The socle in the proof of the Mordell-Lang conjecture: $G^\sharp$ versus socle($G^\sharp$)

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Subtext: model-theoretic proof by Hrushovski of the Mordell-Lang conjecture for function fields

Theorem 1 (ML) *$k \subset K$ algebraically closed fields. G semi-abelian variety over K, X irreducible subvariety of G, $\Gamma \subset G(K)$ finite rank subgroup.*
Assume $Stab_G(X)$ finite and $\Gamma \cap X$ Zariski-dense in X.
Then there are H semiabelian variety over k, Y irreducible subvariety of H, homomorphism $h : H_K \to G$ such that $h(Y_K) = a + X$ for some $a \in G(K)$

Roughly speaking, the situation descends to the small field k.

Content

1. The model-theoretic socle

Setting: Work inside a saturated model of a stable theory $T = T^{eq}$. G type-definable group of finite U-rank.

We use throughout the following immediate consequence of Zilber's indecomposability theorem (in its finite U-rank version).

Theorem 2 *Let $\{Q_i : i \in I\}$ be a family of minimal types in G. Then the subgroup generated by the Q_i's is type-definable and connected.*

Definition 1 *G, defined over B, is rigid if all connected type-definable subgroups (with extra parameters) are type-definable over $acl(B)$*

Definition 2 *Q a minimal type. G is Q-semiminimal if $G \subset acl(F \cup Q)$ for some finite set F.*
G is semipluriminimal if there are minimal types $Q_1, \ldots, Q_n$ and a finite set F such that $G \subset acl(F \cup Q_1 \cup \ldots \cup Q_n)$.

Definition of the socle

Proposition 1 *Q a minimal type of G. There is a largest connected type-definable Q-semiminimal subgroup B_Q of G.*

Proposition/definition 1 *There is a largest connected type-definable semipluriminimal subgroup of G. We denote it by $\underline{S(G), \text{ the socle of } G}$.*

Proposition 2 *$S(G) = B_{Q_1} + \ldots + B_{Q_n}$ for some minimal types $Q_1, \ldots, Q_n$, which can be assumed to be pairwise orthogonal. In particular, every minimal type in G is nonorthogonal to one of the Q_i's.*

The (weak) socle theorem

Theorem 3 *(Hrushovski in the ML paper)*
G type definable over $\emptyset$ as above, and commutative. Assume G is rigid. p a complete stationary type in G, over $\emptyset$. Assume $Stab_G(p)$ is finite. Then there is a translate of $S(G)$ containing (the realizations of) p.

2. Hrushovski's proof (1996)

Setting: K a model of DCF_0 or a saturated model of $SCF_{p,1}$. k subfield of absolute constants of K (k is algebraically closed).
A an abelian variety over K (complete connected algebraic group)
$T = \mathbb{G}_m^r$ torus
$G \in Ext(A,T)$ semiabelian variety: $0 \to T \to G \to A \to 0$.

First reduction: go from $\Gamma \subset G(K)$ subgroup of finite rank (i.e. included in the (prime-to-p) divisible hull of a finitely generated group) to a type-definable group.

Proposition/definition 2 *There is a smallest type-definable Zariski-dense subgroup of $G(K)$, denoted by $G^\sharp$.*

Alternative descriptions:
char. 0: $G^\sharp$ is the Manin kernel of G (at least for G abelian variety), also the Kolchin closure of the torsion points
char. p: $G^\sharp = p^\infty G(K) := \cap_n p^n G(K)$

Properties
$G^\sharp$ has finite U-rank (and finite transcendance rank).
Type-definable connected subgroups of $G^\sharp$ are of the form $H^\sharp$ for H semiabelian subvariety of G.
$G^\sharp$ is rigid, and $S(G^\sharp)$ as well.

From Γ to $S(G^\sharp)$:

Char. 0 case: using Manin maps, $\Gamma \subset H$, for some definable group H of finite Morley rank, containing $G^\sharp$.
Fact: $S(H) = S(G^\sharp)$.

Char. p case: by basic group theory (using the fact that $p\Gamma$ has finite index in Γ), X is contained in a translate of $G^\sharp$.

In both cases, from the socle theorem, we may assume $X \cap S(G^\sharp)$ Zariski-dense in X, up to replacing X by a translate.

Now the core of the proof by Hrushovski:

- $S(G^\sharp)$ is semipluriminimal

- minimal types of finite transcendance degree carry a Zariski geometry (also proved by Delon without the finite transcendance degree hypothesis)

- trichotomy for Zariski geometries (dichotomy in the context of groups)

No substential difference in the proof between abelian varieties and semiabelian varieties.

3. The abelian case

Poincaré's reducibility theorem: A abelian variety and B abelian subvariety. Then there exists a (quasi-)supplement: an abelian subvariety C such that $A = B + C$ and $B \cap C$ is finite.

Consequences:
$A = A_1 + \ldots + A_m$ sum of simple abelian varieties (with pairwise finite intersection).
If A_i is simple, $A_i^\sharp$ has no non trivial connected type-definable subgroups.
From the indecomposability theorem, $A_i^\sharp$ is semiminimal.
Hence $S(A^\sharp) = A^\sharp$.

An alternative argument for ML (without trichotomy for Zariski geometries)

Case where A is an abelian variety over $K_0 = \mathbb{C}(t)^{alg}$ or $K_0 = \mathbb{F}_p(t)^{sep}$).
Reduction to Manin-Mumford theorem (same statement as ML, with $\Gamma =$ torsion points).
Ingredients:
• a criterion from Wagner for elementary substructures of finite Morley rank groups
• "theorem of the Kernel": $A^\sharp(K_0) \subset A_{torsion}$ (assuming A has k-trace 0 in char. 0, i.e. no nonzero homomorphism $B \to A$, with B descending to k)
• in char. p, $A^\sharp$, with its induced structure, has QE, hence finite Morley rank

4. From $G^\sharp$ to $S(G^\sharp)$ in the semiabelian case

Extra difficulties in the semiabelian case. Here $G \in Ext(A,T)$:

- Poincaré's reducibility theorem fails for semiabelian varieties. In G, T has a quasi-supplement if and only if G is isogenous to $A \times T$, which is not the case in general.

- there are examples where the induced sequence $0 \to T^\sharp \to G^\sharp \to A^\sharp \to 0$ is not exact. So one cannot deduce good properties for $G^\sharp$ from the same properties for $A^\sharp$ and $T^\sharp$. Take A (ordinary) abelian variety descending to k, and G an extension of A by T which does not descend to k (it exists from the parametrization of such extensions using the dual abelian variety $\widehat{A}$)

Understanding $S(G^\sharp)$ (without trichotomy for Zariski geometries)

Nonorthogonality classes of minimal types in $G^\sharp$: the generic type of k and the rest.
Q minimal type in $G^\sharp$, $B_Q = H^\sharp$ for some H semiabelian subvariety of G:
• if Q is nonorthogonal to k, H is isogenous to some H_0 semiabelian variety over k
• if Q is orthogonal to k, H is an abelian variety with k-trace 0 (i.e. no nonzero homomorphism $H' \to H$ with H' descending to k)

Corollary 1 $S(G^\sharp) = G_0^\sharp + A_0^\sharp$ *with* G_0 *isogenous to a semiabelian variety over* k*, and* A_0 *abelian variety of* k*-trace* 0*. Furthermore,* $G_0^\sharp \perp A_0^\sharp$ *(and the intersection is finite).*

Cheap reduction from semiabelian varieties to abelian varieties

Sketch of the proof: from the previous reductions, we may assume $X \cap S(G^\sharp)$ is Zariski-dense in X.
Write $S(G^\sharp) = G_0^\sharp + A_0^\sharp$, with $G_0^\sharp \perp A_0^\sharp$.
There is a complete type p dense in $X \cap S(G^\sharp)$, with finite stabilizer in $G^\sharp$.
It follows from orthogonality that there are two complete types $p_1 \subset G_0^\sharp$ and $p_2 \subset A_0^\sharp$ such that $p = p_1 + p_2$.
Let $X_i = \overline{p_i}^{Zar}$, $X_1 \subset G_0$ and $X_2 \subset A_0$.
We can apply ML for abelian varieties to A_0 and X_2. Since A_0 has k-trace 0, it gives that X_2 is a point a.
Hence $X = X_1 + X_2 = a + X_1$, and (G_0, X_1) come from a similar configuration over k.

Note that $S(G^\sharp)$ has the same good properties as $A^\sharp$:

	$A^\sharp$	$G^\sharp$	$S(G^\sharp)$
semipluriminimal	yes	no	yes
QE for the induced structure	yes	no	yes
(finite) relative Morley rank	yes	no	yes
K_0- rational points are torsion points when over $K_0 = \mathbb{F}_p(t)^{sep}$	yes	?	yes

The negative answers can be observed in the previous example where $0 \to T^\sharp \to G^\sharp \to A^\sharp \to 0$ is not exact.
There are also examples where this sequence is exact, hence easily $G^\sharp$ has finite relative Morley rank and satisfies the theorem of the Kernel, but where $G^\sharp \neq S(G^\sharp)$.

5. The algebro-geometric socle

$k \subset K$, k algebraically closed, K separably closed, G semiabelian variety over K.
There exist:

- the largest semiabelian subvariety G_0 of G isogenous to a semi-abelian variety over k
- the largest abelian subvariety A_0 of G of k-trace 0

Define the $\underline{k\text{-socle of } G}$: $S_k(G) := G_0 + A_0$.
The gap between $S_k(G)$ and G reflects the failure of Poincaré's reducibility theorem for semiabelian varieties, up to weak descent to k.

Examples:

- if A is an abelian variety, $S_k(A) = A$
- if G is isogenous to a semiabelian variety over k, $S_k(G) = G$
- if G is isogenous to $T \times A$, $S_k(G) = G$
- if $G \in Ext(A, T)$ with A simple abelian variety descending to k, and G not descending to k, then $S_k(G) = T$
- if $G \in Ext(A, T)$ and A has k-trace 0, $S_k(G) = G$ if and only if the extension G is almost split (i.e. isogneous to $A \times T$)

In the previous context (i.e. K model of DCF_0 or $SCF_{p,1}$, and $k = C_K$), we have exactly $S(G^\sharp) = (S_k(G))^\sharp$, and $S_k(G) = \overline{S(G^\sharp)}^{Zar}$.

In particular, if G can be written $G = G_0 + A_0$ with G_0 and A_0 as above, then $S(G^\sharp) = G^\sharp$.

Passing to Zariski closures, a direct translation of the socle theorem is:

Corollary 2 *K, k, G as above. X subvariety of G such that $Stab_G(X)$ is finite. Γ finite rank subgroup of $G(K)$ such that $X \cap \Gamma$ is Zariski dense in X. Then X is contained in some translate of $S_k(G)$.*

Note that the statement is false without the mention of Γ. This hypothesis is needed in order to pass to the context of DCF_0 or $SCF_{p,1}$, which have an interesting theory of orthogonality in a finite rank context.

Corollary 3 *ML for G reduces to ML for $S_k(G)$.*

We don't know how to obtain this reduction without model-theory. Of course, it is only interesting in the cases where $G \neq S_k(G)$.