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Uniqueness Results, Control and Asymptotic Analysis for Slightly Compressible Fluids

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We prove a unique continuation property for (weak) solutions of slightly compressible fluid equations. We deduce approximate controllability for such equations. We present the asymptotic analysis when the penalty's coefficient tends to infinity in control problems. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: compressible Stokes equations; controllability; uniqueness properties; Carleman estimates; asymptotic analysis

1. Introduction

Consider an open bounded connected and regular set Ω of $\mathbb{R}^n$ ($n \geqslant 2$), a time $T > 0$ and an open non-empty subset ω of Ω. We write $Q = \Omega \times]0, T[$, $q = \omega \times]0, T[$, and $\sum = \partial\Omega \times]0, T[$. Let $a = (a_1, \dots, a_n) \in L^\infty(Q)^n$ and $(b_1, \dots, b_n) \in L^\infty(0, T; W^{1,\infty}(\Omega))^n$.

For $\lambda > 0$, $v = (v_1, \dots, v_n) \in L^2(q)^n$, and $y^0 \in L^2(\Omega))^n$, we denote by $y = y(x,t) = (y_1(x,t), \dots, y_n(x,t))$ the vector valued solution of

$$\begin{cases} y' - \Delta y - \lambda \nabla \operatorname{div} y + (a.\nabla) y + (y.\nabla) b = v\chi_q \quad \text{in } Q, \\ y = 0 \quad \text{on } \sum, \\ y(0) = y^0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $'$ denotes the derivative with respect to time and χ_q is the characteristic function of q.

When λ is large, system (1.1) models the velocity of a slightly compressible fluid since we can write $y' - \Delta y - \lambda \nabla \operatorname{div} y = y' - \Delta y - \nabla P$ with pressure $P = \lambda \operatorname{div} y$ i.e. $\operatorname{div} y = (1/\lambda) P$.

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Using a variational method, one can prove that under the above hypothesis on the data, for every $\lambda > 0$, there exists a unique solution $y \in C([0,T]; L^2(\Omega))^n \cap L^2(0,T; H_0^1(\Omega))^n$. Furthermore, we have $y \in C(]0,T]; H_0^1(\Omega))^n \cap H^1(\delta, T; L^2(\Omega))^n$, for every $\delta > 0$.

Our first purpose is then to study the reachable set at time T which is defined for a fixed $y^0 \in L^2(\Omega)^n$ by

$$R(T) = \{y(x,T), \quad v \in L^2(q)^n, \quad y \text{ solution of } (1.1)\}.$$

Clearly, due to the smoothing effect of equation (1.1), $R(T)$ is a strict subset of $L^2(\Omega)^n$.

It is classical that the density in $L^2(\Omega)^n$ of the reachable set at time T is equivalent to the following continuation property concerning solutions of the adjoint system.

Let $u^0 \in L^2(\Omega)^n$ and let u be the solution of

$$\begin{cases} u' - \Delta u - \lambda \nabla \operatorname{div} u - \partial_k(a_k u) + (\nabla b_k) u_k = 0 & \text{in } Q, \\ u = 0 & \text{on } \sum, \\ u(0) = u^0 & \text{in } \Omega, \\ u = 0 & \text{in } \omega \times (0,T) \end{cases} \tag{1.2}$$

implies $u^0 = 0$.

We will prove:

Theorem 1.1. (i) *Let O be an open and non-empty subset of $\Omega \times (-T, T)$ and let u be solution of*

$$\begin{cases} u' - \Delta u - \lambda \nabla \operatorname{div} u - \partial_k(a_k u) + (\nabla b_k) u_k = 0 & \text{in } \Omega \times (-T,T), \\ u \in L^2_{\text{loc}}(-T,T; H^1_{\text{loc}}(\Omega))^n, \\ u = 0 & \text{in } O, \end{cases} \tag{1.3}$$

then u is null in the horizontal component of O defined by

$$C(O) = \{(x,t), \quad \exists x_0, \quad (x_0,t) \in O\}.$$

(ii) *Let O be an non-empty and open subset of the boundary of $\partial\Omega \times (-T,T)$. If*

$$\begin{cases} u' - \Delta u - \lambda \nabla \operatorname{div} u - \partial_k(a_k u) + (\nabla b_k) u_k = 0 & \text{in } \Omega \times (-T,T), \\ u \in L^2_{\text{loc}}(-T,T; H^2(\Omega))^n, \\ u = \dfrac{\partial u}{\partial v} + \lambda \operatorname{div} u \ v = 0 & \text{in } O, \end{cases} \tag{1.4}$$

then u is null in $C(O)$.

These results have as direct consequences.

Corollary 1.1. *For every $\lambda > 0$, every $y^0 \in L^2(\Omega)^n$ and every $T > 0$, the reachable set $R(T)$ is dense in $L^2(\Omega)^n$.*

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Let then $T > 0$, $\alpha > 0$ and $(y^0, y^1) \in (L^2(\Omega)^n)^2$ be fixed. We denote by $|.|_2$ the $L^2(\Omega)^n$ norm and by (,) the scalar product in $L^2(\Omega)^n$. For every $\lambda > 0$, we denote by U_λ the set of admissible control defined by

$$U_\lambda = \{v \in L^2(q)^n, \text{ such that the solution of (1.1) satisfies } |y(T) - y^1|_2 \leqslant \alpha\}.$$

Corollary 1.1 proves that $U_\lambda \neq \emptyset$ and it is classical that U_λ is a convex and closed set of $L^2(q)^n$. There then exists a unique control $v_\lambda \in U_\lambda$ such that

$$\|v_\lambda\|_{L^2(q)^n} = \min_{v \in U_\lambda} \|v\|_{L^2(q)^n}.$$

A convexity and duality argument developed in [9] proves that $v_\lambda = u_\lambda \chi_q$ where

$$\begin{cases} -u'_\lambda - \Delta u_\lambda - \lambda \nabla \operatorname{div} u_\lambda - \partial_k(a_k u_\lambda) + (\nabla b_k) u_{\lambda k} = 0 & \text{in } Q, \\ u_\lambda = 0 & \text{on } \sum, \\ u_\lambda(T) = u_\lambda^0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

Here u_λ^0 is the minimizer on $L^2(\Omega)^n$ of the functional

$$J_\lambda(u^0) = \frac{1}{2}\int_q |u|^2 \,\mathrm{d}x \,\mathrm{d}t + \alpha |u^0|_2 - (y^1, u^0) + (y^0, u(0)) \tag{1.6}$$

with

$$\begin{cases} -u' - \Delta u - \lambda \nabla \operatorname{div} u - \partial_k(a_k u) + (\nabla b_k) u_k = 0 & \text{in } Q, \\ u = 0 & \text{on } \sum, \\ u(T) = u^0 & \text{in } \Omega. \end{cases} \tag{1.7}$$

Let π denote the orthogonal projection of $L^2(\Omega)^n$ onto H. We recall that π maps continuously $L^2(\Omega)^n$ in H and $H_0^1(\Omega)^n$ in V. We will denote by y_λ, y_λ^r and $y_\lambda^s = y_\lambda - y_\lambda^r$ the solutions of

$$\begin{cases} y'_\lambda - \Delta y_\lambda - \lambda \nabla \operatorname{div} y_\lambda + (a.\nabla) y_\lambda + (y_\lambda.\nabla) b = v_\lambda \chi_q & \text{in } Q, \\ y_\lambda^{r'} - \Delta y_\lambda^r - \lambda \nabla \operatorname{div} y_\lambda^r + (a.\nabla) y_\lambda^r + (y_\lambda^r.\nabla) b = v_\lambda \chi_q & \text{in } Q, \\ y_\lambda = y_\lambda^r = 0 & \text{on } \sum, \\ y_\lambda(0) = y^0, \quad y_\lambda^r(0) = \pi y^0 & \text{in } \Omega. \end{cases} \tag{1.8}$$

We then have

$$|y_\lambda(T) - y^1|_2 \leqslant \alpha.$$

Our second aim is the asymptotic analysis when λ tends to infinity of problems (1.8). Before stating our result, let us introduce some notations and recall some known results.

Let H and V be the closure in $L^2(\Omega)^n$ and in $H_0^1(\Omega)^n$ of

$$E = \{z \in C_0^\infty(\Omega)^n, \quad \operatorname{div} z = 0 \text{ in } \Omega\}, \quad \text{respectively.}$$

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For $v=(v_1,\dots,v_n)\in L^2(q)^n$, and $z^0\in H$, we denote by $z=z(x,t)=(z_1(x,t),\dots,z_n(x,t))$ the vector valued solution of

$$\begin{cases} z'-\Delta z+(a.\nabla)z+(z.\nabla)b+\nabla p=v\chi_q & \text{in } Q,\\ z=0 & \text{on } \sum,\\ \operatorname{div} z=0 & \text{in } Q,\\ z(0)=z^0 & \text{in } \Omega.^{\omega} \end{cases} \tag{1.9}$$

Again, using a variational method, one can prove (see [14]) that under the above hypothesis on the data, there exists a pressure $p\in\mathscr{D}'(0,T)$ and a unique solution $z\in C([0,T];\ L^2(\Omega))^n\cap L^2(0,T;V)$. Furthermore, we have $z\in C(]0,T];V)\cap H^1(\delta,T;L^2(\Omega))^n$, for every $\delta>0$.

The reachable set is defined as before by

$$R(T)=\{z(x,T),\quad v\in L^2(q)^n,\quad z \text{ solution of (1.9)}\}.$$

The density of the reachable set in H was proved in [12] when $a\in C^2(Q)^n$ and $b\in C^1(Q)^n$, in [3] for both a, $b\in L^\infty(0,T;W^{1,\infty}(\Omega))^n$ and in [2] for $a\in L^\infty(Q)^n$ and $b\in L^\infty(0,T;W^{1,\infty}(\Omega))^n$ but under the hypothesis that there exists a pressure $q\in L^2_{\text{loc}}(Q)$ for solutions of the adjoint system of (1.9) which is

$$\begin{cases} -w'-\Delta w-\partial_k(a_k w)+(\nabla b_k)w_k+\nabla q=0 & \text{in } Q,\\ w=0 & \text{on } \sum,\\ \operatorname{div} w=0 & \text{in } Q,\\ w(T)=w^0\in H. \end{cases} \tag{1.10}$$

Notice that we know (see [14]) that there exists a pressure q in $H^{-1}(0,T;L^2(\Omega))$ and that the difficulty when one wants a pressure in $L^2_{\text{loc}}(0,T;L^2(\Omega))$ concerns the regularity with respect to the time variable. In [2], we proved the existence of a pressure q in $L^2_{\text{loc}}(0,T;L^2(\Omega))$ solution of (1.10) in the following cases (see Lemma 3.1, p. 289):

(H1) If $a\in L^\infty(Q)^n$ and $b\in L^\infty(0,T;W^{1,\infty}(\Omega))^n$ and if there exists a compact set K of Ω such that for almost every t, $\operatorname{supp}(a(\cdots;t))\subset K$

(H2) If $a\in L^\infty(Q)^n$, $b\in L^\infty(0,T;W^{1,\infty}(\Omega))^n$ and if w is null in a neighborhood of the boundary of Ω.

(H3) $\begin{cases} \text{if } n=2\ \exists p_0\in\,]2,+\infty[,\ \forall j\in\{1,\dots,n\}\ a_j\in(0,T;W^{1,p_0}(\Omega))\cap L^\infty(Q)\\ \text{if } n>2\ \forall j\in\{1,\dots,n\}\ a_j\in L^2(0,T,W^{1,n}(\Omega))\cap L^\infty(Q). \end{cases}$

(H4) $\begin{cases} \text{if } n=2\ \exists p_0\in\,]2,+\infty[,\ \forall j\in\{1,\dots,n\}\ a_j\in H^1(0,T,L^{p_0}(\Omega))\cap L^\infty(Q)\\ \text{if } n>2\ \forall j\in\{1,\dots,n\}\ a_j\in H^1(0,T,L^n(\Omega))\cap L^\infty(Q). \end{cases}$

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Furthermore, in all these cases, it was proved that there exists a pressure q such that $(T-t)q \in L^2(Q)$ and that if $w^0 \in V$ then $q \in L^2(Q)$.

In what follows, we will denote by (H) the following hypothesis.

(H) $\forall w^0 \in V, \quad \exists p \in L^2(Q)$ solution of (1.10).

Hypothesis (H) is fulfilled if one of (H1), (H3) or (H4) is fulfilled.

We then have [12, 3, 2].

Proposition 1.1. (i) *Suppose* (H) *is satisfied. For every* $z^0 \in H$ *and every* $T>0$, *the reachable set* $R(T)$ *of solutions of* (1.9) *is dense in* (H).

(ii) *Suppose* ω *is a neighborhood of the boundary of* Ω. *For every* $z^0 \in H$ *and every* $T>0$, *the reachable set* $R(T)$ *of solutions of* (1.9) *is dense in* (H).

Remarks. (i) In [4], we used the above characterization of the control with L^2-minimum norm for the study of non-linear heat equations.

(ii) More recently, Imanuvilov [7] has proved the null controllability for solutions of (1.9) when $a=b=a(x) \in W^{2,\infty}(\Omega)$ with hypothesis (H1) on its support.

(iii) In [1], Coron proved the approximate controllability for Navier–Stokes equations with slip boundary conditions. It appears also here that problems arise near the boundary of Ω.

(iv) One has to compare Proposition 1.1 and Corollary 1.1. The approximate controllability for slightly compressible fluids is realised without any conditions on a, b except those ensuring the well posedness of the system. In fact, this is quite natural since equation (1.1) can be seen as a Stokes equation where we have regularized the pressure.

(iv) The last remark tries to explain in what sense our hypothesis (H) is quasi-optimal in order to ensure both control and asymptotic analysis of our problem. Indeed, reading proofs of Theorems 1.1 and 1.2, one can see that we need in an essential manner the existence of a pressure $q \in L^2(Q)$ for the system

$$\begin{cases} z' - \Delta z + \nabla q = f \in L^2(0,T;H^{-1}(\Omega))^n & \text{in } Q, \\ z = 0 \quad \text{on } \sum, \\ \operatorname{div} z = 0 \quad \text{in } Q, \\ z(x,0) = z^0 \in V. \end{cases}$$

It is known (by linearity and [14]) that one suppose $z^0 = 0$ and that there exists $q \in H^{-1}(0,T;L^2(\Omega))$. In [2], we proved that if there exists a compact set K included in Ω, such that for almost every t, $\operatorname{supp} f(.,t) \subset K$, then choosing q with $\int_\Omega q(x,t)\,\mathrm{d}x = 0$. a.e. t, ensure that $q \in L^2(Q)$ with the continuity of the map $f \to q$. Lebeau ([8]) then proved that one can find f such that $q \notin L^2_{\mathrm{loc}}(0,T;\mathscr{D}'(\Omega))$.

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We prove:

Theorem 1.2. *Let* $T > 0$, $(y^0, y^1) \in L^2(\Omega)^n \times H$ *and* $\alpha > 0$. *Then*

(i) *Suppose that* (H) *is satisfied. Controls* $(v_\lambda)_{\lambda>0}$ *strongly converge in* $L^2(q)^n$ *to a function* $v \in L^2(q)^n$ *when* $\lambda \to +\infty$. *Functions* $(y^r_\lambda)_{\lambda>0}$ *strongly converge in* $C([0,T]; L^2(\Omega)^n) \cap L^2(0,T; H^1_0(\Omega))^n$ *to the solution* y *of system*

$$\begin{cases} y' - \Delta y + (a.\nabla)y + (y.\nabla)b = \nabla p + v\chi_q & \text{in } Q, \\ \operatorname{div} y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = \pi y^0 & \text{in } \Omega, \\ |y(T) - y^1|_2 \leqslant \alpha. \end{cases} \tag{1.11}$$

Furthermore, v *is the control of* $L^2(q)^n$ *– minimum norm among the set of admissible control associated to* (1.11). *Functions* $(y^s_\lambda)_\lambda$ *strongly converge to* 0 *in* $L^2(Q)^n$ *and in* $C([\delta,T]; L^2(\Omega)^n) \cap L^2(\delta,T; H^1_0(\Omega))^n$ *for every* $\delta > 0$.

(ii) *If* $y^0 \in H^1_0(\Omega)^n$ *the convergence of* $(y^r_\lambda)_{\lambda>0}$ *is realised in* $C([0,T]; H^1_0(\Omega))^n \cap H^1(0,T; L^2(\Omega))^n$.

(iii) *Suppose* ω *is a neighborhood of the boundary of* Ω. *Controls* $(v_\lambda)_{\lambda>0}$ *weakly converge* (*after extraction of a subsequence*) *in* $L^2(q)^n$ *to a function* $v \in L^2(q)^n$ *when* $\lambda \to +\infty$. *For every* $\delta > 0$, *functions* $(y^s_\lambda)_{\lambda>0}$ *strongly converge in* $C([\delta,T]; L^2(\Omega)^n) \cap L^2(Q)$ *to* 0 *while* y^r_λ *strongly converge in* $C([0,T]; L^2(\Omega)^n) \cap L^2(0,T; H^1_0)$ *to the solution* y *of* (1.11).

Remark. In general, if $y^1 \notin H$, one cannot prove that control (v_λ) are bounded and this is natural since the limit problem (for fix initial data and right-hand side) of (1.1) are functions living in $C([0,T]; H)$, solutions of $y' - \Delta y + (a.\nabla)y + (y.\nabla)b = \nabla p + v\chi_q$ in Q which implies that such a final data cannot be in the reachable set of such functions.

We now describe our plan: in section 2, we prove Theorem 1.1. We prove a Carleman estimate for solutions of (1.3). The tools used are developed in [3]. In section 3, we prove Theorem 1.2. We first do the asymptotic analysis of systems (1.1).

2. Proof of Theorem 1.1

Let us first mention that the second assertion of Theorem 1.1 is a direct consequence of the first one. Indeed, let $(x_0, t_0) \in \partial\Omega \times (-T, T)$ and α, $r > 0$ such that $u = \partial u/\partial v + \lambda \operatorname{div} uv = 0$ in $(B(x_0,t_0) \cap \partial\Omega) \times (t_0 - \alpha, t_0 + \alpha)$. Let $\bar{u}$ be the extension by 0 of u in $(B(x_0,t_0) \cap \Omega^c) \times (t_0 - \alpha, t_0 + \alpha)$. It is easy to see that $\bar{u}$ is solution of (1.3) in $(B(x_0,t_0) \cup \Omega) \times (t_0 - \alpha, t_0 + \alpha)$ with $O = (B(x_0,t_0) \cap \Omega^c) \times (t_0 - \alpha, t_0 + \alpha)$. This implies that $\bar{u} = u = 0$ in the horizontal component of this set which proves (ii).

We now prove (i).

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2.1. *Preliminaries*

The results that we recall in this section are classical and we refer to the books of Robert [10] and St Raymond [11] for details.

We note $D_j = (h/i)\partial_j$, $\lambda(\xi) = (1 + |\xi|^2)^{1/2}$, for $\xi \in \mathbb{R}^n$. The set S^m (where $m \in \mathbb{Z}$) denotes the space of symbols defined on $\mathbb{R}^{2n}$ and it satisfies

$$\forall \alpha, \beta \in \mathbb{N}^n,\ \exists C_{\alpha,\beta} \forall x, \xi, h \quad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leqslant C_{\alpha,\beta} \lambda(\xi)^{m-|\beta|}$$

and

$$\forall j \in \mathbb{N},\ \exists a_j(x, \xi) \in S^{m-j}, \quad a(x, \xi, h) \in \sum_{j=0}^{N} h^j a_j(x, \xi) + h^{N+1} S^{m-N-1}.$$

The principal symbol of a is then a_0. If $a \in S^m$ and if u is $C_0^\infty(\mathbb{R}^n)$, we define the pseudodifferential operator of order m

$$a(x, D, h) u(x) = \frac{1}{(2\pi h)^n} \int e^{ix\cdot\xi/h} a(x, \xi, h) \hat{u}\left(\frac{\xi}{h}\right) d\xi$$

(where $\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\, dx$ is the Fourier transform of u). If $a \in S^m$, $a(x, D, h)$ maps $H^s(\mathbb{R}^n)$ in $H^{s-m}(\mathbb{R}^n)$. We write E^m the space of pseudo-differential operators of order m and $E^{-\infty} = \bigcap_{m \in Z} E^m$.

We write $|v|_0^2 = \int_{\mathbb{R}^n} |v(x)|^2\, dx$, $(u, v) = \int_{\mathbb{R}^n} u\bar{v}$ and $|v|_1^2 = |\lambda(D) v|_0^2$. Thus,

$$|v|_1^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \lambda^2(h\xi) |\hat{u}(\xi)|^2\, d\xi = \int |v|^2\, dx + h^2 \int |\nabla v|^2\, dx.$$

We recall that if $a \in E^m$ and $b \in E^l$ then $a(x, D) \circ b(x, D) \in E^{m+l}$ with a principal part $a_0 b_0$. On the other hand, $(i/h)[a(x, D), b(x, D)] = (i/h)(a(x, D) \circ b(x, D) - b(x, D) \circ a(x, D)) \in E^{m+l-1}$ and its principal part is $\{a_0, b_0\} = \partial_\xi a_0 \partial_x b_0 - \partial_x a_0 \partial_\xi b_0$.

We recall the Garding inequality:

Proposition 2.1. *Let U be an open set of $\mathbb{R}^n$. If $a \in E^2$ satisfies*

$$\exists c_1 > 0,\ \forall (x, \xi) \in U \times \mathbb{R}^n, \quad \operatorname{Re} a_0(x, \xi) \geqslant c_1 \lambda(\xi)^2,$$

then for every compact set K of U, there exists $h_K > 0$ such that

$$\forall u \in H_0^1(K),\ \forall h \in]0, h_K], \quad \operatorname{Re}(a(x, D, h) u, u) \geqslant \frac{c_1}{4} |u|_1^2.$$

The following proposition concerns the inversion on the high frequencies of operators of order 2 which are elliptic on high frequencies.

Proposition 2.2. *If $p \in E^2$ satisfies*

$$\exists c > 0: \quad |p_0(x, \xi)| \geqslant c\lambda(\xi)^2 \quad \forall \xi,\ |\xi| \geqslant R,\ \forall x \in \mathbb{R}^n,$$

then there exists $e \in E^{-2}$ and $\alpha \in E^{-\infty}$ such that $e \circ p = 1 + \alpha + hR$, where $R \in E^{-1}$, and $\alpha(x, \xi) = 0$, if $|\xi| \geqslant R$.

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As we are dealing with evolution equations, the operators under consideration will depend on a parameter (the time) and we will need uniform estimates. Let $I = [-T_0, T_0]$ be a subinterval of $\mathbb{R}$. We will say that a set of operators with symbols of order m, $\{a(t,x,\xi,h)\}_{t\in I}$, is in $L^\infty(I, E^m)$ if the constants $C_{\alpha,\beta}$ appearing in the definition of S^m do not depend on $t \in I$. We recall that (see [10]) if $(a(t))_{t\in I} \in L^\infty(I, E^m)$ and if $(b(t))_{t\in I} \in L^\infty(I, E^l)$ then $(a(t)\circ b(t))_t \in L^\infty(I, E^{m+l})$ and we have

$$\forall s \in \mathbb{R},\ \exists C_s > 0\colon\ \forall t \in I,\ \forall u \in H^s, \quad |a(t)u|_{s-m} \leqslant C_s |u|_s.$$

In the same way, we have the following Garding inequality: if U is an open set of $\mathbb{R}^n$ and if $(a(t))_{t\in I} \in L^\infty(I, E^2)$ satisfies

$$\exists c_1 > 0\colon\ \forall t \in I,\ \forall (x,\xi) \in U \times \mathbb{R}^n, \quad \operatorname{Re} a_0(t,x,\xi) \geqslant c_1 \lambda(\xi)^2,$$

then for every compact set K of U, there exists $h_K > 0$ such that

$$\forall u \in H_0^1(K), \quad \forall (t,h) \in I \times]0, h_K], \quad \operatorname{Re}(a(t,x,D,h)u,u) \geqslant \frac{c_1}{4} |u|_1^2.$$

Finally, we have the analogue of Proposition 2.2: If $(p(t))_t \in L^\infty(I, E^2)$ satisfies

$$\exists c, R > 0\colon \quad |p_0(x,\xi)| \geqslant c\lambda(\xi)^2 \quad \forall \xi, |\xi| \geqslant R, \quad \forall (t,x) \in I \times \mathbb{R}^n$$

then there exist $(e(t))_t \in L^\infty(I, E^{-2})$ and $(\alpha(t))_t \in L^\infty(I, E^{-\infty})$ such that $e(t)\circ p(t) = 1 + \alpha(t) + hR(t)$, with $R(t) \in L^\infty(I, E^{-1})$ and $\alpha(t,x,\xi) = 0$ if $|\xi| \geqslant R$.

Let now $\varphi = \varphi(t,x) \in C_0^\infty(\mathbb{R}^{n+1})$. We write $\varphi_t(x) = \varphi(t,x)$ and $p(t)$ the operator of E^2:

$$p(t)(x,D,h) = -h^2 \mathrm{e}^{\varphi/h} \circ \Delta \circ \mathrm{e}^{-\varphi/h}.$$

$p(t)$ has as principal symbol

$$p_0(t)(x,\xi) = \sum_{j=1}^n \left(\xi_j + \mathrm{i}\frac{\partial \varphi}{\partial x_j}\right)^2.$$

The adjoint operator $p^*(t)$ of $p(t)$ is also of order 2 and its principal symbol is the conjugate of $p_0(t)(x,\xi)$, $\bar{p}_0(t)(x,\xi)$. We then decompose the operator $p(t) = a(t) + \mathrm{i}b(t)$ with ($a*$ denotes the adjoint operator of a): $a(t) = [p(t) + p^*(t)]/2 \in L^\infty(-T, T; E^2)$ with principal part $a_0(t) = \operatorname{Re} p_0(t)$ and $b(t) = [p(t) - p^*(t)]/2i \in L^\infty(-T, T; E^1)$ with principal part $b_0(t) = \operatorname{Im} p_0(t)$.

Let U_0 be a bounded and open set of $\mathbb{R}^n$ and suppose that there exists $C_0 > 0$ such that φ_0 satisfies

$$(x,\xi) \in \bar{U}_0 \times \mathbb{R}^n, \quad a_0(0)(x,\xi) = 0 \Rightarrow \{a_0(0), b_0(0)\}(x,\xi) \geqslant C_0. \tag{2.1}$$

Then there exist $\rho_0 > 0$ and $c_3, d' > 0$ such that for $|t| \leqslant \rho_0$, φ_t satisfies

$$\begin{gathered}(x,\xi) \in \bar{U}_0 \times \mathbb{R}^n, \quad a_0(t)(x,\xi) = 0 \Rightarrow \{a_0(t), b_0(t)\}(x,\xi) \geqslant \frac{C_0}{2}, \\ d'\lambda^{-2}(\xi) a_0(t)^2(x,\xi) + d' b_0(t)^2(x,\xi) + \{a_0(t), b_0(t)\}(x,\xi) \geqslant c_3 \lambda^2(\xi).\end{gathered} \tag{2.2}$$

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We now define general operators of order 1 and 2. For this let $s \in \mathbb{N}$ and for $f = (f_1, \dots, f_s)$ we write

$$L_1(f) = \sum_{(j,k)\in\{1,\dots,n\}\times\{1,\dots,s\}} c_{jk} \frac{\partial f_k}{\partial x_j}$$

with coefficients $c_{jk} \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We have $L_1 \in E^1$.

On the other hand, for $k = (k_{lm})_{1 \leqslant l, m \leqslant s}$, define

$$L_2(k) = \sum_{(l,m)\in\{1,\dots,s\}^2,(l',m')\in\{1,\dots,n\}^2} \frac{\partial}{\partial x_{l'}} \left(d_{lml'm'} \frac{\partial}{\partial x_{m'}} k_{lm} \right),$$

where $d_{lml'm'} \in C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$. We have $L_2 \in E^2$.

2.2. *Carleman inequality*

In the sequel, c and d will denote positive constants that may change line to line and are independent of the parameter h.

Lemma 2.1. *Let U_0 be a bounded and open set of $\mathbb{R}^n$. We suppose that φ satisfies* (2.1). *Then there exists $\rho_0 > 0$, such that for every compact set K included in U_0, there exist $h_1 > 0$ and $c > 0$ such that for every $h \in]0, h_1[$, one has*

$$\begin{aligned}\int_{K\times(-\rho_0,\rho_0)} |z|^2 e^{2\varphi/h} \, dx\, dt \leqslant{}& ch \int_{K\times(-\rho_0,\rho_0)} |f|^2 e^{2\varphi/h} \, dx\, dt \\ &+ \frac{c}{h} \int_{K\times(-\rho_0,\rho_0)} |g|^2 e^{2\varphi/h} \, dx\, dt \\ &+ ch^3 \int_{K\times(-\rho_0,\rho_0)} |z' - \Delta z - L_1 f - L_2 g|^2 e^{2\varphi/h} \, dx\, dt\end{aligned}$$

for every $(z, f, g) \in L^2((U_0 \times (-\rho_0, \rho_0))^3$ with $z' - \Delta z - L_1 f - L_2 g \in L^2(U_0 \times (-\rho_0, \rho_0))$, all these functions being compactly supported in $K \times]-\rho_0, \rho_0[$.

Remark. This result has to be compared with those of [3, 2]. In [3], we have studied the case where $L_2 = 0$ for stationary equations (we obtain then an inequality which involved the 'H^1–' norm of the function. In [2], we studied the case $L_2 = 0$ for heat equations and the stationary version of the above lemma.

Proof of Lemma 2.1. Let ρ_0 be given by (2.1) and its consequences. We write $v = z e^{\varphi/h}$, $H = (z' - \Delta z - L_1 f - L_2 g) e^{\varphi/h}$, $r = g e^{\varphi/h}$ and $q = f e^{\varphi/h}$. We then have

$$h^2 v' + p(t) v = h\varphi' v - h\Delta\varphi v + h\mathscr{L}_1(D) q + \mathscr{L}_2(D) r + hR_0 v + R_1(r) + h^2 H,$$

where $R_0 = -h[e^{\varphi/h}, L_1] e^{-\varphi/h} \in L^\infty(-T, T; E^0)$ and $R_1 = -h^2 [e^{\varphi/h}, L_2] e^{-\varphi/h} \in L^\infty(-T, T; E^1)$. We study separately high and low frequencies.

$$V = \{\xi \in \mathbb{R}^n \colon \exists x \in \mathbb{R}^n, \quad \text{such that} \quad p_0(t)(x, \xi) = 0\}.$$

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We have $V \subset \bar{B}_{\mathbb{R}^n}(0, c_0)$ where $c_0 = |\nabla\varphi|_{L^\infty(R^{n+1})}$. We consider $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leqslant \chi \leqslant 1$ and $\chi = 1$ on a neighbourhood of $\bar{B}_{\mathbb{R}^n}(0, 2c_0^2 + 2)$.

Operators $a(t)$ (which were defined by $(p(t) + p^*(t))/2$) satisfy the hypotheses of Proposition 2.2 hence they are invertible on the high frequencies and therefore there exist operators $d(t)$ of order -2 such that

$$d(t) \circ a(t) = 1 + \alpha(t) + hR(t)$$

with $R(t) \in L^\infty(-T, T; E^{-1})$ and $\alpha\delta(D) \in hL^\infty(-T, T; E^{-1})$.

Since $[a(t), \lambda^{-2}] \in hL^\infty(-T, T; E^{-1})$, there exists $c > 0$ such that

$$\begin{aligned} |\delta(D)v|_0 &\leqslant |d(t) \circ a(t)\delta(D)v|_0 + ch|v|_{-1} \leqslant c(|a(t)\delta(D)v|_{-2} + h|v|_{-1}) \\ &\leqslant c(|\lambda^{-2}a(t)\delta(D)v|_0 + h|v|_{-1}) \leqslant c(|a(t)(\lambda^{-2}\delta(D)v|_0 + h|v|_{-1}). \end{aligned} \tag{2.4}$$

On the other hand, we have

$$\begin{aligned} &h^2\lambda^{-2}\delta(D)v' + \mathrm{i}\lambda^{-2}\delta(D)b(t)v + a(t)\lambda^{-2}\delta(D)v \\ &\quad = \lambda^{-2}\delta(D)[h\varphi'v - h\Delta\varphi v + h\mathscr{L}_1(D)q + \mathscr{L}_2(D)r + R_1(r) \\ &\qquad + hR_0 v + h^2 H] - [\lambda^{-2}\delta(D), a(t)]v \end{aligned} \tag{2.5}$$

with $[\lambda^{-2}\delta(D), a(t)] \in hL^\infty(-T, T; E^{-1})$.

Let $u = \lambda^{-2}\delta(D)v$. Since v belongs to $L^2(-\rho_0, \rho_0; L^2(K))$ and since λ^{-2} is in $L^\infty(E^{-2})$, the function u is in $L^2(-\rho_0, \rho_0; H^2(\mathbb{R}^n))$. Furthermore, using the equation satisfied by u, we can see that u is in $H_0^1(-\rho_0, \rho_0; L^2(\mathbb{R}^n))$. Taking the $L^2(\mathbb{R}^{n+1})$-norm, we then get

$$\begin{aligned} &|h^2\lambda^{-2}\delta(D)v' + \mathrm{i}\lambda^{-2}\delta(D)b(t)v|_0^2 + |a(t)\lambda^{-2}\delta(D)v|_0^2 \\ &\qquad + 2\,\mathrm{Re}(h^2\lambda^{-2}\delta(D)v', a(t)\lambda^{-2}\delta(D)v) + 2\,\mathrm{Re}(\mathrm{i}\lambda^{-2}\delta(D)b(t)v, a(t)\lambda^{-2}\delta(D)v) \\ &\quad = |\lambda^{-2}\delta(D)[h\varphi'v - h\Delta\varphi v + h\mathscr{L}_1(D)q + hR_0 v + h^2 H] - [\lambda^{-2}\delta(D), a(t)]v|_0^2. \end{aligned} \tag{2.6}$$

As $\lambda^{-2}\delta(D)b(t) \in L^\infty(-T, T; E^{-1})$ and $a(t)\lambda^{-2}\delta(D) \in L^\infty(-T, T; E^0)$ we get

$$[\lambda^{-2}\delta(D)b(t), a(t)\lambda^{-2}\delta(D)] \in hL^\infty(-T, T; E^{-2})$$

which easily proves that

$$\begin{aligned} |2\,\mathrm{Re}(\mathrm{i}\lambda^{-2}\delta(D)b(t)v, a(t)\lambda^{-2}\delta(D)v)| &= \mathrm{i}([a(t)\lambda^{-2}\delta(D), \lambda^{-2}\delta(D)b(t)]v, v)| \\ &\leqslant ch|v|_{-1}^2. \end{aligned} \tag{2.7}$$

Now,

$$\begin{aligned} 2\,\mathrm{Re}(h^2u', a(t)u) &= 2h^2\int_{\mathbb{R}^n \times (-\rho_0, \rho_0)} u'(-h^2\Delta u - |\nabla\varphi|^2 u)\,\mathrm{d}x\,\mathrm{d}t \\ &= -h^2\int_{\mathbb{R}^n \times (-\rho_0, \rho_0)} |u|^2(|\nabla\varphi|^2)'\,\mathrm{d}x\,\mathrm{d}t \leqslant ch^2|u|_0^2 \\ &\leqslant ch^2|v|_0^2. \end{aligned} \tag{2.8}$$

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Combining (2.4), (2.6), (2.7) and (2.8), we obtain

$$|\delta(D)v|_0^2 \leqslant c[h|v|_0^2 + h^2|q|_0^2 + |r|_0^2 + h^4|H|_0^2]. \tag{2.9}$$

Let us now study $\chi(D)v$. As χ is compactly supported in ξ, $\chi(D) \in E^{-\infty}$ and $\chi(D) \circ p(t) \in p(t) \circ \chi(D) + hL^\infty(-T, T; E^{-\infty})$. Furthermore, owing to the equation satisfied by $\chi(D)v$, we get $\chi(D)v \in H_0^1(-\rho_0, \rho_0; H^2(\mathbb{R}^n))$.

There then exists $R_{-\infty}(t) \in L^\infty(-T, T; E^{-\infty})$, such that

$$h^2\chi(D)v' + p \circ \chi(D)v = h\chi(D) \circ \mathscr{L}_1(D)g + \chi(D) \circ \mathscr{L}_2(D)r - h^2\chi(D)G + h\chi(D) \circ R_0(g) + \chi(D) \circ R_1(r) + hR_{-\infty}(t)(v). \tag{2.10}$$

Consider $\beta \in C_0^\infty(U)$ such that $\beta = 1$ on K. One has $\chi(D)v = \beta(x)\chi(D)v + [\chi(D), \beta]v$ with $[\chi(D), \beta] \in hL^\infty(-T, T; E^{-\infty})$ and $w = \beta\chi(D)v \in C_0^\infty(U)$. Using the decomposition of $p(t)$ as $a(t) + \mathrm{i}b(t)$, one has

$$|h^2v' + p(t)(v)|_0^2 = |a(t)(x, D)(v)|_0^2 + |h^2v' + b(t)(x, D)(v)|_0^2 + \mathrm{i}((a \circ b - b \circ a)v, v) + 2\,\mathrm{Re}(h^2v', a(t)(x, D)(v)). \tag{2.11}$$

We have $a \circ b - b \circ a = [a, b] \in L^\infty(-T, T; E^2)$ with principal symbol $(h/i)\{a_0, b_0\}$.

Now, using (2.1), we have for $|t| \leqslant \rho_0$,

$$d'\lambda^{-2}(\xi)a_0(t)^2(x, \xi) + d'b_0(t)^2(x, \xi) + \{a_0(t), b_0(t)\}(x, \xi) \geqslant c_3\lambda^2(\xi). \tag{2.12}$$

We apply Garding inequality on U, when K is the support of β and when the operator is $d'a(t) \circ \lambda^{-2} \circ a(t) + d'b(t) \circ b(t) + i/h[a(t), b(t)](x, D)$. We deduce that

$$i([a(t), b(t)]w, w) \geqslant \frac{c_3}{4}\, h|w|_1^2 - d'h|\lambda^{-1} \circ a(t)w|_0^2 - d'h|b(t)w|_0^2. \tag{2.13}$$

On the other hand, taking into account (2.11), (2.13) and $|\lambda^{-1}a(t)(w)|_0^2 \leqslant |a(t)(w)|_0^2$, we obtain for h small enough

$$|a(t)(w)|_0^2 + \mathrm{i}([a, b]w, w) \geqslant h\,\frac{c_3}{4}\,|w|_1^2. \tag{2.14}$$

Using (2.14), the fact that $w = \beta\chi(D)v$, we have

$$|a(t)(v)|_0^2 + \mathrm{i}([a, b]v, v) \geqslant h\,\frac{c_3}{4}\,|v|_1^2 - h^2|v|_0^2. \tag{2.15}$$

Now, as usual, we can compute (integrating by parts) that

$$|2\,\mathrm{Re}(h^2\chi(D)v', a(t)\chi(D)v)| \leqslant ch^2|v|_0^2. \tag{2.16}$$

On the other hand, it is known (see [12] for explicit phase, [3] for any phase with (2.1)) that

$$h|\chi(D)v|_1^2 \leqslant c|h^2\chi(D)v' + p(\chi(D)v)|_0^2,$$

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thus

$$|h^2\chi(D)v' + p(\chi(D)v)|_0^2 \leqslant 8h^2|\chi(D)\circ\mathscr{L}_1(D)g|_0^2 + 8|\chi(D)\circ\mathscr{L}_2(D)r|_0^2 + 8h^4|\chi(D)G|_0^2 + 8h^2|\chi(D)\circ R_0(g)|_0^2 + 8|\chi(D)\circ R_1(r)|_0^2 + 8h^2|R_{-\infty}(t)(v)|_0^2. \quad (2.17)$$

As $\chi(D)\in E^{-\infty}$, all the operators appearing on the right-hand side are in $L^\infty(-T,T;E^0)$, hence there exists $c>0$ such that for h small enough (2.17) implies

$$|\chi(D)v|_1^2 \leqslant c\left[h|g|_0^2 + h^3|G|_0^2 + \frac{1}{h}|r|_0^2 + h|v|_0^2\right]. \quad (2.18)$$

Combining (2.9) and (2.18), we obtain

$$\begin{aligned}|v|_0^2 &\leqslant 2(|\chi(D)v|_0^2 + |\delta(D)v|_0^2)\\ &\leqslant 2c\left[h|g|_0^2 + h^3|G|_0^2 + \frac{1}{h}|r|_0^2 + h|v|_0^2\right]\\ &\quad + 2c(h^2|g|_0^2 + |r|_0^2 + h^4|G|_0^2 + h^2|v|_0^2),\end{aligned} \quad (2.19)$$

which proves the lemma. □

2.3. *Proof of Theorem* 1.1

We follow the steps of the proof of the main result in [3]. We denote by $B(r)$ the open ball in $\mathbb{R}^{n+1}$ centred at $(0,0)$ with radius r. In order to prove Theorem 1.1, it is sufficient to prove that if $u=0$ in a half-neighbourhood (in $\mathbb{R}^{n+1}$) of $(0,0)$ like

$$\{(x,t);\ \psi(x,t)<0\}\cap B(\rho),$$

where ψ is C^∞, $\psi(0,0)=0$, $\nabla_x\psi(0,0)\neq 0$ then u vanishes in a neighbourhood of $(0,0)$.

Without loss of generality, one can suppose that $\nabla_x\psi(0,0)=(0,\ldots,0,1)$. As in [3], we first prove a unique continuation property for a 'radius' $r=1$ and small potentials a_{kl}^j and b_{kl}^j. We write

$$W=\{(x,t),\quad |t|<1,\quad |x|<1\}$$

and

$$A=\max_j |a_j|_{L^\infty(W)},\quad B=\max_j |b_j|_{L^\infty((0,T;W^{1,\infty}(W))}.$$

We then have:

Lemma 2.2. *There exists* $M>0$ *such that for all* $(u,a_j,b_j)\in L^2(-1,1;H^1(|x|<1))\times L^\infty(W)^n\times L^\infty(0,T;W^{1,\infty}(W))^n$ *with*

$$\begin{cases} u \text{ is solution of (1.3)},\\ \max(A,B)\leqslant M,\\ u=0 \text{ in } W\cap[x_n+M(|x'|+|t|)<0],\end{cases} \quad (2.20)$$

one has $u=0$ *in a neighbourhood of* $(0,0)$ *in* $\mathbb{R}^{n+1}$.

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Proof of Lemma 2.2. The choice of φ is the same as in [3] and it is

$$\varphi(x,t) = (x_n + |x'|^2 + t^2 - \delta)^2 \chi,$$

where $\delta > 0$ has to be chosen and $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ satisfies $\chi = 1$ on W. We proved in [3] (Lemma 4.5) that

$$\begin{aligned} &\exists \delta > 0, \quad \exists r_0 > 0, \quad \text{such that } \varphi \text{ satisfies (2.1) on } U_0 = \{x, |x| < r_0\} \\ &\text{with } C_0 = \delta^2. \end{aligned} \tag{2.21}$$

In what follows, δ and r_0 are chosen such that (2.21) holds. We now apply Lemma 2.1 with $U_0 = \{x, |x| < r_0\}$ in order to get the existence of $\rho_0 > 0$ such that the conclusion of this lemma is satisfied. We then fix $r_1 > 0$ small enough besides δ^2, r_0 and ρ_0 such that $B(4r_1) \subset \{(t,x), |t| < \rho_0, |x| < r_0\}$ and we choose $\zeta \in C_0^\infty(B(r_1))$ with $\zeta = 1$ on $B(3r_1/4)$. We write $K = \{|x| \leqslant r_0/2\}$, $\rho_1 = \rho_0/2$ and

$$\Sigma = \operatorname{supp}[\nabla_{x,t}\zeta] \cap \{x_n + M(|x'| + |t|) \geqslant 0\}.$$

Then there exists $M_1 > 0$ such that for all $M \in]0, M_1]$, one has

$$\sup_{(x,t)\in\Sigma} \varphi(x,t) < \varphi(0,0) = \delta^2. \tag{2.22}$$

We define $z = \zeta u$ and $q = \zeta \operatorname{div} u$. We then have

$$\operatorname{div} u' - (1+\lambda)\Delta(\operatorname{div} u) = \operatorname{div}(\partial a_j u) - \operatorname{div}(\partial_i b_j u_j) \quad \text{in } Q.$$

After a simple computation, one can then prove that

$$\begin{aligned} q' - (1+\lambda)\Delta q = {} & \zeta' \operatorname{div} u + \operatorname{div}(2\nabla\zeta \operatorname{div} u) - \Delta\zeta \operatorname{div} u + \operatorname{div} \partial_j(a_j u)) \\ & - \partial_j(\partial_k \zeta a_j u) + \partial_{jk}\zeta a_j u - \partial_j(z \partial_i b_j) + \partial_i \zeta \partial_i b_j u_j. \end{aligned} \tag{2.23}$$

So we obtain an equation of the form

$$q' - (1+\lambda)\Delta q - L_1(\partial_k \zeta a_j u_k; z_j \partial_i b_j) - L_2(a_j u) = \partial_{jk}\zeta a_j u + \partial_i \zeta \partial_i b_j u_j, \tag{2.24}$$

where L_1 (resp. L_2) is a first (resp. second) order differential operator. Let

$$F_{ijk} = \partial_{jk}\zeta a_j u + \partial_i \zeta \partial_i b_j u_j.$$

Applying Lemma 2.1, we can write that

$$\begin{aligned} \int |q|^2 \mathrm{e}^{2\varphi/h} \, \mathrm{d}x \, \mathrm{d}t \leqslant {} & ch \int (|\partial_k \zeta a_j u_k|^2 + |\partial_i b_j|_\infty |z|^2) \mathrm{e}^{2\varphi/h} \, \mathrm{d}x \, \mathrm{d}t \\ & + \frac{c}{h} \int |a|_\infty^2 |z|^2 \mathrm{e}^{2\varphi/h} \, \mathrm{d}x \, \mathrm{d}t + ch^3 \int |F_{ijk}|^2 \mathrm{e}^{2\varphi/h} \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{2.25}$$

On the other hand, we have

$$\begin{aligned} z' - \Delta z &= \zeta(u' - \Delta u) + u\zeta' - 2\nabla\zeta \cdot \nabla u - \Delta\zeta u \\ &= \lambda \nabla q - \nabla\zeta \operatorname{div} u + u\zeta' - 2\nabla\zeta \cdot \nabla u - \Delta\zeta u \\ &\quad - \partial_j(a_j z) + \partial_j \zeta a_j u - \partial_i b_j z \end{aligned} \tag{2.26}$$

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Hence there exist $c > 0$ and $h_1 > 0$ such that for every $h \in]0, h_1]$, we have

$$\int |z|^2 e^{2\varphi/h} \, dx \, dt + h^2 \int |\nabla z|^2 e^{2\varphi/h} \, dx \, dt$$

$$\leqslant ch^3 \int (|\partial_j \zeta a_j u|^2 + |\partial_i b_j z|^2) e^{2\varphi/h} \, dx \, dt + ch^3 \int |\nabla \zeta|^2 |\operatorname{div} u|^2 e^{2\varphi/h} \, dx \, dt$$

$$+ ch \int |a_j z|^2 e^{2\varphi/h} \, dx \, dt + ch \int |q|^2 e^{2\varphi/h} \, dx \, dt. \tag{2.27}$$

Combining these two inequalities, we deduce that there exists $c > 0$ such that for h small enough

$$\int |z|^2 e^{2\varphi/h} \, dx \, dt + h^2 \int |\nabla z|^2 e^{2\varphi/h} \, dx \, dt$$

$$\leqslant ch^3 B^2 \int |\nabla z|^2 e^{2\varphi/h} \, dx \, dt + ch^2 B^2 \int |z|^2 e^{2\varphi/h} dx \, dt$$

$$+ cA^2 \int |z|^2 e^{2\varphi/h} \, dx \, dt + ch^3 \int |H_0|^2 e^{2\varphi/h} \, dx \, dt, \tag{2.28}$$

where $H_0 \in L^2$ does not depend on h and is supported in Σ.

Hence, for $A < 1/2c$, and for h small enough, we get for $h > 0$ small enough,

$$\int |z|^2 e^{2\varphi/h} \, dx \, dt \leqslant \int |H_0|^2 e^{2\varphi/h} \, dx \, dt \tag{2.29}$$

with $G^2 = (|F|^2 + |H_0|^2) + |f_0|^2 \in L^1$ and

$$\operatorname{supp}(G) \subset \Sigma. \tag{2.30}$$

Using (2.22) and letting $h \to 0$, we deduce that $z = 0$ in a neighbourhood of $(0, 0)$ which proves Lemma 2.2. □

The proof of Theorem 1.2 follows by the same change of scale as in [3].

3. Asymptotic analysis

3.1. *Asymptotic analysis of* (1.1)

In this section, we study the convergence when λ tends to $+\infty$ of solutions of

$$\begin{cases} y'_\lambda - \Delta y_\lambda - \lambda \nabla \operatorname{div} y_\lambda + (a \,.\, \nabla) y_\lambda + (y_\lambda \,.\, \nabla) b = f_\lambda & \text{in } Q, \\ y_\lambda = 0 & \text{on } \Sigma, \\ y_\lambda(0) = y^0_\lambda & \text{in } \Omega, \end{cases} \tag{3.1}$$

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when $y^0_\lambda \to y^0$ in $L^2(\Omega)^n$ (or in $H^1_0(\Omega)^n$, $f_\lambda \to f$ in $L^2(Q)^n$, and $(a,b) \in L^\infty(Q)^n \times L^\infty(0,T;W^{1,\infty}(\Omega))^n$ are fixed.

Let us denote by Λ_λ the operator

$$\Lambda_\lambda : L^2(\Omega)^n \times L^2(Q)^n \to X = C([0,T];L^2(\Omega))^n \cap L^2(0,T;H^1_0(\Omega))^n,$$

$$(y^0, f) \to y$$

with

$$\begin{cases} y' - \Delta y - \lambda \nabla \operatorname{div} y + (a.\nabla)y + (y.\nabla)b = f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0) = y^0 \quad \text{in } \Omega. \end{cases} \tag{3.2}$$

The space X is endowed with the natural norm $\|\cdot\|_{C([0,T];L^2(\Omega))^n} + \|\cdot\|_{L^2(0,T;H^1_0(\Omega))^n}$. The operator Λ_λ is linear continuous and

$$\exists c > 0,\ \forall \lambda > 0, \quad \|\Lambda_\lambda(y^0,f)\|_X + |\lambda \operatorname{div} \Lambda_\lambda(y^0,f)|_{L^2(Q)} \leqslant c(|y^0|_2 + |f|_2) \tag{3.3}$$

with $c = O(|a|_\infty + |b|_{L^\infty(0,T;W^{1,\infty}(\Omega))^n} + 1)$.

Furthermore, Λ_λ maps continuously $H^1_0(\Omega)^n \times L^2(Q)^n$ in the space $Y = C([0,T];H^1_0(\Omega))^n \cap H^1(0,T;L^2(\Omega))^n$ with

$$\begin{aligned} &\exists c > 0,\ \forall \lambda > 0, \quad \|\Lambda_\lambda(y^0,f)\|_Y + \lambda |\operatorname{div} \Lambda_\lambda(y^0,f)|_{C([0,T]:L^2(\Omega))} \\ &\quad \leqslant c(\|y^0\|_{H^1_0(\Omega)} + |f|_2) + \lambda |\operatorname{div} y^0|_2, \end{aligned} \tag{3.4}$$

with $c = O(|a|_\infty + |b|_{L^\infty(0,T;W^{1,\infty}(\Omega))^n} + 1)$.

Remark 3.1. Equation (3.3) is obtained by multiplying (3.2) by y and (3.4) is obtained by multiplying by y'.

In (3.1), we have $y_\lambda = \Lambda_\lambda(y^0_\lambda, f_\lambda)$. We write $y_\lambda = y^r_\lambda + y^s_\lambda$ with $y^r_\lambda = \Lambda_\lambda(\pi y^0, f_\lambda)$, and $y^s_\lambda = \Lambda_\lambda(y^0_\lambda - \pi y^0_\lambda, 0)$. We prove:

Proposition 3.1. (i) *If $(y^0_\lambda, f_\lambda) \in V \times L^2(Q)^n$ and converge to (y^0, f) in these spaces then $(y_\lambda)_\lambda$ converge in Y to the solution z of*

$$\begin{cases} z' - \Delta z + (a.\nabla)z + (z.\nabla)b = f + \nabla_q \quad \text{in } Q, \\ \operatorname{div} z = 0 \quad \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \\ z(0) = y^0 \quad \text{in } \Omega. \end{cases} \tag{3.5}$$

Furthermore, $\lambda \operatorname{div} y_\lambda - q \to 0$ in $L^2(Q)$, where the pressure q satisfies $\int_\Omega q(x,t)\,\mathrm{d}x = 0$ for a.e. t.

(ii) *If $(y^0_\lambda, f_\lambda) \in H \times L^2(Q)^n$ and converges to (y^0, f) in these spaces then $(y_\lambda)_\lambda$ strongly converges in X to z and $\lambda \operatorname{div} y_\lambda - q \to 0$ in $H^{-1}(0,T;L^2(\Omega))$.*

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(iii) *Suppose* (H) *satisfied. If* $(y_\lambda^0, f_\lambda) \in L^2(\Omega)^n \times L^2(Q)^n$ (*resp.* $H_0^1(\Omega))^n \times L^2(Q)^n$) *and converge to* (y^0, f) *in these spaces, then* $(y_\lambda - y_\lambda^s)_\lambda$ *strongly converge in* X (*resp.* Y) *to the solution* Z *of* (3.5) *with initial data* $Z(0) = \pi y^0$ while $(y_\lambda^s)_\lambda$ *strongly converge to* 0 *in* $L^2(Q)$ *and in* $C([\delta, T]; L^2(\Omega))^n \cap L^2(\delta, T; H_0^1(\Omega))^n$ *for every* $\delta > 0$.

Proof of Proposition 3.1. We will denote by c any positive constant which depends only on T, Ω, $|a|_\infty$, $|b|_{L^\infty(0,T;W^{1,\infty}(\Omega))^n}$ and that may change line to line.

We first prove (i). Here $y_\lambda^s = 0$. Using (3.4), (y_λ) is bounded in $C([0,T]; H_0^1(\Omega))^n \cap H^1(0,T; L^2(\Omega))^n$ and $\operatorname{div} y_\lambda \to 0$ in $C([0,T]; L^2(\Omega))$ when $\lambda \to +\infty$.

Since $y^0 \in V$, we have (see [14]) $z \in C([0,T]; V) \cap H^1(0,T;H)$ and we can choose

$$q \in L^2(0,T; H^1(\Omega)) \quad \text{with} \quad \int_\Omega q(x,t)\,\mathrm{d}x = 0 \quad \text{for almost every } t.$$

We write $w_\lambda = y_\lambda - z$. We have

$$\begin{cases} w_\lambda' - \Delta w_\lambda + (a.\nabla)w_\lambda + (w_\lambda.\nabla)b = f_\lambda - f + \nabla(\lambda \operatorname{div} y_\lambda - q) & \text{in } Q, \\ w_\lambda = 0 \quad \text{on } \Sigma, \\ w_\lambda(0) = y_\lambda^0 - y^0 \quad \text{in } \Omega. \end{cases} \tag{3.6}$$

Multiplying equation (3.6) by w_λ and integrating by parts over $Q_t = \Omega \times (0,t)$, we obtain (using $\operatorname{div} w_\lambda = \operatorname{div} y_\lambda$)

$$\int_\Omega |w_\lambda(x,t)|^2\,\mathrm{d}x + \int_{Q_t} |\nabla w_\lambda|^2\,\mathrm{d}x\,\mathrm{d}t + \lambda \int_{Q_t} (\operatorname{div} y_\lambda)^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\left(|f_\lambda - f|_2^2 + |y_\lambda^0 - y^0|_2^2 + \frac{1}{2\lambda}|q|_2^2\right) \tag{3.7}$$

hence $w_\lambda \to 0$ in $C([0,T]; L^2(\Omega))^n \cap L^2(0,T; H^1(\Omega))^n$.

Multiplying now equation (3.6) by w_λ' and using (3.7), we have

$$\int_{Q_T} |w_\lambda'|^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega |\nabla w_\lambda(x,T)|^2\,\mathrm{d}x \leqslant c\left(|f_\lambda - f|_2^2 + |\nabla y_\lambda^0 - \nabla y^0|_2^2 + \frac{1}{\lambda}|q|_2^2\right) + \left|\int_{Q_T} \nabla q . w_\lambda'\,\mathrm{d}x\,\mathrm{d}t\right| \tag{3.8}$$

hence (recall that $q \in L^2(0,T; H^1(\Omega))$ (w_λ') is bounded in $L^2(Q_T)^n$. As $w_\lambda \to 0$ in $L^2(Q_T)^n$, we deduce that w_λ' weakly converge in $L^2(Q_T)^n$ to 0. We then have

$$\int_{Q_T} \nabla q . w_\lambda'\,\mathrm{d}x\,\mathrm{d}t \to 0$$

which proves that $w_\lambda \to 0$ in $H^1(0,T; L^2(\Omega))$ and finally (writing now (3.9) over Q_t) in $C([0,T]; H_0^1(\Omega)^n)$.

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Now (3.6) proves that $\nabla(\lambda \operatorname{div} y_\lambda^r - q) \to 0$ in $L^2(0, T; H^{-1}(\Omega))$ which implies (since $\int_\Omega (\lambda \operatorname{div} y_\lambda^r - q)\,dx = 0$ for almost every t and using Lemma 6.1, p. 150 of [14]) that $\lambda \operatorname{div} y_\lambda^r - q \to 0$ in $L^2(Q)$.

Proof of (ii): We keep the same notation as in the proof of (i). We distinguish two cases.

(1) *Case* 1: $y^0 \in V$. We still have $q \in L^2(0, T; H^1(\Omega))$ hence (3.7) is valid so $w_\lambda \to 0$ in $C([0, T]; L^2(\Omega))^n \cap L^2(0, T; H^1(\Omega))^n$.

(2) *Case* 2: $y^0 \in H$. We no more have $q \in L^2(0, T; H^1(\Omega))$ since we lose the regularity with respect to the time variable. Let then $\varepsilon > 0$. For each $\lambda > 0$, we introduce $\bar{y}^\varepsilon \in V$ and $\bar{y}_\lambda^\varepsilon \in H_0^1(\Omega)^n$ with $|\bar{y}_\lambda^\varepsilon - y_\lambda^0|_2 \leqslant \varepsilon$ and $|\bar{y}^\varepsilon - y^0|_2 \leqslant \varepsilon$. Let $Y_\lambda^\varepsilon = \Lambda_\lambda(\bar{y}_\lambda^\varepsilon, f_\lambda)$ and let $(z^\varepsilon, q^\varepsilon) \in C([0, T]; V) \times L^2(0, T; H^1(\Omega))$ be solution of (3.5) with $z^\varepsilon(0) = \bar{y}^\varepsilon$. We have

$$Y_\lambda^\varepsilon - y_\lambda = \Lambda_\lambda(\bar{y}_\lambda^\varepsilon, f_\lambda) - \Lambda_\lambda(y_\lambda^0, f_\lambda) = \Lambda_\lambda(\bar{y}_\lambda^\varepsilon - y_\lambda^0, 0).$$

Using (3.3), we get $\| Y_\lambda^\varepsilon - y_\lambda \|_X \leqslant c\varepsilon$. In the same manner, $\| z^\varepsilon - z \|_X \leqslant c\varepsilon$ hence

$$\| y_\lambda - z \|_X \leqslant 2c\varepsilon + \| Y_\lambda^\varepsilon - z^\varepsilon \|_X. \tag{3.9}$$

Now, (3.7) gives

$$\begin{aligned} \| Y_\lambda^\varepsilon - z^\varepsilon \|_X &\leqslant c\left(|f_\lambda - f|_2 + |\bar{y}_\lambda^\varepsilon - \bar{y}^\varepsilon|_2 + \frac{1}{\lambda} |q^\varepsilon|_2 \right) \\ &\leqslant c\left(|f_\lambda - f|_2 + |y_\lambda^0 - y^0|_2 + \frac{1}{\lambda} |q^\varepsilon|_2 + 2\varepsilon \right). \end{aligned}$$

We deduce that for $\lambda > 2|q^\varepsilon|_2/\varepsilon$ and such that $|f_\lambda - f|_2 + |y_\lambda^0 - y^0|_2 \leqslant \varepsilon$, we have $\| Y_\lambda^\varepsilon - z^\varepsilon \|_X \leqslant 4c\varepsilon$ and then (with (3.9)) $\| y_\lambda - z \|_X \leqslant 6c\varepsilon$ which proves that $y_\lambda \to z$ in X.

Finally, $\lambda \operatorname{div} y_\lambda \to q$ in $H^{-1}(0, T; L^2(\Omega))$ is obtained as in (i) since this time $\nabla \lambda \operatorname{div} y_\lambda \to \nabla q$ in $H^{-1}(Q)$.

Proof of (iii): We have $y_\lambda = y_\lambda^r + y_\lambda^s$. Since π maps continuously $L^2(\Omega)$ in H and $H_0^1(\Omega)$ in V, we have

$$y^0 \in H_0^1(\Omega) \Rightarrow \pi y^0 \in V$$

and

$$y_\lambda^0 \to y^0 \quad \text{in} \quad H_0^1(\Omega)^n \Rightarrow \pi y_\lambda^0 \to \pi y^0 \quad \text{in } V.$$

Steps (i) and (ii) can then be applied to $y_\lambda^r = \Lambda_\lambda(\pi y_\lambda^0, f_\lambda)$.

We still have to study $y_\lambda^s = \Lambda_\lambda(y_\lambda^0 - \pi y_\lambda^0, 0)$. Using (3.3), we can easily prove that (y_λ^s) are bounded in X. Since $t y_\lambda^s = \Lambda_\lambda(0, y_\lambda^s)$, $(t y_\lambda^s)_\lambda$ are bounded in Y (by (3.4)) hence strongly converge in $C([0, T]; L^2(\Omega))$. Let us denote by y the limit. Since $\operatorname{div} y_\lambda^s \to 0$ in $L^2(Q)$ and $y \in C([0, T]; L^2(\Omega))^n$, we have $y \in C([0, T]; H)$. On the other hand, let $u^0 \in V$ and let (u, p) be the solution of (1.10) with $v(T) = u^0$. Owing to hypothesis (H), we have $p \in L^2(Q)$ and

$$(y_\lambda^s(T), u^0) - (y_\lambda^s(0), u(0)) - (\operatorname{div} y_\lambda^s, p)_{L^2(Q)} = 0.$$

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As $\pi y_\lambda^s(0) = 0$ and $u(0) \in H$, we have $(y_\lambda^s(0), u(0)) = 0$. As $\operatorname{div} y_\lambda^s \to 0$ in $L^2(Q)$, we deduce

$$\forall u^0 \in V, \quad (y(T), u^0) = 0.$$

This proves that $\pi y(T) = 0$. As we already had $y(T) \in H$ we conclude $y(T) = 0$. This is of course valid for every $T > 0$. In order to end the proof of (iii), we still have to prove that y_λ^s strongly converge in $L^2(Q)^n$ to 0. Let $g_\lambda(t) = |y_\lambda^s(t)|_2^2$. We just proved that for $0 < t < T$, $g_\lambda(t) \to 0$ and (3.3) gives $0 \leqslant g_\lambda(t) \leqslant c|y^0 - \pi y^0|_2^2 \in L^1(0, T)$. The convergence in L^2 is then a consequence of Lebesgue's Theorem. □

3.2. *Asymptotic analysis of problems* (1.8)

We prove in this section Theorem 1.2. We recall that the control v_λ of $L^2(q)^n$ – minimum norm is given by $v_\lambda = u_\lambda \chi_q$ where

$$\begin{cases} -u_\lambda' - \Delta u_\lambda - \lambda \nabla \operatorname{div} u_\lambda - \partial_k(a_k u_\lambda) + (\nabla b_k) u_{\lambda k} = 0 & \text{in } Q, \\ u_\lambda = 0 & \text{on } \Sigma, \\ u_\lambda(T) = u_\lambda^0 & \text{in } \Omega, \end{cases} \tag{3.10}$$

where u_λ^0 is the minimizer in $L^2(\Omega)^n$ of the functional

$$J_\lambda(u^0) = \frac{1}{2}\int_q |u|^2 \, \mathrm{d}x \, \mathrm{d}t + \alpha |u^0|_2 - (y^1, u^0) + (y^0, u(0)) \tag{3.11}$$

with

$$\begin{cases} -u' - \Delta u - \lambda \nabla \operatorname{div} u - \partial_k(a_k u) + (\nabla b_k) u_k = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(T) = u^0 & \text{in } \Omega. \end{cases} \tag{3.12}$$

On the other hand, for $(z^0, z^1) \in H \times H$, the control v of $L^2(q)$ – minimum norm steering solutions of (3.5) (with $f = v\chi_q$) from z^0 into the ball in H centred at z^1 and with radius α, is given by $v = \bar{u}\chi_q$ where $\bar{u}$ is the solution of (1.10) with initial data $\bar{u}^0$ minimizing over H the functional

$$J(w^0) \overset{\text{def}}{=} \frac{1}{2}\int_q |w|^2 \, \mathrm{d}x \, \mathrm{d}t + \alpha |w^0|_2 - (z^1, w^0) + (z^0, w(0)),$$

where w is the solution of (1.10) with initial data w^0.

We prove:

Lemma 3.1. (i) *Functionals J_λ satisfy*

$$\lim_{|u^0|_2 \to +\infty} J_\lambda(u^0) = +\infty$$

uniformly in $\lambda \geqslant 1$.

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(ii) *Suppose* (H) *valid.* $\forall w^0 \in H$, *we have*

$$\lim_{\lambda \to +\infty} J_\lambda(w^0) = \frac{1}{2}\int_q |w|^2 \,\mathrm{d}x\,\mathrm{d}t + \alpha |w^0|_2 - (y^1, w^0) + (\pi y^0, w(0))$$

$$\stackrel{\text{def}}{=} J_\infty(w^0).$$

Proof of Lemma 3.1. (i): Suppose (i) fails. There then exists $A \in \mathbb{R}$ and a sequence $(w_\lambda^0)_\lambda$ with $|w_\lambda^0|_2 \to +\infty$ when $\lambda \to +\infty$ such that $J_\lambda(w_\lambda^0) \leqslant A$.

Functions $\tilde{w}_\lambda^0 = w_\lambda^0/|w_\lambda^0|_2$ are bounded in L^2 and after extraction of a subsequence, they weakly converge to a $\tilde{w}^0 \in L^2$. Let $\tilde{w}_\lambda$ be the solution of (3.12) with $\tilde{w}_\lambda(T) = \tilde{w}_\lambda^0$. It is easy to see that $\tilde{w}_\lambda$ are bounded in $L^\infty(0,T;L^2(\Omega))^n \cap L^2(0,T;H_0^1(\Omega))^n$ and that $\operatorname{div}\tilde{w}_\lambda \to 0$ in $L^2(Q)$. We write $\tilde{w}_\lambda = \tilde{w}_\lambda^r + \tilde{w}_\lambda^s$ where $\tilde{w}_\lambda^r$ and $\tilde{w}_\lambda^s$ are, respectively, the solutions of (3.12) with $\tilde{w}_\lambda^r(T) = \pi\tilde{w}_\lambda^0$ and $\tilde{w}_\lambda^s(T) = \tilde{w}_\lambda^0 - \pi\tilde{w}_\lambda^0$. Both of these two sequences are bounded in $L^\infty(0,T;L^2(\Omega))^n \cap L^2(0,T;H_0^1(\Omega))^n$ and $\lim \operatorname{div}\tilde{w}_\lambda^r = \lim \operatorname{div}\tilde{w}_\lambda^s = 0$ in $L^2(Q)$.

Let $\tilde{w}$ be now the solution of (1.10) with $\tilde{w}(T) = \pi w^0$ and let $W_\lambda = \tilde{w}_\lambda^r - \tilde{w}$. We have

$$\begin{cases} -W_\lambda' - \Delta W_\lambda - \lambda\nabla \operatorname{div}\tilde{w}_\lambda - \partial_k(a_k W_\lambda) + (\nabla b_k) W_{\lambda k} + \nabla q = 0 & \text{in } Q, \\ W_\lambda = 0, \quad \text{on } \Sigma, \\ W_\lambda(T) = \pi\tilde{w}_\lambda^0 - \pi\tilde{w}^0 & \text{in } \Omega. \end{cases} \tag{3.13}$$

For all $f \in L^2(Q)$, introducing $(z,p) \in Y \times L^2(0,T;H^1(\Omega))$ solution of (3.5) with $z(0) = 0$, we have

$$-(W_\lambda(T), z(T)) + (W_\lambda, f)_{L^2(Q)} - (\operatorname{div}\tilde{w}_\lambda^r, p)_{L^2(Q)} = 0,$$

hence $(W_\lambda, f)_{L^2(Q)} \to 0$. This proves that (after extraction of subsequence) $\tilde{w}_\lambda^r$ weakly converge in $L^2(0,T;H_0^1(\Omega))^n$ and weakly—* in $L^\infty(0,T;L^2(\Omega))$ to $\tilde{w}$.

In order to prove that $\tilde{w}_\lambda^s$ tend to 0 in the same sense, one has to write that

$$-(\tilde{w}_\lambda^s(T), z(T)) + (\tilde{w}_\lambda^s, f) = (\operatorname{div}\tilde{w}_\lambda^s, p)_{L^2(Q)}$$

and remark that $(\tilde{w}^0 - \pi\tilde{w}^0, z(T)) = 0$ since $z(T) \in H$.

On the other hand,

$$\frac{1}{2}|w_\lambda^0|_2 \int_q |\tilde{w}_\lambda|^2 \,\mathrm{d}x\,\mathrm{d}t + \alpha - (y^1, \tilde{w}_\lambda^0) + (y^0, \tilde{w}_\lambda(0)) \leqslant \frac{A}{|w_\lambda^0|_2},$$

hence $\tilde{w}_\lambda \to 0$ in $L^2(q)^n$ and, since $\tilde{w}_\lambda$ weakly converge in $L^2(q)^n$ to $\tilde{w}$, we deduce that $\tilde{w} = 0$ in q. In both cases where (H) is fulfilled or ω is a neighbourhood of the boundary of Ω, we can apply Theorem 1.4 of [2] to deduce that $\tilde{w} = 0$ thus $\pi\tilde{w}^0 = 0$.

As $y^1 \in H$, one has $(y^1, \tilde{w}_\lambda^0) \to (y^1, \tilde{w}^0) = (\pi y^1, \tilde{w}^0) = (y^1, \tilde{w}^0) = 0$.

Let $\bar{y}_\lambda = \Lambda_\lambda(y^0, 0)$. Using Proposition 3.1(iii), we can write

$$(\bar{y}_\lambda(T), \tilde{w}_\lambda^0) = (y^0, \tilde{w}_\lambda(0)) \to (\bar{y}(T), \tilde{w}^0)$$
$$= (\bar{y}(T), \pi\tilde{w}^0) = (\bar{y}(0), \tilde{w}(0)) = (\pi y^0, \tilde{w}(0)) = 0,$$

where $\bar{y}$ is the solution of (3.5) with null right side and initial data πy^0.

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We finally deduce that

$$\liminf_{\lambda \to +\infty} \frac{J_\lambda(\tilde{w}_\lambda^0)}{|\tilde{w}_\lambda^0|_2} \geqslant \alpha$$

which ends lemma's (i) proof.

Let us prove assertion (ii): Suppose first $w^0 \in V$ and let w_λ (resp. $(w, q) \in X \times L^2(Q)$) be the solution of (3.12) (resp. (1.10)) with $w_\lambda(T) = w(T) = w^0$. Since (H) is fulfilled, $q \in L^2(Q)$, and it is then easy to prove (as in Proposition 3.1) that w_λ converges in X to w hence

$$\lim_{\lambda \to \infty} J_\lambda(w^0) = \frac{1}{2} \int_q |w|^2 \, dx \, dt + \alpha |w^0|_2 - (y^1, w^0) + (\pi y^0, w(0))$$

$$\stackrel{\text{def}}{=} J_\infty(w^0). \tag{3.14}$$

Let now $w^0 \in H$ and $\varepsilon > 0$. Multiplying (3.12) by w_λ and integrating over Q_T, we obtain

$$\exists c > 0, \ \forall \lambda > 0, \quad \forall w^0 \in H, \quad \| w_\lambda \|_X \leqslant c |w^0|_2$$

with $c = c(|a|_{L^\infty(Q)} + |b|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + 1)$.

It easily follows that the set $\{J_\infty, J_\lambda, \lambda \geqslant 1\}$ is equicontinuous on H. Let $\beta > 0$ such that for every $\lambda \geqslant 1$, $|w^0 - w^1|_H \leqslant \beta$ implies $|J_\lambda(w^0) - J_\lambda(w^1)| \leqslant \varepsilon$ and $|J_\infty(w^0) - J_\infty(w^1)| \leqslant \varepsilon$.

We choose $w^1 \in V$ with $|w^0 - w^1|_H \leqslant \beta$ and we can then write

$$\begin{aligned} |J_\lambda(w^0) - J_\infty(w^0)| &\leqslant |J_\lambda(w^0) - J_\lambda(w^1)| + |J_\lambda(w^1) - J_\infty(w^1)| \\ &\quad + |J_\infty(w^1) - J_\infty(w^0)| \\ &\leqslant 2\varepsilon + |J_\lambda(w^1) - J_\infty(w^1)| \\ &\leqslant 3\varepsilon \end{aligned}$$

for λ large enough (since $w^1 \in V$). This proves (ii). □

Proof of Theorem 1.2. Lemma 3.1(i) and $J_\lambda(u_\lambda^0) \leqslant 0$ prove that (u_λ^0) are bounded in $L^2(\Omega)^n$. After extraction of a subsequence, they weakly converge in $L^2(\Omega)^n$ to $u^0 \in L^2(\Omega)^n$. With exactly the same argument as above, solutions u_λ weakly converge (after extraction of a sequence) in $L^2(0, T; H_0^1(\Omega))^n$ and weakly—* in $L^\infty(0, T; L^2(\Omega))^n$ to the solution u of (1.10) with $u(T) = \pi u^0$.

Now, for every $w^0 \in H$ we can write (using Lemma 3.1(ii) and $J_\lambda(w^0) \geqslant J_\lambda(u_\lambda^0)$)

$$J_\infty(w^0) = \lim_{\lambda \to \infty} J_\lambda(w^0) \geqslant \limsup_{\lambda \to \infty} J_\lambda(u_\lambda^0) \geqslant \liminf_{\lambda \to \infty} J_\lambda(u_\lambda^0).$$

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On the other hand, we have (recall that $\bar{y}_\lambda = \Lambda_\lambda(y^0, 0)$)

$$\liminf_{\lambda\to\infty} \int_q |u_\lambda|^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant \int_q |u|^2 \,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad \liminf_{\lambda\to\infty} |u_\lambda^0|_2 \geqslant |u^0|_2 \geqslant |\pi u^0|_2,$$

$$\lim_{\lambda\to\infty} (y^1, u_\lambda^0) = (y^1, u^0) = (y^1, \pi u^0), \tag{3.15}$$

$$\lim_{\lambda\to\infty} (y^0, u_\lambda^0(0)) = \lim_{\lambda\to\infty} (\bar{y}_\lambda(T), u_\lambda^0) = (\pi y^0, u(0)) = (\pi y^0, \pi u(0)),$$

hence,

$$\liminf_{\lambda\to\infty} J_\lambda(u_\lambda^0) \geqslant J_\infty(\pi u^0)$$

and πu^0 is the minimizer of J_∞ over H.

Now, take $w^0 = \pi u^0$, to obtain

$$\limsup J_\lambda(u_\lambda^0) = \liminf J_\lambda(u_\lambda^0) = J_\infty(\pi u^0).$$

Using (3.15), $\liminf J_\lambda(u_\lambda^0) = J_\infty(\pi u^0)$ implies that u_λ strongly converges in $L^2(q)$ to u (and u_λ^0 strongly converges in $L^2(\Omega)^n$ to πu^0). Furthermore, u is the solution of (1.10) with as initial data the minimizer of J_∞ over H and this proves that u is the control of $L^2(q)$—minimum norm which steers πy^0 in the ball centered at y^1 and of radius α. Using Proposition 3.1, this ends the proof of Theorem 1.2(i).

When (H) is not fulfilled but ω is a neighbourhood of the whole boundary of Ω, we can just prove the weak convergence of controls v_λ to a function v in $L^2(q)$: this is given by assertion (i) of Lemma 3.1. Assertion (ii) can then be proved with a duality process. □

Remark. If one could prove assertion (ii) of Lemma 3.1, we would have the strong convergence of the controls. As it was seen in Proposition 3.1, our proof of the (strong) convergence of functions w_λ to w (the notations are the same as in Proposition 3.1) needs that one should ensure the existence of a pressure in $L^2(Q)$ for the limit problem satisfied by w.

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