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Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 6 Linear differential systems

- Basic ordinary differential equation

We suppose given $a \in \mathbb{R}$ and $u_0 \in \mathbb{R}$. We search a function $u : [0, +\infty[\longrightarrow \mathbb{R}$ satisfying the two following conditions. On one hand the initial condition $u(0) = u_0$ and on the other hand, the differential equation for the evolution: $\frac{\mathrm{d}u}{\mathrm{d}t} + a\,u(t) = 0$. A dynamical system is given by an initial condition and a dynamical equation as the differential equation in our case.

Proposition: necessary condition. If the function u satisfies both the initial condition $u(0) = u_0$ and the differential equation $\frac{\mathrm{d}u}{\mathrm{d}t} + a\,u(t) = 0$, then we have $u(t) = \exp(-a\,t)\,u_0$.

Proof of the proposition. We introduce the auxiliary function $v(t) = \exp(a\,t)\,u(t)$. Then this function is derivable if the function u is derivable and we have $\frac{\mathrm{d}v}{\mathrm{d}t} = a\exp(a\,t)\,u(t) + \exp(a\,t)\,\frac{\mathrm{d}u}{\mathrm{d}t} = a\exp(a\,t)\,u(t) + \exp(a\,t)\,(-a\,u(t)) = 0$. The function v is a constant on the interval $[0, +\infty[$. Then for each $t \geq 0$, we have $v(t) = v(0) = u_0$. Then $\exp(a\,t)\,u(t) = u_0$ for any arbitrary t and the proposition is established. $\square$

Proposition: sufficient condition. The fonction $u(t) = \exp(-a\,t)\,u_0$ satisfy the initial condition $u(0) = u_0$ and the differential evolution equation $\frac{\mathrm{d}u}{\mathrm{d}t} + a\,u(t) = 0$.

- Ordinary differential equation with a right hand side

We suppose now that a given function f is present in the right hand side of the differential equation and we have $\frac{\mathrm{d}u}{\mathrm{d}t} + a\,u(t) = f(t)$ for each t.

Proposition: a general methodology. If we know a particular solution u_p of the dynamical equation, then the solution of the initial condition $u(0) = u_0$ and of the differential equation $\frac{\mathrm{d}u}{\mathrm{d}t} + a\,u(t) = f(t)$ can be written $u(t) = u_p(t) + \alpha\exp(-a\,t)$: the solution is obtain by adding to the particular solution the general solution of the homogeneous equation. Moreover, the number α is imposed by the relation $u_p(0) + \alpha = u_0$.

If $f(t) \equiv 1$ and $a \neq 0$, a constant function is solution of the equation $\frac{\mathrm{d}u}{\mathrm{d}t} + a\,u(t) = 1$ and we have $u_p(t) = \frac{1}{a}$.
If $f(t) = t$, we can search the function u_p as a polynomial of degree 1: $u_p(t) = \alpha\,t + \beta$. After identification in the equation $\frac{\mathrm{d}u}{\mathrm{d}t} + a\,u(t) = t$, we have $a\,\alpha = 1$ and $\alpha + a\,\beta$ and $u_p(t) = \frac{t}{a} - \frac{1}{a^2}$.
It $f(t) = \exp(-a\,t)$, we have to make attention because this function is solution of the homogeneous equation; we have $\frac{\mathrm{d}f}{\mathrm{d}t} + a\,f(t) = 0$. We can search a function u_p of the form

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$u_p(t) = \alpha t \exp(-at)$. Then this fonction is solution of the differential equation $\frac{du}{dt} + a u(t) = \exp(-at)$ if and only if $\alpha = 1$.

- Linear system of differential equation

We suppose given an integer $n \geq 1$, a column vector $u_0 \in \mathbb{R}^n$ and a fixed real square matrix $A \in \mathscr{M}_n(\mathbb{R})$. We search a vector function $u : [0, +\infty[\longrightarrow \mathbb{R}^n$ satisfying the initial condition $u(0) = u_0$ and the system of differential equations $\frac{du}{dt} + A u(t) = 0$.

Theorem: structure of the solutions of a differential system. The set of solutions of the differential system $\frac{du}{dt} + A u(t) = 0$ is a vector space of functions of dimesion n.
In other words, there exists functions $t \longmapsto \varphi_j(t) \in \mathbb{R}^n$ such that each solution $u(t)$ of the system $\frac{du}{dt} + A u(t) = 0$ can be expanded as a linear combination of the φ_js. There exists n real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that for all $t \geq 0$, we have $u(t) = \sum_{j=1}^{n} \alpha_j \varphi_j(t)$.

We remark that is is true for the diagonal matrix $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The solutions of the system $\frac{du}{dt} + \Lambda u(t) = 0$ can be written $u(t) = \alpha_1 \varphi_1(t) + \alpha_2 \varphi_2(t)$ with
$\varphi_1(t) = \begin{pmatrix} \exp(-t) \\ 0 \end{pmatrix}$ and $\varphi_2(t) = \begin{pmatrix} 0 \\ \exp(t) \end{pmatrix}$.
Moreover, for each $t \geq 0$, the set $(\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))$ is a basis of $\mathbb{R}^n$. In particular for $t = 0$. Then it is possible to decompose the initial datum $u_0 \in \mathbb{R}^n$ in the initial basis $(\varphi_1(0), \varphi_2(0), \ldots, \varphi_n(0))$: there exists n real numbers $\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0$ such that
$u_0 = \sum_{j=1}^{n} \alpha_j^0 \varphi_j(0)$. Then the solution $u(t)$ of the initial condition $u(0) = u_0$ and of the differential system $\frac{du}{dt} + A u(t) = 0$ is written $u(t) = \sum_{j=1}^{n} \alpha_j^0 \varphi_j(t)$.

- Case of a diagonalizable matrix

We suppose now that there exists an invertible matrix P and a diagonal matrix Λ such the matrix A of the differential system $\frac{du}{dt} + A u(t) = 0$ satisfies the relation $P^{-1} A P = \Lambda$. Then this matrix can be decomposed under the form $A = P \Lambda P^{-1}$. We search the unknown vector in the new basis and we set $u(t) = P v(t)$. Then $\frac{du}{dt} + A u(t) = P \big(\frac{dv}{dt} + \Lambda v(t) \big)$. The vector $v(t) = (v_1(t), v_2(t), \ldots v_n(t))^{\mathrm{t}}$ satisfies the system of decoupled equations $\frac{dv}{dt} + \Lambda v(t) = 0$. If we write $\Lambda = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we have $\frac{dv_j}{dt} + \lambda_j v_j(t) = 0$ and $v_j(t) = \alpha_j^0 \exp(-\lambda_j t)$. The coeficients $\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0$ are obtained by solving the linear system $P v(0) = u_0$ relative to the initial condition.

- An important second order differential equation

We supose given $\omega > 0$. An harmonic oscillator satisfies the second order differential equation $\frac{d^2x}{dt^2} + \omega^2 x(t) = 0$ and the initial condition $x(0) = x_0$ and $\frac{dx}{dt}(0) = v_0$. We write this equation as a first order system of two equations, introducing $v(t) = \frac{dx}{dt}$. We have $\frac{du}{dt} + A u(t) = 0$ with $u(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$ and $A = \begin{pmatrix} 0 & -1 \\ \omega^2 & 0 \end{pmatrix}$. We observe that this matrix A is not diagonalizable in the field of real numbers. To avoid the introduction of complex numbers, we introduce a basis of solutions, with $\varphi_1(t) = \begin{pmatrix} \cos(\omega t) \\ -\omega \sin(\omega t) \end{pmatrix}$ and $\varphi_2(t) = \begin{pmatrix} \sin(\omega t) \\ \omega \cos(\omega t) \end{pmatrix}$. Then the general solution of the system $\frac{du}{dt} + A u(t) = 0$ can be written $u(t) = \alpha_1 \varphi_1(t) + \alpha_2 \varphi_2(t)$. Taking the particular value $t=0$, it is easy from the initial condition to explicit the parameters $\alpha_1 = x_0$ and $\alpha_2 = \frac{v_0}{\omega}$. Finally, a solution of the initial problem is explicited with $x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$.

Exercices

- Particular solution when the right hand side is the sinus function

We search a particular solution u_p of the differential equation $\frac{\mathrm{d}u}{\mathrm{d}t} + a\,u(t) = \sin t$.

a) Try a function u_p that is a linear combination of sinus and cosine: $u_p(t) = \alpha \sin t + \beta \cos t$.

b) Solve the linear system with unknowns α and β.

c) Check your results. $\Big[\frac{1}{1+a^2}\,(a \sin t - \cos t)\Big]$

- A system of coupled differential equations

We consider the differential system composed by the two equations $\frac{\mathrm{d}x}{\mathrm{d}t} = y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = x$. It is associated with the initial conditions $x(0) = x_0$ and $y(0) = y_0$. We introduce the column vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

a) Find a matrix A in order to write the system of differential equations under the form $\frac{\mathrm{d}}{\mathrm{d}t} X = A\,X$.

b) Determine the eigenvalues and the eigenvectors of the matrix A. We denote by r_1 and r_2 the two eigenvectors.

Introduce the auxiliary unknowns $\varphi_1(t)$ and $\varphi_2(t)$ in order to decompose the unknown vector under the form $X(t) = \varphi_1(t)\,r_1 + \varphi_2(t)\,r_2$.

c) How the numbers $\varphi_1(0)$ and $\varphi_2(0)$ are related to the initial velues x_0 and y_0 ?

d) What are the differential equations satisfied by the functions φ_1 and φ_2 ?

e) Solve these differential equations.

f) Deduce from the revious questions an explicit formula for the solutions $x(t)$ and $y(t)$ of the initial system composed by the differential equations and the initial conditions.

g) Check your results!

- An other system of coupled differential equations

We consider the differential system composed by the two equations $\frac{\mathrm{d}x}{\mathrm{d}t} = x + y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = x - y$. It is associated with the initial conditions $x(0) = x_0$ and $y(0) = y_0$. We introduce the column vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Follow the methodology explicited in the previous exercice to propose explicit formula for the solutions $x(t)$ and $y(t)$ of this differential system.

$$\Big[x(t) = \frac{1}{2\sqrt{2}}\Big[\big((\sqrt{2}+1)\,x_0 + y_0\big)\exp(t\sqrt{2}) + \big((\sqrt{2}-1)\,x_0 - y_0\big)\exp(-t\sqrt{2})\Big],$$
$$y(t) = \frac{1}{2\sqrt{2}}\Big[\big(x_0 + (\sqrt{2}-1)\,y_0\big)\exp(t\sqrt{2}) + \big(-x_0 + (\sqrt{2}+1)\,y_0\big)\exp(-t\sqrt{2})\Big]\Big]$$

• A variant of the previous approach to avoid complex numbers

We consider the differential system composed by the two equations $\frac{\mathrm{d}x}{\mathrm{d}t} = -y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = x$. It is associated with the real initial conditions $x(0) = x_0$ and $y(0) = y_0$. We introduce the column vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

a) Find a matrix A in order to write the system of differential equations under the form $\frac{\mathrm{d}}{\mathrm{d}t} X = A\,X$.

b) Determine the eigenvalues of the matrix A.

Since the eigenvalues are complex numbers, the methodology suggested in the first exercice conducts to algebraic calculations with complex numbers that are not necessary because all in this problem is composed by real numbers. We suggest to follow a variant of the previous approach.

c) Propose a first solution $X_1(t) = (x_1(t), y_1(t))^{\mathrm{t}}$ of the differential system $\frac{\mathrm{d}}{\mathrm{d}t} X = A\,X$ with the first component $x_1(t)$ equal to a basic trigonometric function.

d) Construct a basis $(X_1(t), X_2(t))$ of all the solutions of the differential system $\frac{\mathrm{d}}{\mathrm{d}t} X = A\,X$ by adding to $X_1(t)$ constructed at the previous question a second solution $X_2(t)$ obtained with the same approach.

e) Express the general solution $X(t)$ of $\frac{\mathrm{d}}{\mathrm{d}t} X = A\,X$ in the basis $(X_1(t), X_2(t))$:
$X(t) = \varphi_1(t)\,X_1(t) + \varphi_2(t)\,X_2(t)$. What are the equations satisfied by the functions $\varphi_1(t)$ and $\varphi_2(t)$?

f) Deduce from the revious questions explicit formulas for the solutions $x(t)$ and $y(t)$ of the initial system composed by the differential equations and the initial conditions.