le cnam

Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 2 Linear algebra

- A fundamental example

We introduce the set P_1 of all affine functions. We say the a map f from $\mathbb{R}$ to $\mathbb{R}$ belongs to the space P_1 if and only if there exits a and b in $\mathbb{R}$ such that for each $t \in \mathbb{R}$, $f(t) = a\,t + b$. In other words $P_1 = \{f : \mathbb{R} \longrightarrow \mathbb{R}, \exists a, b \in \mathbb{R}, \forall t \in \mathbb{R}, f(t) = a\,t + b\}$.

The sum $f+g$ of two affine functions is again an affine function. If $g(t) = \alpha\,t + \beta$, the map $f+g$ is defined by the relation $(f+g)(t) = f(t) + g(t)$. Then $(f+g)(t) = (a+\alpha)\,t + (b+\beta)$ and the sum $f+g$ is again an affine function. The addition of two functions of P_1 is a new function in the space P_1.

The external multiplication of a scalar λ by an element $f \in P_1$ is defined by the relation $(\lambda.f)(t) = \lambda\, f(t)$ for each $t \in \mathbb{R}$. We observe that the result of this external product af a scalar by an affine function is again an affine function because $(\lambda.f)(t) = (\lambda\, a)\,t + (\lambda\, b)$ for every argument $t \in \mathbb{R}$.

- Vector space

A vector space $(E, +, \cdot)$ is the datum of a set of vectors E, an addition $E \times E \longrightarrow E$ associating a unique vector $x+y$ to each pair $(x, y) \in E^2$, and an external multiplication of a scalar by a vector $\mathbb{R} \times E \longrightarrow E$: for each $\lambda \in \mathbb{R}$ and an arbitrary $x \in E$ the vector vecteur $\lambda \cdot x$ belongs to the space E.

The addition in the vector space E defines an commutative group: we have the associativity: $(x+y)+z = x+(y+z)$, the commutativity: $x+y = y+x$, the existence of a neutral element: $x+0 = 0+x = x$ and each vector has en opposite: $x+(-x) = (-x)+x = 0$. Moreover, the external multiplication by a scalar is coherent with the addition and the usual multiplication by numbers: $1 \cdot x = x$, $(\lambda+\mu)\cdot x = (\lambda \cdot x) + (\mu \cdot x)$, $\lambda \cdot (x+y) = (\lambda \cdot x) + (\lambda \cdot y)$ and $\lambda \cdot (\mu \cdot x) = (\lambda\, \mu) \cdot x$.

A space vector allows a lot of calculus. In particular, it extends for spaces of functions the common properties of vectors in the ordinary three-dimensional euclidian space.

For any integer $n \geq 1$, the set $\mathbb{R}^n$ is a vector space on the field of numbers with the usual addition, component by component. We have an analogous property in $\mathbb{C}^n$ with numbers chosen as complex numbers. If $m \geq 1$ is an other integer, the set $\mathscr{M}_{nm}(\mathbb{R})$ of matrices with n lines and

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m columns is also a vector space on the associated field of numbers. The space P_1 introduced previously is also a vector space on real numbers. The associated elements can be named as "vectors", even if they are functions!

- Linear combination

We suppose given an integer $n \geq 1$ and a family $x_1, \ldots, x_n$ of vectors in the vector space E. We suppose also given a family $\lambda_1, \ldots, \lambda_n$ of numbers. A linear combination of these vectors associated with this family of numbers is a vector $x \in E$ that can be written under the form $x = \lambda_1 \cdot x_1 + \ldots + \lambda_n \cdot x_n = \sum_{j=1}^n \lambda_j \cdot x_j$.
For example, if f_0 is the constant function equal to 1 in the space P_1 (this means that $f_0(t) = 1$ for each $t \in \mathbb{R}$), and if f_1 is the linear function $\mathbb{R} \ni t \longmapsto t \in \mathbb{R}$, the linear combination of these two vectors associated with the real numbers b and a is the resulting linear combination $f = b\, f_0 + a\, f_1$; it is simply the affine function $\mathbb{R} \ni t \longmapsto a\,t + b \in \mathbb{R}$.
Remark that when we write $f(t) = a\,t + b$, we write an equality betwen numbers whereas if we write $f = a\, f_1 + b\, f_0$, this is an equality between two functions (or two vectors !).

- Vector subspace

We suppose given a vector space $(E, +, \cdot)$ and a subset $F \subset E$ of this space. This set is a vector subspace if and only if the addition in E and the external multiplication by numbers, well defined in $F \subset E$ allows the triple $(F, +, \cdot)$ to be a vector space.
A necessary and sufficient condition for a subset F of the vector space E to be a vector subspace is first that F containts 0, the neutral element for the addition in E and secondly that any linear combination of vectors in F belongs again in the subset F. This last condition can be also formulated as follows: for each pair of vectors x and y in F, the sum $x + y$ belongs to à F and for each scalar λ and each vector $x \in F$, the product $\lambda \cdot x$ belongs again in F.
For example, the set F_0 of constant functions is a vector subspace of space P_1. Similarly, the set F_1 of all linear functions is also a vector subspace of space P_1.

- Vector subspace generated by a finite family of vectors

We suppose given a finite family $x_1, \ldots, x_n$ of vectors in the vector space E. The set $< x_1, \ldots, x_n >$ of all linear combinations of the form $\sum_{j=1}^n \lambda_j \cdot x_j$ is a subspace of the vector space E. By definition, it is the vector subspace $< x_1, \ldots, x_n >$ generated by this family of n vectors. We have: $< x_1, \ldots, x_n >= \{\sum_{j=1}^n \lambda_j \cdot x_j,\, \lambda_1, \ldots, \lambda_n \in \mathbb{R}\}$.
We have for example with the notations introduced previously $< f_0 >= F_0$ and $< f_1 >= F_1$.

- Basis if a vector space with a finite dimension

We consider a vector space E and an integer $n \geq 1$. A basis $(e_1, e_2, \ldots, e_n)$ of the space E is a family of vectors such every vector $x \in E$ as a linear combination in $< e_1, \ldots, e_n >$ in a unique way: $x = \sum_{j=1}^n x_j \cdot e_j$. The scalar coefficients $x_1, \ldots, x_n$ exist and are unique: $\forall x \in E,\, \exists!\, x_1, \ldots, x_n \in \mathbb{R},\, x = \sum_{j=1}^n x_j \cdot e_j$. The coefficients $x_1, \ldots, x_n$ are called the coordinates of the vector x relatively to the basis $(e_1, e_2, \ldots, e_n)$.
For example, in the space P_1 introduced previously, the family (f_0, f_1) is a basis.

- Dimension of a vector space

If the vector space E admits a basis composed with exactly n vecteurs, we say that the space E if of dimension n: we write $\dim E = n$.
We have for example $\dim P_1 = 2$, $\dim \mathbb{R}^n = n$ and $\dim \mathscr{M}_{nm} = nm$. If $n = m = 2$, we have the decomposition $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ that explicits a basis of spae $\mathscr{M}_{2,2} \equiv \mathscr{M}_2$.

- Linear map

We consider two vector spaces E and F and a map $u : E \longrightarrow F$: for each $x \in E$, there exists a unique vector $y = u(x)$ image of x by the map u. We say that the map u is linear if and only if the two following conditions of compatibility are satisfied. Compatibility with the addition: $\forall x, y \in E$, $u(x+y) = u(x) + u(y)$, and compatibility with the external multiplication:
$\forall \lambda \in \mathbb{R}$, $u(\lambda.x) = \lambda.u(x)$. Examples of such linear maps are proposed in the first exercice of this chapter.
We denote by $\mathscr{L}(E,F)$ the set of all linear maps from E to F. This set if a vector space with an addition defined by $\forall x \in E$, $(u+v)(x) = u(x) + v(x)$ and an external multiplication satisfying $\forall \lambda \in \mathbb{R}$, $\forall x \in E$, $(\lambda.u)(x) = \lambda.u(x)$. If $F = E$, we reduce the notation with $\mathscr{L}(E) \equiv \mathscr{L}(E,E)$, space of endomorphisms of the vector space E.

- Kernel and image

If $u \in \mathscr{L}(E,F)$ is a linear map between the two spaces E and F, the kernel $\mathrm{Ker}\, u$ is defined by $\mathrm{Ker}\, u = \{x \in E,\, u(x) = 0\}$. It is a vector sub-space of the space E.
With the same linear map $u \in \mathscr{L}(E,F)$, the image $\mathrm{Im}\, u$ is included in F and is defined by $\mathrm{Im}\, u = \{y \in F, \exists x \in E, y = u(x)\}$. It is a vector sub-space of the space F.
Observe that if $\dim E = n$, then we have always the relation $\dim(\mathrm{Ker}\, u) + \dim(\mathrm{Im}\, u) = \dim E$.

- Matrix of a linear map relatively to a set of bases

We consider a vector space E of finite dimension n and we introduce a basis $(e_1, e_2, \dots, e_n)$ of this space. We suppose given also a vector space F of dimension p and we introduce a basis $(f_1, f_2, \dots, f_p)$ of the vector space F. For $j = 1, \dots, n$, the vector $u(e_j) \in F$ can be secomposed in a unique way in the basis $(f_1, f_2, \dots, f_p)$: there exists unique coefficients $a_{1j}, a_{2j}, \dots, a_{pj}$ in such a way that $u(e_j) = \sum_{i=1}^{p} a_{ij} \cdot f_i$. We regroup these $n\,p$ coefficients into a matrix $M_u \equiv \big(a_{ij}\big)_{1 \le i \le p, 1 \le j \le n}$ with p lines and n columns. This matrix is the matrix of the linear map u relatively to the bases $(e_1, e_2, \dots, e_n)$ of E and $(f_1, f_2, \dots, f_p)$ of F. We can write it in the following way: $M_u = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pn} \end{pmatrix}$.

- Output of a given vector

With the previous notations, we regroup the components $x_1, x_2, \dots, x_n$ of the vector $x = \sum_{j=1}^{n} x_j \cdot e_j$ in the basis $(e_1, e_2, \dots, e_n)$ of E into a single vector X with one column and n lines: $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

Analogously, the coordinates $y_1, y_2, \dots, y_p$ of the vector $y = u(x) = \sum_{i=1}^{p} y_i \cdot f_i$ in the basis $(f_1, f_2, \dots, f_p)$ of F are presented with a vector Y with one column and p liges :

$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$.

Then the coordinates $y_i = \sum_{j=1}^{n} a_{ij} x_j$ can be expressed with the help of the product of the matrix M_u with the vector X: $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_p \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = M_u \cdot X.$

The coordinates Y of the image vector $u(x)$ are obtained by the mutiplication of the matrice M_u of operator u by the coordinates X of the vector $x \in E$.

- Composition of linear maps and product of matrices

We consider now three vector spaces D, E and F with various dimensions q, n and p and two linear maps $v : D \longrightarrow E$ from D to E and $u : E \longrightarrow F$ from E to F. Thus we have the following diagam $D \xrightarrow{v} E \xrightarrow{u} F$ that allows to define the composed map $u_\circ v$: $(u_\circ v)(\xi) = u\big(v(\xi)\big)$ for an arbitrary vector $\xi \in D$. The compoosition $u_\circ v$ of these two linear maps is also a linear map.

We consider a basis $(d_1, d_2, \dots, d_q)$ of the space D and do not forget the two previous families $(e_1, e_2, \dots, e_n)$ and $(f_1, f_2, \dots, f_p)$ in the spaces E and F respectively. We suppose that $v(d_k) = \sum_{j=1}^{n} b_{jk} e_j$. Then in the bases (d_k) and (e_j), the map v is represented by a matrix M_v with n lines and q columns that can be written $M_v = \big(b_{jk}\big)_{1 \le j \le n,\, 1 \le k \le q}$. We have $(u_\circ v)(d_k) = \sum_{i=1}^{p} \big(\sum_{j=1}^{n} a_{ij} b_{jk}\big) f_i$ an this means that relatively to the bases (d_k) et (f_i), the map $u_\circ v$ obtained by compostion admits a matrix $M_{u_\circ v} = (c_{ik})$ with p lines and q columns with $c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$ for $1 \le i \le p$ and $1 \le k \le q$. We note that the matrix $M_{u_\circ v}$ is equal to the product of the matrices M_u and M_v: $M_{u_\circ v} = M_u\, M_v$, which means

$$\begin{pmatrix} c_{11} & \cdots & c_{1k} & \cdots & c_{1q} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ik} & \cdots & c_{iq} \\ \vdots & & \vdots & & \vdots \\ c_{p1} & \cdots & c_{pk} & \cdots & c_{pq} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{p1} & \cdots & a_{pj} & \cdots & a_{pn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1k} & \cdots & b_{1q} \\ \vdots & & \vdots & & \vdots \\ b_{j1} & \cdots & b_{jk} & \cdots & b_{jq} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nk} & \cdots & b_{nq} \end{pmatrix}.$$

In this product of two matrices, $M_{u\circ v} = M_u\, M_v$, we remark that the number n of columns of the left matrix (these M_u) is equal to the number of lines n of the matrix on the right (here M_v). In the other cases, the product $M_u\, M_v$ cannot be defined.

Exercices

- Matrix of linear operators

We introduce the space P_1 of affine functions, the functions f_0 and f_1 defined by the relations $f_0(t) = 1$ and $f_1(t) = t$ for any arbitrary $t \in \mathbb{R}$. We consider also the vector space $F = \mathbb{R}$. Let u the map that to each $f \in P_1$ of the form $f(t) = a\,t + b$ associates the number a: $u(f) = a$. With the same notations for the function f, we define also the map v such that $v(f) = b$.

a) Recall why the family (f_0, f_1) is a basis of the space P_1.

b) Propose a basis for the space F.

c) What are the dimensions of P_1 and F?

d) Prove that the map u is linear from P_1 to F.

e) What is the matrix M_u of the linear map u relatively to the bases proposed in questions a) and b) ?

f) Prove that the map v is linear from P_1 to F: $v \in \mathscr{L}(P_1, F)$.

g) What is the matrix M_v of the linear map v relatively to the bases used in the previous questions?

- Matrix of an other linear operator

With the notations introduced in the previous exercice for the space P_1 and the basis (f_0, f_1), we introduce the map w defined on P_1 and taking its values in P_1 by the relation
$w(b\,f_0 + a\,f_1) = (2\,a + 3\,b)\,f_1$.

a) For $f \in P_1$, the vector $w(f)$ is also a vector in P_1, and $w(f)$ is an affine function. For an arbitrary $t \in \mathbb{R}$, what is the value of the number $\big(w(f)\big)(t)$ if $f(t) = a\,t + b$?

b) Precise the value of $w(f_0)$.

c) Same question for $w(f_1)$.

d) Prove that the application $w : P_1 \longrightarrow P_1$ is linear.

e) What is the matrix M_w of the linear map w relatively the the basis (f_0, f_1)?

- A family of three vectors in $\mathbb{R}^2$

We set $u_1 = \big(1, 0\big)$, $u_2 = \big(-\frac{1}{2}, \frac{\sqrt{3}}{2}\big)$ and $u_3 = \big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\big)$.

a) Prove that these 3 vectors are linearly dependent.

b) Explicit a linear combination of these 3 vectors that is equal to zero with nontrivial coefficients.

- An example of composition of linear maps

We note $b_1 = (1, 0)$, $b_2 = (0, 1)$ the canonical bases of $\mathbb{R}^2$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ the canonical bases of $\mathbb{R}^3$ and $\varepsilon_1 = (1, 0, 0, 0)$, $\varepsilon_2 = (0, 1, 0, 0)$, $\varepsilon_3 = (0, 0, 1, 0)$, $\varepsilon_4 = (0, 0, 0, 1)$ the canonical bases of $\mathbb{R}^4$.

We consider the linear map f with domain $\mathbb{R}^3$ and codomain $\mathbb{R}^4$ defined by the relations $f(e_1) = 4\,\varepsilon_1 + 2\,\varepsilon_3$, $f(e_2) = 8\,\varepsilon_2 - \varepsilon_3$ and $f(e_3) = \varepsilon_4$.

We consider also the linear map g from $\mathbb{R}^4$ to $\mathbb{R}^2$ defined by $g(\varepsilon_1) = (1, 1)$, $g(\varepsilon_2) = (0, 1)$, $g(\varepsilon_3) = (1, 0)$ and $g(\varepsilon_4) = (-1, -1)$.

a) What is the order and the expression of the matrix M_f of the operator f relatively to the bases (e_1, e_2, e_3) and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$?

b) Same questions for the matrix M_g of the linear map g relatively to the bases $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and (b_1, b_2).

c) What are the domain and the codomain of the map $g_\circ f$?

d) Compute the output vectors $(g_\circ f)(e_j)$ for $j = 1,\ 2,\ 3$ in the basis (b_1, b_2).

e) Deduce from the previous question the matrix $M_{g_\circ f}$ of the mapping $g_\circ f$ relatively to the two given bases.

f) Verify that we have the relation $M_{g_\circ f} = M_g\, M_f$.

- Bases of $\mathbb{R}^3$

We introduce the following three vectors $u_1 = (1, 1, 1)^{\mathrm{t}}$, $u_2 = (0, a, 1)^{\mathrm{t}}$ and $u_3 = (0, 0, b)^{\mathrm{t}}$ of the space $\mathbb{R}^3$, parameterized by the real numbers a and b.

a) Explicit a necessary and sufficient condition to exprees that the family (u_1, u_2, u_3) is a basis $\mathbb{R}^3$.

b) Same questions for the three vectors $v_1 = (0, a, b)^{\mathrm{t}}$, $v_2 = (a, 0, b)^{\mathrm{t}}$ and $v_3 = (a, b, 0)^{\mathrm{t}}$.

- A linear system

a) Solve the following linear system $\left\{ \begin{array}{rcl} 2x - y + 3z &=& 1 \\ x + 2y - z &=& 2 \\ 3x + y + 2z &=& 1\,. \end{array} \right.$

b) What is the kernel of the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$?

c) Check your result of the last question.

- An example of diagonalization

We consider the two matrices $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

a) Prove that the inverse P^{-1} of the matrix P can be written $P^{-1} = \frac{1}{2} P$.

b) Compute the products $P^{-1} A$ et $A\,P$.

c) With two different computations, prove the relation $P^{-1} A\, P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

- Inverse of a product

We consider two square invertible matrices A and B of order n.

a) Recall the properties satisfied by A^{-1} and B^{-1}.

b) Show that $(AB)^{-1} = B^{-1}A^{-1}$.
c) What is the value of $(BA)^{-1}$?

- A zero divisor

We consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
a) Show that $A \neq 0$.
b) What is the value of $A^2 = A \times A$?

- The question of inversion of square matrices of order 2

We consider a 2×2 general matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_2(\mathbb{R})$. We set $\delta = ad - bc$.
a) Prove that if $\delta \neq 0$, we can solve every linear system of the type $AX = Y$, where Y is an arbitrary column matrix with two lines.
b) If $\delta \neq 0$, compute the inverse matrix A^{-1}.
c) Show that if $\delta = 0$, there exists a matrix $B \in \mathscr{M}_2(\mathbb{R})$ such that $AB = BA = 0$.
d) If $A \neq 0$ is a matrix in $\mathscr{M}_2(\mathbb{R})$ such that $ad - bc = 0$, prove that it admits at least a zero divisor that will be explicited.

- A basis for the P_2 finite element

We consider the vector space P_2 composed with polynomials of degree ≤ 2.
a) Show that the family of functions $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$ is a basis of the space P_2.
b) We introduce three functions φ_0, φ_1 and φ_2 that belong in the space P_2 and satisfying the conditions $\varphi_0(0) = 1$, $\varphi_0(1) = 0$, $\varphi_0\left(\frac{1}{2}\right) = 0$, $\varphi_1(0) = 0$, $\varphi_1(1) = 1$, $\varphi_1\left(\frac{1}{2}\right) = 0$, $\varphi_2(0) = 0$, $\varphi_2(1) = 0$ and $\varphi_2\left(\frac{1}{2}\right) = 1$. Explicit these functions relatively to the canonic basis (f_0, f_1, f_2) defined at the previous question.

$$[\varphi_0(x) = (x-1)(2x-1),\ \varphi_1(x) = x(2x-1),\ \varphi_2(x) = 4x(1-x)]$$

c) Prove that for $f \in P_2$, we have the decomposition $f = f(0)\,\varphi_0 + f(1)\,\varphi_1 + f\left(\frac{1}{2}\right)\varphi_2$.