A Quasistatic Viscoplastic Contact Problem with Normal Compliance and Friction

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Abstract. We consider a quasistatic contact problem between a viscoplastic body and an obstacle, the so-called foundation. The contact is modeled with normal compliance and the associated version of Coulomb's law of dry friction. We derive a variational formulation of the problem and, under a smallness assumption on the normal compliance functions, we establish the existence of a weak solution to the model. The proof is carried out in several steps. It is based on a time-discretization method, arguments of monotonicity and compactness, Banach fixed point theorem and Schauder fixed point theorem.

Keywords and Phrases: viscoplastic material, quasistatic process, frictional contact, normal compliance, Coulomb's law of dry friction, variational inequality, fixed point, weak solution.

1 Introduction

We investigated recently a number of problems related to quasistatic contact for viscoplastic materials. In particular, models for frictionless contact were considered in [18], in the case of Signorini contact conditions, and in [8] in the case of normal compliance contact conditions. The analysis of the bilateral contact with Tresca friction law was provided in [2] and its numerical approximation was provided in [5]. An extension of the existence and uniqueness result obtained in [2] in the study of frictional contact conditions modeled with a dissipative potential was establish in [19]. Frictional problems with regularized Coulomb's law were study recently in [1] both in the case of bilateral and Signorini unilateral contact. The results obtained in [1, 2, 8, 18, 19] deal with the variational analysis of the mechanical problems. They involve existence and uniqueness of weak solutions, i.e. solutions which satisfy variational formulations of the corresponding mechanical problems. They were obtained using arguments of evolutionary variational inequalities and fixed point. The results in [5, 8] concern the numerical analysis of the models, including error estimates for the approximate solutions.

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In all the papers above, the materials's behavior was modeled with a rate-type viscoplastic constitutive relation of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + G(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field while $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denote the stress and linearized strain tensor, respectively. Here $\mathcal{E}$ is the forth order tensor of elastic coefficients, G is a nonlinear constitutive function and the dot above a variable represents its time derivative.

Rate-type viscoplastic constitutive laws of the form (1.1) were considered in the literature in order to model the behavior of real materials like rubbers, metals, pastes, rocks and so on. Various results and mechanical interpretation concerning models of this form may be found for instance in [6] and the references therein. A survey of results in the study problems involving (1.1) can be found in [11] in the case of displacement-traction boundary conditions, and in [10] in the case of frictionless or frictional contact conditions.

The aim of this paper is to present a new result in the study of quasistatic frictional contact problems with viscoplastic materials of the form (1.1). The novelty consists in the fact that here we model the contact with a general normal compliance contact condition and the associated version of Coulomb's law. The normal compliance contact condition was first considered in [14] in the study of dynamic problems with linearly elastic and viscoelastic materials. This condition allows the interpenetration of the body's surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities. On occasions, the normal compliance condition has been employed as a mathematical regularization of Signorini's nonpenetration condition and used as such in numerical solution algorithms. Contact problems with normal compliance have been discussed in numerous papers, e.g. [3, 4, 12, 13, 17] and the references therein. In particular, the first existence result in the study of quasistatic contact problems with normal compliance and friction was obtained in [3] in the case of linearly elastic materials and in [17] in the case of nonlinear Kelvin-Voigt viscoelastic materials. In this paper we extend these results to the case of rate-type viscoplastic materials of the form (1.1). To this end we use arguments based on time discretization, monotonicity, compactness, Banach fixed point theorem and Schauder fixed point theorem.

The paper is organized as follows. In Section 2 we introduce some notation and preliminaries. In Section 3 we describe the model for the process, set it in a variational formulation, list the assumption on the data and state our main result, Theorem 3.1. It states the existence of a weak solution to the model, if an appropriate smallness assumption involving the normal compliance contact functions is satisfied. The proof of Theorem 3.1 is provided in Section 6. It is based on the study of two intermediate problems which are presented in Sections 4 and 5.

2 Notation and preliminaries

In this short section we present the notation we shall use and some preliminary material. For further details, we refer the reader to [7, 11, 16].

We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while " $\cdot$ " and $|\cdot|$ will represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent variable. We also use the following notation:

$$H = \{\, \boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\,\}, \qquad Q = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\,\},$$

$$H_1 = \{\, \boldsymbol{u} = (u_i) \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\,\}, \qquad Q_1 = \{\, \boldsymbol{\sigma} \in Q \mid \text{Div}\, \boldsymbol{\sigma} \in H\,\}.$$

Here $\boldsymbol{\varepsilon} : H_1 \longrightarrow Q$ and Div $: Q_1 \longrightarrow H$ are the deformation and the divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces H, Q, H_1 and Q_1 are real Hilbert spaces endowed with the cannonical inner products given by :

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij} \tau_{ij} dx,$$

$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\text{Div}\, \boldsymbol{\sigma}, \text{Div}\, \boldsymbol{\tau})_H.$$

The associated norms on the spaces H, Q, H_1 and Q_1 are denoted by $\|\cdot\|_H$, $\|\cdot\|_Q$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{Q_1}$, respectively.

For every element $\boldsymbol{v} \in H_1$ we also use the notation $\boldsymbol{v}$ to denote the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = v \cdot \nu, \qquad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}.$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in Q_1$, and we note that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu},$$

and the following Green's formula holds :

$$(2.1) \qquad (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\text{Div}\, \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu \cdot \boldsymbol{v}\, da \qquad \forall \boldsymbol{v} \in H_1.$$

Now, let Γ_1 be a measurable part of Γ such that $meas\,\Gamma_1 > 0$ and let V be the closed subspace of H_1 given by

$$V = \{\ \boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1\ \}.$$

Since $meas\,\Gamma_1 > 0$, the following Korn's inequality holds:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \geq c_K\,\|\boldsymbol{v}\|_{H_1} \quad \forall\,\boldsymbol{v} \in V, \tag{2.2}$$

where $c_K > 0$ is a constant depending only on Ω and Γ_1. A proof of Korn's inequality can be found in, for instance, [15] p. 79. Over the space V we consider the inner product given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \tag{2.3}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (2.2) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V. Therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a positive constant c_B depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_B \|\boldsymbol{v}\|_V \qquad \forall v \in V. \tag{2.4}$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0,T]; X)$ for the space of continuous functions from $[0,T]$ to X; $C([0,T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X.$$

For $p \in [1, \infty]$, we use the standard notation for $L^p(0,T;X)$ spaces. We also use the Sobolev space $W^{1,\infty}(0,T;X)$ with the norm

$$\|x\|_{W^{1,\infty}(0,T;X)} = \|x\|_{L^\infty(0,T;X)} + \|\dot{x}\|_{L^\infty(0,T;X)},$$

where a dot now represents the weak derivative with respect to the time variable. A sequence of elements of the space X will be denoted $\{x_n\}$. Finally, everywhere in this paper $X_1 \times X_2$ will represent the product of the real Hilbert spaces X_1 and X_2, whose elements will be denoted (x_1, x_2).

3 Problem statement and variational formulation

In this section we describe the model for the process, present its variational formulation and state our main result, Theorem 3.1.

The physical setting is as follows. A viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a regular boundary Γ that is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\,\Gamma_1 > 0$. Let $T > 0$ and let $[0,T]$ denote the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$ and thus the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0,T)$ and a surface traction of density $\boldsymbol{f}_2$ acts on $\Gamma_2 \times (0,T)$. We assume that the acceleration in the system is negligible and we use (1.1) as constitutive relation. The boundary conditions on the potential contact surface Γ_3 involve normal compliance and friction and will be discussed below.

Under these conditions, the classical formulation of the mechanical problem of frictional contact of the elastic body is the following.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + G(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0,T), \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{3.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.4}$$

$$-\sigma_\nu = p_\nu(u_\nu - g_a), \tag{3.5}$$

$$\left.\begin{array}{l} |\boldsymbol{\sigma}_\tau| \le p_\tau(u_\nu - g_a) \\ |\boldsymbol{\sigma}_\tau| < p_\tau(u_\nu - g_a) \Rightarrow \dot{\boldsymbol{u}}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = p_\tau(u_\nu - g_a) \Rightarrow \boldsymbol{\sigma}_\tau = -\lambda\, \dot{\boldsymbol{u}}_\tau,\ \lambda \ge 0 \end{array}\right\} \quad \text{on } \Gamma_3 \times (0,T), \tag{3.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in } \Omega. \tag{3.7}$$

Here (3.1) is the viscoplastic constitutive law, (3.2) represents the equilibrium equation, (3.3) and (3.4) are the displacement-traction boundary conditions and, finally, the functions $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ in (3.7) denote the initial displacement and the initial stress, respectively.

We make some comments on the contact conditions (3.5) and (3.6) in which σ_ν denotes the normal stress, $\boldsymbol{\sigma}_\tau$ represents the tangential traction, u_ν is the normal displacement and $\dot{\boldsymbol{u}}_\tau$ represents the tangential velocity. The equality (3.5) represents the *normal compliance* contact condition in which p_ν is a prescribed nonnegative function and g_a denotes the gap between the potential contact surface Γ_3 and the foundation, measured along the direction of the outward normal $\boldsymbol{\nu}$. When positive, $u_\nu - g_a$ represents the penetration of the surface asperities into those of the foundation. In this condition the interpenetration is allowed but penalized. An example of a normal compliance function p_ν is

$$p_\nu(r) = c_\nu r_+, \tag{3.8}$$

where c_ν is a positive constant and $r_+ = \max\{0, r\}$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$.

The relations (3.6) represent a version of Coulomb's law of dry friction in which p_τ is a prescribed nonnegative function, the so-called *friction bound*. According to (3.6) the tangential shear cannot exceed the maximal frictional resistance $p_\tau(u_\nu - g_a)$. Then, if the strict inequality holds, the surface adheres to the foundation and is in the so-called *stick* state, and when equality holds there is relative sliding, the so-called *slip* state. Therefore, at each time instant the potential contact surface Γ_3 is divided into three zones: the stick zone, the slip zone and the zone of separation, in which $u_\nu < g_a$ and there is no contact. The boundaries of these zones are unknown a priori and form free boundaries. The choice

$$p_\tau = \mu p_\nu, \tag{3.9}$$

leads to the usual Coulomb's law, and $\mu \geq 0$ is the coefficient of friction (see, e.g., [7] or [16]). More recently, a modified version of the Coulomb friction law was derived in [20, 21] from thermodynamic considerations. It consists of using the friction law (3.6) with

$$p_\tau = \mu p_\nu (1 - \delta p_\nu)_+, \tag{3.10}$$

where δ is a small positive material constant related to the wear and hardness of the surface. Contact and frictional boundary conditions of the form (3.5), (3.6) were considered in [17] in the study of quasistatic process for Kelvin-Voigt viscoelastic materials.

In the study of the mechanical problem (3.1)–(3.7) we assume that $\mathcal{E} : \Omega \times S_d \to S_d$ is a bounded symmetric positive definite fourth order tensor, i.e.

$$\begin{cases} (a) \;\; \mathcal{E}_{ijkl} \in L^\infty(\Omega), \;\; 1 \leq i,j,k,l \leq d. \\ (b) \;\; \mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{E}\boldsymbol{\tau}, \;\; \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_d, \;\; \text{a.e. in } \Omega. \\ (c) \;\; \text{There exists } m > 0 \text{ such that} \\ \qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m|\boldsymbol{\tau}|^2 \;\; \forall \boldsymbol{\tau} \in S_d, \;\; \text{a.e. in } \Omega. \end{cases} \tag{3.11}$$

$G : \Omega \times S_d \times S_d \to S_d$ has the properties:

$$\begin{cases} (a) \;\; \text{There exists an } L_G > 0 \text{ such that } \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \;\; \text{a.e. in } \Omega, \\ \qquad |G(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - G(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)| \leq L_G \left(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \right). \\ (b) \;\; \text{For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S_d, \; \boldsymbol{x} \mapsto G(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable.} \\ (c) \;\; \text{The mapping } \boldsymbol{x} \mapsto G(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}) \in Q. \end{cases} \tag{3.12}$$

The functions $p_r : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+ \;\; (r = \nu, \tau)$ satisfy:

$$\begin{cases} (a) \;\; \text{There exists } \; L_r > 0 \; \text{ such that} \\ \qquad |p_r(\boldsymbol{x}, u_1) - p_r(\boldsymbol{x}, u_2)| \leq L_r |u_1 - u_2| \\ \qquad \forall\, u_1, \, u_2 \in \mathbb{R}, \;\; \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ (b) \;\; \boldsymbol{x} \longmapsto p_r(\boldsymbol{x}, u) \; \text{ is Lebesgue measurable on } \Gamma_3 \\ \qquad \forall\, u \in \mathbb{R}. \\ (c) \;\; \boldsymbol{x} \longmapsto p_r(\boldsymbol{x}, u) = 0 \;\; \text{for} \;\; u \leq 0, \;\; \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{3.13}$$

The assumptions (3.13) on p_ν and p_τ are fairly general. The main restriction is the requirement that asymptotically the functions grow at most linearly. Clearly, the function defined in (3.8) satisfies this condition. We also observe that if the functions p_ν and p_τ are related by (3.9) or (3.10) and p_ν satisfies condition (3.13)(a), then p_τ does too with $L_\tau = \mu\, L_\nu$.

The forces and tractions are assumed to satisfy

$$\boldsymbol{f}_0 \in W^{1,\infty}(0,T;H), \qquad \boldsymbol{f}_2 \in W^{1,\infty}(0,T;L^2(\Gamma_2)^d), \tag{3.14}$$

and the gap function satisfies

$$g_a \in L^2(\Gamma_3), \qquad g_a \geq 0 \quad \text{a.e.} \quad \text{on} \quad \Gamma_3. \tag{3.15}$$

Next we define the functional $\varphi : V \times V \to \mathbb{R}$ by

$$\varphi(\boldsymbol{v},\boldsymbol{w}) = \int_{\Gamma_3} p_\nu(v_\nu - g_a)\, w_\nu\, da + \int_{\Gamma_3} p_\tau(v_\nu - g_a)\, |\boldsymbol{w}_\tau|\, da. \tag{3.16}$$

Using the conditions (3.13) and (3.15) it follows that for all $\boldsymbol{v} \in V$ the functions $\boldsymbol{x} \mapsto p_r(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x}) - g(\boldsymbol{x}))$ $(r = \nu, \tau)$ belong to $L^2(\Gamma_3)$ and therefore the integrals in (3.16) are well defined.

Let $\boldsymbol{f} : [0,T] \to V$ given by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \tag{3.17}$$

for all $\boldsymbol{v} \in V$, $t \in [0,T]$. We note that conditions (3.14) imply

$$\boldsymbol{f} \in W^{1,\infty}(0,T;V). \tag{3.18}$$

Finally we assume that the initial data satisfy

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q_1, \tag{3.19}$$

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \varphi(\boldsymbol{u}_0, \boldsymbol{v}) \geq (\boldsymbol{f}(0), \boldsymbol{v})_V \qquad \forall \boldsymbol{v} \in V. \tag{3.20}$$

It is straightforward to see that if $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ are sufficiently smooth functions satisfying (3.2)–(3.6), then $\boldsymbol{u}(t) \in V$, $\boldsymbol{\sigma}(t) \in Q_1$ and, using Green's formula (2.1), it follows that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + \varphi(\boldsymbol{u}(t), \boldsymbol{v}) - \varphi(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V,$$

for all $\boldsymbol{v} \in V$ and $t \in [0,T]$. Therefore, using (3.1), (3.7) and the previous inequality yields to the following variational formulation of the problem P.

Problem P_V. *Find a displacement field $\boldsymbol{u} : [0,T] \to V$ and a stress field $\boldsymbol{\sigma} : [0,T] \to Q_1$ such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + G(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{a.e. } t \in (0,T), \tag{3.21}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + \varphi(\boldsymbol{u}(t), \boldsymbol{v}) - \varphi(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T), \tag{3.22}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \tag{3.23}$$

The main result of this paper is the following

Theorem 3.1. *Assume that conditions* (3.11)–(3.15), (3.19) *and* (3.20) *hold. Then there exists* $L_0 > 0$ *depending only on* Ω, Γ_1, Γ_2, Γ_3, $\mathcal{E}$, G *and* T *such that if* $L_\nu + L_\tau < L_0$, *then problem* P_V *has at least a solution. Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{1,\infty}(0,T;V), \quad \boldsymbol{\sigma} \in W^{1,\infty}(0,T;Q_1). \tag{3.24}$$

The proof of Theorem 3.1 will be provided in Section 6. It is carried out in several steps and it is based on the study of two intermediate problems that we present in Section 4 and 5, respectively. Everywhere in the rest of the paper c will denote a positive generic constant which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{E}$, G and T, and whose value may change from place to place.

We end this section with some comments on Theorem 3.1. First, we conclude by this theorem that problem (3.1)–(3.7) has at least a *weak solution* provided $L_\nu + L_\tau$ is sufficiently small. Next, the critical value L_0 depends on the constitutive functions, on the geometry of the problem and on the duration of the process, but does not depend on the external forces, nor on the initial data. The verification of the condition $L_\nu + L_\tau < L_0$ which guarantees the solvability of problem P_V as well as its physical interpretation depends on the specific mechanical problem. For example, consider the mechanical problem P in which the function p_ν is given by (3.8) and the function p_τ is given by (3.9) or by (3.10). It follows that assumption (3.13)(a) is satisfied with $L_\nu = c_\nu$ and $L_\tau = \mu c_\nu$ and therefore the condition $L_\nu + L_\tau < L_0$ holds if $c_\nu(1+\mu) < L_0$, which may be interpreted as a smallness assumption involving the coefficients c_ν and μ. Notice also that the important question of uniqueness of the solution to problem P_V is left open. This is do even for the local elastic problem with normal compliance treated in [3], when the coefficient of friction and the loads are assumed to be sufficiently small, as well as for the global elastic problem with normal compliance and friction studied in [4]. We finally remark that in the case of viscoelastic materials the unique solvability of quasistatic problems with normal compliance and friction may be proved without any smallness assumption on the data, see for example [17].

4 Intermediate elastic problem

In this section we solve the contact problem in the particular case when the viscoplastic part of the stress tensor, the normal stress and the friction bound are known. To this end, everywhere in this section we consider two functions $\boldsymbol{\eta}$ and $\boldsymbol{g}$ such that

$$\boldsymbol{\eta} \in L^\infty(0,T;Q), \tag{4.1}$$

$$\boldsymbol{g} = (g_1, g_2) \in W^{1,\infty}(0,T;L^2(\Gamma_3)^2), \tag{4.2}$$

$$\boldsymbol{g}(0) = \boldsymbol{g}_0, \tag{4.3}$$

where $\boldsymbol{g}_0$ is the element of $L^2(\Gamma_3)^2$ given by

$$\boldsymbol{g}_0 = (p_\nu(u_{0\nu} - g_a), p_\tau(u_{0\nu} - g_a)). \tag{4.4}$$

We denote by $\boldsymbol{z}_\eta \in W^{1,\infty}(0,T;Q)$ and $j(\boldsymbol{g}(t),\cdot) : V \to \mathbb{R}$ the functions defined by

$$\boldsymbol{z}_\eta(t) = \int_0^t \boldsymbol{\eta}(s)ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall t \in [0,T], \tag{4.5}$$

$$j(\boldsymbol{g}(t),\boldsymbol{v}) = \int_{\Gamma_3} |g_1(t)|\, v_\nu\, da + \int_{\Gamma_3} |g_2(t)|\, |\boldsymbol{v}_\tau|\, da \quad \forall \boldsymbol{v} \in V, \tag{4.6}$$

and we consider the following intermediate variational problem:

Problem $P_V^{\eta g}$. *Find a displacement field $\boldsymbol{u}_{\eta g} : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_{\eta g} : [0,T] \to Q_1$ such that*

$$\boldsymbol{\sigma}_{\eta g}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta g}(t)) + \boldsymbol{z}_\eta(t) \quad \forall t \in [0,T], \tag{4.7}$$

$$\begin{aligned}(\boldsymbol{\sigma}_{\eta g}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\eta g}(t)))_Q + j(\boldsymbol{g}(t),\boldsymbol{v}) - j(\boldsymbol{g}(t),\dot{\boldsymbol{u}}_{\eta g}(t))\\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{\eta g}(t))_V \quad \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T),\end{aligned} \tag{4.8}$$

$$\boldsymbol{u}_{\eta g}(0) = \boldsymbol{u}_0. \tag{4.9}$$

Notice that equality (4.7) represent an elastic-like constitutive law. For this reason we refer in the sequel to problem $P_V^{\eta g}$ as an intermediate elastic problem. In the study of this problem we have the following result.

Theorem 4.1. *Assume that conditions* (3.11), (3.18)–(3.20), (4.1)–(4.3) *hold. Then problem $P_V^{\eta g}$ has a unique solution with regularity*

$$\boldsymbol{u}_{\eta g} \in W^{1,\infty}(0,T;V), \quad \boldsymbol{\sigma}_{\eta g} \in W^{1,\infty}(0,T;Q_1). \tag{4.10}$$

Moreover, there exists a constant $c > 0$ such that

$$\begin{aligned}\|\boldsymbol{u}_{\eta g}\|_{W^{1,\infty}(0,T;V)} + \|\boldsymbol{\sigma}_{\eta g}\|_{W^{1,\infty}(0,T;Q_1)} \leq c\,(\|\boldsymbol{g}\|_{W^{1,\infty}(0,T;L^2(\Gamma_3)^2)} +\\ \|\boldsymbol{f}\|_{W^{1,\infty}(0,T;V)} + \|\boldsymbol{z}_\eta\|_{W^{1,\infty}(0,T;Q)} + \|\boldsymbol{u}_0\|_V).\end{aligned} \tag{4.11}$$

Proof. The proof of Theorem 4.1 is carried out in several steps, by using arguments similar to those used in [1, 2] and [9]. Since the modifications are straightforward we omit the details. Everywhere below we use the bilinear form $a : V \times V \to \mathbb{R}$ given by

$$a(\boldsymbol{u},\boldsymbol{v}) = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q. \tag{4.12}$$

The steps of the proof are the following.

i) Incremental time-discretized problems.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the time interval $[0,T]$ such that $t_n = nh$ for $n = 1,\ldots,N$ where $h = T/N$. For a continuous function $t \mapsto \boldsymbol{w}(t)$ we use the notation $\boldsymbol{w}_n = \boldsymbol{w}(t_n)$. For a sequence $\{\boldsymbol{w}_n\}_{n=0}^N$, we denote $\Delta\boldsymbol{w}_n = \boldsymbol{w}_n - \boldsymbol{w}_{n-1}$ for the difference, and $\delta\boldsymbol{w}_n = \Delta\boldsymbol{w}_n/h$ the corresponding divided difference. No summation is implied over the repeated index n.

Using standard arguments of elliptic variational inequalities we prove that there exists a unique sequence $\{\boldsymbol{u}_n^{\eta g}\}_{n=0}^N \subset V$ such that $\boldsymbol{u}_0^{\eta g} = \boldsymbol{u}_0$ and, for $n = 1,\ldots,N$,

$$\begin{aligned} &a(\boldsymbol{u}_n^{\eta g}, \boldsymbol{v} - \delta\boldsymbol{u}_n^{\eta g}) + (\boldsymbol{z}_{\eta n}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{\eta g}))_Q + \\ &j(\boldsymbol{g}_n, \boldsymbol{v}) - j(\boldsymbol{g}_n, \delta\boldsymbol{u}_n^{\eta g}) \geq (\boldsymbol{f}_n, \boldsymbol{v} - \delta\boldsymbol{u}_n^{\eta g})_V \quad \forall\, \boldsymbol{v} \in V. \end{aligned} \tag{4.13}$$

Moreover, the following estimates hold:

$$\|\boldsymbol{u}_n^{\eta g}\|_V \leq c\,(\|\boldsymbol{g}_n\|_{L^2(\Gamma_3)^2} + \|\boldsymbol{f}_n\|_V + \|\boldsymbol{z}_{\eta n}\|_Q), \quad 0 \leq n \leq N, \tag{4.14}$$

$$\|\Delta\boldsymbol{u}_n^{\eta g}\|_V \leq c\,(\|\Delta\boldsymbol{g}_n\|_{L^2(\Gamma_3)^2} + \|\Delta\boldsymbol{f}_n\|_V + \|\Delta\boldsymbol{z}_{\eta n}\|_Q), \quad 1 \leq n \leq N. \tag{4.15}$$

Notice that, in order to prove the estimate (4.14) for $n = 0$, we use the compatibility assumption

$$a(\boldsymbol{u}_0, \boldsymbol{v}) + (\boldsymbol{z}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{g}(0), \boldsymbol{v}) \geq (\boldsymbol{f}(0), \boldsymbol{v})_V,$$

which follows from (3.20),(4.2)–(4.4) and the definition of functionals φ and j.

ii) A weak $*$ convergence result.

We then define a piecewise linear interpolant $\boldsymbol{u}_N^{\eta g}$ from the sequence $\{\boldsymbol{u}_n^{\eta g}\}_{n=0}^N \subset V$ by the formula

$$\boldsymbol{u}_N^{\eta g}(t) = \boldsymbol{u}_{n-1}^{\eta g} + \frac{t - t_{n-1}}{h}\,\Delta\boldsymbol{u}_n^{\eta g}, \quad t \in [t_{n-1}, t_n], \quad 1 \leq n \leq N. \tag{4.16}$$

Obviously, the functions $\boldsymbol{u}_N^{\eta g}$ belong to the space $W^{1,\infty}(0,T;V)$ and assertions (4.14) and (4.15) prove that the sequence $\{\boldsymbol{u}_N^{\eta g}\}_N$ is a bounded sequence in this space. Then, there exists a function $\boldsymbol{u}_{\eta g} \in W^{1,\infty}(0,T;V)$ and a subsequence of $\{\boldsymbol{u}_N^{\eta g}\}_N$, still denoted $\{\boldsymbol{u}_N^{\eta g}\}_N$, such that

$$\boldsymbol{u}_N^{\eta g} \to \boldsymbol{u}_{\eta g} \quad \text{weak } * \text{ in} \quad W^{1,\infty}(0,T;V). \tag{4.17}$$

Notice that we will prove in part v) of this proof that the whole sequence $\{\boldsymbol{u}_N^{\eta g}\}_N$ converges to $\boldsymbol{u}_{\eta g}$.

Moreover, there exists $c > 0$ such that

$$\begin{aligned} \|\boldsymbol{u}_{\eta g}\|_{W^{1,\infty}(0,T;V)} &\leq c\,(\|\boldsymbol{g}\|_{W^{1,\infty}(0,T;L^2(\Gamma_3)^2)} + \\ &\|\boldsymbol{f}\|_{W^{1,\infty}(0,T;V)} + \|\boldsymbol{z}_\eta\|_{W^{1,\infty}(0,T;Q)} + \|\boldsymbol{u}_0\|_V). \end{aligned} \tag{4.18}$$

The proof of (4.18) is based again on inequalities (4.14) and (4.15), which provide estimates of the functions $\boldsymbol{u}_N^{\eta g}$ in the norm of the space $W^{1,\infty}(0,T;V)$.

iii) Convergence and semicontinuity results.

Let $\boldsymbol{u}_{\eta g}$ denote an element of $W^{1,\infty}(0,T;V)$ provided in step *ii)* as the weak $*$ limit of a subsequence of the sequence $\{\boldsymbol{u}_N^{\eta g}\}_N$, that we still denote $\{\boldsymbol{u}_N^{\eta g}\}_N$.

We introduce the piecewise constant functions $\tilde{\boldsymbol{u}}_N^{\eta g} : [0,T] \to V$, $\tilde{\boldsymbol{z}}_N^{\eta} : [0,T] \to Q$, $\tilde{\boldsymbol{g}}_N : [0,T] \to L^2(\Gamma_3)^2$, and $\tilde{\boldsymbol{f}}_N : [0,T] \to V$ by the formulas

$$\tilde{\boldsymbol{u}}_N^{\eta g}(t) = \boldsymbol{u}_{\eta g n}, \quad \tilde{\boldsymbol{z}}_N^{\eta}(t) = \boldsymbol{z}_{\eta n}, \quad \tilde{\boldsymbol{g}}_N(t) = \boldsymbol{g}_n, \quad \tilde{\boldsymbol{f}}_N(t) = \boldsymbol{f}_n, \quad \forall t \in (t_{n-1}, t_n], \quad n = 1, \ldots, N. \tag{4.19}$$

As for almost every $t \in (0,T)$, we have

$$\|\tilde{\boldsymbol{u}}_N^{\eta g}(t) - \boldsymbol{u}_N^{\eta g}(t)\|_V \leq \frac{T}{N} \|\dot{\boldsymbol{u}}_N^{\eta g}(t)\|_V, \tag{4.20}$$

and, since $\{\dot{\boldsymbol{u}}_N^{\eta g}\}_N$ is bounded in $L^\infty(0,T;V)$, we deduce that

$$\tilde{\boldsymbol{u}}_N^{\eta g} \to \boldsymbol{u}_{\eta g} \quad \text{weak } * \text{ in} \quad L^\infty(0,T;V). \tag{4.21}$$

Furthermore, since $\boldsymbol{z}_\eta \in W^{1,\infty}(0,T;Q)$, $\boldsymbol{g} \in W^{1,\infty}(0,T;L^2(\Gamma_3)^2)$ and $\boldsymbol{f} \in W^{1,\infty}(0,T;V)$, we deduce that

$$\tilde{\boldsymbol{z}}_N^{\eta} \to \boldsymbol{z}_\eta \quad \text{strongly in} \quad L^2(0,T;Q), \tag{4.22}$$

$$\tilde{\boldsymbol{g}}_N \to \boldsymbol{g} \quad \text{strongly in} \quad L^2(0,T;L^2(\Gamma_3)^2), \tag{4.23}$$

$$\tilde{\boldsymbol{f}}_N \to \boldsymbol{f} \quad \text{strongly in} \quad L^2(0,T;V). \tag{4.24}$$

Using (4.13), we have for almost every $t \in (0,T)$ and every $\boldsymbol{v} \in L^2(0,T;V)$,

$$a(\tilde{\boldsymbol{u}}_N^{\eta g}(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_N^{\eta g}(t)) + (\tilde{\boldsymbol{z}}_N^{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_N^{\eta g}(t)))_Q$$
$$+ j(\tilde{\boldsymbol{g}}_N(t), \boldsymbol{v}(t)) - j(\tilde{\boldsymbol{g}}_N(t), \dot{\boldsymbol{u}}_N^{\eta g}(t)) \geq (\tilde{\boldsymbol{f}}_N(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_N^{\eta g}(t))_V$$

or else

$$\begin{aligned} &\int_0^T a(\tilde{\boldsymbol{u}}_N^{\eta g}(t), \boldsymbol{v}(t))dt + \int_0^T (\tilde{\boldsymbol{z}}_N^{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_N^{\eta g}(t)))_Q dt \\ &\quad + \int_0^T j(\tilde{\boldsymbol{g}}_N(t), \boldsymbol{v}(t))dt \geq \int_0^T (\tilde{\boldsymbol{f}}_N(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_N^{\eta g}(t))_V dt \\ &\quad + \int_0^T a(\tilde{\boldsymbol{u}}_N^{\eta g}(t), \dot{\boldsymbol{u}}_N^{\eta g}(t))dt + \int_0^T j(\tilde{\boldsymbol{g}}_N(t), \dot{\boldsymbol{u}}_N^{\eta g}(t))dt. \end{aligned} \tag{4.25}$$

In order to be able to pass to the lower limit in (4.25), we need some convergence results. On one hand, with (4.21)–(4.24) and the weak $*$ convergence of $\{\dot{\boldsymbol{u}}_N^{\eta g}\}_N$ to $\dot{\boldsymbol{u}}_{\eta g}$, we get

$$\int_0^T a(\tilde{\boldsymbol{u}}_N^{\eta g}(t), \boldsymbol{v}(t))\, dt \to \int_0^T a(\boldsymbol{u}_{\eta g}(t), \boldsymbol{v}(t))\, dt, \tag{4.26}$$

$$\int_0^T (\tilde{\boldsymbol{z}}_N^{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_N^{\eta g}(t)))_Q\, dt \to \int_0^T (\boldsymbol{z}_{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\eta g}(t)))_Q\, dt, \tag{4.27}$$

$$\int_0^T (\tilde{\boldsymbol{f}}_N(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_N^{\eta g}(t))_V\, dt \to \int_0^T (\boldsymbol{f}(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_{\eta g}(t))_V\, dt. \tag{4.28}$$

$$\int_0^T j(\tilde{\boldsymbol{g}}_N(t), \boldsymbol{v}(t)) dt \to \int_0^T j(\boldsymbol{g}(t), \boldsymbol{v}(t)) dt \tag{4.29}$$

for all $\boldsymbol{v} \in L^2(0,T;V)$, as $N \to \infty$.

On another hand, writing

$$j(\tilde{\boldsymbol{g}}_N(t), \dot{\boldsymbol{u}}_N^{\eta g}(t)) = j(\tilde{\boldsymbol{g}}_N(t), \dot{\boldsymbol{u}}_N^{\eta g}(t)) - j(\boldsymbol{g}(t), \dot{\boldsymbol{u}}_N^{\eta g}(t)) + j(\boldsymbol{g}(t), \dot{\boldsymbol{u}}_N^{\eta g}(t)),$$

with

$$\lim_N \int_0^T (j(\tilde{\boldsymbol{g}}_N(t), \dot{\boldsymbol{u}}_N^{\eta g}(t)) - j(\boldsymbol{g}(t), \dot{\boldsymbol{u}}_N^{\eta g}(t))) dt = 0,$$

we get (with a standard semicontinuity result)

$$\liminf_N \int_0^T j(\tilde{\boldsymbol{g}}_N(t), \dot{\boldsymbol{u}}_N^{\eta g}(t))\, dt \geq \int_0^T j(\boldsymbol{g}(t), \dot{\boldsymbol{u}}_{\eta g}(t))\, dt. \tag{4.30}$$

Furthermore, writing $\tilde{\boldsymbol{u}}_N^{\eta g}(t) = (\tilde{\boldsymbol{u}}_N^{\eta g}(t) - \boldsymbol{u}_N^{\eta g}(t)) + \boldsymbol{u}_N^{\eta g}(t)$ and keeping in mind that $\boldsymbol{u}_N^{\eta g}(0) = \boldsymbol{u}_0$ we obtain

$$\begin{aligned}\int_0^T a(\tilde{\boldsymbol{u}}_N^{\eta g}(t), \dot{\boldsymbol{u}}_N^{\eta g}(t)) dt = & \int_0^T a(\tilde{\boldsymbol{u}}_N^{\eta g}(t) - \boldsymbol{u}_N^{\eta g}(t), \dot{\boldsymbol{u}}_N^{\eta g}(t)) dt \\ & + \tfrac{1}{2}\, a(\boldsymbol{u}_N^{\eta g}(T), \boldsymbol{u}_N^{\eta g}(T)) - \tfrac{1}{2}\, a(\boldsymbol{u}_0, \boldsymbol{u}_0).\end{aligned}$$

Using the pointwise weak convergence in V of $\boldsymbol{u}_N^{\eta g}$ to $\boldsymbol{u}_{\eta g}$, (4.20), the boundedness of the sequence $\{\dot{\boldsymbol{u}}_N^{\eta g}\}_N$ in $L^\infty(0,T;V)$ and a lower semicontinuity argument, we find $\boldsymbol{u}_{\eta g}(0) = \boldsymbol{u}_0$ and

$$\begin{aligned}\liminf_N \int_0^T a(\tilde{\boldsymbol{u}}_N^{\eta g}(t), \dot{\boldsymbol{u}}_N^{\eta g}(t))\, dt &\geq \frac{1}{2}\, a(\boldsymbol{u}_{\eta g}(T), \boldsymbol{u}_{\eta g}(T)) - \frac{1}{2}\, a(\boldsymbol{u}_0, \boldsymbol{u}_0) \\ &= \int_0^T a(\boldsymbol{u}_{\eta g}(t), \dot{\boldsymbol{u}}_{\eta g}(t))\, dt.\end{aligned} \tag{4.31}$$

iv) Existence.

Assertion (4.25) and convergences (4.26)–(4.31) lead to

$$a(\boldsymbol{u}_{\eta g}(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_{\eta g}(t)) + (\boldsymbol{z}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\eta g}(t)))_Q + j(\boldsymbol{g}(t), \boldsymbol{v}) - j(\boldsymbol{g}(t), \dot{\boldsymbol{u}}_{\eta g}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_{\eta g}(t))_V \quad \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T). \tag{4.32}$$

Moreover, recall that $\boldsymbol{u}_{\eta g}(0) = \boldsymbol{u}_0$, and therefore (4.9) holds. Let $\boldsymbol{\sigma}_{\eta g} \in W^{1,\infty}(0,T;Q)$ be the element given by (4.7). We use (4.32) and (4.12) to find (4.8). Notice also that (4.8) implies

$$\operatorname{Div} \boldsymbol{\sigma}_{\eta g} + \boldsymbol{f}_0 = 0 \quad \text{a.e. in } \Omega \times (0,T) \tag{4.33}$$

and, therefore, it follows from (3.14) that $\boldsymbol{\sigma}_{\eta g} \in W^{1,\infty}(0,T;Q_1)$. We conclude that the couple $(\boldsymbol{u}_{\eta g}, \boldsymbol{\sigma}_{\eta g})$ is a solution of Problem $P_V^{\eta g}$ with regularity (4.10).

v) Uniqueness and boundness.

The uniqueness part of the theorem follows from the unique solvability of the Cauchy problem (4.32), (4.9) which can be obtained using standard arguments. Moreover this proves that the whole sequence $\{\boldsymbol{u}_N^{\eta g}\}_N$ is weak $*$ convergent in $W^{1,\infty}(0,T;V)$ to the element $\boldsymbol{u}_{\eta g}$. Finally, the estimate (4.11) follows from (4.18), (4.7), (4.33) and (3.11). □

5 Intermediate viscoplastic problem

In this section we solve the contact problem for the fully viscoplastic law, in the case when the normal stress and the friction bound on the contact surface are given. To this end we shall use the Banach fixed point theorem. Let $\boldsymbol{g}$ be a given function which satisfies (4.2), (4.3) and consider the following intermediate problem.

Problem P_V^g. *Find a displacement field $\boldsymbol{u}_g : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_g : [0,T] \to Q_1$ such that*

$$\dot{\boldsymbol{\sigma}}_g(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_g(t)) + G(\boldsymbol{\sigma}_g(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_g(t))) \quad \text{a.e. } t \in (0,T), \tag{5.1}$$

$$(\boldsymbol{\sigma}_g(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_g(t)))_Q + j(\boldsymbol{g}(t), \boldsymbol{v}) - j(\boldsymbol{g}(t), \dot{\boldsymbol{u}}_g(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_g(t))_V \quad \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T), \tag{5.2}$$

$$\boldsymbol{u}_g(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}_g(0) = \boldsymbol{\sigma}_0. \tag{5.3}$$

We have following existence and uniqueness result.

Theorem 5.1. *Assume that conditions* (3.11), (3.12), (3.18)–(3.20), (4.2), (4.3) *hold. Then problem* P_V^g *has a unique solution with regularity*

$$\boldsymbol{u}_g \in W^{1,\infty}(0,T;V), \quad \boldsymbol{\sigma}_g \in W^{1,\infty}(0,T;Q_1). \tag{5.4}$$

Moreover, there exists a constant $c > 0$ *such that*

$$\|\boldsymbol{u}_g\|_{W^{1,\infty}(0,T;V)} + \|\boldsymbol{\sigma}_g\|_{W^{1,\infty}(0,T;Q_1)} \le c\,(\|\boldsymbol{g}\|_{W^{1,\infty}(0,T;L^2(\Gamma_3)^2)} + \|\boldsymbol{f}\|_{W^{1,\infty}(0,T;V)} + \|\boldsymbol{u}_0\|_V + \|\boldsymbol{\sigma}_0\|_{Q_1} + \|G(\boldsymbol{0},\boldsymbol{0})\|_Q). \tag{5.5}$$

Proof. The proof of Theorem 5.1 is carried out in three steps which are described below.

i) The Banach fixed point.

We consider the operator $\Lambda_g : L^\infty(0,T;Q) \to L^\infty(0,T;Q)$ defined by

$$\Lambda_g \boldsymbol{\eta} = G(\boldsymbol{\sigma}_{\eta g}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta g})) \quad \forall\, \boldsymbol{\eta} \in L^\infty(0,T;Q), \tag{5.6}$$

where $(\boldsymbol{u}_{\eta g}, \boldsymbol{\sigma}_{\eta g})$ is the solution of the intermediate elastic problem $P_V^{\eta g}$ provided by Theorem 4.1. We shall prove that the operator Λ_g has a unique fixed point $\boldsymbol{\eta}_g^* \in L^\infty(0,T;Q)$. To this end, consider $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in L^\infty(0,T;Q)$ and denote $\boldsymbol{u}_{ig} = \boldsymbol{u}_{\eta_i g}$, $\boldsymbol{\sigma}_{ig} = \boldsymbol{\sigma}_{\eta_i g}$, and $\boldsymbol{z}_i = \boldsymbol{z}_{\eta_i}$ for $i = 1, 2$. Rewrite (4.32) as

$$a(\boldsymbol{u}_{1g}, \boldsymbol{v} - \dot{\boldsymbol{u}}_{1g}) + (\boldsymbol{z}_1, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{1g}))_Q + j(\boldsymbol{g}, \boldsymbol{v}) - j(\boldsymbol{g}, \dot{\boldsymbol{u}}_{1g}) \ge (\boldsymbol{f}, \boldsymbol{v} - \dot{\boldsymbol{u}}_{1g})_V,$$

$$a(\boldsymbol{u}_{2g}, \boldsymbol{v} - \dot{\boldsymbol{u}}_{2g}) + (\boldsymbol{z}_2, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{2g}))_Q + j(\boldsymbol{g}, \boldsymbol{v}) - j(\boldsymbol{g}, \dot{\boldsymbol{u}}_{2g}) \ge (\boldsymbol{f}, \boldsymbol{v} - \dot{\boldsymbol{u}}_{2g})_V$$

for all $\boldsymbol{v} \in V$, a.e. on $(0,T)$. We take $\boldsymbol{v} = \dot{\boldsymbol{u}}_{2g}$ in the first inequality, $\boldsymbol{v} = \dot{\boldsymbol{u}}_{1g}$ in the second inequality, and add the two inequalities to obtain

$$\frac{1}{2}\frac{d}{dt}\, a(\boldsymbol{u}_{1g} - \boldsymbol{u}_{2g}, \boldsymbol{u}_{1g} - \boldsymbol{u}_{2g}) \le -(\boldsymbol{z}_1 - \boldsymbol{z}_2, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2))_Q \quad \text{a.e. on } (0,T).$$

Let $t \in [0,T]$. Integrate the previous inequality from 0 to t:

$$\begin{aligned}
&\frac{1}{2}\, a(\boldsymbol{u}_{1g}(t) - \boldsymbol{u}_{2g}(t), \boldsymbol{u}_{1g}(t) - \boldsymbol{u}_{2g}(t)) \\
&\quad \le -(\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_{1g}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_{2g}(t)))_Q \\
&\quad\quad + \int_0^t (\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_{1g}(s)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_{2g}(s)))_Q ds.
\end{aligned}$$

Then,

$$\begin{aligned}
c\,\|\boldsymbol{u}_{1g}(t) - \boldsymbol{u}_{2g}(t)\|_V^2 &\le \|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\|_Q \|\boldsymbol{u}_{1g}(t) - \boldsymbol{u}_{2g}(t)\|_V \\
&\quad + \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V ds.
\end{aligned}$$

Now

$$\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t) = \int_0^t (\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s))\, ds,$$

so

$$\|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\|_Q \le \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q ds.$$

Therefore, we have

$$\begin{aligned}\|\boldsymbol{u}_{1g}(t) - \boldsymbol{u}_{2g}(t)\|_V^2 \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds \\ +c \int_0^t \|\boldsymbol{u}_{1g}(s) - \boldsymbol{u}_{2g}(s)\|_V^2 ds.\end{aligned}$$

Applying the Gronwall inequality, we obtain

$$\|\boldsymbol{u}_{1g}(t) - \boldsymbol{u}_{2g}(t)\|_V^2 \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds. \tag{5.7}$$

By the definition (4.7) for $\boldsymbol{\sigma}_{1g}$ and $\boldsymbol{\sigma}_{2g}$, we have

$$\begin{aligned}\boldsymbol{\sigma}_{1g}(t) - \boldsymbol{\sigma}_{2g}(t) &= \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{1g}(t) - \boldsymbol{u}_{2g}(t)) + \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t) \\ &= \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{1g}(t) - \boldsymbol{u}_{2g}(t)) + \int_0^t (\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s))\, ds.\end{aligned}$$

Then

$$\|\boldsymbol{\sigma}_{1g}(t) - \boldsymbol{\sigma}_{2g}(t)\|_Q^2 \le c\,\|\boldsymbol{u}_{1g}(t) - \boldsymbol{u}_{2g}(t)\|_V^2 + c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds.$$

Using (5.7), we have

$$\|\boldsymbol{\sigma}_{1g}(t) - \boldsymbol{\sigma}_{2g}(t)\|_Q^2 \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds. \tag{5.8}$$

Finally,

$$\Lambda_g\boldsymbol{\eta}_1(t) - \Lambda_g\boldsymbol{\eta}_2(t) = G(\boldsymbol{\sigma}_{1g}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_{1g}(t))) - G(\boldsymbol{\sigma}_{2g}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_{2g}(t))).$$

Using the assumptions (3.12), and the bounds (5.7) and (5.8), we find that

$$\|\Lambda_g\boldsymbol{\eta}_1(t) - \Lambda_g\boldsymbol{\eta}_2(t)\|_Q^2 \le c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q^2 ds. \tag{5.9}$$

¿From (5.9) and an application of Banach's fixed point theorem to a suitable iterative of the map Λ_g we deduce that the operator Λ_g has a unique fixed point $\boldsymbol{\eta}_g^* \in L^\infty(0, T; Q)$.

ii) Existence.

Let $(\boldsymbol{u}_g^*, \boldsymbol{\sigma}_g^*)$ be the solution of problem $P_V^{\eta g}$ for $\boldsymbol{\eta} = \boldsymbol{\eta}_g^*$, that is $\boldsymbol{u}_g^* = \boldsymbol{u}_{\eta_g^* g}$ and $\boldsymbol{\sigma}_g^* = \boldsymbol{\sigma}_{\eta_g^* g}$. Using (4.7) and (4.5) we have

$$\dot{\boldsymbol{\sigma}}_g^*(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_g^*(t)) + \boldsymbol{\eta}_g^*(t) \quad \text{a.e. } t \in (0, T)$$

and, using (5.6) yields

$$\boldsymbol{\eta}_g^*(t) = \Lambda_g \boldsymbol{\eta}_g^*(t) = G(\boldsymbol{\sigma}_g^*(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_g^*(t))) \quad \text{a.e. } t \in (0,T).$$

Combining the previous two equalities we find that $(\boldsymbol{u}_g^*, \boldsymbol{\sigma}_g^*)$ satisfies (5.1). Moreover, from (4.5), (4.7) and (4.9) it follows that (5.3) holds and, finally, (5.2) is a consequence of (4.8). We conclude that $(\boldsymbol{u}_g^*, \boldsymbol{\sigma}_g^*)$ is a solution of problem P_V^g with regularity (5.4).

iii) Uniqueness.

The uniqueness part follows from the uniqueness of the fixed point of the operator Λ_g defined by (5.6). The proof is an application of Gronwall's lemma, details can be found in [2].

iv) Boundness.

Using (5.1)–(5.3) and arguments similar to those used in the proof of Lemma 3.1 in [1], after a tedious calculus it can be shown that

$$\begin{aligned} \|\boldsymbol{u}_g^*(t)\|_V + \|\boldsymbol{\sigma}_g^*(t)\|_{Q_1} \leq c\,(\|\boldsymbol{g}\|_{W^{1,\infty}(0,T;L^2(\Gamma_3)^2)} + \|\boldsymbol{f}\|_{W^{1,\infty}(0,T;V)} \\ +\|\boldsymbol{u}_0\|_V + \|\boldsymbol{\sigma}_0\|_{Q_1} + \|G(\mathbf{0},\mathbf{0})\|_Q) \quad \forall t \in [0,T]. \end{aligned} \tag{5.10}$$

Now, let $\boldsymbol{z}_g^* = \boldsymbol{z}_{\eta_g^*}$. Using (4.5) and (5.6) we deduce

$$\boldsymbol{z}_g^*(t) = \int_0^t G(\boldsymbol{\sigma}_g^*(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_g^*(s)))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall t \in [0,T]. \tag{5.11}$$

Writing

$$G(\boldsymbol{\sigma}_g^*(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_g^*(s))) = \Big[G(\boldsymbol{\sigma}_g^*(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_g^*(s))) - G(\mathbf{0},\mathbf{0})\Big] + G(\mathbf{0},\mathbf{0})$$

and using condition (3.12), it follows from (5.11) that

$$\begin{aligned} \|\boldsymbol{z}_g^*(t)\|_Q \leq c\,\Big(\|\boldsymbol{u}_0\|_V + \|\boldsymbol{\sigma}_0\|_{Q_1} + \|G(\mathbf{0},\mathbf{0})\|_Q + \\ + \int_0^t (\|\boldsymbol{u}_g^*(s)\|_V + \|\boldsymbol{\sigma}_g^*(s)\|_Q)ds\Big) \quad \text{a.e. } t \in (0,T), \end{aligned} \tag{5.12}$$

$$\|\dot{\boldsymbol{z}}_g^*(t)\|_Q \leq c\,(\|\boldsymbol{u}_g^*(t)\|_V + \|\boldsymbol{\sigma}_g^*(t)\|_Q + \|G(\mathbf{0},\mathbf{0})\|_Q) \quad \forall t \in [0,T]. \tag{5.13}$$

Now, keeping in mind (5.10), (5.12) and (5.13) we find

$$\begin{aligned} \|\boldsymbol{z}_g^*\|_{W^{1,\infty}(0,T;Q)} \leq c\,(\|\boldsymbol{u}_0\|_V + \|\boldsymbol{\sigma}_0\|_{Q_1} + \|G(\mathbf{0},\mathbf{0})\|_Q \\ +\|\boldsymbol{g}\|_{W^{1,\infty}(0,T;L^2(\Gamma_3)^2)} + \|\boldsymbol{f}\|_{W^{1,\infty}(0,T;V)}) \quad \forall t \in [0,T]. \end{aligned} \tag{5.14}$$

We use now (4.11) with $\boldsymbol{\eta} = \boldsymbol{\eta}_g^*$ and (5.14) to obtain (5.5). □

6 Proof of Theorem 3.1

In this section we provide the proof of Theorem 3.1 which is based on the Schauder fixed point theorem. To this end, everywhere in the sequel we denote by $Lip^{\,0}_{k}$ the set

$$(6.1)\quad Lip^{\,0}_{k} = \{\, \boldsymbol{g} \in W^{1,\infty}(0,T;L^2(\Gamma_3)^2) \mid \|\dot{\boldsymbol{g}}\|_{L^\infty(0,T;L^2(\Gamma_3)^2)} \le k \text{ and } \boldsymbol{g}(0) = \boldsymbol{g}_0 \,\}$$

where $\boldsymbol{g}_0$ is the element of $L^2(\Gamma_3)^2$ given by (4.3) and $k > 0$.

Given $k > 0$ and a function $\boldsymbol{g} \in Lip^{\,0}_{k}$, we denote by $(\boldsymbol{u}_g, \boldsymbol{\sigma}_g)$ the solution of the intermediate viscoplastic problem P_V^g obtained in Theorem 5.1. Keeping in mind assumption (3.13) on the normal compliance contact functions it follows that $p_r(u_{g\nu} - g_a) \in C([0,T];L^2(\Gamma_3)^2)$ for $r = \nu, \tau$, and therefore we are allowed to consider the operator $\mathcal{T} : Lip^{\,0}_{k} \to C([0,T];L^2(\Gamma_3)^2)$ given by

$$(6.2)\qquad \mathcal{T}\boldsymbol{g} = (p_\nu(u_{g\nu} - g_a), p_\tau(u_{g\nu} - g_a)).$$

We now investigate the properties of the operator $\mathcal{T}$.

Lemma 6.1. *There exists L_0 depending only on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{E}$, G and T such that if $L_\nu + L_\tau < L_0$, then there exists $k > 0$ with the following property:*

$$(6.3)\qquad \boldsymbol{g} \in Lip^{\,0}_{k} \implies \mathcal{T}\boldsymbol{g} \in Lip^{\,0}_{k}.$$

Proof. Let $k > 0$, $\boldsymbol{g} \in Lip^{\,0}_{k}$, and let t_1, $t_2 \in [0,T]$. Using (6.2), (3.13) and (2.4) we obtain

$$\begin{aligned} \|\mathcal{T}\boldsymbol{g}(t_1) - \mathcal{T}\boldsymbol{g}(t_2)\|_{L^2(\Gamma_3)^2} &\le c(L_\nu + L_\tau)\|\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)\|_V \\ &\le c(L_\nu + L_\tau)\|\dot{\boldsymbol{u}}_g\|_{L^\infty(0,T;V)}|t_1 - t_2|, \end{aligned}$$

and, keeping in mind (5.5), we find

$$(6.4)\quad \|\mathcal{T}\boldsymbol{g}(t_1) - \mathcal{T}\boldsymbol{g}(t_2)\|_{L^2(\Gamma_3)^2} \le c(L_\nu + L_\tau)(\|\boldsymbol{g}\|_{W^{1,\infty}(0,T;L^2(\Gamma_3)^2)} + \theta)|t_1 - t_2|,$$

where θ is a positive quantity which does not depend on k. It follows from (6.1) that

$$(6.5)\qquad \|\boldsymbol{g}\|_{W^{1,\infty}(0,T;L^2(\Gamma_3)^2)} \le k(T+1) + \|\boldsymbol{g}_0\|_{L^2(\Gamma_3)^2}$$

and therefore (6.4) implies that

$$\|\mathcal{T}\boldsymbol{g}(t_1) - \mathcal{T}\boldsymbol{g}(t_2)\|_{L^2(\Gamma_3)^2} \le c(L_\nu + L_\tau)(k(T+1) + \|\boldsymbol{g}_0\|_{L^2(\Gamma_3)^2} + \theta)|t_1 - t_2|.$$

We conclude that $\mathcal{T}g \in W^{1,\infty}(0,T;L^2(\Gamma_3)^2)$ and, moreover,

$$\Big\|\frac{d}{dt}\mathcal{T}\boldsymbol{g}\Big\|_{L^\infty(0,T;L^2(\Gamma_3)^2)} \le c(L_\nu + L_\tau)(k(T+1) + \|\boldsymbol{g}_0\|_{L^2(\Gamma_3)^2} + \theta).$$

Let $L_0 = \frac{1}{c(T+1)}$, it is straightforward to see that if $L_\nu + L_\tau < L_0$ we can find $k > 0$ such that $c(L_\nu + L_\tau)(k(T+1) + \|\boldsymbol{g}_0\|_{L^2(\Gamma_3)^2} + \theta) \leq k$ and therefore

$$\Big\|\frac{d}{dt}\mathcal{T}\boldsymbol{g}\Big\|_{L^\infty(0,T;L^2(\Gamma_3)^2)} \leq k.$$

Since by (5.3) and (4.4) we have $\mathcal{T}\boldsymbol{g}(0) = \boldsymbol{g}_0$, we conclude that $\mathcal{T}\boldsymbol{g} \in Lip_k^0$, which proves the lemma. □

In the sequel we assume that $L_\nu + L_\tau < L_0$ and we choose $k > 0$ such that (6.3) holds. It is straightforward to see that Lip_k^0 is a nonempty closed bounded convex set of the Banach space $C([0,T]; L^2(\Gamma_3)^2)$. Moreover, our choice of k guarantees that $\mathcal{T}(Lip_k^0) \subset Lip_k^0$. We investigate in what follows the properties of the operator $\mathcal{T} : Lip_k^0 \to Lip_k^0$.

Lemma 6.2. *i) For every $k > 0$, there exists $c_k > 0$ such that for every $(\boldsymbol{g}_1, \boldsymbol{g}_2) \in (Lip_k^0)^2$, we have*

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{C([0,T];V)} + \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{C([0,T];Q_1)} \leq c_k \sqrt{\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{C([0,T];L^2(\Gamma_3)^d)}}\ , \tag{6.6}$$

where $\boldsymbol{u}_i$ $(i = 1,2)$ is the displacement solution of problem $P_V^{g_i}$ provided by Theorem 5.1.

ii) $\mathcal{T}(Lip_k^0)$ is a relatively compact subset of $C([0,T]; L^2(\Gamma_3)^2)$.

Proof. *i)* Let $(\boldsymbol{g}_1, \boldsymbol{g}_2) \in (Lip_k^0)^2$ and let $t \in [0,T]$. Using $P_V^{g_i}$, one can see that displacements $\boldsymbol{u}_i$ satisfies the inequality

$$\begin{aligned} a(\boldsymbol{u}_i(t), \boldsymbol{v}) + j(\boldsymbol{g}_i(t), \boldsymbol{v}) \geq (\boldsymbol{f}(t), \boldsymbol{v})_V - \\ (\boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \int_0^t G(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_i(s)))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q. \end{aligned} \tag{6.7}$$

Using standard arguments, from (6.7) we obtain

$$\begin{aligned} a(\boldsymbol{u}_2(t) - \boldsymbol{u}_1(t), \boldsymbol{u}_2(t) - \boldsymbol{u}_1(t)) \leq j(\boldsymbol{g}_1(t), \boldsymbol{u}_2(t) - \boldsymbol{u}_1(t)) + j(\boldsymbol{g}_2(t), \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)) \\ +(\int_0^t [G(\boldsymbol{\sigma}_2(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_2(s))) - G(\boldsymbol{\sigma}_1(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)))]ds, \boldsymbol{\varepsilon}(\boldsymbol{u}_1(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2(t)))_Q. \end{aligned}$$

and, with (3.11),(3.12), we deduce

$$\begin{aligned} \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V^2 \leq c\,[\,\|\boldsymbol{g}_1(t)\|_{L^2(\Gamma_3)^d}\,\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{L^2(\Gamma_3)^d} + \\ \|\boldsymbol{g}_2(t)\|_{L^2(\Gamma_3)^d}\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{L^2(\Gamma_3)^d} + \\ \int_0^t (\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V^2 + \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q^2)ds]. \end{aligned} \tag{6.8}$$

We use now (6.8) and (6.5) to obtain

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 \le c\,[M_k\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{C([0,T];L^2(\Gamma_3)^d)} + \int_0^t (\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2 + \|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_Q^2)ds], \tag{6.9}$$

where $M_k = 2[k(T+1) + \|\boldsymbol{g}_0\|_{L^2(\Gamma_3)^2}]$.

On another hand,

$$\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_1-\boldsymbol{u}_2)(t) + \int_0^t (G(\boldsymbol{\sigma}_1(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s))) - G(\boldsymbol{\sigma}_2(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_2(s))))ds$$

and

$$\mathrm{Div}(\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)) = 0,$$

thus

$$\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_{Q_1} \le c\,[\,\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V + \int_0^t (\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V + \|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_Q)ds],$$

which implies

$$\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_{Q_1}^2 \le c\,[\,\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 + \int_0^t (\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2 + \|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_Q^2)ds].$$

Keeping in mind (6.9), from the previous inequality we find

$$\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_{Q_1}^2 \le c\,[M_k\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{C([0,T];L^2(\Gamma_3)^d)} + \int_0^t (\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2 + \|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_Q^2)ds]. \tag{6.10}$$

Adding (6.9) and (6.10) and using Gronwall's lemma, we deduce (6.6) with $c_k = \mathcal{O}(k)+1$.

ii) Let $\{\boldsymbol{g}_n\}$ be a sequence of elements of Lip_k^0 and, for all $n\in\mathbb{N}$, denote by $(\boldsymbol{u}_n,\boldsymbol{\sigma}_n)$ the solution of Problem $P_V^{g_n}$ provided by Theorem 5.1. Using (6.5) it follows that $\{\boldsymbol{g}_n\}$ is a bounded sequence in the space $W^{1,\infty}(0,T;L^2(\Gamma_3)^2)$ and, therefore, the estimate (5.5) shows that $\{\boldsymbol{u}_n\}$ is a bounded sequence in $W^{1,\infty}(0,T;V)$. Using Arzela-Ascoli theorem, it follows that the set of the traces on Γ_3 of the displacements $\{\boldsymbol{u}_n\}_n$ is relatively compact in $C([0,T];L^2(\Gamma_3)^d)$. Thus, we can extract a subsequence of $\{\boldsymbol{u}_n\}_n$ such that their traces on Γ_3 converge strongly in $C([0,T];L^2(\Gamma_3)^d)$ and therefore represent a Cauchy sequence in that space. Using the point *i)* of Lemma 6.1, we deduce that the corresponding subsequences of $\{\boldsymbol{u}_n\}_n$ and $\{\boldsymbol{\sigma}_n\}_n$ are Cauchy sequences in the spaces $C([0,T];V)$ and $C([0,T];Q_1)$, respectively, and therefore they converge strongly in these spaces. Let

$\boldsymbol{u} \in C([0,T];V)$ be the limit in $C([0,T];V)$ of the corresponding subsequence $\{\boldsymbol{u}_n\}_n$, as $n \to \infty$. We use now (2.4) and (3.13) to see that

$$p_\nu(u_{n\nu} - g_a) \to p_\nu(u_\nu - g_a), \quad p_\tau(u_{n\nu} - g_a) \to p_\tau(u_\nu - g_a) \quad \text{in} \quad C([0,T];L^2(\Gamma_3)),$$

which shows that the sequence $\{\mathcal{T}\boldsymbol{g}_n\}$ converges in $C([0,T];L^2(\Gamma_3)^2)$. This ends the proof of the lemma. □

Lemma 6.3. *The operator $\mathcal{T}$ is continuous.*

Proof. Let $\boldsymbol{g} \in Lip_k^0$ and let $\{\boldsymbol{g}_n\}_n$ be a sequence of elements of Lip_k^0 such that $\boldsymbol{g}_n \to \boldsymbol{g}$ in $C([0,T];L^2(\Gamma_3)^2)$. For all $n \in \mathbb{N}$ denote by $(\boldsymbol{u}_n, \boldsymbol{\sigma}_n)$ the solution of Problem $P_V^{g_n}$ provided by Theorem 5.1. Using (6.5) and (5.5) it follows that $\{\boldsymbol{u}_n, \boldsymbol{\sigma}_n\}$ is a bounded sequence in $W^{1,\infty}(0,T;V \times Q_1)$. Therefore, using arguments similar to those used in the proof of Lemma 6.2, we deduce that there exists an element $(\boldsymbol{u}, \boldsymbol{\sigma}) \in W^{1,\infty}(0,T;V \times Q_1)$ such that, extracting a subsequence denoted $\{\boldsymbol{u}_{n_p}, \boldsymbol{\sigma}_{n_p}\}$, we have

$$(\boldsymbol{u}_{n_p}, \boldsymbol{\sigma}_{n_p}) \to (\boldsymbol{u}, \boldsymbol{\sigma}) \quad \text{weak } * \quad \text{in} \quad W^{1,\infty}(0,T;V \times Q_1), \tag{6.11}$$

$$(\boldsymbol{u}_{n_p}, \boldsymbol{\sigma}_{n_p}) \to (\boldsymbol{u}, \boldsymbol{\sigma}) \quad \text{in} \quad C([0,T];V \times Q_1). \tag{6.12}$$

Using now (5.1)–(5.3) we obtain

$$\boldsymbol{\sigma}_{n_p}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n_p}(t)) + \boldsymbol{z}_{n_p}(t) \quad \forall t \in [0,T], \tag{6.13}$$

$$\begin{aligned} a(\boldsymbol{u}_{n_p}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{n_p}(t)) + (\boldsymbol{z}_{n_p}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{n_p}(t))_Q +& \\ j(\boldsymbol{g}_{n_p}(t), \boldsymbol{v}) - j(\boldsymbol{g}(t), \dot{\boldsymbol{u}}_{n_p}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{n_p}(t))_V& \\ \forall \boldsymbol{v} \in V, \ \text{a.e. } t \in (0,T),& \end{aligned} \tag{6.14}$$

where

$$\boldsymbol{z}_{n_p}(t) = \int_0^t G(\boldsymbol{\sigma}_{n_p}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_{n_p}(s)) \, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall t \in [0,T]. \tag{6.15}$$

Moreover,

$$\boldsymbol{u}_{n_p}(0) = \boldsymbol{u}_0. \tag{6.16}$$

Passing to the limit in (6.13)–(6.15) as $n \to \infty$ and using the convergences (6.11), (6.12) we find that

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \boldsymbol{z}(t) \quad \forall t \in [0,T], \tag{6.17}$$

$$\begin{aligned} a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + (\boldsymbol{z}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))_Q +& \\ j(\boldsymbol{g}(t), \boldsymbol{v}) - j(\boldsymbol{g}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V& \\ \forall \boldsymbol{v} \in V, \ \text{a.e. } t \in (0,T),& \end{aligned} \tag{6.18}$$

where

$$\boldsymbol{z}(t) = \int_0^t G(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)) \, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall t \in [0,T]. \tag{6.19}$$

Indeed, using (3.12) and (6.12) it follows that $\boldsymbol{z}_{n_p} \to \boldsymbol{z}$ in $C([0,T];Q)$ and, therefore, (6.13) and (3.11) imply (6.17). To prove (6.18) we integrate (6.14) on $[0,T]$, employ arguments similar to those used in the proofs of Theorem 4.1 (see (4.26)–(4.30)) and then perform a localization argument, based on a classical application of Lebesgue point for L^1 functions.

It follows now from (6.16)–(6.19) that $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a solution of problem P_V^g. We use now (6.12) and (2.4) to see that $u_{n_p\nu} \to u_\nu$ in $C([0,T];L^2(\Gamma_3))$ and, keeping in mind (6.2) and (3.13), it follows that $\mathcal{T}\boldsymbol{g}_{n_p} \to \mathcal{T}\boldsymbol{g}$ in $C([0,T];L^2(\Gamma_3)^2)$. Our arguments above show that the element $\mathcal{T}\boldsymbol{g}$ is the limit of all convergent subsequences $\{\mathcal{T}\boldsymbol{g}_{n_p}\} \subset \{\mathcal{T}\boldsymbol{g}_n\}$. Keeping in mind that the sequence $\mathcal{T}\boldsymbol{g}_n$ is relatively compact (see Lemma 6.2 *ii*)), we deduce that the whole sequence is convergent, i.e. $\mathcal{T}\boldsymbol{g}_n \to \mathcal{T}\boldsymbol{g}$ in $C([0,T];L^2(\Gamma_3)^2)$, which concludes the proof. □

We have now all the ingredients to prove Theorem 3.1.

Proof of Theorem 3.1. It follows from Lemmas 6.1, 6.2, 6.3 and Schauder's fixed point theorem that, if $L_\nu + L_\tau < L_0$, there exists an element $\boldsymbol{g}^* = (g_1^*, g_2^*) \in Lip_k^0$ such that $\mathcal{T}\boldsymbol{g}^* = \boldsymbol{g}^*$. Let $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*)$ denote the solution of Problem P_V^g for $\boldsymbol{g} = \boldsymbol{g}^*$. Keeping in mind that p_ν and p_τ are positive functions, from (6.2) it follows that

$$|g_1^*(t)| = p_\nu(u_\nu^*(t) - g_a), \quad |g_2^*(t)| = p_\tau(u_\nu^*(t) - g_a) \quad \forall t \in [0,T]$$

and, using (4.6) and (3.16) we find

$$j(\boldsymbol{g}^*(t), \boldsymbol{v}) = \varphi(\boldsymbol{u}^*(t), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V, \quad \forall t \in [0,T].$$

Using now (5.1)–(5.4) we deduce that $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*)$ is a solution of problem (3.21)–(3.23) with regularity (3.24), which concludes the proof. □

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