le cnam

Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 12 Green formula

- Integration by parts in one space dimension

We give ourselves two real numbers a and b so that $a < b$. Recall that the classical relation $\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = f(b) - f(a)$ can be written by introducing the exterior normal $n(x)$ at the two points a and b of the boundary $\partial([a,b]) = \{a,b\}$ of the interval $[a,b]$: $n(a) = -1$ and $n(b) = +1$. Then $\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = \sum_{x\in\partial[a,b]} f(x)\,n(x)$.

- Integrating a derivative in a rectangle

We introduce $a > 0$, $b > 0$ and the rectangle $\Omega =]0,a[\times]0,b[$. Its edges are parallel to the coordinate axes and the boundary $\partial\Omega$ is composed by four sides a_j for $1 \leq j \leq 4$. We have $a_1 = [0,a]\times\{0\}$, $a_2 = \{a\}\times[0,b]$, $a_3 = [0,a]\times\{b\}$ and $a_4 = \{0\}\times[0,b]$. Along each of these sides, an external unit normal n_j pointing outside the domaine can be defined and we have $n^1 = (0,-1)$, $n^2 = (1,0)$, $n^3 = (0,1)$ and $n^4 = (-1,0)$.
We evaluate the integral $I = \iint_\Omega \frac{\partial f}{\partial x}\,\mathrm{d}x\,\mathrm{d}y$ for a differentialble function f. Thanks to Fubini's theorem, we have
$I = \int_0^b \mathrm{d}y \int_0^a \mathrm{d}x \frac{\partial f}{\partial x} = \int_0^b \mathrm{d}y \big[f(a,y) - f(0,y)\big] = \int_{a_2} f\,n_x^2\,\mathrm{d}y + \int_{a_4} f\,n_x^4\,\mathrm{d}y$ and we can add to this expression the double sum $\int_{a_1} f\,n_x^1\,\mathrm{d}x + \int_{a_3} f\,n_x^3\,\mathrm{d}x$ because it is composed by two null terms. Then $I = \int_{\partial\Omega} f(M)\,n_x(M)\,\mathrm{d}s(M)$. We have a first conclusion in this specific case: $\iint_\Omega \frac{\partial f}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} f\,n_x\,\mathrm{d}s$. We proceed in the same way with the integral $J = \iint_\Omega \frac{\partial f}{\partial y}\,\mathrm{d}x\,\mathrm{d}y$.
With the other expression of Fubini's theorem, we have
$J = \int_0^a \mathrm{d}x \int_0^b \mathrm{d}y \frac{\partial f}{\partial y} = \int_0^a \mathrm{d}x \big[f(x,b) - f(x,0)\big] = \int_{a_3} f\,n_y^3\,\mathrm{d}x + \int_{a_1} f\,n_y^1\,\mathrm{d}y$. As previously, we add the two integrals $\int_{a_2} f\,n_y^2\,\mathrm{d}y$ and $\int_{a_4} f\,n_y^4\,\mathrm{d}y$ that do not contribute because $n_y^2 = n_y^4 = 0$. Then $J = \int_{\partial\Omega} f(M)\,n_y(M)\,\mathrm{d}s(M)$ and we have finally in this second case $\iint_\Omega \frac{\partial f}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} f\,n_y\,\mathrm{d}s$.
In conclusion, when we integrate a derivative, we obtain an integral on the boundary of the domain.

- Integration by parts in a bounded domain in two space dimensions

In the case of two dimensions, we give ourselves a bounded domain Ω of the plane $\mathbb{R}^2$: it is included in a sufficiently large rectangle. We assume that the boundary $\partial\Omega$ of Ω is fairly regular curve, which is sometimes noted Γ. For example, if Ω is the disk D of center the origin and radius $R > 0$, then its boundary ∂D is the circle of center the origin O and radius R. We

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denote $\overline{\Omega} = \Omega \cup \partial\Omega$ the union of Ω and its boundary $\partial\Omega$. We say that $\overline{\Omega}$ is the adherence of Ω. For a point $x \in \partial\Omega$ on the boundary of Ω, we denote $n(x)$ the directed normal vector that points towards the exterior of Ω. Recall that $n(x)$ is a unit vector and that the point x is at any position of the edge.

In the case of the disk D, a point on the boundary can be written $x = (x_1, x_2) = (R\cos\theta, R\sin\theta)$ with $\theta \in [0, 2\pi]$ the usual polar angle. Then the normal vector $n(x)$ has very simple coordinates in this case: $n(x) = (\cos\theta, \sin\theta)$.

Finally, we give ourselves a regular application f defined on the adherence $\overline{\Omega}$: $f\colon\, \overline{\Omega} \longrightarrow \mathbb{R}$. Then the theorem of integration by parts expresses that the integral of a derivative of the function f in the domain Ω reduces to a curvilinear integral on the boundary $\partial\Omega$:
$\iint_\Omega \frac{\partial f}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} f\, n_x\,\mathrm{d}s$ and $\iint_\Omega \frac{\partial f}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} f\, n_y\,\mathrm{d}s$. We express this property synthetically as $\iint_\Omega \frac{\partial f}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} f\, n_j\,\mathrm{d}s$ for the two components $(j = 1, 2)$ or even $\iint_\Omega \partial_j f\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} f\, n_j\,\mathrm{d}s$ with $\partial_j \equiv \frac{\partial}{\partial x_j}$.

A common consequence of this relation is obtained with $f = uv$. With the Leibniz rule of differentiation of the product of two functions, we have from the previous considerations the identity $\iint_\Omega \frac{\partial u}{\partial x_j}\, v\,\mathrm{d}x\,\mathrm{d}y = -\iint_\Omega u\,\frac{\partial v}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}y + \int_{\partial\Omega} u\,v\,n_j\,\mathrm{d}s$.

- Integral of the divergence of a vector field

We can combine these two relations by introducing a vector field $\Phi\colon\, \overline{\Omega} \ni (x, y) \longmapsto \Phi(x, y) \equiv (\Phi_x, \Phi_y) \in \mathbb{R}^2$ regular on the adherence of Ω. The divergence of the vector field Φ is by definition the scalar field $\mathrm{div}\,\Phi$ defined by $\mathrm{div}\,\Phi \equiv \frac{\partial \Phi_x}{\partial x} + \frac{\partial \Phi_y}{\partial y}$. If we note exceptionally with a point the scalar product $\Phi.n \equiv \Phi_x n_x + \Phi_y n_y$ along of the boundary, the previous two relations can be written as $\iint_\Omega \mathrm{div}\,\Phi\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} (\Phi.n)\,\mathrm{d}s$.

Exercices

- Integration by parts of a vector field

Let D be a two-dimensional domain ($D \subset \mathbb{R}^2$) and n the external normal along the boundary ∂D. Let $\Phi\colon\, D \longrightarrow \mathbb{R}^2$ a regular vector field.

a) Prove that we have the relation $\int_D \mathrm{div}\Phi\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial D} (\Phi, n)\,\mathrm{d}s$.

b) Deduce from the previous question that we have $\int_D \mathrm{d}x\,\mathrm{d}y = \int_{\partial D} x\,n_x\,\mathrm{d}s = \int_{\partial D} x\,\mathrm{d}y$.

c) Deduce from the first question that we have $\int_D \mathrm{d}x\,\mathrm{d}y = \int_{\partial D} y\,n_y\,\mathrm{d}s = -\int_{\partial D} y\,\mathrm{d}x$.

- Computation of surfaces

We denote by K the square $[0, a] \times [0, a]$.

a) With the help of the second question of the previous exercice, recover the surface $|K|$ of the square K.

b) Same question using the third question of the previous exercice.

- A curious expression of the surface

Let Ω be a subset of the plane $\mathbb{R}^2$. We suppose that Ω is bounded and that the boundary $\partial\Omega$ is regular. The external normal along the boundary $\partial\Omega$ is denoted by $n(x)$. We suppose that the tangent vector $\tau(x) \equiv \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right)$ is such that the local basis $(n(x), \tau(x))$ is orthonormal and

direct in the plane $\mathbb{R}^2$ for the canonical scalar product. We set
$I = \int_{\partial\Omega} (-y \sin^2 x \, \mathrm{d}x + \frac{1}{2}(x + \sin x \cos x) \, \mathrm{d}y)$.

a) Introduce the two components n_x et n_y of the external normal and the curvilinear abscissa $\mathrm{d}s$ in the expression of the curvilinear integral I.

b) Show that we have the relation $I = \int_{\partial\Omega} (\Phi_x n_x + \Phi_y n_y) \, \mathrm{d}s$ for a vector field $\Phi \equiv (\Phi_x, \Phi_y)$ that is to precise.

c) What is the value of $\delta \equiv \mathrm{div}\Phi$?

d) Prove that the surface $|\Omega|$ of the set Ω can be computed with the help of the expression $|\Omega| = \int_{\partial\Omega} (-y \sin^2 x \, \mathrm{d}x + \frac{1}{2}(x + \sin x \cos x) \, \mathrm{d}y)$.

- Variational formulation for an elliptic problem

The domain Ω is a bounded part in $\mathbb{R}^2$. We denote by $\Gamma \equiv \partial\Omega$ its boundary. It is supposed to be regular. If v and w are two regular scalar functions defined on the set $\Gamma \equiv \partial\Omega$ and n the external boundary to Γ, we recall that $\frac{\partial v}{\partial n} \equiv \nabla v \,.\, n \equiv \sum_j \frac{\partial v}{\partial x_j} n_j$.

a) Give some examples of such a domain. Precise the geometrical nature of the boundary Γ and give some information about the external normal n.

Let f be a regular given scalar function defined on the set $\Omega \cup \Gamma$. We consider the following problem: search a scalar funtion u defined Ω such that $-\Delta u = f$ in Ω and $u = 0$ on the boundary Γ. We recall that $\Delta u \equiv \sum_j \frac{\partial^2 u}{\partial x_j^2}$. This problem is called the homogeneous Dirichlet problem for the Poisson equation.

b) Show that $-\int_\Omega \Delta v \, w \, \mathrm{d}x = \int_\Omega \nabla v \,.\, \nabla w \, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial v}{\partial n} \, w \, \mathrm{d}\gamma$. Let u and v be two functions that are both solution of the problem $-\Delta \zeta = f$ in Ω and $\zeta = 0$ on Γ.

c) What is the system of equations satisfied by the difference $\varphi \equiv u - v$?

d) Deduce from the previous questions that for an arbitrary funtion w identically equal to zero on the boundary Γ, we have $\int_\Omega \nabla \varphi \,.\, \nabla w \, \mathrm{d}x = 0$.

e) Deduce from the previous questions that the function φ is identically null and that the Dirichlet problem $-\Delta u = f$ in Ω and $u = 0$ on Γ admits at most one regular solution.