Une méthode de pénalisation par face pour l’approximation des équations de Navier-Stokes à nombre de Reynolds élevé

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The addition of a term penalizing the jump of the gradient over element edges
\[
J(u_h, v_h) = \sum_K \int_{\partial K \setminus \partial \Omega} \gamma h_\partial^K [\nabla u_h \cdot n][\nabla v_h \cdot n] \, ds
\]
to the standard Galerkin formulation may be used to stabilize
- transport operators
- Stokes like systems
- symmetric Friedrichs systems

Error analysis for linear problems leads to (quasi) optimal apriori error estimates for continuous finite element spaces
\[
V_h = \{ \nu : \nu \in C^0(\Omega); \nu|_K \in P_k(K), \forall K \in T_h \}.
\]
State of the art: face penalty methods

The addition of a term penalizing the jump of the gradient over element edges

\[ J(u_h, v_h) = \sum_K \int_{\partial K \setminus \partial \Omega} \gamma h_\partial h^\alpha \langle \nabla u_h \cdot n \rangle \langle \nabla v_h \cdot n \rangle \, ds \]

to the standard Galerkin formulation may be used to stabilize

- transport operators
- Stokes like systems
- symmetric Friedrichs systems

**Theorem (Burman-Fernández-Hansbo (2004))**

There exists an interpolation operator \( \pi_h^* \) on \( V_h^k \) such that

\[ \| h^{1/2} \left( I - \pi_h^* \right) \nabla v_h \|_{0, \Omega}^2 \leq \gamma \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 \left[ \nabla v_h \cdot n \right]^2, \]

with \( \left[ v \right] \overset{\text{def}}{=} v^+ - v^- \) if \( \partial K \subset \Omega \) and \( \left[ v \right] \overset{\text{def}}{=} 0 \) if \( \partial K \subset \partial \Omega \).
Brief history

- Burman & Fernández, the incompressible Navier-Stokes equations, semidiscretization in space (2005).

Important related work:

Knowing the exact solution \((u, p)\) we could compute the ideal projection \((\pi_h u, \pi_h p) \in [V_h]^d \times V_h\).

Since the exact solution is unknown we have to do with a working projection given by a discrete scheme (typically Galerkin FEM).

The working projection should be stable and accurate uniformly in the Reynolds number: *standard Galerkin has to be modified.*

Assumption:
- the Bernoulli hypothesis: *all fine to coarse interaction is dissipative.*

We choose \(\pi_h\), (the ideal projection) to be the \(L^2\)-projection.
Let \( W = H(\text{div}) \times L^2_0, \) \( U = (u, p), \) \( L(\mathbf{w})U = (\mathbf{w} \cdot \nabla)u + \nabla p, \)
\( \pi^\perp = (I - \pi^*_h). \) Assume \( f \in [V_h]^d. \) Find \( U \in W \) such that
\[
(\partial_t u + L(u)U, v) + (\nabla \cdot u, q) = (f, v), \quad \forall (v, q) \in W.
\]
Stabilized methods based on scale separation, the Euler equations

1. Find $U \in W$ such that

$$(\partial_t u + L(u)U, v) + (\nabla \cdot u, q) = (f, v), \forall (v, q) \in W.$$  

2. Scale separation $U = U_h + \tilde{U}$, $U_h = \pi_h U$
   - $\pi_h U$ is the $L^2$-projection of $U$ onto $W_h = [V_h]^d \times V_h$
   - $\tilde{U}$ orthogonal to the finite element space (c.f. Codina).
Stabilized methods based on scale separation, the Euler equations

1. Find $U \in W$ such that

$$(\partial_t u + L(u) U, v) + (\nabla \cdot u, q) = (f, v), \forall (v, q) \in W.$$ 

2. Scale separation $U = U_h + \tilde{U}, U_h = \pi_h U$

3. Inserting $U_h + \tilde{U}$ yields the formulation

$$(\partial_t u_h + L(u_h) U_h, v_h) + (\nabla \cdot u_h, q_h) = (f, v_h)$$

$$+(T^{-1}(\pi^\perp L(u_h) U_h, \pi^\perp \nabla \cdot u_h), (\pi^\perp L(u_h) V_h, \pi^\perp \nabla \cdot v_h))$$

$$+((\tilde{u} \cdot \nabla) u, v_h) \quad \forall V = (v, q) \in W.$$ 

4. $T^{-1}$ is the solution operator for the fine scale equation

$$(\partial_t \tilde{u} + (u \cdot \nabla) \tilde{u} + (\tilde{u} \cdot \nabla) u_h + \nabla \tilde{p}, \tilde{v}) + (\nabla \cdot \tilde{u}, \tilde{q})$$

$$= (\pi^\perp L(u_h) U_h, \tilde{v}) + (\pi^\perp \nabla \cdot u_h, \tilde{q}).$$
Simplifications leading to edge oriented stabilization

1. We drop the fine to coarse interaction terms $((\tilde{u} \cdot \nabla)u, v_h)$.

2. Bernoulli hypothesis: approximate $T^{-1}$ with a scaled diagonal matrix.

3. Stabilized FEM based on the projected residual: Find $U_h \in W_h$ such that

$$
\begin{align*}
(\partial_t u_h + L(u_h) U_h, v_h) + (\nabla \cdot u_h, q_h) &= (f, v_h) \\
- (\delta_u \pi^\perp L(u_h) U_h, \pi^\perp L(u_h) V_h) - (\delta_{\text{div}} \pi^\perp \nabla \cdot u_h, \pi^\perp \nabla \cdot v_h) &= 0
\end{align*}
$$

4. ∀$V_h = (v_h, q_h) \in W_h$.

4. Equivalent dissipation (recall $\pi^\perp = (I - \pi^*_h)$):

$$
\| \pi^\perp L(u_h) U_h \|_K^2 \leq \sum_{e \in E(K)} \int_e \gamma_{hK} [L(u_h) U_h]^2 \, ds \\
\leq \sum_{e \in E(K)} \int_e \gamma_{hK} \left\{ [(u_h \cdot \nabla)u_h]^2 + [\nabla p_h]^2 \right\} \, ds
$$
For all $t \in (0, T)$, find $(u_h(t), p_h(t)) \in [V_h]^d \times V_h$ such that

\[
\begin{cases}
(\partial_t u_h, v_h) + a(u_h; u_h, v_h) + b(p_h, v_h) = (f, v_h), \\
- b(q_h, u_h) = 0,
\end{cases}
\]

\[
uh(0) = \pi_h u_0,
\]

for all $(v_h, q_h) \in [V_h]^d \times V_h$, with

\[
a(u_h; u_h, v_h) \overset{\text{def}}{=} (u_h \cdot \nabla u_h, v_h) + (\nu \nabla u_h, \nabla v_h) + \frac{1}{2}(\nabla \cdot u_h, u_h \cdot v_h) + \text{bd terms}
\]

\[
b(p_h, v_h) \overset{\text{def}}{=} -(p_h, \nabla \cdot v_h) + \text{bd terms}
\]
For all $t \in (0, T)$, find $(u_h(t), p_h(t)) \in [V_h]^d \times V_h$ such that

\[
\begin{aligned}
&\left(\partial_t u_h, v_h\right) + a(u_h, u_h, v_h) + b(p_h, v_h) + j_{u_h}(u_h, v_h) = (f, v_h), \\
&- b(q_h, u_h) + j(p_h, q_h) = 0,
\end{aligned}
\]

for all $(v_h, q_h) \in [V_h]^d \times V_h$, with

\[
\begin{aligned}
&j_{u_h}(u_h, v_h) \overset{\text{def}}{=} \sum_{K \in T_h} \int_{\partial K} \gamma h_K^2 (1 + |u_h \cdot n|^2) \langle \nabla u_h \rangle : \langle \nabla v_h \rangle, \\
&j(p_h, q_h) \overset{\text{def}}{=} \sum_{K \in T_h} \int_{\partial K} \gamma h_K^2 \langle \nabla p_h \rangle \cdot \langle \nabla q_h \rangle.
\end{aligned}
\]
Convergence (selected results)

- $J[u_h; (u_h, p_h), (v_h, q_h)] = j_u(u_h, v_h) + j(p_h, q_h)$
- Triple-norm: $\| (v_h, q_h) \|_{w_h}^2 \overset{\text{def}}{=} \| \nu^{\frac{1}{2}} \nabla v_h \|_{0,\Omega}^2 + J[w_h; (v_h, q_h), (v_h, q_h)]$.

Theorem (Velocity convergence, Burman & Fernández, 2005)

The following estimates hold (when $\nu < h$)

\[
\| \pi_h u - u_h \|_{L^\infty((0,T);L^2(\Omega))} \leq h^{\frac{3}{2}} C(u, p)e^{c(u)T},
\]

\[
\left( \int_0^T \| (\pi_h u - u_h, \pi_h p - p_h) \|_{u_h}^2 \, dt \right)^{\frac{1}{2}} \leq h^{\frac{3}{2}} C(u, p, T)e^{c(u)T},
\]

\[
\int_0^T J[u_h, (u_h, p_h), (u_h, p_h)] \, dt \leq h^3 C(u, p)e^{c(u)T}
\]

with $c(u)$ depending on $\| u \|_{L^\infty(0,T;W^{1,\infty}(\Omega))}$ and $C(u, p)$ depending on $\| u \|_{L^2(0,T;H^2(\Omega))}$, $\| p \|_{L^2(0,T;H^2(\Omega))}$, $\| u \|_{L^\infty(0,T;W^{1,\infty}(\Omega))}$.
Energy consistency: monitoring artificial dissipation

For the Navier-Stokes equations there holds

\[ \| \mathbf{u}(T) \|^2 + \int_0^T \| \nu^{\frac{1}{2}} \nabla \mathbf{u} \|^2 \, dt = \| \mathbf{u}(0) \|^2 + (\mathbf{f}, \mathbf{u}). \]

Any reasonable numerical method will satisfy

\[ \| \mathbf{u}_h(T) \|^2 + \int_0^T \left\{ \| \nu^{\frac{1}{2}} \nabla \mathbf{u}_h \|^2 + S(\mathbf{u}_h, \mathbf{p}_h) \right\} \, dt = \| \mathbf{u}_h(0) \|^2 + (\mathbf{f}, \mathbf{u}_h). \]

\( S(\mathbf{u}_h, \mathbf{p}_h) \) the artificial dissipation added for the method to remain stable.

- Define \( D = \frac{\int_0^T S(\mathbf{u}_h, \mathbf{p}_h) \, dt}{\int_0^T \| \nu^{\frac{1}{2}} \nabla \mathbf{u}_h \|^2 \, dt} : \)

\[ \| \mathbf{u}_h(T) \|^2 + (1 + D) \int_0^T \| \nu^{\frac{1}{2}} \nabla \mathbf{u}_h \|^2 \, dt = \| \mathbf{u}_h(0) \|^2 + (\mathbf{f}, \mathbf{u}_h). \]
The interior penalty operator can be seen as a subgrid viscosity: starting from polynomial order 3 and onward the kernel is a $C^1$ space with approximation properties.

Scale separation by polynomial order instead of hierarchic meshes.

The dissipation ratio $D$ measures the energy consistency and is (related to) an a posteriori error estimator.

For high Reynolds number flow theory predicts (P1 elements and sufficiently regular solution):

$$\text{computational error} \leq \text{numerical dissipation} = \text{stabilization} \leq Ch^3$$
Let us now assume $f = 0$ and consider the projection on mesh $\mathcal{T}_h$ of the exact solution

$$
\|\pi_h u(T)\|^2 + \|(I - \pi_h)u(T)\|^2 + \int_0^T \|\nu^{\frac{1}{2}} \nabla u\|^2 dt = \|u(0)\|^2,
$$

but $(I - \pi_h)u(T)$ represents the unresolved scales and hence

$$
\|(I - \pi_h)u(T)\|^2 \approx \int_{\xi_h}^\infty E(\xi) \, d\xi
$$

(where $E(\xi)$ denotes the energy distribution over the wave numbers)
Let us now assume $\mathbf{f} = 0$ and consider the projection on mesh $\mathcal{T}_h$ of the exact solution

$$\| \pi_h \mathbf{u}(T) \|^2 + \| (I - \pi_h) \mathbf{u}(T) \|^2 + \int_0^T \left\| \nu^\frac{1}{2} \nabla \mathbf{u} \right\|^2 dt = \| \mathbf{u}(0) \|^2,$$

but $(I - \pi_h) \mathbf{u}(T)$ represents the unresolved scales and hence

$$\| (I - \pi_h) \mathbf{u}(T) \|^2 \approx \int_{\xi_h}^{\infty} E(\xi) \, d\xi$$

(where $E(\xi)$ denotes the energy distribution over the wave numbers) leading to

$$\frac{\| \mathbf{u}(0) \|^2 - \| \pi_h \mathbf{u}(T) \|^2}{\int_0^T \left\| \nu^\frac{1}{2} \nabla \mathbf{u} \right\|^2 dt} \approx 1 + \frac{\int_{\xi_h}^{\infty} E(\xi) \, d\xi}{\int_0^T \left\| \nu^\frac{1}{2} \nabla \mathbf{u} \right\|^2 dt}$$
The continuous case:

\[
\frac{\|u(0)\|^2 - \|\pi_h u(T)\|^2}{\int_0^T \|\nu^{1/2} \nabla u\|^2 dt} \approx 1 + \frac{\int_0^\infty E(\xi) \, d\xi}{\int_0^T \|\nu^{1/2} \nabla u\|^2 dt}
\]

The discrete case:

\[
\frac{\|u_h(0)\|^2 - \|u_h(T)\|^2}{\int_0^T \|\nu^{1/2} \nabla u_h\|^2 dt} = 1 + D
\]
The continuous case:

$$\frac{\|u(0)\|^2 - \|\pi_h u(T)\|^2}{\int_0^T \|\nu^{\frac{1}{2}} \nabla u\|^2 dt} \approx 1 + \frac{\int_{\xi_h}^{\infty} E(\xi) \, d\xi}{\int_0^T \|\nu^{\frac{1}{2}} \nabla u\|^2 dt}$$

The discrete case:

$$\frac{\|u_h(0)\|^2 - \|u_h(T)\|^2}{\int_0^T \|\nu^{\frac{1}{2}} \nabla u_h\|^2 dt} = 1 + D$$

We conclude that if \(u_h \approx \pi_h u\) is to hold then

$$D \approx \frac{\int_{\xi_h}^{\infty} E(\xi) \, d\xi}{\int_0^T \|\nu^{\frac{1}{2}} \nabla u\|^2 dt}$$
Scale separation and the energy inequality

- The continuous case:

\[
\frac{\| u(0) \|^2 - \| \pi_h u(T) \|^2}{\int_0^T \| \nu^{1/2} \nabla u \|^2 dt} \approx 1 + \frac{\int_{\xi_h}^{\infty} E(\xi) \, d\xi}{\int_0^T \| \nu^{1/2} \nabla u \|^2 dt}
\]

- The discrete case:

\[
\frac{\| u_h(0) \|^2 - \| u_h(T) \|^2}{\int_0^T \| \nu^{1/2} \nabla u_h \|^2 dt} = 1 + D
\]

- We conclude that if \( u_h \approx \pi_h u \) is to hold then

\[
D \approx \frac{\int_{\xi_h}^{\infty} E(\xi) \, d\xi}{\int_0^T \| \nu^{1/2} \nabla u \|^2 dt}
\]

Remark: for standard Galerkin \( D = 0 \)!
Definition in 2D: \( E(\xi) \sim \xi |\hat{u}(\xi)|^2 \)

In 2D there holds for isotropic decaying turbulence: \( E(\xi) \sim \xi^{-3} \) (Kraichnan, 1967).

If \( \xi_h \approx h^{-1} \) is in the inertial range where \( E(\xi) \approx \xi^{-3} \) then

\[
D \approx C \frac{\int_{\xi_h}^{\xi_v} \xi^{-3} \, d\xi}{\int_0^T \| \nu^{\frac{1}{2}} \nabla u \|^2 \, dt} \approx C \frac{\xi_h^{-2} - \xi_v^{-2}}{\int_0^T \| \nu^{\frac{1}{2}} \nabla u \|^2 \, dt}.
\]

Assuming \( \xi_v^{-2} \) negligible we expect

\[
D \sim h^2
\]
Numerical Results: Turek benchmark $Re = 100$ flow around a cylinder, P1/P1

![Flow around a cylinder](image)

<table>
<thead>
<tr>
<th>NoDOFs</th>
<th>dt</th>
<th>$C_{D_{\text{max}}}$</th>
<th>$C_{L_{\text{max}}}$</th>
<th>$St$</th>
<th>$\Delta P$</th>
<th>D</th>
<th>$O(h^\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8667</td>
<td>0.01</td>
<td>3.2518</td>
<td>1.0438</td>
<td>0.2994</td>
<td>2.4989</td>
<td>0.1031</td>
<td></td>
</tr>
<tr>
<td>33132</td>
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<td>1.0377</td>
<td>0.3016</td>
<td>2.4875</td>
<td>0.0230</td>
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<td>131784</td>
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<td>3.2308</td>
<td>1.0262</td>
<td>0.3008</td>
<td>2.4697</td>
<td>0.0035</td>
<td>2.72</td>
</tr>
<tr>
<td>lower</td>
<td>-</td>
<td>3.22</td>
<td>0.99</td>
<td>0.2950</td>
<td>2.46</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>upper</td>
<td>-</td>
<td>3.24</td>
<td>1.01</td>
<td>0.3050</td>
<td>2.50</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Unit square, $u_\infty = 1$, $\sigma = \frac{1}{28}$, $\nu = 3.571 \cdot 10^{-6} \rightarrow Re_\sigma = 10000$.
Lesieur et al. proposed this problem as a model case for decaying 2D turbulence.
They showed numerically that $E(\xi)$ decays between $\xi^{-4}$ and $\xi^{-3}$ for the streamwise velocity component (Fourier transform only in the $x$-variable).
we expect: $c_1 h^3 < D < c_2 h^2$ to be consistent with Lesieur and $D \sim h^2$ to be consistent with Kraichnan.
Numerical Results: $Re = 10000$ mixing layer

The convergence of $D$ implies $E(\xi) \sim \xi^{-3}$ coherent with the scaling law of Kraichnan and with the numerical results of Lesieur.

<table>
<thead>
<tr>
<th>P1 el.</th>
<th>$D$</th>
<th>$O(h^{\alpha_D})$</th>
<th>$J(u_h, p_h)$</th>
<th>P2 el.</th>
<th>$D$</th>
<th>$O(h^{\alpha_D})$</th>
<th>$J(u_h, p_h)$</th>
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<tbody>
<tr>
<td>80</td>
<td>5.6</td>
<td>-</td>
<td>6E-4</td>
<td>40</td>
<td>0.38</td>
<td>-</td>
<td>5E-5</td>
</tr>
<tr>
<td>160</td>
<td>1.4</td>
<td>2.0</td>
<td>2E-4</td>
<td>80</td>
<td>0.1098</td>
<td>1.79</td>
<td>1.5E-5</td>
</tr>
<tr>
<td>320</td>
<td>0.3</td>
<td>2.22</td>
<td>4E-5</td>
<td>160</td>
<td>0.025</td>
<td>2.0</td>
<td>3.6E-6</td>
</tr>
</tbody>
</table>
Face oriented interior penalty methods work for incompressible flow at high Reynolds number.

Interaction with turbulence?

Future work focuses on complex flow problems such as:
- Incompressible flow in 3D at high Reynolds number (turbulence)
- Viscoelastic flow
- Freesurface flow
- Compressible flow
Mixing layer, Reynolds 10000, P1/80 $\times$ 80, P2/32 $\times$ 32, P2/160 $\times$ 160, $t=50,80,100$
Mixing layer, Reynolds 10000, P1/320 × 320, P2/32 × 32, P2/160 × 160, t=50,80,100
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IP for Navier-Stokes
P2/P2, 160 × 160, t=20,30,50,70,80,100,120,140,200

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IP for Navier-Stokes
P1/P1, 40 × 40, t=20,30,50,70,80,100,120,140,200
P1/P1, $80 \times 80$, $t=20, 30, 50, 70, 80, 100, 120, 140, 200$
P1/P1, 320 × 320, t=20,30,50,70,80,100,120,140,200