids.

Neue Artikel

- I. Aavatsmark, G.T. Eigestad and R.A. Klausen, Numerical convergence of the MPFA O-method for general quadrilateral grids in two and three dimensions, in: Compatible spatial discretizations, *IMA Vol. Ser.* **142**, Springer, 2006, 1-21.
- R.A. Klausen and R. Winther, Robust convergence of multi point flux approximations on rough grids, *Numer. Math.* **104** (2006), 317–337.
- J.M. Nordbotten, I. Aavatsmark and G.T. Eigestad, Monotonicity of control volume methods, To appear in *Numer. Math.*
- I. Aavatsmark, G.T. Eigestad, J.M. Nordbotten and B.T. Mallison, A compact MPFA method with improved monotonicity, *Numer. Methods Partial Diff. Eqns.* Submitted 2006.