FAMILIES OF RATIONAL CURVES ON HOLOMORPHIC SYMPLECTIC VARIETIES

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Abstract. We study families of rational curves on certain irreducible holomorphic symplectic varieties. In particular, we prove that projective holomorphic symplectic fourfolds of $K3^{[2]}$-type contain uniruled divisors and rational Lagrangian surfaces.

1. Introduction

Let S be a $K3$ surface, and let H be an ample divisor on S. By a theorem of Bogomolov-Mumford [MM83], H is linearly equivalent to a sum of rational curves. The goal of this paper is to investigate the extent to which this result can be generalized to the higher-dimensional setting.

Let X be a complex manifold. We say that X is irreducible holomorphic symplectic – in the text, we will often simply refer to such manifolds as holomorphic symplectic – if X is simply connected and $H^0(X, \Omega^2_X)$ is spanned by an everywhere non-degenerate form of degree 2. These objects were introduced by Beauville in [Bea83]. The holomorphic symplectic surfaces are the$K3$ surfaces.

It is to be noted that holomorphic symplectic varieties are not hyperbolic. This has been recently proved by Verbitsky, cf [Ver14], using among other things his global Torelli theorem [Ver09]. Much less seems to be known on rational curves on (projective) holomorphic symplectic varieties.

A striking application of the existence of rational curves on projective $K3$ surfaces has been given by Beauville and Voisin in [BV04]. They remark that picking any point on a rational curve gives a canonical zero-cycle of degree 1, and show that this has remarkable consequences on the study of the Chow group of $K3$ surfaces.

In the case of projective holomorphic symplectic varieties of higher dimension, Beauville has stated in [Bea07] conjectures that predict a similar behavior of the Chow group of algebraic cycles modulo rational equivalence. These questions have been studied on Hilbert schemes of points of $K3$ surfaces and Fano variety of lines on cubic fourfolds [Voi08, Fu13b]. These

Date: January 23, 2014.
Key-words : rational curves; irreducible symplectic varieties.
A.M.S. classification : 14J40 (controler).
The research in this paper was partially supported by the University of Strasbourg Institute for Advanced Study (USIAS) as part of a USIAS Fellowship, and by the ANR project "CLASS" no. ANR-10-JCJC-0111.

papers rely on the existence of explicit families of rational curves on the relevant varieties. Tthe conjecture also has been proved for generalized Kummer varieties in [Fu13a].

Motivated in part by these papers, we investigate the existence of families of rational curves on general irreducible holomorphic manifolds of dimension $2n$. We give criteria for the existence of uniruled divisors, as well as for subvarieties of codimension 2 whose maximal rationally connected quotient – that we shall henceforth abbreviate by MRC quotient as it is customary – has dimension $2n-4$ – the smallest possible value.

In particular, we prove the following result. Recall that a holomorphic symplectic manifold is said to be of $K3^{[n]}$-type if it is a deformation of the Hilbert scheme that parametrizes zero-dimensional subschemes of length n on some $K3$ surface. We say that a subscheme Y of a holomorphic symplectic manifold X is Lagrangian if at any smooth point y of Y^{red}, the tangent space $T_{Y^{red},y}$ of Y^{red} at y is a maximal isotropic subspace of $T_{X,y}$ with respect to the holomorphic symplectic form of X.

Theorem 1.1. *Let X be a projective holomorphic symplectic fourfold of $K3^{[2]}$-type. Then the following holds.*

(1) *There exists an ample uniruled divisor on X.*
(2) *There exists a rational Lagrangian surface on X.*

This result was already known by work of Voisin [Voi04] and Amerik-Voisin [AV08] in the case of the Fano variety of lines on a smooth cubic fourfold. As in [BV04], we are able to show that the existence of such a rational surface makes it possible to define a canonical zero-cycle of degree 1 on X (cf. Corollary 4.7). The theorem above suggests a natural extension to general projective holomorphic symplectic varieties in the following way.

Question 1.2. *Let X be a projective holomorphic symplectic variety of dimension $2n$, and let k be an integer between 0 and n. Does there exist a subscheme Y_k of X of pure dimension $2n-k$ such that the MRC quotient of Y has dimension $2n-2k$?*

Again, as shown in section 3, $2n-2k$ is the smallest possible dimension for the MRC quotient of such a Y.

Rational curves have appeared in important recent works on the positive cone of holomorphic symplectic varieties that culminated in the papers [BM13] and [BHT13]. While some of our arguments are related to – and have certainly been influenced by – this line of work, our results are quite different in spirit, as they are relevant even for varieties which are very general in their moduli space, and have Picard number 1.

Organisation of the paper. This paper relies on the global Torelli theorem for holomorphic symplectic manifolds as proved by Verbitsky in [Ver09], as well as on results of Markman on monodromy groups [Mar10]. Consequences of these results for moduli spaces of polarized holomorphic symplectic manifolds of $K3^{[n]}$-type have been studied by Apostolov in [Apo11]. We describe these results in section 2, that does not contain any new results. Section 3 contains general results on the deformation of families of stable curves of genus zero on holomorphic symplectic manifolds of dimension $2n$. We give criterion for the existence of uniruled divisors, as well as codimension 2 subschemes with MRC quotient of dimension $2n-4$. In section 4, we give examples – in the $K3^{[n]}$ case – where these criteria can be applied. Using the results of section 2, we prove Theorem 1.1 and construct a canonical zero-cycle of degree 1 in the $K3^{[2]}$ case. Finally, the last section is devoted to some open questions.

Acknowledgements. We thank Daniel Huybrechts and Claire Voisin for interesting discussions on the subject of this paper. We are especially grateful to Eyal Markman for enlightening discussions and comments, and for sharing with us some unpublished notes of his. The cohomology computations at the end of section 4 are due to him.

While finishing the redaction of this paper, independent work of Amerik and Verbitsky [AV14] appeared and seems to have some overlap with some arguments of section 3. Since the goals and the results of the two papers seem to be quite different, we did not try to eliminate similar discussions.

We always work over the field $\mathbb{C}$ of complex numbers.

2. Deformations of irreducible holomorphic symplectic manifolds

This section is devoted to recalling important results regarding the deformation theory of irreducible holomorphic symplectic varieties.

The foundational results on irreducible holomorphic symplectic varieties are due to Beauville in his seminal paper [Bea83]. In this paper, we will make crucial use of connectedness properties of moduli spaces of certain irreducible holomorphic symplectic varieties. These are due to Apostolov [Apo11], and they rely on the global Torelli theorem of Verbitsky [Ver09] and on results of Markman [Mar10] on the monodromy group of varieties of $K3^{[n]}$-type.

Let S be a complex $K3$ surface, and let n be an integer strictly bigger than 1. Let $S^{[n]}$ be the Hilbert scheme – or the Douady space if S is not projective – of zero-dimensional subschemes of length n on S. By [Fog68], the complex variety $S^{[n]}$ is smooth. General properties of Hilbert schemes show that $S^{[n]}$ is projective as soon as S is projective.

By [Bea83], $S^{[n]}$ is an irreducible holomorphic symplectic variety. As shown by Beauville, the second cohomology group of $S^{[n]}$ is related to that of S as follows. Let $S^{(n)}$ be the n-th symmetric product of S, and let $\epsilon : S^{[n]} \to S^{(n)}$ be the Hilbert-Chow morphism. The map

$$\epsilon^* : H^2(S^{(n)}, \mathbb{Z}) \to H^2(S^{[n]}, \mathbb{Z})$$

is injective. Furthermore, let $\pi : S^n \to S^{(n)}$ be the canonical map, and let $p_1, \ldots, p_n$ be the projections from S^n to S. There exists a unique map

$$i : H^2(S, \mathbb{Z}) \to H^2(S^{[n]}, \mathbb{Z})$$

such that for any $\alpha \in H^2(S, \mathbb{Z})$, $i(\alpha) = \epsilon^*(\beta)$, where $\pi^*(\beta) = p_1^*(\alpha) + \ldots + p_n^*(\alpha)$. The map i is an injection.

Finally, let E be the exceptional divisor in $S^{[n]}$, that is, the divisor that parametrizes non-reduced points. The cohomology class of E in $H^2(S^{[n]}, \mathbb{Z})$ is uniquely divisible by 2. Let δ be the element of $H^2(S^{[n]}, \mathbb{Z})$ such that $2\delta = [E]$. Then we have

$$H^2(S^{[n]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta, \tag{2.1}$$

where the embedding of $H^2(S, \mathbb{Z})$ into $H^2(S^{[n]}, \mathbb{Z})$ is the one given by the map i above.

Finally, the second cohomology group $H^2(S^{[n]}, \mathbb{Z})$ is endowed with a canonical quadratic form q, the *Beauville-Bogomolov form.* The decomposition (2.1) is orthogonal with respect to q, and the restriction of q to $H^2(S, \mathbb{Z})$ is the canonical quadratic form on the second cohomology group of a surface induced by cup-product.

If α and β are elements of $H^2(S^{[n]}, \mathbb{Z})$, we will write $\alpha.\beta$ for $q(\alpha, \beta)$. We have

$$\delta.\delta = -2(n-1).$$

It follows from this discussion that the lattice $H^2(S^{[n]}, \mathbb{Z})$ is isomorphic to the lattice

$$\Lambda_n = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} \oplus < -2(n-1) > .$$

Let X be an irreducible holomorphic symplectic variety, and assume that X is of $K3^{[n]}$-type. Recall that this means that X is deformation-equivalent, as a complex variety, to a complex variety of the form $S^{[n]}$, where S is a $K3$ surface.

Suppose from now on that X is polarized. Let h be the first chern class in $H^2(X, \mathbb{Z})$ of the chosen primitive ample line bundle on X. We will be interested in the deformations of the pair (X, h). Note that the deformations of the pair (X, h) can be identified with the deformation of the pair (X, rh) for any $r \in \mathbb{Q}^*$.

Remark 2.1. While our assumption on X is simply that there is a deformation of X to some $S^{[n]}$ as complex varieties, an argument using period domains and Verbitsky's global Torelli theorem shows that if X is polarized, there exists a deformation of the pair (X, h) to some $(S^{[n]}, h')$, where S is a projective $K3$ surface and h' is the class of a polarization on $S^{[n]}$ – see Proposition 7.1 of [Mar11]. The goal of this section is to control the cohomology class h' with respect to the decomposition (2.1). □

We now describe results by Apostolov [Apo11] that give information on the possible deformation types of the pair (X, h). To such a pair we can associate two numerical invariants. The first one is the *degree*, namely, the even number h^2. The second one is the *divisibility* of h, that is, the positive integer t such that $h.H^2(X, \mathbb{Z}) = t\mathbb{Z}$.

If $n = 1$, X is a $K3$ surface and $H^2(X, \mathbb{Z})$ is a unimodular lattice. As a consequence, the divisibility of the primitive polarization h is always 1. However, $H^2(X, \mathbb{Z})$ is not unimodular as soon as $n > 1$, and the divisibility can be different from 1 accordingly.

Lemma 2.2. *Let n be an integer at least equal to 2, let S be a $K3$ surface, and let $h \in H^2(S^{[n]}, \mathbb{Z})$ be a primitive polarization. Using the canonical decomposition of (2.1), write*

$$h = \lambda h' + \mu\delta,$$

where h' is a primitive element of $H^2(S, \mathbb{Z})$. Then the divisibility of h is

$$t = \gcd(2(n-1), \lambda).$$

Proof. Since $H^2(S, \mathbb{Z})$ is unimodular, $h'.H^2(S^{[n]}, \mathbb{Z}) = h'.H^2(S, \mathbb{Z}) = \mathbb{Z}$. As a consequence, $\lambda h'.H^2(S^{[n]}, \mathbb{Z}) = \lambda\mathbb{Z}$. Similarly, $\mu\delta.H^2(S^{[n]}, \mathbb{Z}) = 2\mu(n-1)\mathbb{Z}$. Since δ and h' are orthogonal, this implies that $h.H^2(S^{[n]}, \mathbb{Z}) = \gcd(2\mu(n-1), \lambda)\mathbb{Z}$. The polarization h is primitive, which implies that λ and μ are relatively prime, and proves the result. □

Corollary 2.3. *Let (X,h) be a primitively polarized variety of $K3^{[2]}$-type. Then the divisibility t of h is either 1 or 2. Furthermore, in the latter case, the degree $2d$ of h is congruent to -2 modulo 8.*

Proof. Since the divisibility of a polarization does not vary under deformations, we can assume that X is of the form $S^{[2]}$ for some $K3$ surface S, for which the result is a consequence of the preceding proposition. □

The two cases above are referred to as the *split* case if $t=1$ and the *non-split* case if $t=2$, see [GHS10].

Definition 2.4. *A polarization type for varieties of $K3^{[n]}$-type is the isomorphism class of pairs (Λ, x), where Λ is a lattice isomorphic to Λ_n and x is a primitive element of Λ. We say that a holomorphic symplectic variety X with a primitive polarization h has polarization type (Λ, x) if the pairs $(H^2(X,\mathbb{Z}),h)$ and (Λ, x) are isomorphic.*

The degree and the divisibility of a primitive polarization h of a variety X of $K3^{[2]}$-type determine the isomorphism class of the pair $(H^2(X,\mathbb{Z}),h)$ among pairs (Λ, x) where Λ is a lattice isomorphic to Λ_2 and x is a primitive element of Λ by Corollary 3.7 of [GHS10].

As in [Ver09, Huy11], it is possible to construct the moduli space of marked primitively polarized varieties of $K3^{[n]}$-type with a fixed polarization type. This is a smooth complex variety that is a fine moduli space for triples (X,h,φ), where X is a variety of $K3^{[n]}$-type with a primitive polarization h of the polarization type considered, and $\varphi : H^2(X,\mathbb{Z}) \to \Lambda_n$ is an isometry.

The key result of Apostolov we need is the following.

Theorem 2.5 ([Apo11], Corollary 2.5)**.** *The moduli space of marked primitively polarized varieties of $K3^{[2]}$-type and fixed polarization type is connected.*

This theorem relies on two important results, namely the global Torelli theorem of Verbitsky [Ver09] and the computation of monodromy groups for varieties of $K3^{[n]}$ due to Markman [Mar10]. We will use it in the following form.

Corollary 2.6. *Let (X,h) be a primitively polarized variety of $K3^{[2]}$-type. Then there exists a $K3$ surface S, with a primitive polarization $h' \in H^2(S,\mathbb{Z})$, two coprime integers λ and μ such that $(\lambda,\mu) \in \{(1,0),(2,1)\}$, a smooth projective morphism $\pi : \mathscr{X} \to B$ of complex smooth quasi-projective varieties, two points b and b' of B, and a global section α of $R^2\pi_*\mathbb{Z}$ such that*

(1) $(\mathscr{X}_b, \alpha_b) \simeq (X,h)$;
(2) $(\mathscr{X}_{b'}, \alpha_{b'}) \simeq (S^{[2]}, \lambda h' - \mu\delta)$.

In other words, the pair (X,h) is deformation-equivalent to the pair $(S^{[2]}, \lambda h' - \mu\delta)$.

Proof. Let $2d = h^2$ and let t be the divisibility of h. If $t=1$, let (S,h') be a $K3$ surface with a primitive polarization h' of degree $2d$ and let $(\lambda,\mu)=(1,0)$.

If $t = 2$, by Corollary 2.3, we can write $2d = 8d' - 2$. Let (S, h') be a $K3$ surface with a primitive polarization of degree $2d'$ and let $(\lambda, \mu) = (2, 1)$.

Let $\widetilde{h} = \lambda h' + \mu\delta$. Then $\widetilde{h}$ is primitive and we have

$$\widetilde{h}^2 = 2d$$

and

$$\widetilde{h}.H^2(S^{[2]}, \mathbb{Z}) = t\mathbb{Z}.$$

The class $\widetilde{h} \in H^2(S^{[2]}, \mathbb{Z})$ might not be ample. However, since its square is positive, either $\widetilde{h}$ or $-\widetilde{h}$ becomes ample on any small deformation of $(S^{[2]}, \widetilde{h})$ with Picard number 1, by Theorem 3.11 of [Huy99] (see also [Huy03]). Intersecting with curves of the form $C + s_2 + \ldots + s_n$ on $S^{[n]}$ where C is an ample curve of class h' on S and the s_i are distinct varying points of S not lying on C shows that $-\widetilde{h}$ is not the class of an effective divisor. This proves that there exists a pair $(Y, \widetilde{h}_Y)$, where Y is a holomorphic symplectic variety and $\widetilde{h}_Y$ is a primitive ample class in $H^2(Y, \mathbb{Z})$ that is deformation equivalent to $(S^{[n]}, \widetilde{h})$.

By construction, (X, h) and $(Y, \widetilde{h}_Y)$ have the same polarization type. By Theorem 2.5, they are deformation-equivalent. This in turn shows that (X, h) is deformation-equivalent to $(S^{[2]}, \lambda h' + \mu\delta)$. □

3. Deforming rational curves and Lagrangian subvarieties of holomorphic symplectic varieties

Let $\pi : \mathscr{X} \to B$ be a smooth projective morphism of complex quasi-projective varieties of relative dimension d, and let α be a global section of the local system $R^{2d-2}\pi_*\mathbb{Z}$. Fixing such a section α, we can consider the relative Kontsevich moduli stack of genus zero stable curves $\overline{\mathscr{M}_0}(\mathscr{X}/B, \alpha)$. We refer to [BM96, FP97, AV02] for details and constructions.

The space $\overline{\mathscr{M}_0}(\mathscr{X}/B, \alpha)$ parametrizes maps $f : C \to X$ from genus zero stable curves to fibers $X = \mathscr{X}_b$ of π such that $f_*[C] = \alpha_b$. The map $\overline{\mathscr{M}_0}(\mathscr{X}/B, \alpha) \to B$ is proper. If f is a stable map, we denote by $[f]$ the corresponding point of the Kontsevich moduli stack.

For the remainder of this section, we work in the following situation. We fix a smooth projective holomorphic symplectic variety X of dimension $2n$ and $f : C \to X$ an unramified map from a stable curve C of genus zero to X. Let $\mathscr{X} \to B$ be a smooth projective morphism of smooth connected quasi-projective varieties and let 0 be a point of B such that $\mathscr{X}_0 = X$. Let α be a global section of $R^{4n-2}\pi_*\mathbb{Z}$ such that $\alpha_0 = f_*[C]$ in $H^{4n-2}(X, \mathbb{Z})$.

Proposition 3.1. *Let $\mathscr{M}$ be an irreducible component of $\overline{\mathscr{M}_0}(X, f_*[C])$ containing the point corresponding to f. Then stack $\mathscr{M}$ has dimension at least $2n - 2$. If $\mathscr{M}$ has dimension $2n - 2$, then any irreducible component of the Kontsevich moduli stack $\overline{\mathscr{M}_0}(\mathscr{X}/B, \alpha)$ that contains $[f]$ dominates B. In other words, the stable map $f : C \to X$ deforms over a finite cover of B.*

Related results have been obtained by Ran, see for instance Example 5.2 of [Ran95].

Proof. Let $\widetilde{\mathscr{X}} \to S$ be a local family of deformations of X such that B is the Noether-Lefschetz locus associated to $f_*[C]$ in S. In particular, B is a smooth divisor in S.

Since the canonical bundle of X is trivial, a standard dimension count shows that any component of the deformation space of the stable map f over S has dimension at least

$$\dim S + 2n - 3 = \dim B + 2n - 2.$$

Since the image in S of any component of the deformation space of the stable map f is contained in B, the fibers of such a component all have dimension at least $2n-2$. If any fiber has dimension $2n-2$, it also follows that the corresponding component has to dominate B, which shows the result. $\square$

In order to use the criterion above in concrete situations, we will study the locus spanned by a family of rational curves. The following gives a strong restriction on this locus.

Proposition 3.2. *Let X be a projective manifold of dimension $2n$ endowed with a symplectic form, and let Y be a closed subvariety of codimension k of X. Then the MRC quotient of Y is at least $(2n-2k)$-dimensional.*

Before proving the proposition, we prove the following easy fact from linear algebra.

Lemma 3.3. *Let (F, ω) be a symplectic vector space of dimension $2n$ and V a subspace of codimension k of F. Then V contains a subspace V' of dimension at least $2n-2k$ such that the 2-form $\omega_{|V'}$ is symplectic on V'. In particular, $\omega_V^{2n-2k} \neq 0$.*

Proof of Lemma 3.3. Let $V^\perp$ be the orthogonal to V with respect to the symplectic form ω. Since ω is non-degenerate, we have

$$\dim(V^\perp) = k,$$

which implies that $\dim(V \cap V^\perp) \leq k$.

Let V' be a subspace of V such that V' and $V \cap V^\perp$ are in direct sum in V. Then $\dim(V') \geq 2n-2k$. It is readily seen that the restriction of ω to V' is non degenerate. $\square$

Proof of Proposition 3.2. We argue by contradiction and suppose the MRC quotient of Y has dimension at most $2n-2k-1$. Let $\mu : \widetilde{Y} \to Y \subset X$ be a resolution of the singularities of Y. Mumford's theorem on zero cycles – see for instance Proposition 22.24 of [Voi02] – implies that

$$H^0(\widetilde{Y}, \Omega^m_{\widetilde{Y}}) = 0, \quad \forall m \geq 2n-2k,$$

However, if ω is the symplectic form on X, then

$$\mu^*(\omega^{n-k}) \neq 0,$$

by Lemma 3.3, which is a contradiction. $\square$

The results above allow us to give a simple criterion for the existence of uniruled divisors on polarized deformations of a given holomorphic symplectic variety.

Corollary 3.4. *Let $\mathscr{M}$ be an irreducible component of $\overline{\mathscr{M}_0}(X, f_*[C])$ containing $[f]$, and let Y be the subscheme of X covered by the deformations of f parametrized by $\mathscr{M}$.*

If Y is a divisor in X, then

(1) *The stable map $f : C \to X$ deforms over a finite cover of B.*
(2) *Let b be a point of B. Then $\mathscr{X}_b$ contains a uniruled divisor.*

Proof. Assume that $\dim(\mathscr{M}) > 2n-2$. By a dimension count, this implies that if y is any point of Y, the family of curves parametrized by $\mathscr{M}$ passing through y is at least 1-dimensional, which in turn shows that the MRC quotient of Y has dimension at most $\dim(Y) - 2 = 2n-3$ and contradicts Proposition 3.2. As a consequence, $\mathscr{M}$ has dimension $2n-2$ and (1) holds by Proposition 3.1.

To show statement (2), we can assume that B has dimension 1. By assumption, the deformations of f in X cover a divisor in X. Let $\mathscr{M}'$ be an irreducible component of $\overline{\mathscr{M}_0}(\mathscr{X}/B, \alpha)$ containing $\mathscr{M}$. Then $\mathscr{M}'$ dominates B by Proposition 3.1. Let $\mathscr{Y}' \subset \mathscr{X} \to B$ be the locus in $\mathscr{X}$ covered by the deformations of f parametrized by $\mathscr{M}'$. Since $\mathscr{M}'$ dominates B, any irreducible component of $\mathscr{Y}'$ dominates B. Since the fiber of $\mathscr{Y}' \to B$ over 0 is a divisor in $\mathscr{X}_0 = X$, the fiber of $\mathscr{Y}' \to B$ at any point b is a divisor, which is uniruled by construction. $\square$

Remark 3.5. It might not be true that Y itself deforms over a finite cover of B.

The criterion above can be extended to the existence of subschemes of codimension 2 with MRC quotient of dimension $2n-4$ – by Proposition 3.2, this is the minimal possible dimension.

Proposition 3.6. *Assume that there exists*

- *a projective scheme Z of dimension $2n-2$ together with a morphism $\varphi : Z \to X$ birational onto its image.*
- *A stable map of genus zero $g : C \to Z$. We assume furthermore that each irreducible component of C meets the open subset of Z where φ is birational onto its image.*

satisfying the following conditions:

- *$f = \varphi \circ g$, and the deformations of g cover Z.*
- *There exists a $(2n-2)$-dimensional irreducible component $\mathscr{M}_Z$ of the space $\overline{\mathscr{M}_0}(Z, g_*[C])$ containing $[g]$.*

Let $\mathscr{M}$ be the image of $\mathscr{M}_Z$ in $\overline{\mathscr{M}_0}(X, f_*[C])$ and let Y be the reduced subscheme of X covered by the deformations of f parametrized by $\mathscr{M}$.

Then the following hold:

(1) *The space $\mathscr{M}$ has dimension $2n-2$, and it is an irreducible component of $\overline{\mathscr{M}_0}(X, f_*[C])$.*
(2) *The stable map $f : C \to X$ deforms over a finite cover of B.*
(3) *Let b be a point of B. Then $\mathscr{X}_b$ contains a subscheme Y_b of codimension 2 such that the MRC quotient of Y_b has dimension $2n-4$.*

While the statement above is somewhat long, it gives a practical criterion, as it makes it possible to replace deformation-theoretic computations on X by computations on the auxiliary variety Z. In particular, one does not need to determine the locus in X spanned by the deformations of the stable curve C. This is apparent in the following application which will be generalized in Proposition 4.3.

Corollary 3.7. *Assume that X has dimension 4, and that there exists a rational map $\psi : \mathbb{P}^2 \dashrightarrow X$, birational onto its image, such that the stable map f can be written as $i \circ \psi : \mathbb{P}^1 \to X$, where $i : \mathbb{P}^1 \to \mathbb{P}^2$ is the inclusion of a line in $\mathbb{P}^2$ that does not meet the indeterminacy locus of ψ. Then*

1. *The stable map $f : C \to X$ deforms over a finite cover of B.*
2. *Let b be a point of B. Then $\mathscr{X}_b$ contains a Lagrangian subvariety, i.e., an isotropic subvariety of dimension 2, that is rational.*

Proof. We can find a suitable blow-up Z of $\mathbb{P}^2$ such that the composition $Z \to \mathbb{P}^2 \to X$ is a morphism and such that $Z \to \mathbb{P}^2$ is an isomorphism outside the indeterminacy locus of $\psi : \mathbb{P}^2 \dashrightarrow X$. We write $g : \mathbb{P}^1 \to Z$ for the unique lift of i. It is readily seen that Z and i satisfy the hypothesis of Proposition 3.6. □

Proof of Proposition 3.6. The assumptions on φ and g ensure that the natural map from $\overline{\mathscr{M}_0}(Z, g_*[C])$ to $\overline{\mathscr{M}_0}(X, f_*[C])$ is an immersion at f. As a consequence, the dimension of $\mathscr{M}$ is $2n-2$. Let $\widetilde{\mathscr{M}}$ be an irreducible component of $\overline{\mathscr{M}_0}(X, f_*[C])$ containing $\mathscr{M}$, and assume that $\widetilde{\mathscr{M}} \neq \mathscr{M}$. Then the dimension of $\widetilde{\mathscr{M}}$ is strictly bigger than $2n-2$.

From the definition, it is readily seen that Y is the image of Z in X. Since by assumption the map from Z to X is birational onto its image, Y has codimension 2 in X. Let Y' be the reduced closed subscheme of X covered by the deformations of f parametrized by $\widetilde{\mathscr{M}}$. Since $\widetilde{\mathscr{M}}$ is irreducible, Y' is irreducible. We also have $Y \subset Y'$. By corollary 3.4, Y' cannot be a divisor as $\widetilde{\mathscr{M}}$ has dimension strictly greater than $2n-2$. As a consequence, $Y' = Y$.

Since φ is a birational isomorphism from Z to Y, and since every component of $f(C)$ meets the locus above which p is an isomorphism, any deformation of f whose image is in Y lifts to a deformation of g in Z. This shows that $\widetilde{\mathscr{M}}$ has the same dimension as $\mathscr{M}_Z$, i.e., that $\widetilde{\mathscr{M}}$ has dimension $2n-2$. This shows statement (1).

Statement (2) follows from (1) and Proposition 3.1. We prove the third statement. We can assume that B is a curve. Let $\mathscr{M}'$ be an irreducible component of $\overline{\mathscr{M}_0}(\mathscr{X}/B, \alpha)$ containing $[f]$.

Let b be a point of B. As in Corollary 3.4 and by upper semicontinuity of dimensions, the deformations of f parametrized by $\mathscr{M}'$ in $\mathscr{X}_b$ cover a codimension 2 subscheme of $\mathscr{X}_b$.

Since this codimension 2 subscheme is covered by a 2-dimensional family of stable curves, one of its irreducible components has MRC quotient of dimension at most $2n-4$. By Proposition 3.2, this dimension is exactly $2n-4$, which shows (3). □

Remark 3.8. As in Corollary 3.4, it might not be true that Y itself deforms over a finite cover of B: one needs to consider the union of Y and some other components of the locus covered by the deformations of f in X, as the fiber over 0 of the irreducible component $\mathscr{M}'$ above might not be irreducible.

In this section, we give examples of applications of Corollary 3.4 and Proposition 3.6. In particular, we prove Theorem 1.1. For ease of exposition, we only consider varieties of $K3^{[n]}$-type.

As in section 2, if S is a $K3$ surface and n a positive integer, we consider $H^2(S,\mathbb{Z})$ as a subspace of $H^2(S^{[n]},\mathbb{Z})$. If $n \geq 2$, we denote by δ half of the cohomology class of the exceptional divisor of $S^{[n]}$.

Proposition 4.1. *Let S be a $K3$ surface, and let $h \in H^2(S,\mathbb{Z})$ be an ample cohomology class. Let $n \geq 2$ be an integer. Let X be a holomorphic symplectic variety together with a class $h_X \in H^2(X,\mathbb{Q})$ such that the pair (X, h_X) is deformation-equivalent to the pair $(S^{[n]}, h - \frac{\mu}{n-1}\delta)$ where $\mu \in \{0,1\}$ and $\mu \neq 0$ if $h^2 = 2$. Then X contains a uniruled divisor.*

Remark 4.2. If $\mu = 0$, the class h cannot be ample on $S^{[n]}$ as $h.\delta = 0$. However, Theorem 3.1 of [Huy99] shows that X is projective as it contains a class with positive square. The class of the cohomology divisor above has to be proportional to h_X, as can be seen by deforming (X, h_X) to a variety with Picard number 1.

Proof. We first assume $\mu = 0$. Up to deforming the pair (S,h) and by a theorem of Chen [Che02], we can find an integral nodal rational curve on S with cohomology class h. We can also assume that the Néron-Severi group of S has rank 1.

Let $\varphi : \mathbb{P}^1 \to S$ be the corresponding map with image R. Let $s_2, \ldots, s_n$ be $n-1$ distinct points in S that do not lie on R. The stable map

$$f : \mathbb{P}^1 \to S^{[n]}, t \mapsto \varphi(t) + s_2 + \ldots + s_n$$

satisfies the assumptions of Corollary 3.4, as the deformations of f cover the divisor D in $S^{[n]}$ corresponding to length n zero-dimensional subschemes of $S^{[n]}$ that meet R.

To conclude the proof, we need to show that the class $f_*[\mathbb{P}^1]$ remains a Hodge class along the deformation space of $(S^{[n]}, h)$. Let $\alpha \in H^2(S^{[n]},\mathbb{Q})$ be the dual of $f_*[\mathbb{P}^1]$ with respect to the Beauville-Bogomolov form. In other words, α is characterized by the identity $\alpha.v = f_*[\mathbb{P}^1] \cup v$ for any $v \in H^2(S^{[n]},\mathbb{Q})$. The class α is a Hodge class. We need to show that the class α remains a Hodge class along the deformation space of $(S^{[n]}, h)$.

Since the Néron-Severi group of S has rank 1, any Hodge class on $H^2(S^{[n]},\mathbb{Q})$ is a linear combination of h and δ. Since the image of f does not meet the exceptional divisor, $\alpha.\delta = 0$, so α is proportional to h. This concludes the proof.

Now assume that $\mu = 1$. Again, up to deforming (S,h) and using [MM83, Che02], we can assume that S has Picard number 1 and contains a pencil of curves of genus 1 whose general member has only nodal singularities. Let $\mathscr{C} \to S$ be the corresponding family.

Let C be a general member of the pencil of curves above. The normalization C' of C is a smooth curve of genus 1. Let $s_2, \ldots, s_n$ be $n-2$ distinct points of S that do not lie on C. Any g^1_2 on C' induces a morphism $\varphi : \mathbb{P}^1 \to S^{[2]}$. Consider the stable map

$$f : \mathbb{P}^1 \to S^{[n]}, t \mapsto \varphi(t) + s_2 + \ldots + s_n$$

associated to such a general g^1_2. Then the class $f_*[\mathbb{P}^1]$ is dual to the class $h - \frac{1}{n-1}\delta$ by the computation of Lemma 2.1 in [CK12].

Varying the g^1_2, the curve C in the family $\mathscr{C} \to S$, and the points s_i, we see that the deformations of f cover a divisor in $S^{[n]}$. Corollary 3.4 allows us to conclude. $\square$

We now turn to codimension 2 subvarieties, i.e., to the application of Proposition 3.6.

Proposition 4.3. *Let S be a K3 surface, and let $h \in H^2(S, \mathbb{Z})$ be an ample cohomology class. Let $n \geq 2$ be an integer. Let X be a holomorphic symplectic variety together with a class $h_X \in H^2(X, \mathbb{Q})$ such that the pair (X, h_X) is deformation-equivalent to the pair $(S^{[n]}, h - \frac{\mu}{2(n-1)}\delta)$ where $\mu \in \{0, 1\}$. Then X contains a subscheme Y of codimension 2 such that the MRC quotient of Y has dimension $2n-4$.*

Proof. The proof is a variation on Corollary 3.7. As above, we can assume that the Néron-Severi group of S has rank 1, and that we can find a nodal rational curve R on S with cohomology class h. Let $\varphi : \mathbb{P}^1 \to S$ be the corresponding map with image R.

Let $s_3, \ldots, s_n$ be $n-2$ distinct points in S that do not lie on R. The symmetric product of $\mathbb{P}^1$ is isomorphic to $\mathbb{P}^2$, and we denote by ψ the rational map

$$\psi : \mathbb{P}^2 = (\mathbb{P}^1)^{[2]} \dashrightarrow S^{[n]},\ t_1 + t_2 \mapsto t_1 + t_2 + s_3 + \ldots + s_n.$$

The diagonal Δ in $(\mathbb{P}^1)^{[2]} = \mathbb{P}^2$ is a smooth conic. Let l be a line in $\mathbb{P}^2$ that does not pass through the finitely many points of indeterminacy of ψ and that intersects Δ in two distinct points x and y. The point x corresponds to a zero-dimensional non-reduced subscheme of length 2 of $S^{[2]}$ lying on R. As a consequence, $x + s_3 + \ldots + s_n$ is a well-defined point of $S^{[n]}$. The locus of length n subschemes of S having the same support as $x + s_3 + \ldots + s_n$ is a smooth rational curve C_x in $S^{[n]}$ contained in the exceptional divisor E of $S^{[n]}$. The curve C_x is a fiber of the Hilbert-Chow morphism $S^{[n]} \to S^{(n)}$.

We define a stable map $f_\mu : C_\mu \to S^{[n]}$ of genus 0 as follows.

- If $\mu = 1$, let $f_1 : C_1 = \mathbb{P}^1 \to S^{[n]}$ be the composition of ψ and the inclusion of the line l in $\mathbb{P}^2$.
- If $\mu = 0$, let $f_0 : C_0 \to S^{[n]}$ be the stable map of genus zero obtained by glueing f_1 with the smooth rational curve C_x at the point $x + s_3 + \ldots + s_n$.

We define schemes Z_μ mapping to $S^{[n]}$ as follows.

- If $\mu = 1$, let T_1 be the product $\mathbb{P}^2 \times S^{[n-2]}$. The rational map

  $$\psi_1 : T_1 \dashrightarrow S^{[n]},\ (t_1 + t_2, P) \mapsto t_1 + t_2 + P$$

  is defined as long as the support of the subscheme $P \subset S$ of length $n-2$ does not meet R. Blowing up along the indeterminacy locus, we get a map

  $$p_1 : Z_1 \to S^{[n]}$$

  that is birational onto its image.

- If $\mu = 0$, let E_0 be the product $\mathbb{P}(\varphi^* T_S) \times S^{[n-2]}$ – recall that $\varphi : \mathbb{P}^1 \to S$ is the rational curve we are considering. We identify the diagonal Δ with $\mathbb{P}(T_{\mathbb{P}^1}) \subset \mathbb{P}(\varphi^* T_S)$. The natural morphism $c : \mathbb{P}(\varphi^* T_S) \to S^{[2]}$ induces a rational morphism
$$\psi'_0 : E_0 = \mathbb{P}(\varphi^* T_S) \times S^{[n-2]} \to S^{[n]}, (x, P) \mapsto c(x) + P$$
that is defined as long as the support of the subscheme $P \subset S$ of length $n-2$ does not meet R.

  Let T_0 be the projective scheme obtained by gluing T_1 and E_0 along their common closed subscheme $\Delta \times S^{[n-2]}$. The rational morphisms ψ_1 and ψ'_0 glue together to induce a rational morphism
$$\psi_0 : T_0 \dashrightarrow S^{[n]}.$$
  Again, blowing up along the indeterminacy locus, we get a map
$$p_0 : Z_1 \to S^{[n]}$$
  that is birational onto its image.

By definition, the map f_μ factors as

$$C_\mu \xrightarrow{h_\mu} T_\mu \xrightarrow{\psi_\mu} S^{[n]}$$

and the map $C_\mu \to T_\mu$ does not meet the indeterminacy locus of $\psi_\mu : T_\mu \to S^{[n]}$. As a consequence, f_μ lifts to a map $g_\mu : C_\mu \to Z_\mu$ such that $f_\mu = p_\mu \circ g_\mu$, and the local deformations of g_μ in Z_μ are the same as the local deformations of h_μ in T_μ. Certainly, the deformations of G_μ cover Z_μ.

It is readily seen from the definition that the local deformation space of h_μ in T_μ has dimension $2n-2$ – either directly or by computing normal bundles. This shows that f_μ, Z_μ and g_μ satisfy the assumptions of Proposition 3.6.

To conclude the proof, we need to show as in the proof of Proposition 4.1 that the dual of the class $f_{\mu *}[C_\mu]$ with respect to the Beauville-Bogomolov form is proportional to $h - \frac{\mu}{2(n-1)}\delta$.

Let H be a generic hyperplane section of S with class h. Let H_n be the divisor in $S^{[n]}$ corresponding to the subschemes of S of length n whose support intersects H. Then the class of H_n in $H^2(S^{[n]}, \mathbb{Z})$ is h.

With the notations above, it can be checked that the intersection number of the image of l with H_n is h^2 – the image of l can be deformed to a rational curve of the form $t \mapsto \varphi(t) + t_2 + s_3 + \ldots + s_n$, where t_2 is a smooth point of R that does not belong to H_n. Similarly, the intersection number of C_x with H_n is zero.

We readily check that the intersection number of the image of l with the exceptional divisor E is 2: the images of x and y are the two transverse intersection points.

Finally, the intersection number of C_x with the exceptional divisor E is -1, see for instance Example 4.2 of [HT10].

Since $\delta = \frac{1}{2}[E]$ and $\delta^2 = -2(n-1)$, this shows that
$$(f_\mu)_*[C_\mu] \cup \delta = \mu$$

and, finally, that $(f_\mu)_*[C_\mu]$ is dual to $h - \frac{\mu}{2(n-1)}\delta$. $\square$

Remark 4.4. In the construction above, we could have dealt with the case $\mu = -1$ in a similar way by considering the stable map obtained by glueing l with C_x and C_y. However, it is possible to show that the sign of μ does not change the deformation type of a pair $(S^{[n]}, h + \mu\delta)$ – indeed, the reflection with respect to the class of the exceptional divisor of $S^{[n]}$ changes the sign of μ and belongs to the monodromy group of $S^{[n]}$ in the terminology of Markman, as shown in [Mar13] relying on work of Druel [Dru11], which implies the statement by Corollary 7.4 of [Mar11].

The results above can be applied to the theorem we stated in the introduction.

Proof of Theorem 1.1. Let h be a primitive polarization on a projective fourfold X of $K3^{[2]}$-type. By Corollary 2.6, we can find a $K3$ surface S, an ample cohomology class h' and $\mu \in \{0, 1\}$ such that (X, h) is deformation equivalent to $(S^{[2]}, \lambda h' - \mu\delta)$ for $(\lambda, \mu) \in \{(1, 0), (2, 1)\}$. Up to dividing by 2 if $\mu = 1$, which does not change the deformation type, we can apply Propositions 4.1 and 4.3 to conclude – note that rationally connected irreducible surfaces are rational. $\square$

The results above do not give directly the cohomology classes of the subschemes we constructed. However, cohomological arguments give the following refinement of Theorem 1.1. The argument that allows us to pass from Theorem 1.1 to Theorem 4.5 is due to Eyal Markman, to whom we are very grateful for sharing his unpublished notes [Mar] with us.

Theorem 4.5. *Let X be a projective holomorphic symplectic fourfold of $K3^{[2]}$-type and let h be an ample class in $H^2(X, \mathbb{Z})$. In Theorem 1.1, we can assume that the cohomology class of the uniruled divisor on X is a multiple of h, and that the cohomology class of the rational surface is a multiple of $5h \cup h - \frac{1}{6}q(h)c_2(X)$.*

Remark 4.6. For the sake of clarity, we did not use the notation h^2 in the formula above.

Proof. We can assume that the pair (X, h) is very general. In that case, the Picard number of X is one, so the cohomology class of any divisor on X is proportional to h.

Results of Markman in [Mar] show that the group of Hodge classes in $H^4(X, \mathbb{Z})$ is generated by $h \cup h$ and $c_2(X)$, and that any Lagrangian surface in X has cohomology class a multiple of $5h \cup h - \frac{1}{6}q(h)c_2(X)$. $\square$

The refinement above allows us to construct a canonical zero-cycle of degree 1 on projective varieties of $K3^{[2]}$-type.

Corollary 4.7. *Let X be a projective holomorphic symplectic fourfold of $K3^{[2]}$-type. All points of X lying on some rational surface with cohomology class a multiple of $5h \cup h - \frac{1}{6}q(h)c_2(X)$ have the same class c_X in $CH_0(X)$.*

Proof. We only need to show that any two such surfaces have non-empty intersection. For this, it is enough to show that the square of $5h \cup h - \frac{1}{6}q(h)c_2(X)$ is nonzero.

By computations of [Mar], we have, in the cohomology ring of X,

$$h^4 = 3q(h)^2,\ c_2(X)^2 = 828 \text{ and } h \cup h \cup c_2(X) = 30q(h).$$

This shows that

$$(5h \cup h - \frac{1}{6} q(h) c_2(X))^2 = 48 q(h) \neq 0.$$

□

5. Some open questions

We briefly discuss some questions raised by our results above. In view of our constructions, it seems natural to hope for a positive answer for Question 1.2. Note that this is the case if X is of the form $S^{[n]}$ for some $K3$ surface S as follows by taking Y to be the closure in $S^{[n]}$ of the locus of points $s_1 + \ldots + s_n$, where the s_i are distinct points of S, k of which lie on a given rational curve of S.

It would be interesting to refine Question 1.2 to specify the expected cohomology classes of the subschemes Y_k.

The particular case of middle-dimensional subschemes seems of special interest in view of the study of rational equivalence on holomorphic symplectic varieties.

Question 5.1. *Let X be a projective holomorphic symplectic variety of dimension $2n$. Does there exist a rationally connected subvariety Y of X such that Y has dimension n and nonzero self-intersection ?*

A positive answer to question 5.1 implies as in Corollary 4.7 the existence of a canonical zero-cycle of degree 1 on X. This raises the following question.

Question 5.2. *Assume that Question 5.1 has a positive answer for X and let y be any point of Y. Let H be an ample divisor on X, and let $k_0, \ldots, k_{2n}$ be nonnegative integers such that $2k_0 + \sum_i 2ik_i = 2n$.*

Is the zero-cycle $H^{k_0} \Pi_i c_i(X)^{k_i} \in CH_0(X)$ proportional to the class of y in $CH_0(X)$?

Even in the case of a general polarized fourfold of $K3^{[2]}$-type, we do not know the answer to the preceding question.

Finally, Question 1.2 raises a counting problem as in the case of the Yau-Zaslow conjecture for rational curves on $K3$ surfaces [YZ96], which was solved in [KMPS10]. We do not know of a precise formulation for this question.

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