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Numerical Simulation of Gravity Waves

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We present a new spectral method to simulate numerically the water-wave problem in a channel for a fluid of finite or infinite depth. It is formulated in terms of the free surface elevation η and the velocity potential φ. The numerical method is based on the reduction of this problem to a lower-dimensional computation involving surface variables alone. To accomplish this, we describe the Taylor expansion of the Dirichlet Neumann operator in homogeneous powers of the surface elevation η. Each term is a concatenation of Fourier multipliers with powers of η and its derivatives and is valid uniformly in wavenumber. These are easily calculated using the fast Fourier transform. The method is illustrated by computing the long time evolution of modulated wave packets and of approximations to the Stokes steady wave train. By imposing a surface pressure we observe surface steepening in large amplitude evolution, and wake and bow wave development for flows with a close to critical Froude number. Finally, we give an example of nonlinear evolution of the distribution of energy among normal modes.

1. INTRODUCTION

The water-wave problem describes the evolution of an incompressible ideal, irrotational fluid with a free surface under the influence of gravity. The study of nonlinear gravity waves has been the object of numerous theoretical, numerical, and experimental investigations. These have classical origins in the work last century of Boussinesq, Korteweg and deVries, and Stokes [12]. Today, work on the subject can be divided roughly into two categories, the analysis of steady waves, which progress by translation at constant velocity without change of form, and the initial value problem for time dependent nonlinear evolution. This paper presents a new numerical method for calculation of the latter problem. The relevance of numerical studies is to problems of nonlinear wave interaction and of breaking and in practical applications in ocean and naval hydrodynamics.

A study of time dependent water waves can focus either on a regime governed by a small parameter, or on large amplitude fully nonlinear evolution. In a perturbative regime one typically reduces the problem to nonlinear partial differential equations in the horizontal variables. In the long wave regime this leads to the Boussinesq and Korteweg–deVries equations. In a modulational regime, one describes the evolution of wave packets through the nonlinear Schrödinger equation for the complex wavetrain envelope (Zakharov [18], Hasimoto and Ono [7], Craig, Sulem, and Sulem [5]). There have been several extensions to higher order; in particular there are studies of the Zakharov equations, where third-order interactions are taken into account, and the modified Zakharov equations, where fourth- and fifth-order interactions are kept [11]. Yuen and Lake [17] survey the development of the theory of modulational waves and their instabilities. Of course, the validity of the description of the evolution given in these approaches is restricted to the appropriate perturbative regime.

In Ref. [9], Longuet-Higgins and Cokelet developed a mixed Eulerian Lagrangian numerical method that describes large amplitude waves in deep water. Their calculations are not limited to a perturbative regime, and they are able to track steep waves up to overturning. The method describes the full problem of ideal Euler flow within the fluid region. That is, for incompressible irrotational flow the fluid velocity can be described as the gradient of the velocity potential, $\mathbf{u} = \nabla\varphi$, where φ is harmonic within the fluid region. In this formulation, the nonlinearity and the time dependence of the problem appear explicitly only in the boundary conditions on the free surface. The main computational effort is to describe the fluid velocity on the free surface; in [9] the relationship between the velocity potential φ and its normal derivative $\partial_n\varphi$ on the free surface is

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given by an integral equation. In subsequent work, Baker, Meiron, and Orszag [1] computed free surface evolution in two dimensions using a vortex method to describe $\mathbf{u} = \nabla\varphi$. Alternatively, West, Brueckner, and Janda [14] and Dommermuth and Yue [6] developed perturbation methods to describe φ and extended Zakharov's mode-coupling description of the nonlinear interaction, to perform numerical simulations of the water-wave problem in deep water. Other authors who compute the evolution of free surfaces include Vinje and Brevig [13] and New, McIver, and Peregrine [10].

In this paper, we present a spectral method to compute time dependent spatially periodic surface waves in water which can be of finite or infinite depth. The independent variables are evaluated at the free surface only and the velocity potential and its derivatives in the interior are not used (although they could be easily computed from the surface values by Cauchy's theorem). The main contribution is a new and numerically efficient description of the Dirichlet–Neumann operator $G(\eta)$, the linear operator relating the boundary values of φ on the surface η to the values of $\partial_n\varphi$. The method is based on the Taylor series expansion of $G(\eta)$ in homogeneous powers of the surface elevation $\eta(x)$, giving a result which is valid uniformly in wavenumber. For two-dimensional problems the expansions are rigorously justified by a theorem of Coifman and Meyer [2] on the analyticity of integral operators of Cauchy type, over curves of controlled Lipschitz norm. Furthermore, we show that $G(\eta) = \sum_{j=0}^{\infty} G_j(\eta)$, where each $G_j(\eta)$ is homogeneous of degree j in η and is a sum of ordered products of $\eta(x)$ and the Fourier multipliers $D = -i\partial_x$ and $\tanh(hD)$ (where h is the average depth of the fluid). This expansion differs from the spectral methods of both West *et al.* [14] and Dommermuth and Yue [6], where both φ and η are assumed to be $O(\varepsilon)$ quantities, and the expansion is not uniform in wavenumber.

In this paper, we restrict ourselves to the case of two-dimensional flows, with finite horizontal bottom boundaries, although these are not inherent limitations of the method. The terms G_j of the Taylor expansion can be obtained from a recursion formula [3] which is described in Section 3; they are easily numerically computed using fast Fourier transform algorithms. We have tested the efficiency of the method on several model problems, which are described in Section 4. The first calculation is of slowly modulated oscillatory wave packets, where the uniformity of the method in wavenumber produces the correct phase and group velocities in the numerical solution. A second set of calculations is of approximations to the steady Stokes wavetrain, for both small and large amplitudes. We also compute waves as they steepen on their approach to a breaking point, imposing a surface pressure to increase their energy. To model the surface motion due to a ship, a localized pressure distribution moving with constant velocity is applied to the free surface. Both subcritical and near critical computations are presented, the latter describing the emergence of solitons in the bow wave. Finally, we describe the nonlinear time evolution of the distribution of energy among the normal modes. All the computations are carried on a Vax or a RISC-IBM and in practice run very rapidly on these machines. Our method is conceptually straightforward, is very fast in practice, and is based upon approximations to the free surface problem that are mathematically rigorous. This approach to the computation of ideal fluids with free surfaces can be extended to waves of larger amplitudes, problems with bottom topography, and three-dimensional flows, projects which we are currently pursuing.

2. MATHEMATICAL FORMULATION

We consider the motion of a free surface of a two-dimensional fluid in a horizontal channel under the influence of gravity. The bottom is fixed at a constant depth $-h$. The fluid is assumed to be incompressible, inviscid, and irrotational. The velocity potential φ satisfies

$$\Delta\varphi = 0 \tag{2.1}$$

in the fluid region bounded by $\{y = -h\}$ and the free surface $\Gamma(t) = \{y = \eta(x, t)\}$ with the boundary conditions

$$\partial\varphi/\partial n = 0 \qquad \text{on} \quad y = -h \tag{2.2}$$

and

$$\varphi_t + \tfrac{1}{2}(\varphi_x^2 + \varphi_y^2) + g\eta = 0 \tag{2.3a}$$

$$\eta_t + \eta_x\varphi_x - \varphi_y = 0 \tag{2.3b}$$

on $y = \eta(x, t)$.

We assume periodic boundary conditions in the x-direction. Let $\xi(x, t) = \varphi(x, \eta(x, t), t)$ be the potential on the top surface. We define the Dirichlet–Neumann operator by

$$G(\eta)\xi = (1 + \eta_x^2)^{1/2}\frac{\partial\varphi}{\partial n}. \tag{2.4}$$

In Eqs. (2.3), the only reference to the velocity potential interior to the fluid is through the normal derivative $\partial_n\varphi = (1 + \eta_x^2)^{-1/2}(\varphi_y - \eta_x\varphi_x)$. Additionally, $\xi_x = \varphi_x + \eta_x\varphi_y$, so that, using (2.4), φ_y on the surface can be described in terms of ξ_x and $G(\eta)\xi$ by $\varphi_y = (1 + \eta_x^2)^{-1}(G(\eta)\xi + \eta_x\xi_x)$. Noting that $\varphi_t = \xi_t - \varphi_y\eta_t$, Eqs. (2.3) are rewritten:

$$\eta_t - G(\eta)\xi = 0 \tag{2.5a}$$

$$\xi_t + \frac{1}{2(1+\eta_x^2)}\left(\xi_x^2 - (G(\eta)\,\xi)^2 - 2\eta_x\xi_x G(\eta)\xi\right) + g\eta = 0. \tag{2.5b}$$

These equations are Hamilton's canonical equations in Zakharov's formulation of the water-wave problem as a Hamiltonian system.

In [2], Coifman and Meyer prove that, if η has supremum-norm and Lipschitz-norm smaller than a certain constant C_1, then G is an analytic function of η. It follows that G can be written in terms of a convergent Taylor expansion

$$G(\eta) = \sum_j G_j(\eta, hD). \tag{2.6}$$

For $\eta = 0$, $G(0)$ is the convolution operator,

$$\begin{aligned} G(0)\,\xi(x) &= \frac{1}{\pi h}\, vp \int_{-\infty}^{+\infty} \frac{\partial_x \xi(y)}{\sinh(y-x)/h}\, dy \\ &= \frac{1}{2\pi} \iint e^{ik(x-y)} k \tanh(hk)\, \xi(y)\, dy\, dk \\ &= D \tanh(hD)\, \xi(x), \end{aligned} \tag{2.7}$$

where $D = -i\partial_x$. For a nontrivial surface deformation η, G_j is a sum of ordered products of η and the Fourier multipliers D and $\tanh(hD)$ and is homogeneous of degree j in η. The expansion is obtained from a recursion formula in the following way.

Consider the harmonic functions $\varphi_p(x, y) = e^{ipx} \cosh(p(y+h))$, which satisfy additionally the bottom boundary condition $\partial_y \varphi_p(x, -h) = 0$. Their gradients are $\nabla\varphi_p = (ipe^{ipx}\cosh(p(y+h)),\ pe^{ipx}\sinh(p(y+h)))^{\mathrm{T}}$; thus the relationship (2.4) between $\varphi_p(x, \eta(x))$ and $\partial_n \varphi_p$ on the surface is

$$\frac{\partial \varphi_p}{\partial y} - \eta_x \frac{\partial \varphi_p}{\partial x} = G(\eta)\,\varphi_p. \tag{2.8}$$

We seek a homogeneous expansion of $G(\eta) = \sum_{j=0}^{\infty} G_j(\eta)$. For this purpose, we write the Taylor expansions of $\cosh(p(\eta(x)+h))$ and $\sinh(p(\eta(x)+h))$ about $\eta = 0$, and substitute in the identity (2.8):

$$\begin{aligned} &\sum_{j\ \mathrm{even}} \frac{1}{j!}(p\eta)^j (p\sinh(ph) - i\eta_x p\cosh(ph))\, e^{ipx} \\ &\quad + \sum_{j\ \mathrm{odd}} \frac{1}{j!}(p\eta)^j (p\cosh(ph) - i\eta_x p\sinh(ph))\, e^{ipx} \\ &= \left(\sum_{l=0}^{\infty} G_l(\eta)\right)\left(\sum_{j\ \mathrm{even}} \frac{1}{j!}(p\eta)^j \cosh(ph)\, e^{ipx} \right. \\ &\quad \left. + \sum_{j\ \mathrm{odd}} \frac{1}{j!}(p\eta)^j \sinh(ph)\, e^{ipx}\right). \end{aligned} \tag{2.9}$$

Identifying terms of the same degree in η, we obtain an expansion of the operator $G(\eta)$. For $j = 0$, we have

$$G(0)\, e^{ipx} = p \tanh(hp)\, e^{ipx}. \tag{2.10}$$

Fourier analysis describes a general function $\zeta(x)$ which is sufficiently well behaved as a sum of such terms; hence

$$G(0)\,\zeta(x) = D\tanh(hD)\,\zeta(x). \tag{2.11}$$

The higher terms in the expansion are derived from (2.9) similarly: For $j > 0$ even,

$$\begin{aligned} G_j(\eta) &= \frac{1}{j!}(\eta^j D^{j+1}\tanh(hD) - i(\eta^j)_x D^j \tanh(hD)) \\ &\quad - \sum_{\substack{l<j \\ l\ \mathrm{even}}} G_l(\eta) \frac{1}{(j-l)!} \eta^{j-l} D^{j-l} \\ &\quad - \sum_{\substack{l<j \\ l\ \mathrm{odd}}} G_l(\eta) \frac{1}{(j-l)!} \eta^{j-l} D^{j-l} \tanh(hD) \end{aligned} \tag{2.12}$$

For j odd,

$$\begin{aligned} G_j(\eta) &= \frac{1}{j!}(\eta^j D^{j+1} - i(\eta^j)_x D^j) \\ &\quad \sum_{\substack{l<j \\ l\ \mathrm{odd}}} G_l(\eta) \frac{1}{(j-l)!} \eta^{j-l} D^{j-l} \\ &\quad - \sum_{\substack{l<j \\ l\ \mathrm{even}}} G_l(\eta) \frac{1}{(j-l)!} \eta^{j-l} D^{j-l} \tanh(hD). \end{aligned} \tag{2.13}$$

This is the recursion formula for $G(\eta)$. The first several terms of this expansion are

$$\begin{aligned} G_0 &= D\tanh(hD) \\ G_1(\eta) &= D(\eta - \tanh(hD)\eta\tanh(hD))D \\ G_2(\eta) &= -\tfrac{1}{2} D(D\eta^2 \tanh(hD) + \tanh(hD)\,\eta^2 D \\ &\quad - 2\tanh(hD)\,\eta D \tanh(hD)\eta\tanh(hD))D. \end{aligned} \tag{2.14}$$

In practice, this expansion is able to be computed to many terms with relatively little effort.

Remarks. (i) To obtain the expansion of the operator G in the case of fluids of infinite depth (that is $h = \infty$), one replaces the Hilbert transform for the strip $i\tanh(hD)$ by the Hilbert transform for the half-plane $i\,\mathrm{sgn}(D)$ in the above expressions.

(ii) The recursion procedure outlined above carries through in higher spatial dimensions as well, giving expansions for the homogeneous operators similar to (2.13).

(iii) We emphasize that G_j is a sum of products of η and the Fourier multipliers D and $\tanh(hD)$. Each term of the expansion is therefore easy to compute numerically when using a pseudospectral method. Products with $\eta(x)$ are performed in physical space, while the Fourier multipliers D and $D\tanh(hD)$ are computed in spectral space. The transformation from one space to the other is efficiently carried out using the fast Fourier transform.

3. DESCRIPTION OF THE NUMERICAL METHOD

To integrate numerically the system (2.5), we separate the linear and nonlinear terms in the form:

$$\eta_t - G(0)\xi = G(\eta)\xi - G(0)\xi \tag{3.1a}$$

$$\xi_t + g\eta = -\frac{1}{2(1+\eta_x^2)}(\xi_x^2 - (G(\eta)\xi)^2 - 2\eta_x\xi_x G(\eta)\xi). \tag{3.1b}$$

In the spatial variable x, the functions are initially expanded in Fourier series. To approximate the operator $G(\eta)$ we use the recursion formula to compute analytically the first M terms in the Taylor series expansion:

$$G(\eta) \approx G(0) + G_1(\eta) + \cdots + G_M(\eta) \equiv G^{(M)}(\eta).$$

For given $\eta(x)$, the result is calculated by a concatenation of the operations of multiplication and the fast Fourier transform.

For the time stepping, the linear part is solved exactly and we use a second-order Adams–Bashforth scheme for the nonlinear terms. We are thus led to solve the system

$$\begin{aligned}\eta(t+\delta t) &= \cos(\sqrt{gG(0)}\,\delta t)\,\eta(t) \\ &+ \sqrt{G(0)/g}\,\sin(\sqrt{gG(0)}\,\delta t)\,\xi(t) \\ &+ \frac{1}{\sqrt{gG(0)}}\sin(\sqrt{gG(0)}\,\delta t) \\ &\times (1.5F_1(t) - 0.5F_1(t-\delta t)) \\ &+ \frac{1}{g}(1-\cos(\sqrt{gG(0)}\,\delta t)) \\ &\times (1.5F_2(t) - 0.5F_2(t-\delta t))\end{aligned} \tag{3.2a}$$

$$\begin{aligned}\xi(t+\delta t) &= -\sqrt{g/G(0)}\,\sin(\sqrt{gG(0)}\,\delta t)\,\eta(t) \\ &+ \cos(\sqrt{gG(0)}\,\delta t)\,\xi(t) \\ &- \frac{1}{G(0)}(1-\cos(\sqrt{gG(0)}\,\delta t)) \\ &\times (1.5F_1(t) - 0.5F_1(t-\delta t)) \\ &+ \frac{1}{\sqrt{gG(0)}}\sin(\sqrt{gG(0)}\,\delta t) \\ &\times (1.5F_2(t) - 0.5F_2(t-\delta t)),\end{aligned} \tag{3.2b}$$

where

$$F_1(t) = (G^{(M)}(\eta) - G_0)\,\xi(t) \tag{3.3a}$$

$$\begin{aligned}F_2(t) = &-\tfrac{1}{2}(1+\eta_x^2)^{-1} \\ &\times (\xi_x^2 - (G^{(M)}(\eta)\xi)^2 - 2\eta_x\xi_x G^{(M)}(\eta)\xi).\end{aligned} \tag{3.3b}$$

It is in the implementation of the nonlinear terms F_1 and F_2 that the expressions (2.12) and (2.13) for the Dirichlet–Neumann operator are used. In the numerical computations to date we used terms in the expansion of $G(\eta)$ up to degree $M = 3$, 4, and 5, using recursion formulae (2.12)–(2.13) to compute them. Spatially, we use a pseudospectral method with a resolution of 32 or 64 collocation points in the interval $[0, 2\pi[$. Our computations were run for times typically as long as several periods of a Stokes wavetrain.

The numerical scheme described above includes no explicit dissipation. After sufficiently long computation time, we have observed oscillations in the wave profile due to an onset of instability initiated by some growth for high wave numbers. In the calculations of Longuet-Higgins and Cokelet [9] and Dommermuth and Yue [6], similar instabilities were observed. To remove them, we used the five-point smoothing function proposed in the latter reference and equivalent in the wavenumber space to the low-pass filter [6]

$$A(k_n) = \frac{1}{8}\left[5 + 4\cos\left(\frac{\pi\,|k_n|}{k_N}\right) - \cos\left(\frac{2\pi\,|k_n|}{k_N}\right)\right],$$

where k_N is the maximum wave number. It allowed our computations to proceed to significantly longer intervals of time.

The validity of the computation is checked by the conservation of total energy

$$E = \tfrac{1}{2}\int (\xi G(\eta)\xi + g\eta^2)\,dx$$

and of momentum

$$I = \int \xi\eta_x\,dx.$$

We note that the functional $E = E(\eta, \xi)$ is Zakharov's Hamiltonian for the water-wave problem.

4. NUMERICAL RESULTS

Our goal in the computations presented in this paper is not to present detailed fluid dynamical computations, rather it is to show that this numerical approach to computing the evolution of free surfaces is flexible and robust and that it can be adapted to many different time dependent problems. We have chosen several different model problems to illustrate the method.

Our first simulation is to compute the evolution of the slowly modulated wave packet

$$\eta_0(x) = 0.01 \exp - \tfrac{4}{3}(x-\pi)^2 \cos(10x)$$

with zero initial velocity potential in a channel of average depth $h=1$. We carried the computation during a long time (about 100 fundamental periods) and the run could be continued longer. In Fig. 1a, $\eta(x,t)$ is represented as a function of (x,t) at short intervals of times, namely from $t=0$ to $t=5$ with $\Delta t = 0.1$. To visualize the evolution of the wave for longer times, we have plotted in Fig. 1b, $y(x,t)+\alpha t$ at $t = 0, 1, 2, 3, ..., 50$ as a function of x (α was chosen equal to 0.01). We observe numerically that the solution has well-defined group velocity $C_g = \omega'(k) = \partial_k \sqrt{gG_0(10)}$, and phase velocity $C_p = (\omega/k) = \sqrt{gG_0(10)}/10$, which is approximately twice C_g (where k and ω are related by the dispersion relation $\omega^2 = gk \tanh(hk)$). This is in accordance with the uniformity in wavenumber of our expansion of $G(\eta)$. The computation was performed with a resolution of 64 collocation points and an expansion of G up to fourth order, which required a computation time of less that 0.1 s per time step on an RISC-IBM 6000.

We also take the second-order approximation to the Stokes wavetrain [15, Chap. 13] as initial conditions:

$$\begin{aligned} \eta_0(x) &= a\cos(kx) + \mu_2 a^2 \cos(2kx) \\ \xi_0(x) &= \nu_1 a \cosh(k(\eta_0+h))\sin(kx) \\ &\quad + \nu_2 a^2 \cosh(2k(\eta_0+h))\sin(2kx) \end{aligned} \tag{4.1}$$

with

$$\mu_2 = \frac{1}{2} k \coth(hk)\left(1+\frac{3}{2\sinh^2 kh}\right)$$

$$\nu_1 = \frac{\omega}{k\sinh(hk)}, \qquad \nu_2 = \frac{3}{8}\frac{\omega}{\sinh^4(hk)},$$

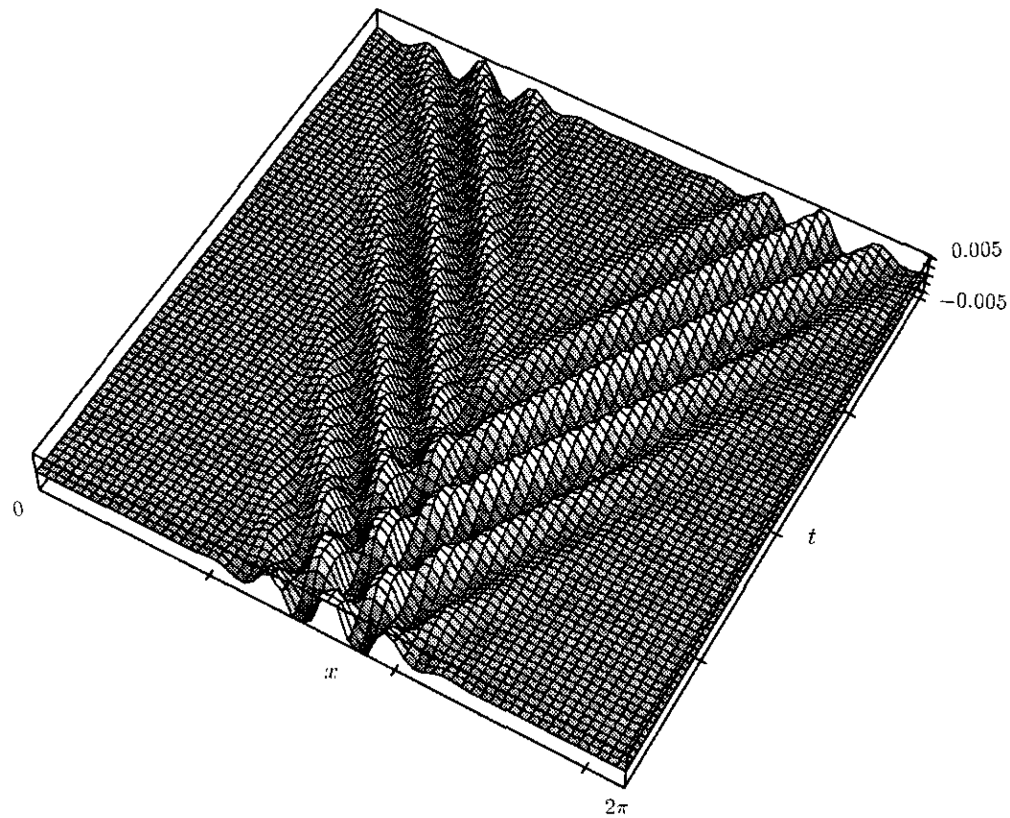

FIG. 1a. Evolution of the free surface $\eta(x,t)$ in the (x,t) plane, at $t=0, 0.1, 0.2, ..., 5$ for the initial condition $\eta_0(x) = 0.01 \exp -(4(x-\pi)^2/3)\cos(10x)$; $\xi_0(x)=0$.

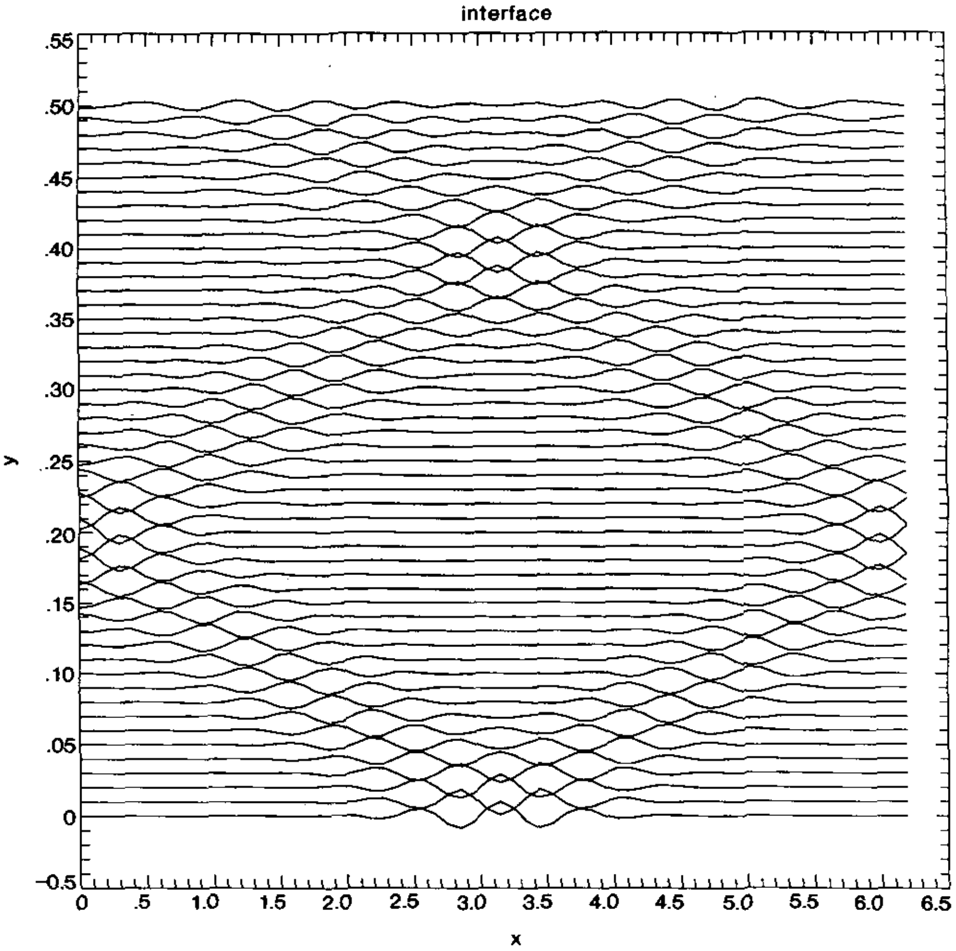

FIG. 1b. Same initial conditions as Fig. 1a. Evolution of the free surface $\eta(x, t) + 0.01t$ vs x at $t = 0, 1, 2, ..., 50a$.

and ω given by the dispersion relation $\omega^2 = gk \tanh(hk)$. We have run several calculations with increasing values of the amplitude parameter. The first was a wave of small amplitude ($k = 5$, $a = 0.065$) in a channel of average depth $h = 1$. This is close to linear regime ($|\eta_0|_\infty = 0.1378h$) and the computation was performed up to $t = 50$ (that is, over 18 fundamental periods). Figure 2a shows the evolution of the surface between $t = 0$ and $t = 12$ in the (x, t)-plane. Two terms in the expansion of the Stokes wave is sufficiently accurate to produce a wave which is virtually steady, although slow oscillations in width of the crest are visible. The second calculation is again the two-term expansion of the Stokes wavetrain, with, however, $k = 1$, $h = 0.4$, and $a = 0.07$. The amplitude $|\eta_0|_\infty = 0.33h$ is in a fully nonlinear regime. The calculation was carried out to $t = 24$ and could again have been carried further with no apparent singularities. The wave is far from steady, yet progresses with remarkable similarity in overall detail to the steady Stokes wavetrain of the same amplitude. Furthermore, the speed of the wave in the computation is within 1 % of that of the steady wavetrain. For cases with larger amplitude, the solution eventually breaks. Indeed, for $a = 0.08$ ($|\eta_0|_\infty = 0.58h$), the wave increases in amplitude and starts to break at $t = 10.2$.

Now we turn to nonlinear, unsteady waves that may

TABLE I

t	Momentum	Energy
0.00	$-0.73644\ 10^{-2}$	$0.10050\ 10^{-1}$
0.40	$-0.78401\ 10^{-2}$	$0.10720\ 10^{-1}$
0.80	$-0.91915\ 10^{-2}$	$0.12839\ 10^{-1}$
1.2	$-0.11249\ 10^{-1}$	$0.16543\ 10^{-1}$
1.6	$-0.13812\ 10^{-1}$	$0.21516\ 10^{-1}$
2.0	$-0.16649\ 10^{-1}$	$0.27071\ 10^{-1}$
2.4	$-0.19362\ 10^{-1}$	$0.31714\ 10^{-1}$
2.8	$-0.21314\ 10^{-1}$	$0.34713\ 10^{-1}$
3.2	$-0.21997\ 10^{-1}$	$0.35449\ 10^{-1}$
3.6	$-0.21975\ 10^{-1}$	$0.35440\ 10^{-1}$
4.0	$-0.21967\ 10^{-1}$	$0.35432\ 10^{-1}$
4.1	$-0.21962\ 10^{-1}$	$0.35418\ 10^{-1}$
4.2	$-0.21956\ 10^{-1}$	$0.35413\ 10^{-1}$
4.3	$-0.21958\ 10^{-1}$	$0.35418\ 10^{-1}$

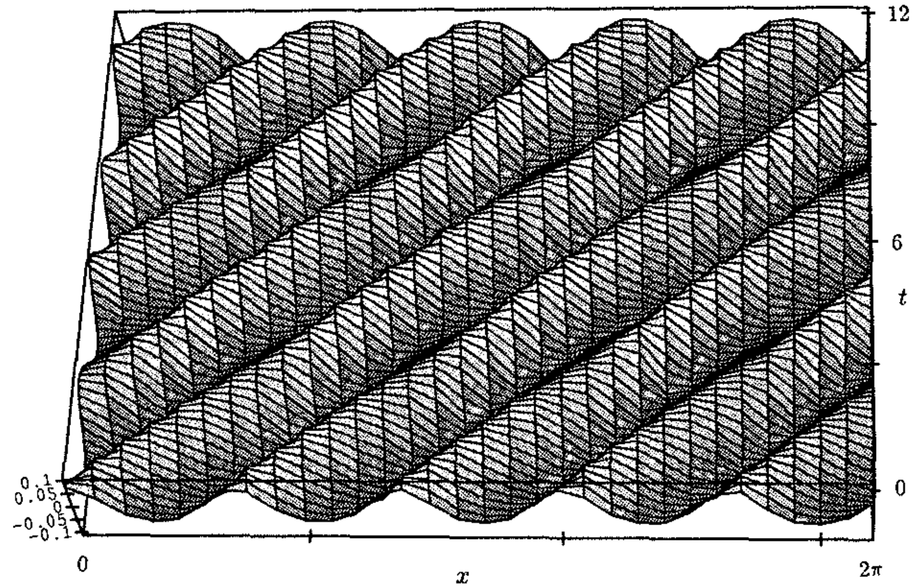

Fig. 2a. Evolution of the second-order approximation Stokes wavetrain, with the parameters: $k=5$, $a=0.065$, $h=1$, at $t=0, 0.2, 0.4, \ldots, 12$.

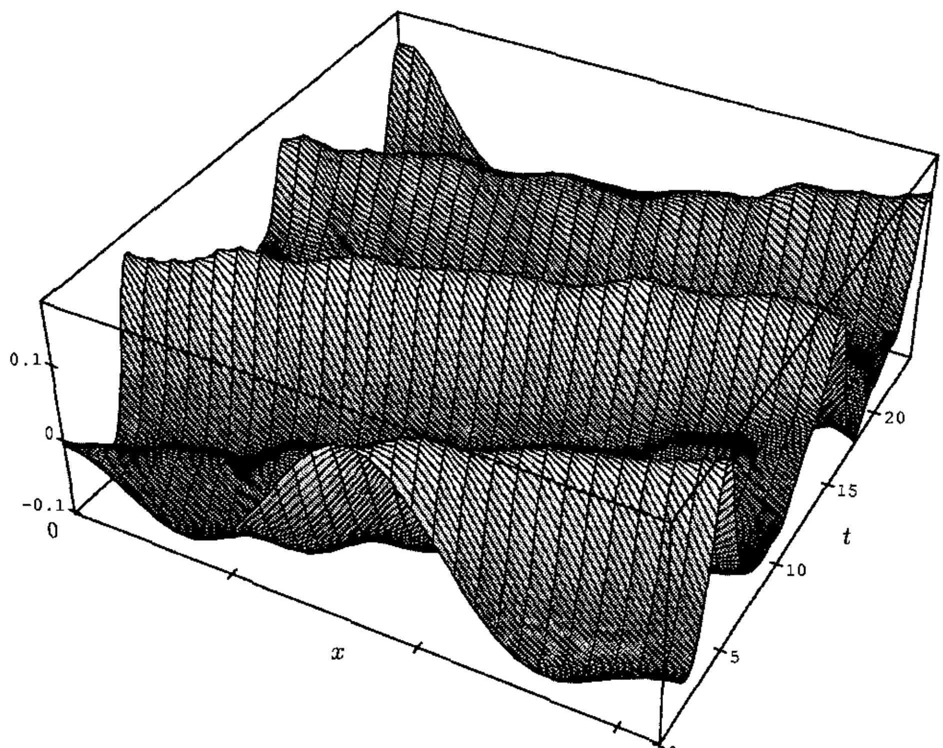

FIG. 2b. Evolution of the second-order approximation Stokes wavetrain, with the parameters: $k=1$, $a=0.07$, $h=0.4$, at $t=0, 0.2, 0.4, \ldots, 24$.

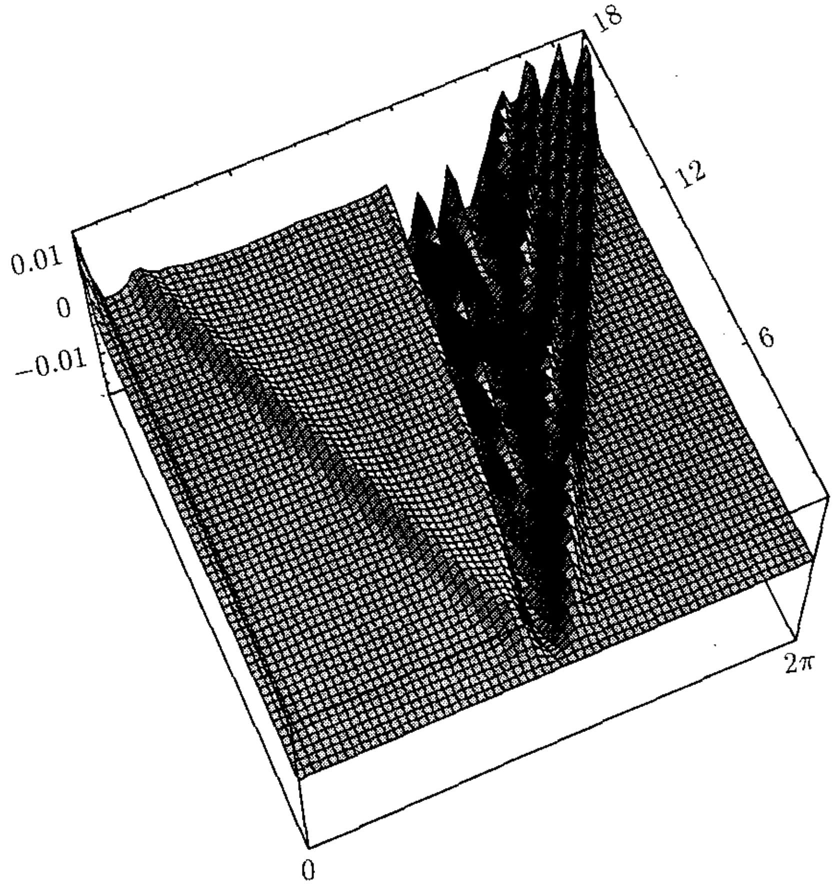

FIG. 4b. Localized surface pressure with Froude number $F = 0.8$.

wake is relatively small, but in the bow wave we observe the development of structures similar to solitons described by Wu [16].

Our final model calculation is an example of the distribution of energy among the normal modes in the time evolution of interacting periodic waves. The initial data are concentrated in modes $k = 1$ and $k = 2$. Figure 5 exhibits the amplitudes of the Fourier transform of η. One sees the exchange of energy between modes and significant mode coupling to at least $k = 6$ over the time interval $t = 0$ to $t = 10$.

5. CONCLUDING REMARKS

We have developed a numerical method to study nonlinear, unsteady free-surface waves. It is presented for fluids in a channel of finite depth but is easily adapted to fluids of infinite depth. Essentially, it involves the expansion of the Dirichlet–Neumann operator in terms of the free surface. The fluid dynamical results in this paper are preliminary, in the sense that we have calculated two-dimensional time-dependent problems where other methods are also available. Nonetheless, we have shown that the method is high order accurate and very fast, as well as being mathematically justified and easy to implement. This approach is not inherently restricted to two-dimensional flows, as opposed to many methods in the study of free surface dynamics. With little modification, one can write the expansion of the Dirichlet–Neumann operator in three dimensions or for flows with bottom topography. Indeed, three space dimensions, time-dependent free surface computations are currently under way. We are hopeful that our method could make problems of this type more amenable.

We plan also to address other questions such as surface waves with bottom topography for which a modification of this method is suitable; again, computations can be reduced to the free surface with a Fourier decomposition used to implement the Dirichlet–Neumann operator. We are also interested in a detailed study of the process of breakdown and the formation of singularities.

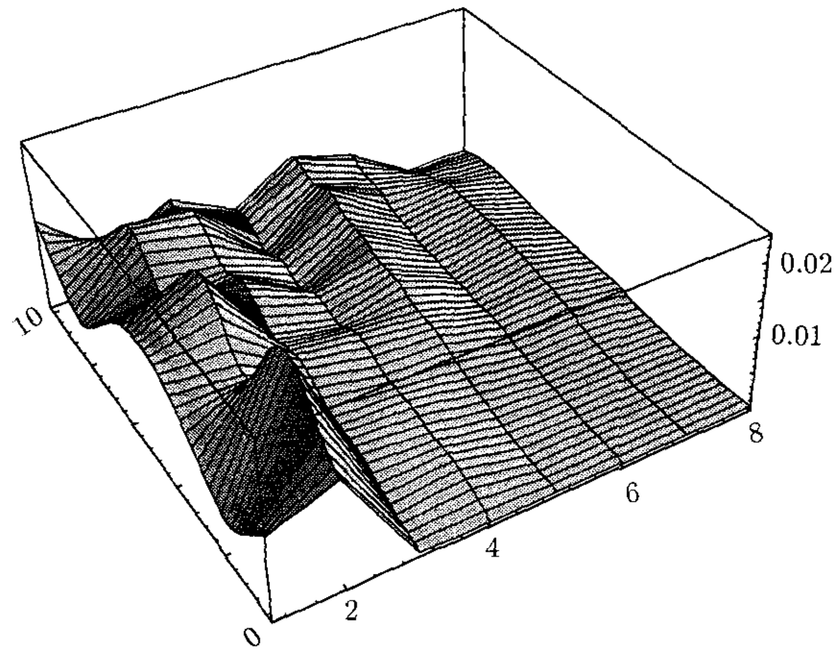

FIG. 5. Time evolution of the amplitudes of the Fourier transform of η for interacting periodic waves.

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