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On Variational Formulations for Steady Water Waves

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1. Introduction

1.1. Background and recent developments

The water-wave problem is the study of two-dimensional, inviscid, irrotational fluid flow in a domain of finite or infinite depth, bounded above by a free surface and driven by the effects of gravity and possibly surface tension. The first contribution to the study of variational formulations of this problem was made by Luke (1967), who published a formal variational principle that recovers the water-wave equations. A Hamiltonian formulation of the problem was reported by Zakharov (1968), and was pursued independently by Broer (1974, 1975), Broer, van Groesen & Timmers (1976) and Miles (1977) (see also Milder (1977)). Since then there has been a great deal of activity concerning variational formulations of exact and model water-wave equations, notably by Miles (1981) (an important summary of early developments in Hamiltonian water-wave theory), Benjamin & Olver (1982) (who used the Hamiltonian structure to classify the symmetries and conservation laws of the water-wave problem), Benjamin (1984) (a treatise on variational principles for Hamiltonian systems and in particular (§6.1) an alternative Hamiltonian formulation for water waves) and Radder (1992) (a further Hamiltonian formulation for water waves).

The past few years have seen much interest in variational formulations of the two-dimensional *steady* water-wave problem in which the free-surface flows are stationary relative to a uniformly translating frame of reference. The most well-known examples are periodic- and solitary-wave flows, which have been the subject of extensive research since the pioneering work of Stokes and Lord Rayleigh in the mid-nineteenth century (see Lamb (1945, §§250–255) and Whitham (1974, §§13.12–13.13) for a thorough discussion of the work of these authors). In a new approach to the question of the existence of travelling waves of small amplitude, Kirchgässner (1988) has recently introduced a powerful method of reducing the water-wave problem to an equivalent finite-dimensional system of ordinary differential equations. The analysis of the reduced problem has lead to important results, in particular to the first existence theorems for solitary waves in the presence of surface tension (AMICK & KIRCHGÄSSNER (1989), IOOSS & KIRCHGÄSSNER (1990)). One may regard Kirchgässner's method as a centre-manifold reduction of a special kind which involves writing a system in which, strictly speaking, there is no time-like variable, as the analogy of a dynamical system. MIELKE (1991) has discussed centre-manifold reductions of dynamical systems in general and has shown that if the full system has a Hamiltonian structure then so does the reduced system. As a consequence of this work, MIELKE (1991, Ch. 9), BAESENS & MACKAY (1992) and BRIDGES (1992, 1994) have studied Hamiltonian formulations of the steady water-wave problem written as a dynamical system. MIELKE discusses the case of finite depth and non-zero surface tension, while BAESENS & MACKAY treat the problem with infinite depth and zero or non-zero surface tension. Although the equivalence between the exact Euler equations for an inviscid, irrotational steady flow with a free boundary and the reduced system is valid only for solutions of small amplitude, the centre-manifold approach motivated BAESENS & MACKAY to make some important observations concerning the bifurcations of large-amplitude periodic Stokes waves, previously discovered numerically by CHEN & SAFFMAN (1980), based upon analogies with bifurcation theory for finite-dimensional Hamiltonian systems. BRIDGES was concerned with giving a general account of the Hamiltonian formalism for steady travelling waves based upon physical considerations and without recourse to specific differentiable manifolds. His discussion, which extends and generalises that of BENJAMIN (1972, Appendix C) and BENJAMIN (1984, §6), includes a natural interpretation of the symplectic form and Hamiltonian in terms of familiar physical quantities.

The present contribution concerns the mathematical framework of the BAESENS-MACKAY and MIELKE theories. It seeks to highlight certain novel aspects of the above problems and to lay the foundation for further work (BUFFONI, GROVES & TOLAND (1996)).

1.2. The steady water-wave problem

The classical time-dependent problem for surface waves on a two-dimensional expanse of water with undisturbed depth $h < \infty$ is formulated as follows. Let (x, y) denote the usual Cartesian coordinates. The fluid occupies the domain $D_\eta = \{(x, y) : x \in \mathbb{R},\ y \in (0, \eta(x, t))\}$, where $\eta > 0$ is a function of the spatial coordinate x and time t that is equal to h when the fluid is at rest. In terms of an Eulerian velocity potential $\phi(x, y, t)$, the mathematical problem is to solve Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \qquad \text{in } D_\eta \tag{1.1}$$

with boundary conditions

$$\phi_y = 0 \quad \text{on } y = 0, \tag{1.2}$$

$$\eta_t = \phi_y - \eta_x\phi_x \quad \text{on } y = \eta(x, t), \tag{1.3}$$

$$\phi_t = -\frac{1}{2}\phi_x^2 - \frac{1}{2}\phi_y^2 - g(\eta - h) + \sigma\left[\frac{\eta_x}{\sqrt{1 + \eta_x^2}}\right]_x \qquad \text{on } y = \eta(x, t), \tag{1.4}$$

in which $\sigma \geqq 0$ is the coefficient of surface tension (e.g., see BENJAMIN (1974), LAMB (1945, Ch. IX), WHITHAM (1974, Ch. 13)). Equation (1.2) is the kinematic condition that water cannot penetrate the rigid horizontal boundary at $y = 0$, while (1.3), (1.4) are respectively the kinematic and dynamic conditions at the free surface.

Waves that are steady with respect to a uniformly translating frame of reference are described by solutions of the special form $\eta(x,t) = \eta(x-ct)$, $\phi(x,y,t) = \phi(x-ct,y)$. Substituting this form of η, ϕ into (1.1)–(1.4) and using a new perturbed velocity potential $\tilde{\phi} = \phi - cx$, one finds that

$$\phi_{xx} + \phi_{yy} = 0 \qquad \text{for } 0 < y < \eta(x), \tag{1.5}$$

$$\phi_y = 0 \qquad \text{on } y = 0, \tag{1.6}$$

$$\phi_y - \eta_x \phi_x = 0 \qquad \text{on } y = \eta(x), \tag{1.7}$$

$$\frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + g(\eta - h) - \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x - \frac{1}{2}c^2 = 0 \quad \text{on } y = \eta(x), \tag{1.8}$$

where x is now a shorthand for the variable $x - ct$ and the tilde has been dropped for notational convenience. (No particular boundary conditions as $x \to \pm\infty$ are specified and both periodic and solitary steady waves are included as special cases.) For the moment, let us concentrate on the case when $\sigma > 0$.

In order to write the free-boundary problem (1.5)–(1.8) as the analogy of a dynamical system, one regards x as a time-like variable, so that η is a real-valued function of x and $\phi(x,\cdot)$ is an element of $X_{\eta(x)}$, where $\{X_\eta\}_{\eta\in\mathbb{R}}$ is a one-parameter family of infinite-dimensional Hilbert spaces whose elements are functions defined on $(0,\eta(x))$. The operation of differentiation with respect to y is then regarded as a densely-defined linear operation on $X_{\eta(x)}$, while differentiations with respect to x are time-like differentiations as in a general dynamical system. The set $V = \bigcup_{x\in(a,b)}\{x\} \times \mathbb{R} \times X_{\eta(x)}$ is a vector bundle upon which various differentiable structures may be imposed by the use of suitable local-coordinate charts (see below). In the context of (1.5)–(1.8), the variables $(x, \eta(x), \phi(x)) \in V$ will henceforth be referred to as the *physical* variables in order to distinguish them from local coordinates on a chart used to define the differentiable structure.

BRIDGES (1992, 1994) works with these physical variables in his construction of a 'spatial Hamiltonian evolution equation' for the steady water-wave problem. In addition, in specifying his phase space he includes a variable Φ, defined to be the trace of ϕ at the free boundary. The present treatment, in common with that of MIELKE (1991, Ch. 9) and BAESENS & MACKAY (1992), avoids such use of a supplementary variable and introduces local-coordinate charts in order to make the differentiable structure of the vector bundle V explicit. This process is easily accomplished by introducing the new variables

$$\tilde{\Phi}(x,z) = \phi(x,y), \qquad y = z\eta(x),$$

so that z belongs to the fixed interval $(0,1)$. It is clear that V may be modelled upon the single local-coordinate chart $\mathbb{R}\times\mathbb{R}\times X_1$, and this construction endows the physical variables with a natural differentiable structure. In terms of these coordinates, equations (1.5)–(1.8) transform into the system

$$\tilde{\Phi}_{xx} - \frac{2z\eta_x}{\eta}\tilde{\Phi}_{xz} + \frac{2z\eta_x^2}{\eta^2}\tilde{\Phi}_z + \frac{z^2\eta_x^2}{\eta^2}\tilde{\Phi}_{zz} - \frac{z\eta_{xx}}{\eta}\tilde{\Phi}_z + \frac{1}{\eta^2}\tilde{\Phi}_{zz} = 0 \quad \text{for } 0 < z < 1, \quad (1.9)$$

$$\tilde{\Phi}_z = 0 \qquad \text{on } z = 0, \quad (1.10)$$

$$\frac{1}{\eta}\tilde{\Phi}_z - \eta_x\tilde{\Phi}_x + \frac{\eta_x^2}{\eta}\tilde{\Phi}_z = 0 \qquad \text{on } z = 1, \quad (1.11)$$

$$\frac{1}{2}\left[\tilde{\Phi}_x - \frac{\eta_x}{\eta}\tilde{\Phi}_z\right]^2 + \frac{\tilde{\Phi}_z^2}{2\eta^2} + g(\eta - h) - \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x - \frac{c^2}{2} = 0 \quad \text{on } z = 1. \quad (1.12)$$

Equations (1.9)–(1.12) are used as the basis for the theory given by MIELKE (1991, Ch. 9), BAESENS & MACKAY (1992) and also for the present theory.

1.3. The present approach

The present tasks are to examine how a solution of the partial differential equation (1.9) and the nonlinear free-boundary conditions (1.10)–(1.12) may be obtained from a trajectory of a particular Hamilitonian system (different when $\sigma > 0$ and $\sigma = 0$), and to show how these systems follow directly from the natural Lagrangian formulations.

The sequel adheres to the strict view that a Hamiltonian system is completely specified once a differentiable manifold M, a 2-form Ω and a differentiable functional H on M have been identified. In any Hamiltonian formulation of the steady water-wave problem, the specification of (M, Ω, H) must yield the boundary-value problem (1.9)–(1.12) without recourse to extra variables, supplementary regularity or boundary conditions not implicit in membership of M.

Since the free-boundary conditions (1.7) and (1.8) or, equivalently, (1.11) and (1.12) lie at the heart of the steady water-wave problem, it is particularly important to see their role in any particular treatment. Clarification of this issue is one of the main aims of the present analysis of the BAESENS-MACKAY and MIELKE Hamiltonian theories. In general, when specifying a Hilbert manifold M, one is not permitted to use boundary conditions which require more regularity than membership of M implies (for example neither $u(0)$ nor $u(1)$ can be specified for a function $u \in L^2(0, 1)$). In the present context this fact means that neither kinematic nor dynamic boundary conditions may be specified a priori. The calculations show that when $\sigma = 0$, the dynamic free-surface condition emerges as a necessary condition for membership of the domain of the Hamiltonian vector field while the kinematic boundary condition is satisfied because of Hamilton's equations. By contrast, when $\sigma > 0$, the kinematic free-surface condition is required for membership of the domain of the Hamiltonian vector field and the dynamic condition follows by Hamilton's equations. It is noteworthy that BAESENS & MACKAY (1992) require elements of their manifold M to satisfy the different boundary conditions appropriate to the domain of the Hamiltonian vector field in both cases. Although this procedure for defining the manifold M is inadmissable because of the regularity problems outlined above, the present analysis does show that the domain of the Hamiltonian vector field $\mathscr{D}(v_H)$ is composed of functions with sufficient regularity to satisfy the boundary conditions which BAESENS & MACKAY demand for membership of M. However, M and $\mathscr{D}(v_H)$ do not coincide.

MIELKE (1991, Ch. 9) has an approach to the Hamiltonian formulation of the steady water-wave problem whose end result is close to that given here. MIELKE introduces a priori a subset of the manifold M (a 'manifold domain') upon which the Hamiltonian vector field v_H is defined. The present approach requires no such a priori specification, but it turns out a posteriori that the domain of v_H which emerges as an essential part of the theory is a manifold domain in MIELKE's sense.

The Lagrangian formulation of the steady water-wave problem is considered in Section 2.3. Following BRIDGES (1992, p. 303) one introduces the obvious functional whose formal Euler-Lagrange equations (in physical coordinates) are (1.5)–(1.8). Introducing local coordinates as described above, one obtains a Lagrangian function for the system (1.9)–(1.12). It is then an elementary calculation to understand how the Legendre transform, properly understood in the context of convex (and not necessarily differentiable) functionals, yields complete Hamiltonian formulations of the steady water-wave problems for both zero and non-zero values of σ. When $\sigma = 0$, the functional constraint defining the manifold N introduced by BAESENS & MACKAY (1992) (see Section 2.2 below) emerges simply as a necessary condition for the Legendre transform to be finite. This analysis shows how the manifolds, the 2-forms and the Hamiltonians for steady water-wave problems emerge naturally from Lagrangian formulations and elementary duality theory.

A feature of the Hamiltonian formulation (N, Ω_1, H_1) of the steady water-wave problem when $\sigma = 0$ (Section 2.2) is that the closed 2-form Ω_1 is degenerate at certain points of N. BAESENS & MACKAY (1992, §7) mention that these points are connected with the existence of steady waves of extreme form (TOLAND (1978), AMICK, FRAENKEL & TOLAND (1982)) and this point is taken up in the present paper. Here Theorem 2.4 shows how the Hamiltonian vector field is well-defined in spite of the degeneracy of Ω_1 and that the 2-form is degenerate at precisely those points in the domain of the Hamiltonian vector field that have stagnation points (points on the free surface where the fluid velocity is zero relative to the moving frame). The free-surface elevation has its maximum value at such points and may have a corner: for example the famous Stokes wave of greatest height (STOKES (1880)) has a $2\pi/3$ corner. The degeneracy of Ω_1 at the stagnation points allows such flows to come under the umbrella of the general Hamiltonian theory.

Section 3 indicates what adjustments are needed to treat the case of infinite depth.

1.4. Nomenclature

All manifolds M are modelled on a single coordinate chart C in a Hilbert space $\mathscr{H}$. The tangent space to M at m (ABRAHAM, MARSDEN & RATIU (1988, Definition 3.3.3)) is denoted by $TM|_m$, and the element of $TM|_m$ represented by a continuously differentiable path γ on M with $\gamma(x) = m$ is identified with the element $\dot{\gamma}(x) \in \mathscr{H}$ and referred to as the *tangent vector to γ at $\gamma(x)$*. The cotangent space to M at m (ABRAHAM, MARSDEN & RATIU (1988, Definition 5.2.9)) is denoted by

$T^*M|_m$. Observe that $TM|_m$ is isomorphic to $\mathscr{H}$ and $T^*M|_m$ is isomorphic to $\mathscr{H}^*$. The *tangent* and *cotangent bundles* of M are denoted by TM and T^*M and treated as manifolds modelled on the single coordinate charts $C \times \mathscr{H}$ and $C \times \mathscr{H}^*$.

A *k-form* α is an assignment of a bounded alternating k-linear map $\alpha|_m : (TM|_m)^k \to \mathbb{R}$ to each point m of its domain $\mathscr{D}(\alpha) \subseteq M$. The *differential* $\mathbf{d}f|_m$ of a continuously-differentiable function $f : M \to \mathbb{R}$ at the point $m \in M$ is the element $df[m]$ of $T^*M|_m$, so that $\alpha = \mathbf{d}f$ is a 1-form defined on the whole of M. A 2-form Ω is *weakly nondegenerate* at a point $m \in M$ if the fact that $\Omega|_m(v_1, v_2) = 0$ for all $v_1 \in TM|_m$ implies that $v_2 = 0$, and is *closed* if its exterior derivative is zero. For the present purposes it suffices to note that the closure condition is always satisfied if $\Omega|_m$ is the same for each m (see ABRAHAM, MARSDEN & RATIU (1988, Ch. 9) and MIELKE (1991, p. 11).

A Hamiltonian system is a triple (M, Ω, H), where M is a manifold, $H : M \to \mathbb{R}$ is a continuously-differentiable function called the *Hamiltonian* and Ω is a symplectic 2-form (a 2-form that is closed and weakly nondegenerate at each point $m \in M$). A *vector field* v on M is an assignment of a tangent vector $v|_m \in TM|_m$ to each point m of its domain $\mathscr{D}(v) \subseteq M$. Each Hamiltonian system has an associated *Hamiltonian vector field* v_H, which is defined as follows. Let $\mathscr{D}(v_H)$ denote the subset of M whose elements m have the property that there exists a tangent vector $v_H|_m \in TM|_m$ such that

$$\mathbf{d}H|_m(v|_m) = \Omega|_m(v_H|_m, v|_m) \tag{1.13}$$

for all $v|_m \in TM|_m$. The weak nondegeneracy of Ω ensures that if $v_H|_m$ exists for a given m, then it is unique. This construction therefore defines a unique vector field v_H with domain $\mathscr{D}(v_H) \subseteq M$. In infinite-dimensional settings $\mathscr{D}(v_H)$ may not be the whole of M, even though $\mathbf{d}H|_m$ exists for each $m \in M$, as the hydrodynamic examples in this paper demonstrate. Since the weak nondegeneracy of Ω is required only to ensure that the Hamiltonian vector field is uniquely defined, one may generalise the above definition of a Hamiltonian system by specifying only that $\Omega|_m$ is weakly nondegenerate at points $m \in \mathscr{D}(v_H)$. In fact, there are important cases where one must even relax the condition of weak nondegeneracy (Section 2.2).

A *trajectory* $\gamma : (a, b) \to \mathscr{D}(v) \subseteq M$ of a vector field v on M is a continuously differentiable path whose tangent vector $\dot{\gamma}(x)$ coincides with $v|_{\gamma(x)}$ at each point $x \in (a, b)$. *Hamilton's equations* for the Hamiltonian system (M, Ω, H) are the differential equations

$$\dot{\gamma}(x) = v_H|_{\gamma(x)} \tag{1.14}$$

which determine the trajectories $\gamma : (a, b) \to \mathscr{D}(v_H) \subset M$ of the Hamiltonian vector field.

2. Surface waves on water of uniform finite depth

This section examines how two-dimensional steady surface-wave problems formulated in Sobolev spaces may be regarded as a Hamiltonian or Lagrangian system, and includes an account of the canonical duality between the two corresponding variational principles.

2.1. Hamiltonian formulation with surface tension

When $\sigma > 0$, the manifold

$$M = \{(\eta, \omega, \Phi, \Psi) : \eta \in (0, \infty), \omega \in \mathbb{R}, \Phi \in H^1(0,1), \Psi \in L^2(0,1), |W| < \sigma\}, \tag{2.1}$$

where

$$W = \omega + \frac{1}{\eta} \int_0^1 z\Psi\Phi_z \, dz, \tag{2.2}$$

is an open subset of the Hilbert space $\mathbb{R} \times \mathbb{R} \times H^1(0,1) \times L^2(0,1)$. A path on M is a function $\gamma : (a,b) \subset \mathbb{R} \to M$, which may be written as

$$\gamma(x) = (\eta(x), \omega(x), \Phi(x), \Psi(x)).$$

If γ is continuously differentiable, its tangent vector $\dot{\gamma}(x) \in TM|_{\gamma(x)} \cong \mathbb{R} \times \mathbb{R} \times H^1(0,1) \times L^2(0,1)$ is

$$\dot{\gamma}(x) = (\dot{\eta}(x), \dot{\omega}(x), \dot{\Phi}(x), \dot{\Psi}(x)).$$

The 2-form Ω on M defined by

$$\Omega|_m((\eta_1, \omega_1, \Phi_1, \Psi_1), (\eta_2, \omega_2, \Phi_2, \Psi_2)) = \omega_2\eta_1 - \eta_2\omega_1 + \int_0^1 (\Psi_2\Phi_1 - \Phi_2\Psi_1)\, dz \tag{2.3}$$

is symplectic: it is closed because it does not depend on m and is easily seen to be weakly nondegenerate at each point of M. Finally, let $H : M \to \mathbb{R}$ be the function

$$H(\eta, \omega, \Phi, \Psi) = \frac{1}{2\eta} \int_0^1 (\Psi^2 - \Phi_z^2)\, dz - \frac{g(\eta - h)^2}{2} + \frac{c^2}{2}(\eta - h) + \sigma - \sqrt{\sigma^2 - W^2}.$$

The following theory shows how trajectories of this Hamiltonian system yield solutions of the steady water-wave problem (1.5)–(1.8) with $\sigma > 0$. It explains how the domain of v_H consists of points of M that have extra regularity and satisfy natural boundary conditions (so that $\mathscr{D}(v_H)$ is not a submanifold of M). No boundary condition is imposed a priori. The kinematic boundary conditions emerge as a consequence of the definition of v_H and the dynamic boundary condition follows from Hamilton's equations.

Theorem 2.1. *Consider the Hamiltonian system (M, Ω, H) described above. The Hamiltonian vector field v_H corresponding to this system is given by*

$$\begin{pmatrix} \eta \\ \omega \\ \Phi \\ \Psi \end{pmatrix} \mapsto \begin{pmatrix} \frac{W}{\sqrt{\sigma^2 - W^2}} \\ \frac{1}{2\eta^2} \int_0^1 (\Psi^2 - \Phi_z^2)\, dz + \frac{W}{\eta^2\sqrt{\sigma^2 - W^2}} \int_0^1 z\Psi\Phi_z\, dz + g(\eta - h) - \frac{1}{2}c^2 \\ \frac{\Psi}{\eta} + \frac{zW\Phi_z}{\eta\sqrt{\sigma^2 - W^2}} \\ -\frac{\Phi_{zz}}{\eta} + \frac{W}{\eta\sqrt{\sigma^2 - W^2}}(z\Psi)_z \end{pmatrix}.$$

The domain $\mathscr{D}(v_H)$ of v_H is the set of all $(\eta, \omega, \Phi, \Psi) \in M$ such that $(\eta, \omega, \Phi, \Psi)$ belongs to $(0, \infty) \times \mathbb{R} \times H^2(0, 1) \times H^1(0, 1)$ *and satisfies*

$$\Phi_z = 0 \quad \text{on } z = 0,$$
$$\Phi_z - \frac{W\Psi}{\sqrt{\sigma^2 - W^2}} = 0 \quad \text{on } z = 1.$$

Proof 2.1. Let $\bar{v}|_m = (\bar{\eta}, \bar{\omega}, \bar{\Phi}, \bar{\Psi})$ belong to $TM|_m$, where m is the element $(\eta, \omega, \Phi, \Psi)$ of M. The point m belongs to $\mathscr{D}(v_H)$ with $v_H|_m = \bar{v}|_m$ if and only if

$$\Omega|_m(\bar{v}|_m, v_1|_m) = \mathbf{d}H|_m(v_1|_m) \tag{2.4}$$

for all tangent vectors $v_1|_m = (\eta_1, \omega_1, \Phi_1, \Psi_1) \in TM|_m$, which is the case if and only if

$$\begin{aligned}\bar{\eta}\omega_1 - \bar{\omega}\eta_1 + \int_0^1 (\bar{\Phi}\Psi_1 - \bar{\Psi}\Phi_1)\,dz = \\ -\frac{\eta_1}{2\eta^2}\int_0^1 (\Psi^2 - \Phi_z^2)\,dz + \frac{1}{\eta}\int_0^1 (\Psi\Psi_1 - \Phi_z\Phi_{1z})\,dz - g(\eta - h)\eta_1 + \frac{1}{2}c^2\eta_1 \\ + \frac{W}{\sqrt{\sigma^2 - W^2}}\left[\omega_1 - \frac{\eta_1}{\eta^2}\int_0^1 z\Psi\Phi_z\,dz + \frac{1}{\eta}\int_0^1 (z\Psi_1\Phi_z + z\Psi\Phi_{1z})\,dz\right].\end{aligned} \tag{2.5}$$

If this formula is true for all $v_1|_m \in TM|_m$, then in particular it must be true when η_1, Φ_1, Ψ_1 are zero and when ω_1, Φ_1, Ψ_1 are zero. Therefore

$$\bar{\eta} = \frac{W}{\sqrt{\sigma^2 - W^2}}, \tag{2.6}$$

$$\bar{\omega} = \frac{1}{2\eta^2}\int_0^1 (\Psi^2 - \Phi_z^2)\,dz + \frac{W}{\eta^2\sqrt{\sigma^2 - W^2}}\int_0^1 z\Psi\Phi_z\,dz + g(\eta - h) - \frac{1}{2}c^2. \tag{2.7}$$

It follows that (2.5) holds for all $v_1 \in TM|_m$ if and only if (2.6), (2.7) are satisfied and

$$\begin{aligned}\int_0^1 (\bar{\Phi}\Psi_1 - \bar{\Psi}\Phi_1)\,dz = \\ \frac{1}{\eta}\int_0^1 (\Psi\Psi_1 - \Phi_z\Phi_{1z})\,dz + \frac{W}{\eta\sqrt{\sigma^2 - W^2}}\int_0^1 (z\Psi_1\Phi_z + z\Psi\Phi_{1z})\,dz\end{aligned} \tag{2.8}$$

for all $\Phi_1 \in H^1(0, 1)$, $\Psi_1 \in L^2(0, 1)$. Setting first Φ_1 and then Ψ_1 to zero, one finds that (2.8) is satisfied if and only if both integrals on the left-hand side are zero, so that

$$\int_0^1 \Psi_1 \left(\frac{1}{\eta}\Psi + \frac{Wz\Phi_z}{\eta\sqrt{\sigma^2 - W^2}} - \bar{\Phi} \right) dz = 0, \tag{2.9}$$

$$\int_0^1 \left\{ \bar{\Psi}\Phi_1 + \Phi_{1z} \left(-\frac{\Phi_z}{\eta} + \frac{Wz\Psi}{\eta\sqrt{\sigma^2 - W^2}} \right) \right\} dz = 0. \tag{2.10}$$

Equation (2.9) is satisfied for all $\Psi_1 \in L^2(0,1)$ if and only if

$$\frac{1}{\eta}\Psi + \frac{Wz\Phi_z}{\eta\sqrt{\sigma^2 - W^2}} = \bar{\Phi} \in H^1(0,1). \tag{2.11}$$

The condition that (2.10) holds for all $\Phi_1 \in H^1(0,1)$ implies that $\bar{\Psi}$ is the weak derivative of $-\Phi_z/\eta + Wz\Psi/(\eta(\sigma^2 - W^2)^{\frac{1}{2}})$, so that

$$\left[-\frac{1}{\eta}\Phi_z + \frac{Wz\Psi}{\eta\sqrt{\sigma^2 - W^2}} \right]_z = \bar{\Psi} \in L^2(0,1), \tag{2.12}$$

and equations (2.11), (2.12) show that $\Phi \in H^2(0,1)$, $\Psi \in H^1(0,1)$. The existence of the weak derivative on the left-hand side of (2.12) implies that natural boundary conditions are satisfied. Taking equation (2.10) and integrating the second term by parts (which is permissible because the expression $-\Phi_z/\eta + Wz\Psi/(\eta(\sigma^2 - W^2)^{\frac{1}{2}})$ belongs to $H^1(0,1)$ and is therefore continuous on $[0,1]$), one finds that

$$\int_0^1 \bar{\Psi}\Phi_1 \, dz + \left[\Phi_1 \left(-\frac{\Phi_z}{\eta} + \frac{Wz\Psi}{\eta\sqrt{\sigma^2 - W^2}} \right) \right]_{z=0}^{z=1}$$
$$- \int_0^1 \Phi_1 \left[-\frac{\Phi_z}{\eta} + \frac{Wz\Psi}{\eta\sqrt{\sigma^2 - W^2}} \right]_z dz = 0.$$

The first and third terms in this equation cancel because of (2.10) and the remaining term vanishes for all $\Phi_1 \in H^1(0,1)$ if and only if the natural boundary conditions

$$\Phi_z = 0 \quad \text{on } z = 0,$$
$$\Phi_z - \frac{W\Psi}{\sqrt{\sigma^2 - W^2}} = 0 \quad \text{on } z = 1$$

are satisfied.

One concludes that the relationship (2.4) holds for a particular m if and only if m belongs to the set $\mathscr{D}(v_H)$ specified in the statement of the theorem, and $\bar{v}|_m$ is then given by (2.6), (2.7), (2.11), (2.12). □

The continuously differentiable path $\gamma : (a,b) \to \mathscr{D}(v_H) \subset M$ given by

$$\gamma(x) = (\eta(x), \omega(x), \Phi(x)(\cdot), \Psi(x)(\cdot)).$$

is a trajectory of v_H if it satisfies Hamilton's equations

$$\dot{\eta}(x) = \frac{W}{\sqrt{\sigma^2 - W^2}}, \tag{2.13}$$

$$\dot{\omega}(x) = \frac{1}{2\eta^2}\int_0^1 (\Psi^2 - \Phi_z^2)\,dz + \frac{W}{\eta^2\sqrt{\sigma^2 - W^2}}\int_0^1 z\Psi\Phi_z\,dz$$

$$+ g(\eta - h) - \frac{1}{2}c^2, \tag{2.14}$$

$$\dot{\Phi}(x)(z) = \frac{\Psi}{\eta} + \frac{zW\Phi_z}{\eta\sqrt{\sigma^2 - W^2}}, \tag{2.15}$$

$$\dot{\Psi}(x)(z) = \left[-\frac{\Phi_z}{\eta} + \frac{Wz\Psi}{\eta\sqrt{\sigma^2 - W^2}}\right]_z \tag{2.16}$$

and the boundary conditions

$$\Phi_z(x)(0) = 0, \tag{2.17}$$

$$\Phi_z(x)(1) - \frac{W\Psi(x)(1)}{\sqrt{\sigma^2 - W^2}} = 0. \tag{2.18}$$

Here $W(x)$ is defined by equation (2.2), z is the dummy variable associated with the Sobolev spaces $H^m(0,1)$, the subscript z indicates weak differentiation with respect to z and the vector $(\dot{\eta}(x), \dot{\omega}(x), \dot{\Phi}(x)(z), \dot{\Psi}(x)(z))$ is, as usual, the tangent vector to γ at the point $\gamma(x)$.

The following theorem describes the relationship between γ and functions in Sobolev spaces defined on $\{(x,z) : x \in (x_1, x_2), z \in (0,1)\}$, where $a < x_1 < x_2 < b$. Its proof, which follows by a result of FRIEDMAN (1969, Part II, Lemma 10.1) and a bootstrap argument, is relatively straightforward and the details are omitted here.

Theorem 2.2. *Suppose that $\gamma : (a,b) \to M$ is a continuously-differentiable path with*

$$\gamma(x) = (\eta(x), \omega(x), \Phi(x), \Psi(x)).$$

There exist functions $\tilde{\Phi}, \tilde{\Psi}$ that are measurable on $(a,b) \times (0,1)$ and have the property that for all $x \in (a,b)$

$$\tilde{\Phi}(x,z) = \Phi(x)(z), \qquad \tilde{\Psi}(x,z) = \Psi(x,z)$$

for almost all $z \in (0,1)$. For all x_1, x_2 with $a < x_1 < x_2 < b$, the weak derivatives

$$\frac{\partial\tilde{\Phi}}{\partial z}, \quad \frac{\partial\tilde{\Phi}}{\partial x}, \quad \frac{\partial^2\tilde{\Phi}}{\partial x\partial z}, \quad \frac{\partial\tilde{\Psi}}{\partial x},$$

exist in $L^2[(x_1,x_2) \times (0,1)]$, and for almost all $x \in (x_1,x_2)$ the identities

$$\Phi_z(x)(z) = \frac{\partial\tilde{\Phi}}{\partial z}(x,z), \qquad \dot{\Phi}(x)(z) = \frac{\partial\tilde{\Phi}}{\partial x}(x,z),$$

$$\dot{\Phi}_z(x)(z) = \frac{\partial^2\tilde{\Phi}}{\partial x\partial z}(x,z), \qquad \dot{\Psi}(x)(z) = \frac{\partial\tilde{\Phi}}{\partial x}(x,z)$$

hold for almost all $z \in (0,1)$.

The next result derives a system of partial differential equations from the system (2.13)–(2.18). Its proof is again straightforward and is omitted.

Theorem 2.3. *Suppose that $\gamma : (a,b) \to \mathscr{D}(v_H) \subset M$ is a continuously differentiable path that satisfies Hamilton's equations for the Hamiltonian system (M, Ω, H). For each x_1, x_2 with $a < x_1 < x_2 < b$, the functions $\tilde{\Phi}$ and $\tilde{\Psi}$ defined in Theorem 2.2 belong to $H^2[(x_1,x_2) \times (0,1)]$ and $H^1[(x_1,x_2) \times (0,1)]$ respectively. The functions η and ω are twice continuously differentiable in x and satisfy the equations*

$$\eta_x = \frac{\tilde{W}}{\sqrt{\sigma^2 - \tilde{W}^2}}, \qquad \eta > 0, \tag{2.19}$$

$$\omega_x = \frac{1}{2\eta^2}\int_0^1 (\tilde{\Psi}^2 - \tilde{\Phi}_z^2)\,dz + \frac{\tilde{W}}{\eta^2\sqrt{\sigma^2 - \tilde{W}^2}}\int_0^1 z\tilde{\Psi}\tilde{\Phi}_z\,dz + g(\eta - h) - \frac{1}{2}c^2, \tag{2.20}$$

where $|\tilde{W}| < \sigma$ and

$$\tilde{W} = \omega + \frac{1}{\eta}\int_0^1 z\tilde{\Psi}\tilde{\Phi}_z\,dz.$$

The functions $\tilde{\Phi}$, $\tilde{\Psi}$ satisfy the equations

$$\tilde{\Phi}_x = \frac{1}{\eta}\left(\tilde{\Psi} + \frac{z\tilde{W}\tilde{\Phi}_z}{\sqrt{\sigma^2 - \tilde{W}^2}}\right), \tag{2.21}$$

$$\tilde{\Psi}_x = \frac{1}{\eta}\left[-\tilde{\Phi}_z + \frac{\tilde{W}z\tilde{\Psi}}{\sqrt{\sigma^2 - \tilde{W}^2}}\right]_z \tag{2.22}$$

in the weak sense on $(x_1,x_2) \times (0,1)$ with boundary conditions

$$\tilde{\Phi}_z(x,0) = 0, \tag{2.23}$$

$$\tilde{\Phi}_z(x,1) = \frac{\tilde{W}\tilde{\Psi}(x,1)}{\sqrt{\sigma^2 - \tilde{W}^2}}. \tag{2.24}$$

The next step is to reduce (2.19)–(2.24) to a set of differential equations that involve only η and $\tilde{\Phi}$. The use of (2.19) to eliminate $\tilde{W}$ from (2.21) results in the equation

$$\tilde{\Psi} = \eta\tilde{\Phi}_x - z\eta_x\tilde{\Phi}_z. \tag{2.25}$$

Eliminating $\tilde{W}$ and $\tilde{\Psi}$ from (2.22) using (2.19) and (2.25), one finds that

$$\tilde{\Phi}_{xx} - \frac{2z\eta_x}{\eta}\tilde{\Phi}_{xz} + \frac{2z\eta_x^2}{\eta^2}\tilde{\Phi}_z + \frac{z^2\eta_x^2}{\eta^2}\tilde{\Phi}_{zz} - \frac{z\eta_{xx}}{\eta}\tilde{\Phi}_z + \frac{1}{\eta^2}\tilde{\Phi}_{zz} = 0 \tag{2.26}$$

in $L^2[(x_1,x_2) \times (0,1)]$. The variables $\tilde{W}$ and $\tilde{\Psi}$ may be similarly eliminated from (2.23), (2.24), which become

$$\tilde{\Phi}_z = 0 \quad \text{on } z = 0, \tag{2.27}$$

$$\frac{1}{\eta}\tilde{\Phi}_z - \eta_x\tilde{\Phi}_x + \frac{\eta_x^2}{\eta}\tilde{\Phi}_z = 0 \quad \text{on } z = 1. \tag{2.28}$$

The remaining equation (2.20) is dealt with as follows. Equation (2.19) implies that

$$\tilde{W} = \frac{\sigma\eta_x}{\sqrt{1+\eta_x^2}}.$$

It then follows from the definition of $\tilde{W}$ and Theorem 2.2 that

$$\left[\frac{\sigma\eta_x}{\sqrt{1+\eta_x^2}}\right]_x = \omega_x - \frac{\eta_x}{\eta^2}\int_0^1 z\tilde{\Psi}\tilde{\Phi}_z\,dz + \frac{1}{\eta}\int_0^1 z(\tilde{\Psi}_x\tilde{\Phi}_z + \tilde{\Psi}\tilde{\Phi}_{xz})\,dz. \tag{2.29}$$

Integrating by parts, one finds that

$$\begin{aligned}
&\int_0^1 z\tilde{\Phi}_{xz}\tilde{\Psi}\,dz \\
&= z\tilde{\Phi}_x\tilde{\Psi}\big|_{z=1} - \int_0^1 \tilde{\Phi}_x(z\tilde{\Psi})_z\,dz \\
&= \frac{1}{\eta}\left(\frac{1}{2}\tilde{\Psi}^2 + \eta_x\tilde{\Phi}_z\tilde{\Psi}\right)\bigg|_{z=1} - \frac{1}{2\eta}\int_0^1 \tilde{\Psi}^2\,dz - \frac{\eta_x}{\eta}\int_0^1 z\tilde{\Phi}_z(z\tilde{\Psi})_z\,dz,
\end{aligned} \tag{2.30}$$

where the second line follows by (2.19), (2.21) and a further integration by parts. Equations (2.19) and (2.22) imply that

$$\begin{aligned}
\int_0^1 z\tilde{\Phi}_z\tilde{\Psi}_x\,dz &= -\frac{1}{\eta}\int_0^1 z\tilde{\Phi}_z\tilde{\Phi}_{zz}\,dz + \frac{\eta_x}{\eta}\int_0^1 z\tilde{\Phi}_z(z\tilde{\Psi})_z\,dz \\
&= -\frac{1}{2\eta}\tilde{\Phi}_z^2\Big|_{z=1} + \frac{1}{2\eta}\int_0^1 \tilde{\Phi}_z^2\,dz + \frac{\eta_x}{\eta}\int_0^1 z\tilde{\Phi}_z(z\tilde{\Psi})_z\,dz.
\end{aligned} \tag{2.31}$$

Putting together equations (2.19), (2.20), (2.29)–(2.31), one finds that

$$\left[\frac{\sigma\eta_x}{\sqrt{1+\eta_x^2}}\right]_x = \frac{1}{2\eta^2}\left(\tilde{\Psi}^2 - \tilde{\Phi}_z^2 + 2\eta_x\tilde{\Phi}_z\tilde{\Psi}\right)\big|_{z=1} + g(\eta - h) - \frac{1}{2}c^2,$$

which by use of (2.19), (2.24), (2.25) may be manipulated into the form

$$\frac{1}{2}\left[\tilde{\Phi}_x - \frac{\eta_x}{\eta}\tilde{\Phi}_z\right]^2 + \frac{1}{2\eta^2}\tilde{\Phi}_z^2 + g(\eta - h) - \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x - \frac{1}{2}c^2 = 0 \quad \text{on } z = 1. \tag{2.32}$$

The final step is to transform equations (2.26)–(2.28), (2.32) into the steady water-wave problem (1.5)–(1.8). The necessary change of variable is

$$\phi(x, y) = \tilde{\Phi}(x, z), \quad \text{where } y = z\eta(x).$$

Observe that $\phi(x, y)$ belongs to $H^2[\{(x, y) : x \in (x_1, x_2), y \in (0, \eta(x))\}]$. Its weak derivatives are calculated by using the chain rule in the usual fashion. Using this change of variable, one may write (2.26)–(2.28), (2.32) in terms of ϕ to obtain the weak form of the steady water-wave problem (1.5)–(1.8). Finally notice that because ϕ is a weak solution of Laplace's equation in the flow domain $\{(x, y) : x \in (x_1, x_2), y \in (0, \eta(x))\}$, it is real-analytic in this domain.

2.2. *Hamiltonian formulation without surface tension*

This section discusses a triple (N, Ω_1, H_1), where N is a manifold, $H_1 : N \to \mathbb{R}$ is a continuously differentiable function and Ω_1 is a closed 2-form on N (see Section 2.4 for an account of how it arises from the Lagrangian formulation.) The manifold N, first introduced by Baesens & MacKay (1992), is the submanifold of M consisting of those elements $(\eta, \omega, \Phi, \Psi)$ that satisfy the constraint $\omega = F(\eta, \Phi, \Psi)$, where

$$F(\eta, \Phi, \Psi) = -\frac{1}{\eta}\int_0^1 z\Phi_z\Psi \, dz.$$

Observe that N is modelled on the single coordinate chart $C = \{(\eta, \Phi, \Psi) \in (0, \infty) \times H^1(0, 1) \times L^2(0, 1)\}$ with coordinate map χ defined by

$$\chi^{-1}(\eta, \Phi, \Psi) = (\eta, F(\eta, \Phi, \Psi), \Phi, \Psi).$$

The Hamiltonian function is defined by

$$H_1 = \frac{1}{2\eta}\int_0^1 (\Psi^2 - \Phi_z^2)\, dz - \frac{g(\eta - h)^2}{2} + \frac{1}{2}c^2(\eta - h)$$

and Ω_1 is the restriction of the symplectic 2-form Ω on M to N; it is given by the explicit formula

$$\begin{aligned}
\Omega_1|_n(v_1, v_2) &= \Omega|_{\chi^{-1}(n)}((\eta_1, dF[n](v_1), \Phi_1, \Psi_1), (\eta_2, dF[n](v_2), \Phi_2, \Psi_2)) \\
&= -\frac{\eta_1}{\eta}\int_0^1 z(\Phi_{2z}\Psi + \Phi_z\Psi_2)\, dz + \frac{\eta_2}{\eta}\int_0^1 z(\Phi_{1z}\Psi + \Phi_z\Psi_1)\, dz \\
&\quad + \int_0^1 (\Psi_2\Phi_1 - \Phi_2\Psi_1)\, dz.
\end{aligned}$$

It is a standard result that the restriction of a closed 2-form on a manifold M to a submanifold N of M is again closed (e.g., see Mielke (1991, p. 28)). Notice, however, that Ω_1 is *not* weakly nondegenerate at all points of N. Nondegeneracy of a symplectic 2-form for a Hamiltonian system is required only to ensure that the Hamiltonian vector field is uniquely defined on its domain. Since Ω_1 is degenerate at certain points of N, one proceeds by allowing a vector field to associate a set T of tangent vectors in the space $TN|_n$ with each point n of its domain $\mathscr{D}(v_{H_1}) \subset N$. The *essential domain* $\mathscr{D}_{\mathrm{ess}}(v)$ of a vector field is then the subset of elements $n \in \mathscr{D}(v)$ for which $v|_n$ is a single tangent vector. An element $n \in N$ is said to belong to $\mathscr{D}(v_{H_1})$ if there exists a tangent vector $\bar{v} \in TN|_n$ with the property that

$$\mathbf{d}H_1|_n(v) = \Omega_1|_n(\bar{v}, v)$$

for all $v \in TN|_n$. If the tangent vector $\bar{v}$ is unique, one writes $\bar{v} = v_{H_1}|_n$; otherwise, one writes $\bar{v} \in v_{H_1}|_n$.

That Ω_1 is not weakly-nondegenerate at all points of N has vital significance for the steady water-wave problem without surface tension and is intimately connected with the existence of the Stokes wave of greatest height (TOLAND (1978), AMICK, FRAENKEL & TOLAND (1982)). To begin the discussion of this pont, let us make the following definition.

Definition 2.1. A point $n = (\eta, \Phi, \Psi) \in N$ with $\Phi \in H^2(0,1), \Psi \in H^1(0,1)$ is a *stagnation point of* N if $\Phi_z(1) = \Psi(1) = 0$ and $g(\eta - h) = c^2/2$.

Stagnation points on trajectories of Hamilton's equations will presently be shown to correspond to spatial positions x at which the solution (η, ϕ) of the steady water-wave problem satisfies $(\phi_x^2 + \phi_y^2)|_{y=\eta} = 0$, so that the fluid at the free surface is at rest relative to the uniformly-translating frame of reference. The dynamic boundary condition (1.8) with $\sigma = 0$ implies that η attains its maximum value of $c^2/(2g) + h$ at such points. It follows that at these points either $\eta_x = 0$ or, as in the case when the free surface has a corner, that η_x is discontinuous and η cannot be a component of a solution of a differential equation. (These remarks do not pertain to the case when $\sigma > 0$ because then the dynamic free-surface boundary condition involves second derivatives of η.) The following analysis shows that Ω_1 is degenerate at stagnation points on N and allows in a natural fashion for jumps in η_x.

Theorem 2.4. *Consider the Hamiltonian system* (N, Ω_1, H_1) *defined above.*

1. *The set* $\mathscr{D}(v_{H_1})$ *consists of the elements* $n = (\eta, \Phi, \Psi) \in N$ *such that* $\Phi \in H^2(0,1)$, $\Psi \in H^1(0,1)$ *with*

$$\Phi_z(0) = 0,$$
$$\frac{1}{2\eta^2}(\Psi^2(1) + \Phi_z^2(1)) = \frac{1}{2}c^2 - g(\eta - h) \tag{2.33}$$

and which have the property that $\Psi(1) = 0$ *implies* $\Phi_z(1) = 0$.

2. *For each* $n \in \mathscr{D}(v_{H_1})$ *the following are equivalent:*
   (i) *n belongs to* $\mathscr{D}_{\mathrm{ess}}(v_{H_1})$*;*
   (ii) *n is not a stagnation point of N;*
   (iii) $\Psi(1) \neq 0$.

3. *For each $n \in \mathscr{D}_{\text{ess}}(v_{H_1})$, the Hamiltonian vector field v_{H_1} is defined by*

$$v_{H_1}|_n = \left(\bar{\eta}, \frac{1}{\eta}(\Psi + \bar{\eta} z \Phi_z), \frac{1}{\eta}(\bar{\eta}(z\Psi)_z - \Phi_{zz}) \right),$$

where $\bar{\eta} = \Phi_z(1)/\Psi(1)$.

4. *For each stagnation point $n \in \mathscr{D}(v_{H_1})$, the Hamiltonian vector field v_{H_1} is the one-dimensional set*

$$v_{H_1}|_n = \left\{ \left(\bar{\eta}, \frac{1}{\eta}(\Psi + \bar{\eta} z \Phi_z), \frac{1}{\eta}(\bar{\eta}(z\Psi)_z - \Phi_{zz}) \right) : \bar{\eta} \in \mathbb{R} \right\}.$$

Proof 2.2. The point $n = (\eta, \Phi, \Psi) \in N$ belongs to $\mathscr{D}(v_{H_1})$ if and only if there exists an element $\bar{v} = (\bar{\eta}, \bar{\Phi}, \bar{\Psi}) \in TN|_n$ such that

$$\Omega_1|_n(\bar{v}|_n, v_1|_n) = \mathbf{d}H|_n(v_1|_n)$$

for all $v_1|_n = (\eta_1, \Phi_1, \Psi_1) \in TN|_n$, which is the case if and only if

$$-\frac{\bar{\eta}}{\eta} \int_0^1 z(\Phi_{1z}\Psi + \Phi_z \Psi_1)\, dz + \frac{\eta_1}{\eta} \int_0^1 z(\bar{\Phi}_z \Psi + \Phi_z \bar{\Psi})\, dz + \int_0^1 (\Psi_1 \bar{\Phi} - \Phi_1 \bar{\Psi})\, dz$$

$$= -\frac{\eta_1}{2\eta^2} \int_0^1 (\Psi^2 - \Phi_z^2)\, dz + \frac{1}{\eta} \int_0^1 (\Psi \Psi_1 - \Phi_z \Phi_{1z})\, dz - g(\eta - h)\eta_1 + \frac{1}{2} c^2 \eta_1. \quad (2.34)$$

This formula is true for all $v_1|_n \in TN|_n$ if and only if each of the following identities hold for all $(\eta_1, \Phi_1, \Psi_1) \in TN|_n$:

$$\frac{1}{\eta} \int_0^1 z(\bar{\Phi}_z \Psi + \Phi_z \bar{\Psi})\, dz = -\frac{1}{2\eta^2} \int_0^1 (\Psi^2 - \Phi_z^2)\, dz - g(\eta - h) + \frac{1}{2} c^2, \quad (2.35)$$

$$\int_0^1 \Psi_1 \left(\frac{1}{\eta} \Psi + \frac{\bar{\eta}}{\eta} z \Phi_z - \bar{\Phi} \right) dz = 0, \quad (2.36)$$

$$\int_0^1 \left\{ \bar{\Psi} \Phi_1 + \Phi_{1z} \left(-\frac{\Phi_z}{\eta} + \frac{\bar{\eta}}{\eta} z \Psi \right) \right\} dz = 0. \quad (2.37)$$

Equation (2.36) is satisfied for all $\Psi_1 \in L^2(0,1)$ if and only if

$$\frac{1}{\eta} \Psi + \frac{\bar{\eta}}{\eta} z \Phi_z = \bar{\Phi} \in H^1(0,1). \quad (2.38)$$

The condition that (2.37) holds for all $\Phi_1 \in H^1(0,1)$ implies that

$$\bar{\Psi} = \left[-\frac{1}{\eta} \Phi_z + \frac{\bar{\eta}}{\eta} z \Psi \right]_z \in L^2(0,1). \quad (2.39)$$

It now follows from (2.38), (2.39) that $\Phi \in H^2(0,1)$, $\Psi \in H^1(0,1)$. Taking equation (2.37) and integrating the second term by parts (which is permissible because $-\Phi_z/\eta + \bar{\eta}z\Psi/\eta \in H^1(0,1)$), one finds that

$$\int_0^1 \bar{\Psi}\Phi_1\,dz + \left[\Phi_1\left(-\frac{\Phi_z}{\eta} + \frac{\bar{\eta}}{\eta}z\Psi\right)\right]_{z=0}^{z=1} - \int_0^1 \Phi_1\left(-\frac{\Phi_z}{\eta} + \frac{\bar{\eta}z\Psi}{\eta}\right)_z dz = 0.$$

The first and third terms in this equation cancel because of (2.39) and the remaining term vanishes for all $\Phi_1 \in H^1(0,1)$ if and only if the natural boundary conditions

$$\Phi_z = 0 \quad \text{on } z = 0, \tag{2.40}$$

$$\Phi_z - \bar{\eta}\Psi = 0 \quad \text{on } z = 1 \tag{2.41}$$

are satisfied. Finally notice that (2.41) shows that if $\Psi(1) = 0$, then $\Phi_z(1) = 0$.

The integral on the left-hand side of (2.35) may be rewritten by using (2.38), (2.39) as follows. Substituting for $\bar{\Phi}$, $\bar{\Psi}$ from (2.38), (2.39), one finds that

$$\begin{aligned}
&\int_0^1 (z\bar{\Phi}_z\Psi + z\Phi_z\bar{\Psi})\,dz \\
&= \frac{1}{\eta}\int_0^1 \left\{z\Psi(\Psi + \bar{\eta}z\Phi_z)_z + z\Phi_z(\bar{\eta}z\Psi - \Phi_z)_z\right\} dz \\
&= \frac{1}{\eta}\Psi(1)(\Psi(1) + \bar{\eta}\Phi_z(1)) - \frac{1}{\eta}\int_0^1 (z\Psi_z + \Psi)(\Psi + \bar{\eta}z\Phi_z)\,dz \\
&\quad + \frac{1}{\eta}\int_0^1 z\Phi_z(\bar{\eta}z\Psi_z + \bar{\eta}\Psi - \Phi_{zz})\,dz \\
&= \frac{1}{\eta}(\Psi^2(1) + \bar{\eta}\Psi(1)\Phi_z(1)) - \frac{1}{\eta}\int_0^1 \left\{z\Phi_z\Phi_{zz} + z\Psi\Psi_z + \Psi^2\right\} dz \\
&= \frac{1}{\eta}\left[\frac{1}{2}\Psi^2(1) - \frac{1}{2}\Phi_z^2(1) + \bar{\eta}\Psi(1)\Phi_z(1)\right] - \frac{1}{2\eta}\int_0^1 (\Psi^2 - \Phi_z^2)\,dz \\
&= \frac{1}{2\eta}[\Psi^2(1) + \Phi_z^2(1)] - \frac{1}{2\eta}\int_0^1 (\Psi^2 - \Phi_z^2)\,dz,
\end{aligned} \tag{2.42}$$

where the last line follows by (2.41). Using (2.42), one may write equation (2.35) as

$$\frac{1}{2\eta^2}(\Psi^2(1) + \Phi_z^2(1)) = \frac{1}{2}c^2 - g(\eta - h).$$

This calculation shows that the conditions specified in part (1) of the theorem hold if n belongs to $\mathscr{D}(v_{H_1})$. Conversely, suppose that the specified conditions are satisfied for some $n \in N$. Equation (2.41) has at least one solution for $\bar{\eta} \in \mathbb{R}$ because $\Psi(1) = 0$ implies that $\Phi_z(1) = 0$. Using this $\bar{\eta}$, define $\bar{\Phi}$, $\bar{\Psi}$ by (2.38), (2.39). It is a straightforward matter to work backwards through the preceding analysis to verify that $n \in \mathscr{D}(v_{H_1})$ and $(\bar{\eta}, \bar{\Phi}, \bar{\Psi}) \in v_{H_1}|_n$.

To prove part (2), suppose that (2)(i) holds, so that n belongs to $\mathscr{D}_{\text{ess}}(v_{H_1})$. The properties specified in part (1) hold and $v_{H_1}|_n$ is a single point; in particular the solution $\bar{\eta}$ of (2.41) is unique. It follows that $\Psi(1)$ is non-zero, and n is therefore not a stagnation point. Now suppose that (2)(ii) is true, so that $n \in \mathscr{D}(v_{H_1})$ is not a stagnation point. By definition, at least one of $\Psi(1)$ and $\Phi_z(1)$ is nonzero. Part (1) of the theorem implies that $\Psi(1)$ is non-zero. Finally suppose that (2)(iii) is true. The solution of $\bar{\eta}$ of (2.41) is then unique, and may be used to define a unique $\bar{\Phi}$ and $\bar{\Psi}$ by (2.38), (2.39). Since $\bar{v}$ is unique, it follows that $n \in \mathscr{D}_{\text{ess}}(v_{H_1})$.

The Hamiltonian vector field $v_{H_1}|_n$ at any point $n \in \mathscr{D}(v_{H_1})$ is found by solving (2.41) for $\bar{\eta}$ and then defining $\bar{\Phi}$, $\bar{\Psi}$ by (2.38), (2.39). If $n \in \mathscr{D}_{\text{ess}}(v_{H_1})$, the solution $\bar{\eta}$ is unique and equal to $\Phi_z(1)/\Psi(1)$; otherwise $\bar{\eta}$ can take any value in $\mathbb{R}$. A substitution of the appropriate value of $\bar{\eta}$ into (2.38), (2.39) establishes the formulae in parts (3) and (4). □

This theorem shows that, in the absence of surface tension, the equation (2.33) corresponding to the dynamic free-surface boundary condition emerges as a criterion for membership of the domain of the Hamiltonian vector field, in contrast to the case of non-zero surface tension where the dynamic free-surface boundary condition is obtained from Hamilton's equations. Also, one must be careful when writing down an appropriate version of Hamilton's equations. Because v_{H_1} is multi-valued, Hamilton's equations (1.14) might be replaced by

$$\dot{\gamma}(x) \in v_{H_1}|_{\gamma(x)}, \tag{2.43}$$

where the right-hand side of this equation is single-valued except at values of x for which $\gamma(x)$ is a stagnation point (see AUBIN & CELLINA (1984)). Such an approach is, however, outside the scope of the present paper. Let us simply note that when $\gamma(x) \in \mathscr{D}_{\text{ess}}(v_{H_1})$ for all x one may use the procedure explained in Section 2.1 to relate trajectories of the vector field v_{H_1} to the steady water-wave problem (1.5)–(1.8) with $\sigma = 0$.

2.3. Lagrangian formulation

A *Lagrangian system* is a pair (P, L), where P is a manifold and $L \in C^1[TP \to \mathbb{R}]$ is a function called the *Lagrangian.* Suppose that P is an open subset of a Hilbert space, so that $B = C^1[[a, b] \to P]$, where $[a, b]$ is a closed interval of $\mathbb{R}$, is a Banach manifold modelled on a single coordinate chart, and define a continuously-differentiable function $\mathscr{L} : B \to \mathbb{R}$ by the formula

$$\mathscr{L}(\gamma) = \int_a^b L(\gamma(x); \dot{\gamma}(x))\, dx.$$

Consider the variational problem of finding an element $\gamma \in B$ with the property that

$$\mathbf{d}\mathscr{L}|_{\gamma}(\gamma_1) = 0 \tag{2.44}$$

for all $\gamma_1 \in C_0^1[[a,b] \to \mathscr{H}]$. It is clear that (2.44) holds if and only if

$$\int_a^b \{d_1L[\gamma(x);\dot{\gamma}(x)](\gamma_1(x)) + d_2L[\gamma(x);\dot{\gamma}(x)](\dot{\gamma}_1(x))\}\, dx = 0 \tag{2.45}$$

for all $\gamma_1 \in C_0^1[[a,b] \to \mathscr{H}]$. Equation (2.45) is the generalised form of the classical *Lagrange's equation* for a Lagrangian system.

It is readily verified (cf. BRIDGES (1992, p. 303)) that equations (1.5)–(1.8) follow from the formal variational principle

$$\delta\left(\int_a^b \left\{ \int_0^{\eta(x)} \frac{1}{2}(\phi_x^2 + \phi_y^2)\, dy \right.\right.$$
$$\left.\left. + \frac{1}{2}g(\eta - h)^2 + \sigma[\sqrt{1+\eta_x^2} - 1] - \frac{1}{2}c^2(\eta - h)\right\}\, dx\right) = 0.$$

Leaving aside the meanings of the terms (η_x, ϕ_x) for a moment, one sees that the functions (η, ϕ) appearing within the braces belong to the set $\{(\eta,\phi) \in (0,\infty) \times H^1(0,\eta)\}$, which is a Hilbert manifold modelled on the single coordinate chart $P = \{(\eta, \Phi) \in (0,\infty) \times H^1(0,1)\}$; the coordinate map is simply a scaling $y = \eta z$, $y \in (0,\eta)$, $z \in (0,1)$. Using this coordinate scaling, and interpreting the formal term (η_x, Φ_x) as a tangent vector, one obtains a Lagrangian $L : TP \to \mathbb{R}$ defined by

$$L(\eta, \Phi; \mu, \theta) = \frac{1}{2\eta}\int_0^1 \left\{[\eta\theta - \mu z\Phi_z]^2 + \Phi_z^2\right\}\, dz + \frac{1}{2}g(\eta - h)^2$$
$$- \frac{1}{2}c^2(\eta - h) + \sigma[\sqrt{1+\mu^2} - 1], \tag{2.46}$$

where $(\eta, \Phi) \in P$, $(\mu, \theta) \in TP_{(\eta,\Phi)} \cong \mathbb{R} \times H^1(0,1)$. The remainder of this section is concerned with the Lagrangian system (P, L).

The continuously differentiable path $\gamma = (\eta, \Phi) : [a,b] \to P$ satisfies (2.45) if

$$\int_a^b \int_0^1 \left\{\frac{1}{\eta}[-(\eta\dot{\Phi} - \dot{\eta}z\Phi_z)\dot{\eta}z + \Phi_z]\Phi_{1z} + [\eta\dot{\Phi} - \dot{\eta}z\Phi_z]\dot{\Phi}_1\right\}\, dz\, dx$$
$$+ \int_a^b \left[-\frac{1}{2\eta^2}\int_0^1 \left\{[\eta\dot{\Phi} - \dot{\eta}z\Phi_z]^2 + \Phi_z^2\right\}\, dz\right]\eta_1\, dx$$
$$+ \int_a^b \left[\frac{1}{\eta}\int_0^1 [\eta\dot{\Phi} - \dot{\eta}z\Phi_z]\dot{\Phi}\, dz + g(\eta - h) - \frac{1}{2}c^2\right]\eta_1\, dx$$

$$+\int_a^b \left[\frac{\sigma\dot{\eta}}{\sqrt{1+\dot{\eta}^2}} - \frac{1}{\eta}\int_0^1 [\eta\dot{\Phi} - \dot{\eta}z\Phi_z]z\Phi_z\,dz \right] \dot{\eta}_1\,dx = 0 \tag{2.47}$$

for all $\gamma_1 = (\eta_1, \Phi_1) \in C_0^1[[a,b] \to \mathscr{H}]$. In the case when γ is twice continuously differentiable (questions about the a priori regularity of η, including highest waves of extreme form, are outside the scope of the present discussion), it follows from an obvious extension of Theorem 2.2 that there are functions $\tilde{\Phi}, \tilde{\Phi}_1 \in L^2[(a,b)\times(0,1)]$ with the property that for all $x \in (a,b)$

$$\tilde{\Phi}(x,z) = \Phi(x)(z), \qquad \tilde{\Phi}_1(x,z) = \Phi_1(x)(z)$$

for almost all $z \in (0,1)$. The functions $\tilde{\Phi}, \tilde{\Phi}_1$ have all the weak derivatives necessary for the arguments used in Theorems 2.2, 2.3 to be applicable in the present context. In particular, equaton (2.47) holds when $\eta_1 = 0$, so that

$$\left[\left(\frac{z\eta_x^2+1}{\eta}\right)\tilde{\Phi}_z\right]_z + (\eta\tilde{\Phi}_x)_x - (\eta_x z\tilde{\Phi}_z)_x - (\eta_x z\tilde{\Phi}_x)_z = 0 \tag{2.48}$$

holds in $L^2[(a,b)\times(0,1)]$. This equation implies the existence of $\tilde{\Phi}_{zz}$ in $L^2[(a,b)\times(0,1)]$ (because all other derivatives in (2.48) are known to exist), so that $\tilde{\Phi}$ belongs to $H^2[(a,b)\times(0,1)]$. After an integration by parts it follows that the natural boundary conditions

$$\tilde{\Phi}_z(x)(0) = 0, \tag{2.49}$$

$$\tilde{\Phi}_z(x)(1) = [\eta(x)\tilde{\Phi}_x(x)(1) - \eta_x(x)\tilde{\Phi}_z(x)(1)]\eta_x(x) \tag{2.50}$$

are satisfied.

The choice of $\Phi_1 = 0$ in (2.47) implies that the equation

$$\left[\frac{\sigma\eta_x}{\sqrt{1+\eta_x^2}}\right]_x - \left[\frac{1}{\eta}\int_0^1 [\eta\tilde{\Phi}_x - \eta_x z\tilde{\Phi}_z]z\tilde{\Phi}_z\,dz\right]_x - g(\eta-h) + \frac{1}{2}c^2$$
$$+\frac{1}{2\eta^2}\int_0^1 \left\{[\eta\tilde{\Phi}_x - \eta_x z\tilde{\Phi}_z]^2 + \tilde{\Phi}_z^2\right\}dz - \frac{1}{\eta}\int_0^1 [\eta\tilde{\Phi}_x - \eta_x z\tilde{\Phi}_z]\tilde{\Phi}_x\,dz = 0 \tag{2.51}$$

is satisfied in $L^2[(a,b)\times(0,1)]$. A straightforward calculation now shows that (2.48)–(2.51) reduce to (2.26)–(2.28), (2.32). Finally, one uses the change of variables given at the end of Section 2.1 to obtain the steady water-wave problem (1.5)–(1.8).

2.4. Hamiltonian-Lagrangian duality

The purpose of this section is to show that the Hamiltonian and Lagrangian formulations of the steady water-wave problem are equivalent by convex duality theory. The starting point of the following theory is the Lagrangian formulation. One takes the Lagrangian function, defined on its natural domain, and using a rigorous variational principle infers both the Hamiltonian function and the manifold upon which it is defined. This manifold is a subset of the cotangent bundle T^*P, where P is the manifold described in the previous section. The specification of the Hamiltonian system is completed by using the canonical 2-form on T^*P.

Let $\mathscr{X}$ and $\mathscr{Y}$ be Hilbert spaces and X be a subset of $\mathscr{X}$. Suppose that $L : X \times \mathscr{Y} \to \mathbb{R}$ has the property that the map $p \mapsto L(q,p)$ is a convex function of p for each fixed $q \in X$. One may then define a function $H : S \subset X \times \mathscr{Y}^* \to \mathbb{R}$ by

$$H(q,p^*) = \sup\{p^*(p) - L(q,p) : p \in \mathscr{Y}\}. \tag{2.52}$$

The domain S of H is the set of $(q,p^*) \in X \times \mathscr{Y}^*$ for which the supremum is finite. In classical Legendre-transform theory (e.g., see Abraham & Marsden (1978, §3.6)), the supremum of $p^*(p) - L(q,p)$ is obtained by looking for its stationary values. The approach to be used here is rather more general, being a direct method based upon the fundamental definition (2.52) which avoids assumptions about the behaviour of L (see Ekeland & Temam (1976) for a discussion of variational problems in this context). The Hamiltonian for the steady water-wave problem emerges, together with its manifold, as an easy consequence of (2.52). The method unifies the cases of zero and and non-zero values of σ and avoids problems implicit in previous attempts at classical Legendre transforms (Baesens & MacKay (1992, p. 337)).

Recall that the Lagrangian for the steady water-wave problem is the restriction to $TP \cong (0,\infty) \times H^1(0,1) \times \mathbb{R} \times H^1(0,1)$ of the function L defined by (2.46) whose natural domain is $(0,\infty) \times H^1(0,1) \times \mathbb{R} \times L^2(0,1)$. In order to carry out the duality theory described above, define $\mathscr{X} = \mathbb{R} \times H^1(0,1)$, $\mathscr{Y} = \mathbb{R} \times L^2(0,1)$, $X = (0,\infty) \times H^1(0,1)$, and set

$$H(\eta,\Phi;\omega,\Psi) = \sup\left\{\omega\mu + \int_0^1 \Psi\theta\,dz - L(\eta,\Phi;\mu,\theta) : (\mu,\theta) \in \mathbb{R} \times L^2(0,1)\right\}, \tag{2.53}$$

where $\mathscr{Y}^*$ has been identified with $\mathscr{Y}$ in the usual manner. Writing out (2.53) explicitly, one finds that

$$\begin{aligned} H(\eta,\Phi;\omega,\Psi) = &\\ \sup_{\mu\in\mathbb{R}} &\left[\omega\mu - \sigma\sqrt{1+\mu^2} + \sup_{\theta\in L^2(0,1)} \int_0^1 \left\{\Psi\theta - \frac{1}{2\eta}(\eta\theta - \mu z\Phi_z)^2\right\} dz\right] \\ &- \frac{1}{2}g(\eta-h)^2 + \frac{1}{2}c^2(\eta-h) + \sigma - \frac{1}{2\eta}\int_0^1 \Phi_z^2\,dz. \end{aligned} \tag{2.54}$$

For each fixed $\mu \in \mathbb{R}$, the supremum

$$\sup_{\theta\in L^2(0,1)} \int_0^1 \left\{\Psi\theta - \frac{1}{2\eta}(\eta\theta - \mu z\Phi_z)^2\right\} dz$$

is clearly attained when the integrand is maximised for each value of $z \in (0, 1)$, that is when

$$\theta = \frac{1}{\eta}(\Psi + \mu z \Phi_z) \in L^2(0, 1).$$

With this choice of θ, equation (2.54) becomes

$$H(\eta, \Phi; \omega, \Psi) = \frac{1}{2\eta} \int_0^1 (\Psi^2 - \Phi_z^2)\, dz - \frac{1}{2} g(\eta - h)^2 + \frac{1}{2} c^2(\eta - h)$$
$$+ \sigma + \sup_{\mu \in \mathbb{R}} \left(W\mu - \sigma\sqrt{1 + \mu^2} \right),$$

where

$$W = \omega + \frac{1}{\eta} \int_0^1 z \Phi_z \Psi\, dz.$$

For fixed W, the function $W\mu - \sigma\sqrt{1 + \mu^2}$ is a real-valued function of a single real variable. By elementary techniques one finds that for any $\sigma \geqq 0$ its supremum is finite if and only if $|W| \leqq \sigma$, in which case its value is $-\sqrt{\sigma^2 - W^2}$ and so

$$H(\eta, \Phi; \omega, \Psi) = \frac{1}{2\eta} \int_0^1 (\Psi^2 - \Phi_z^2)\, dz - \frac{g(\eta - h)^2}{2} + \frac{1}{2} c^2(\eta - h) + \sigma - \sqrt{\sigma^2 - W^2}. \tag{2.55}$$

Observe that when $\sigma > 0$, the interior of the domain of H is

$$M = \{(\eta, \Phi, \omega, \Psi) \in (0, \infty) \times H^1(0, 1) \times \mathbb{R} \times L^2(0, 1) : |W| < \sigma\},$$

so that M and H are precisely the manifold and Hamiltonian used in Section 2.1. When $\sigma = 0$, the domain of H is

$$N = \{(\eta, \Phi, \omega, \Psi) \in (0, \infty) \times H^1(0, 1) \times \mathbb{R} \times L^2(0, 1) : W = 0\},$$

again the same as that used in Section 2.2. Because $\sigma = 0$, the Hamiltonian H above is defined when $W = 0$, and it then coincides with the Hamiltonian H_1 used in Section 2.2.

The cotangent bundle T^*P to the manifold P described in the previous section is isomorphic to $(0, \infty) \times H^1(0, 1) \times \mathbb{R}^* \times H^1(0, 1)^*$. One may endow T^*P with a symplectic structure using the *canonical symplectic* 2*-form* Ω_C, where $\Omega_C|_\rho : (T(T^*P)|_\rho)^2 \to \mathbb{R}$, $\rho = (\eta, \Phi, \omega, \Psi) \in T^*P$ is defined by

$$\Omega_C|_\rho((\eta_1, \Phi_1, \omega_1, \Psi_1), (\eta_2, \Phi_2, \omega_2, \Psi_2) = \omega_2(\eta_1) + \Psi_2(\Phi_1) - \omega_1(\eta_2) - \Psi_1(\Phi_2).$$

The sets M and N are clearly isomorphic to subsets S_1, S_2 of T^*P. Restricting Ω_C to S_1, S_2 and using the isomorphism, one obtains the 2-forms Ω on M and Ω_1 on N employed in Sections 2.1 and 2.2.

3. The case of infinite depth

Attention is now briefly turned to the problem for steady water waves on fluid of infinite depth, the case considered by Baesens & MacKay (1992). The hydrodynamic problem for time-dependent motion is (1.1)–(1.4) with $D_\eta = \{(x, y) : x \in \mathbb{R}, y \in (-\infty, \eta(x, t))\}$. Seeking solutions of the special form $\eta(x, t) = \eta(x - ct)$, $\phi(x, y, t) = \phi(x - ct, y)$, one finds that

$$\begin{aligned} \phi_{xx} + \phi_{yy} &= 0 \qquad \text{for } -\infty < y < \eta(x), \\ \phi_y &\to 0 \qquad \text{as } y \to -\infty, \\ -c\eta_x &= \phi_y - \eta_x\phi_x \qquad \text{on } y = \eta(x), \end{aligned}$$

$$\frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 - c\phi_x + g\eta - \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x = 0 \quad \text{on } y = \eta(x),$$

where x is now a shorthand for $x - ct$. The theory is set in Sobolev spaces such as $H^m[\{(x, y) : x \in (x_1, x_2), y \in (-\infty, \eta(x))\}]$, but $\tilde{\phi} = \phi - cx$, the velocity potential relative to the moving frame (see Section 1.2), does not tend to zero as $y \to -\infty$ and therefore does not belong to any of these spaces. One therefore works with the original function ϕ rather than $\tilde{\phi}$. This choice of variable and the fact that $z = y - \eta$ rather than $z = y/\eta$ is used to map the physical flow domain to a fixed region of $\mathbb{R}^2$ are the main changes from the theory in Section 2. The algebraic manipulations change correspondingly, but the proofs of the theorems are essentially unchanged.

The Hamiltonian system appropriate for the case $\sigma > 0$ is (M, Ω, H), where

$$M = \{(\eta, \omega, \Phi, \Psi) : \eta, \omega \in \mathbb{R}, \Phi \in H^1(-\infty, 0), \Psi \in L^2(-\infty, 0), |W| < \sigma\},$$

with

$$W = \omega + \int_{-\infty}^{0} \Psi\Phi_z \, dz - c\Phi(0),$$

$$\Omega|_m((\eta_1, \omega_1, \Phi_1, \Psi_1), (\eta_2, \omega_2, \Phi_2, \Psi_2)) = \omega_2\eta_1 - \eta_2\omega_1 + \int_{-\infty}^{0} (\Psi_2\Phi_1 - \Phi_2\Psi_1) \, dz,$$

$$H(\eta, \omega, \Phi, \Psi) = \frac{1}{2}\int_{-\infty}^{0} (\Psi^2 - \Phi_z^2) \, dz - \frac{1}{2}g\eta^2 + \sigma - \sqrt{\sigma^2 - W^2}.$$

Solutions of the steady water-wave problem are obtained from trajectories of this Hamiltonian system by a straightforward analogy of the method explained in Section 2.1.

When $\sigma = 0$, one studies the triple (N, Ω_1, H_1), where N is the submanifold of M consisting of those elements $(\eta, \omega, \Phi, \Psi)$ that satisfy the constraint $\omega = F(\eta, \Phi, \Psi)$, in which

$$F(\eta, \Phi, \Psi) = -\int_{-\infty}^{0} \Phi_z\Psi \, dz + c\Phi(0),$$

and Ω_1, H_1 are the restrictions of Ω and H to N. One readily obtains the explicit formulae

$$H_1 = \frac{1}{2}\int_{-\infty}^{0} (\Psi^2 - \Phi_z^2)\, dz - \frac{1}{2} g\eta^2,$$

$$\begin{aligned}\Omega_1|_n(v_1, v_2) = -\eta_1 \int_{-\infty}^{0} (\Phi_{2z}\Psi + \Phi_z\Psi_2)\, dz + \eta_2 \int_{-\infty}^{0} (\Phi_{1z}\Psi + \Phi_z\Psi_1)\, dz \\ + \int_{-\infty}^{0} (\Psi_2\Phi_1 - \Phi_2\Psi_1)\, dz + c[\eta_1\Phi_2(0) - \eta_2\Phi_1(0)].\end{aligned}$$

A straightfoward analogy of the theory in Section 2.2 applies to obtain the steady water-wave problem when $\sigma = 0$; waves of extreme form fit into the theory as before.

Finally, the steady water-wave problem with infinite depth and $\sigma \geqq 0$ is represented by the Lagrangian system (P, L), where $P = \mathbb{R} \times H^1(-\infty, 0)$ and

$$L(\eta, \Phi; \mu, \theta) = \frac{1}{2}\int_{-\infty}^{0} \left\{[\theta - \mu\Phi_z]^2 + \Phi_z^2\right\}\, dz + \frac{1}{2} g\eta^2 + \sigma[\sqrt{1+\mu^2} - 1] + c\mu\Phi(0),$$

where $(\eta, \Phi) \in P$, $(\mu, \theta) \in TP|_{(\eta,\Phi)} \cong \mathbb{R} \times H^1(-\infty, 0)$. The Langragian formulation and Hamiltonian-Lagrangian duality theory detailed in Sections 2.3 and 2.4 immediately adapt to the present context.

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