le cnam

Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 10 Double integral

- Recall on the simple integral

We suppose given two reals $a < b$ and a function $f \,:\, [a,\, b] \longrightarrow \mathbb{R}$. The integral $\int_a^b f(x)\, \mathrm{d}x$ of f on the interval $[a,\, b]$, denoted also as $\int_{[a,\,b]} f$, is a real number satisfying the following properties :

⋆ Length. If $f(x) = 1$ for all x, then $\int_a^b \mathrm{d}x = b - a$.

⋆ Linearity. If f and g are two functions $[a,\, b] \longrightarrow \mathbb{R}$ and λ is a real number, we have $\int_a^b \big(f(x) + g(x)\big)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x + \int_a^b g(x)\, \mathrm{d}x$ and $\int_a^b \big(\lambda\, f(x)\big)\, \mathrm{d}x = \lambda\, \int_a^b f(x)\, \mathrm{d}x$.

⋆ Positivity. If f is a positive function, *id est* $f(x) \geq 0$ for all x, then the corresponding integral is positive : $\int_a^b f(x)\, \mathrm{d}x \geq 0$. We remark that this property can be in defect if we do not suppose $a < b$.

⋆ Additivity relative to the domain (Chasles's relation). If $a < c < b$, then $\int_a^b f(x)\, \mathrm{d}x = \int_a^c f(x)\, \mathrm{d}x + \int_c^b f(x)\, \mathrm{d}x$.

These properties can be extended to the double integral. It is not the case for the following ones.

- Specific properties of the simple integral

⋆ Fundamental theorem of Analysis and integration by parts. We suppose that f a continuous function $[a,\, b] \longmapsto \mathbb{R}$. For example, f can be a polynomial, a sinus or cosinus function, an exponential function, the absolute value or the composite of two continuous function. But f can not be the Heaviside function H proposed initially by Oliver Heaviside (1850-1925) and defined by $H(x) = 0$ for $x \leq 0$ and $H(x) = 1$ when $x > 0$. Then the mapping ψ defined according to $\psi(x) = \int_a^x f(\xi)\, \mathrm{d}\xi$ is a derivable function of the variable x and $\frac{\mathrm{d}}{\mathrm{d}x}\big(\int_a^x f(\xi)\, \mathrm{d}\xi\big) = f(x)$.
In consequence, $\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\, \mathrm{d}\xi = f(b) - f(a)$ if f is a continuously derivable function. The proof of this result introduces the function g defined according to $g(x) = f(x) - \int_a^x \frac{\mathrm{d}f}{\mathrm{d}\xi}\, \mathrm{d}\xi$. Then g is derivable and $\frac{\mathrm{d}g}{\mathrm{d}x} = 0$ on the interval $[a,\, b]$. In consequence, the function g is a constant function : $g(a) = g(b)$. This relation means that $f(a) = f(b) - \int_a^b \frac{\mathrm{d}f}{\mathrm{d}\xi}\, \mathrm{d}\xi$ and the result is proven. □

In practice, this result is expressed as follows : If we suppose given a primitive function F of the function f (that is $\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$), then $\int_a^b f(x)\, \mathrm{d}x = F(b) - F(a)$.

François Dubois, december 2023.

⋆ Change of variable. We suppose that the interval $[a,\, b]$ is parametrized by a one to one increasing function φ from $[\alpha,\, \beta]$ onto $[a,\, b]$: $x = \varphi(t)$ with $t \in [\alpha,\, \beta]$. Then $\int_a^b f(x)\, \mathrm{d}x = \int_\alpha^\beta f\big(\varphi(t)\big)\, \varphi'(t)\, \mathrm{d}t$.

⋆ Calculus of surfaces. If the function f is positive from $[a,\, b]$ to $\mathbb{R}$ (with $a < b$), then the integral $\int_a^b f(x)\, \mathrm{d}x$ is equal to the area $|\Omega|$ of the domain Ω between the abscissae a and b on one hand, the abscissa axis and the curve $y = f(x)$ on the other hand : $\Omega = \{(x,\, y) \in \mathbb{R}^2,\, a \leq x \leq b,\, 0 \leq y \leq f(x)\}$. Then we have $\int_a^b f(x)\, \mathrm{d}x = |\Omega|$.

- Fundamental properties of the double integral

We supose given a bounded subset Ω of the plane $\mathbb{R}^2$ and a bounded function $f : \Omega \longrightarrow \mathbb{R}$. The double integral of the function f on the domain Ω is a real number. It is noted $\int_\Omega f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y$ or $\iint_\Omega f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y$ and often more simply $\int_\Omega f\, \mathrm{d}x\, \mathrm{d}y$ or $\int_\Omega f$.

⋆ Surface : double integral of the function "one". We introduce four real numbers a, b, c and d such that $a < b$ and $c < d$. We consider the rectangle $\Omega =]a,\, b[\times]c,\, d[$ of the plane $\mathbb{R}^2$. The double integral of the function $f(x,\, y) \equiv 1$ is simply the area $(b-a)\,(d-c)$ of the rectangle : $\int_{]a,\, b[\times]c,\, d[}\, \mathrm{d}x\, \mathrm{d}y = (b-a)\,(d-c)$.
More generally, if Ω is a bounded part of the plane, that is if Ω is included in a large rectangle, the double integral on Ω of the function $f(x,\, y) \equiv 1$ is exactly the surface $|\Omega|$ of the domain : $\int_\Omega\, \mathrm{d}x\, \mathrm{d}y = |\Omega|$.

⋆ Linearity. We suppose that the double integral $\int_\Omega f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y$ of the function f is known and we give a number λ. Then $\int_\Omega (\lambda\, f)(x,\, y)\, \mathrm{d}x\, \mathrm{d}y = \lambda \int_\Omega f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y$. If we give also the double integral $\int_\Omega g(x,\, y)\, \mathrm{d}x\, \mathrm{d}y$ of the function g, then $\int_\Omega (f+g)(x,\, y)\, \mathrm{d}x\, \mathrm{d}y = \int_\Omega f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y + \int_\Omega g(x,\, y)\, \mathrm{d}x\, \mathrm{d}y$.

⋆ Positivity. We suppose that the function f is positive on Ω: $f(x,\, y) \geq 0,\, \forall\, (x,\, y) \in \Omega$. Then $\int_\Omega f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y \geq 0$. If $f \leq g$ on Ω that is $f(x,\, y) \leq g(x,\, y)$ for all $(x,\, y) \in \Omega$, then $\int_\Omega f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y \leq \int_\Omega g(x,\, y)\, \mathrm{d}x\, \mathrm{d}y$ [exercice].

⋆ Additivity relative to the domain. We suppose the domain Ω is decomposed into a finite set of simpler sub-domains: Ω_i: $\Omega = \cup_{i=1}^N \Omega_i$. Moreover, the intersection $\Omega_i \cap \Omega_j$ has a null area if $i \neq j$: $|\Omega_i \cap \Omega_j| = 0$. Then the integral on Ω of any function f is the sum of the integrals on each sub-domain Ω_i of this function : $\int_\Omega f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y = \sum_{i=1}^N \int_{\Omega_i} f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y$.

- Integral of a tiered function

We suppose given a decomposition of the domain Ω as above and a tired function f on Ω, that is a constant function in each part Ω_i: $\forall\, i = 1, \ldots, N,\ \exists\, \lambda_i \in \mathbb{R},\ \forall\, (x,\, y) \in \Omega_i,\ f(x,\, y) = \lambda_i$. The calculus of the integral of f on Ω is explicit : $\int_\Omega f(x,\, y)\, \mathrm{d}x\, \mathrm{d}y = \sum_{i=1}^N \lambda_i\, |\Omega_i|$ [exercice].

- Integral of a continuous function

We consider again a bounded domain $\Omega \subset \mathbb{R}^2$ and $f \in \mathscr{C}^0\big(\overline{\Omega}\big)$ a continuous function on Ω up to the boundary. Then the integral of f on Ω is well defined ; it is a real number, or eventually a complex number.

- Fubini theorem [Guido Fubini (1879-1943)]

We consider a bounded domain $\Omega \subset \mathbb{R}^2$ and a bounded function defined on Ω with real or eventually complex values : $\exists\, M \geq 0, \forall\, (x,\, y) \in \Omega,\, |f(x,\, y)| \leq M$. Then the integral of the absolute value of f is finite : $\iint_\Omega |f(x, y)|\, \mathrm{d}x\, \mathrm{d}y < \infty$. Moreover, the double integral of f in the domain Ω is well defined and we have the inequality $|\iint_\Omega f(x, y)\, \mathrm{d}x\, \mathrm{d}y| \leq \iint_\Omega |f(x, y)|\, \mathrm{d}x\, \mathrm{d}y$. The Fubini theorem says that it is always possible to integrate this function "in the order we want".

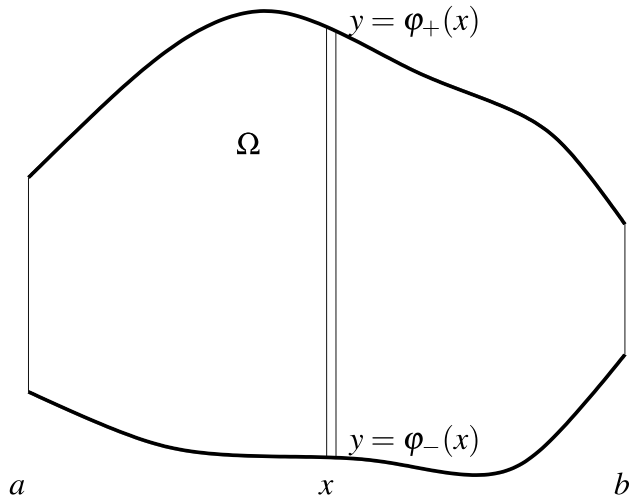

Figure 1. Calculus of the double integral in the domain Ω, first by the integration of the function f relative to y between $\varphi_-(x)$ and $\varphi_+(x)$, and secondly by simple integration relative to x of the result, between a and b.

More precisely, if Ω is between two regular curves of the form $y = \varphi(x)$ as in Figure 1, that is $\Omega = \{(x, y) \in \mathbb{R}^2, a \leq x \leq b, \varphi_-(x) \leq y \leq \varphi_+(x)\}$, we have
$\int_\Omega f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_a^b \mathrm{d}x \big[\int_{\varphi_-(x)}^{\varphi_+(x)} \mathrm{d}y\, f(x, y) \big]$.
If Ω is included between two curves of the type $x = \psi(y)$ like in Figure 2, *id est*
$\Omega = \{(x, y) \in \mathbb{R}^2, c \leq y \leq d, \psi_-(y) \leq x \leq \psi_+(y)\}$, we have
$\int_\Omega f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_c^d \mathrm{d}y \big[\int_{\psi_-(y)}^{\psi_+(y)} \mathrm{d}x\, f(x, y) \big]$. When the domain Ω can be parametrized in two ways, the double integral can be computed by one relation or the other and we have
$\int_\Omega f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_a^b \mathrm{d}x \big[\int_{\varphi_-(x)}^{\varphi_+(x)} \mathrm{d}y\, f(x, y) \big] = \int_c^d \mathrm{d}y \big[\int_{\psi_-(y)}^{\psi_+(y)} \mathrm{d}x\, f(x, y) \big]$.

- A first example of using Fubini theorem

We suppose given two real numbers $a < b$ and an integrable function $g : [a, b] \longrightarrow \mathbb{R}$. We suppose that the function g is positive : $\forall x \in [a, b],\ g(x) \geq 0$. We consider the domain
$\Omega = \{(x, y) \in \mathbb{R}^2, a \leq x \leq b, 0 \leq y \leq g(x)\}$ considered previously with the study of simple integrals. Then the Fubini theorem, with $\varphi_- = 0$, $\varphi_+ = g$ and $f(x, y) \equiv 1$ for each $(x, y) \in \Omega$, allows to conclude that $|\Omega| = \int_a^b g(x)\, \mathrm{d}x$. We recover the link between the simple integral and the calculus of surfaces.

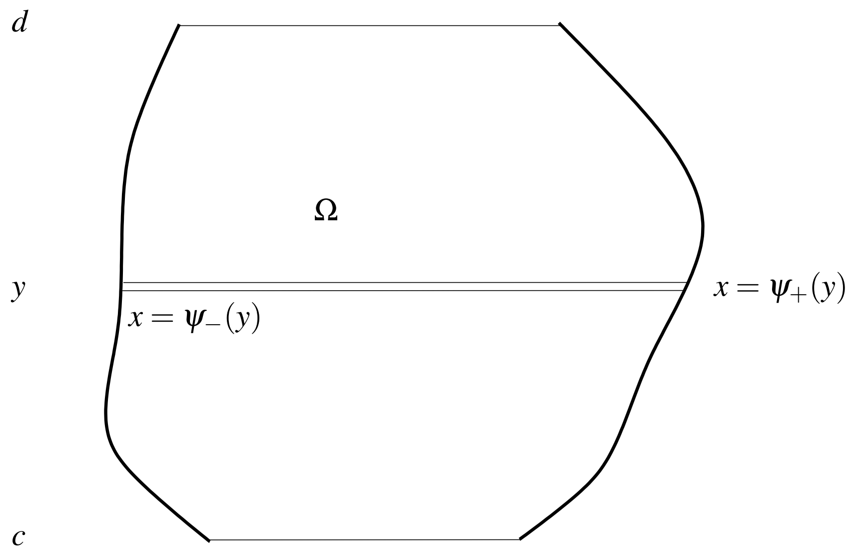

Figure 2. Calculus of the double integral in the domain Ω, first by the integration of the function f relative to x between $\psi_-(y)$ and $\psi_+(y)$, and secondly by simple integration relative to y of the result, between c and d.

Exercices

• Inequalities

a) If $f \le g$ on Ω that is $f(x, y) \le g(x, y)$ for all $(x, y) \in \Omega$, prove that we have the following inequality between numbers $\int_\Omega f(x, y)\, \mathrm{d}x\, \mathrm{d}y \le \int_\Omega g(x, y)\, \mathrm{d}x\, \mathrm{d}y$.

If f is a bounded function in the bounded domain Ω, we introduce the two positive functions $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

b) Prove that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

c) Deduce from the previous question that we have $|\iint_\Omega f(x, y)\, \mathrm{d}x\, \mathrm{d}y| \le \iint_\Omega |f(x, y)|\, \mathrm{d}x\, \mathrm{d}y$.

• Rectangular domains

a) Let D be the domain $D = \{(x, y) \in \mathbb{R}^2, 0 \le x \le 1,\ 0 \le y \le 2\}$.
Compute the double integral $\int_D x\, y\, \mathrm{d}x\, \mathrm{d}y$. [1]

b) Same question with the domain $\Delta = \{(x, y) \in \mathbb{R}^2, 0 \le x \le \pi,\ 0 \le y \le \frac{\pi}{2}\}$ and the integral $J = \int\int_\Delta x \sin(x+y)\, \mathrm{d}x\, \mathrm{d}y$. [$\pi - 2$]

• Double integral in a triangle

We suppose given two real numbers $a > 0$ and $b > 0$ and the triangle
$T = \{(x, y) \in \mathbb{R}^2, x \ge 0, y \ge 0, \frac{x}{a} + \frac{y}{b} \le 1\}$. We consider the function $f(x, y) = x - y$.

a) Prove that the double integral of the absolute value $|f|$ on the triangle T is finite.

b) Compute the double integral $I = \iint_T f(x, y)\, \mathrm{d}x\, \mathrm{d}y$ with the first approach suggested by the Fubini theorem: $I = \iint_T f(x, y)\, \mathrm{d}x\, \mathrm{d}y = \int_0^a \mathrm{d}x \big[\int_?^? \mathrm{d}y\, (x - y)\big]$ with an appropriate substitution of the question marks with an algebraic expression.

c) Compute the double integral I with the second approach suggested by the Fubini theorem: $I = \iint_T f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^b \mathrm{d}y \left[\int_?^? \mathrm{d}x\,(x-y)\right]$. The question marks will be replaced with a corect algebraic expression.

d) Compare your results of questions b) and c). Are they identical? $[\frac{1}{6}\,a\,b\,(a-b)]$

- Exchanging the order of integration

We consider a function f well defined for all reals x and y.
Complete the two expressions of the double integral $\int_0^1 \mathrm{d}y \int_y^{\sqrt{y}} \mathrm{d}x\, f(x,y) = \int_?^? \mathrm{d}x \int_?^? \mathrm{d}y\, f(x,y)$ obtained after the exchange of the order of the integrals.

[remark that $\Omega = \{(x,y) \in \mathbb{R}^2, 0 \le y \le 1, y \le x \le \sqrt{y}\}$]