JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 116, No. 3, pp. 465–483, March 2003 (© 2003)

Analysis of a Viscoelastic Unilateral Contact Problem Involving the Coulomb Friction Law

A. AMASSAD[1] AND C. FABRE[2]

Communicated by R. Glowinski

Abstract. We give an existence result concerning the description of a contact problem between a viscoelastic body and a rigid foundation. We assume that a quasistatic process is valid. The contact is unilateral and involves friction between the two bodies. The friction law that we consider is a regularization of the Coulomb law. We present a weak formulation of the problem involving variational inequalities and establish an existence result, using a discretization method and a fixed-point property. The discretization method leads to the study of optimization problems on convex sets which depend on the discretization step.

Key Words. Quasistatic frictional contact, unilateral contact, Coulomb friction law, viscoelasticity, discretization, fixed points.

1. Introduction and Notations

This work concerns a quasistatic problem modelling the unilateral contact between a viscoelastic body and a rigid foundation. We will consider a nonlinear constitutive relationship and a nonlinear friction law. Before stating the scientific context and our results, first we introduce some notations that will be used in the paper.

Let Ω be a bounded and regular open set of $\mathbb{R}^d$ with boundary Γ. We suppose that Γ is divided in three disjoint parts

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

[1]Assistant Professor, Laboratoire de Mathématiques J.A. Dieudonné, UMR 66 21 du CNRS, Université de Nice, Parc Valrose, Nice, France.

[2]Professor, Laboratoire de Mathématiques J.A. Dieudonné, UMR 66 21 du CNRS, Université de Nice, Parc Valrose, Nice, France.

0022-3239/03/0300-0465/0

with Γ_1 being of nonzero measure. We denote by ν is the unit exterior normal on Γ.

The setting is as follows. A viscoelastic body occupies the domain Ω and is acted upon by given forces and tractions. We assume that a quasistatic process is valid. The constitutive relationship of a viscoelastic material can be written as

$$\sigma = \mathscr{A}(\dot{\epsilon}) + G(\epsilon), \tag{1}$$

in which σ denotes the stress tensor, $\epsilon = \epsilon(u)$ represents the small strain, $\mathscr{A}$ is the viscosity operator, and G is an elasticity map. The dot above represents the derivative with respect to the time variable. The body is clamped on $\Gamma_1 \times (0, T)$ and the surface traction φ_2 acts on $\Gamma_2 \times (0, T)$. Moreover, a volume force of density φ_1 acts on the body in $\Omega \times (0, T)$.

The solid is in frictional contact with a rigid obstacle on $\Gamma_3 \times (0, T)$ and this is where our main interest lies. We consider in this paper the case of a unilateral contact which involves no penetration between the body and the rigid foundation and is modeled with the Signorini conditions. We will consider a nonlocal Coulomb friction law and in fact a regularization of it in order that the boundary terms in the formulation of our problem make sense. In the sequel, R will represent a smoothing operator that is a linear and continuous operator $R: H^{-1/2}(\Gamma) \rightarrow L^2(\Gamma)$. Furthermore, we will suppose that

$$(R\sigma)_\nu \leq 0 \text{ on } \Gamma_3, \qquad \text{if } \sigma_\nu \leq 0 \text{ on } \Gamma_3.$$

Notice that we do not make any hypothesis on the compactness property of the operator R.

Under these hypotheses, the boundary conditions on Γ_3 can be written as

$$u_\nu \leq 0, \qquad \sigma_\nu \leq 0, \qquad u_\nu \sigma_\nu = 0, \tag{2a}$$

$$|\sigma_\tau| \leq \mu |R\sigma_\nu|, \tag{2b}$$

with

$$|\sigma_\tau| < \mu |R\sigma_\nu| \Rightarrow \dot{u}_\tau = 0, \tag{2c}$$

$$|\sigma_\tau| = \mu |R\sigma_\nu| \Rightarrow \exists \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau, \tag{2d}$$

where u_ν and u_τ denote the normal and tangential displacements, σ_ν is the normal stress, σ_τ represents the tangential force on the contact boundary, and $\mu = \mu(x) \in L^\infty(\Gamma_3)$ is a nonnegative function called the coefficient of friction.

With these assumptions, the mechanical problem of frictional contact of the viscoelastic body may be formulated as follows (Refs. 1–3): Find a displacement field $u: \Omega \times [0, T] \to \mathbb{R}^d$ and a stress field $\sigma: \Omega \times [0, T] \to S_d$ such that

$$\sigma = \mathscr{A}(\epsilon(\dot{u})) + G(\epsilon(u)), \qquad \text{in } \Omega \times (0, T), \tag{3a}$$

$$\operatorname{Div} \sigma + \varphi_1 = 0, \qquad \text{in } \Omega \times (0, T), \tag{3b}$$

$$u = 0, \qquad \text{on } \Gamma_1 \times (0, T), \tag{3c}$$

$$\sigma\nu = \varphi_2, \qquad \text{on } \Gamma_2 \times (0, T), \tag{3d}$$

$$u_\nu \leq 0, \qquad \sigma_\nu \leq 0, \qquad u_\nu \sigma_\nu = 0, \qquad \text{on } \Gamma_3 \times (0, T), \tag{3e}$$

$$|\sigma_\tau| \leq \mu |R\sigma_\nu|, \tag{3f}$$

with

$$|\sigma_\tau| < \mu |R\sigma_\nu| \Rightarrow \dot{u}_\tau = 0, \tag{3g}$$

$$|\sigma_\tau| = \mu |R\sigma_\nu| \Rightarrow \exists \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau, \tag{3h}$$

$$u(0) = u_0, \qquad \text{in } \Omega. \tag{3i}$$

As usual, solving the system (3) underlies the obtention of a weak formulation for which we need additional notations. We denote by S_d, $d = 2$ or 3, the space of symmetric tensors of order d on $\mathbb{R}^d$ endowed with its natural scalar product.

If v is a vector in $\mathbb{R}^d$, we write

$$v_\nu = v.\nu \quad \text{and} \quad v_\tau = v - v_\nu \nu,$$

the normal and tangential decomposition of the vector v. In a same way, we write

$$\sigma_\nu = \sigma\nu \cdot \nu \quad \text{and} \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$

the normal and tangential components of the vector $\sigma\nu$ for a tensor σ.

We consider the following spaces (repeated convention indexes are used):

$$H = [L^2(\Omega)]^d,$$

$$H_1 = [H^1(\Omega)]^d,$$

$$\mathscr{H} = \{(\sigma_{ij}) | \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\} = [L^2(\Omega)]_s^{d^2},$$

$$\mathscr{H}_1 = H(\operatorname{Div}, \Omega) = \{\sigma \in \mathscr{H} | (\sigma_{ij,j}) \in H\}.$$

All these spaces are endowed with their natural norms and scalar products,

$$(u, v)_H = \int_\Omega u_i v_i \, dx,$$

$$(\sigma, \tau)_{\mathscr{H}} = \int_\Omega \sigma_{ij} \tau_{ij} \, dx,$$

$$((u, v))_{H_1} = (u, v)_H + (\epsilon(u), \epsilon(v))_{\mathscr{H}},$$

$$((\sigma, \tau))_{\mathscr{H}_1} = (\sigma, \tau)_{\mathscr{H}} + (\mathrm{Div}\, \sigma, \mathrm{Div}\, \tau)_H.$$

When no confusion can be made, we will omit the index in the writing of these scalar products. Furthermore, the notation $\langle \, , \rangle_{X',X}$ will always denote the duality bracket between a space X and its dual X'.

We recall the Green formula (valid in regular cases),

$$(\sigma, \in(v))_{\mathscr{H}} + (\mathrm{Div}\, \sigma, v)_H = \langle \sigma \nu, v \rangle_{H^{-1/2}(\Gamma)^d, H^{1/2}(\Gamma)^d}, \qquad \forall v \in H_1, \tag{4}$$

which allows us to define

$$\sigma \nu \in H^{-1/2}(\Gamma)^d, \qquad \text{for } \sigma \in \mathscr{H}_1,$$

in order that the Green formula still holds.

Let V denote the closed subspace of H_1 defined by

$$V = \{v \in H_1 \,|\, v = 0, \text{ on } \Gamma_1\}.$$

We note that the Korn inequality holds, since

$$\mathrm{meas}(\Gamma_1) > 0;$$

thus,

$$|\epsilon(u)|_{\mathscr{H}} \geq C |u|_{H_1}, \qquad \forall u \in V. \tag{5}$$

Here and below, C represents a positive generic constant which may depend on Ω, Γ, $\mathscr{A}$, G, T, and does not depend on the time, or on the input data φ_1, φ_2, or u_0, and whose value may change from line to line.

Let

$$\langle u, v \rangle_V = \langle \epsilon(u), \epsilon(v) \rangle_{\mathscr{H}}$$

be the inner product on V. Then, by (5), the norms $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent on V, and $(V, |\cdot|_V)$ is a Hilbert space.

Next, we denote by $f(t)$ the element of V' given by

$$\langle f(t), v\rangle_{V', V} = \langle \varphi_1(t), v\rangle_H + \langle \varphi_2(t), \gamma_0 v\rangle_{L^2(\Gamma_2)^d}, \qquad \forall v \in V,\ t \in [0, T], \tag{6}$$

where $\gamma_0 v$ is the trace over Γ of the vector v, and we denote by $j\colon \mathscr{H}_1 \times V \to \mathbb{R}$ the friction functional

$$j(\sigma, v) = \int_{\Gamma_3} \mu(a)|R\sigma_\nu(a)||v_\tau(a)|\, da. \tag{7}$$

We write

$$\mu = |\mu|_{L^\infty(\Gamma_3)},$$

the L^∞ norm of μ. Now, let us introduce the convex set K of admissible displacements, defined by

$$K = \{v \in V | v_\nu \leq 0, \text{ on } \Gamma_3\}.$$

Following Ref. 4 for the formulation of the unilateral contact condition, we introduce the space $H(\Gamma_3)$ as the set of restrictions to Γ_3 of the $H^{1/2}(\Gamma)$ functions which are null on Γ_1. For every $\sigma \in \mathscr{H}_1$, let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H(\Gamma_3)$ and its dual, with

$$\langle \sigma_\nu, v_\nu \rangle = \int_{\Gamma_3} \sigma_\nu v_\nu\, da, \qquad \forall v \in V,$$

in regular situations and which can be defined rigorously by duality with the Green formula in nonsmooth cases.

Existence and uniqueness results for the bilateral viscoelastic frictional problem with regularized contact stress were studied by Shillor and Sofonea in Ref. 2 in the quasistatic case. We refer also to the bibliography of this paper for variant friction laws.

Even for the simplest friction law (as the Tresca law) the question of the existence of a solution was an open problem in the case of unilateral contact and viscoelastic law. It is here (to our knowledge) the first result for both unilateral contact and viscoelastic law.

In Ref. 6, Cocu, Pratt, and Raous solved the existence question for a quasistatic unilateral contact problem with a Coulomb friction law within the linearized elasticity constitutive relationship. We refer to this paper concerning the way of modelling the unilateral contact condition and the link between its variational formulation and the boundary conditions (2) above. A recent derivation of dynamic thermoviscoelastic frictional contact models from thermodynamical considerations can be found in Refs. 5–7.

Finally, we refer to Refs. 8–11 for modelization and analysis of other systems related to friction and particularly to Ref. 11 where a discretization method is used.

We now give a description of what follows. Our main result is stated in Section 2. In Section 3, we present and solve a discretization method with respect to the time variable: we will see that it leads to auxiliary static problems which involve optimization on convex sets that depend on the discretization step. In Section 4, we construct a sequence of vector functions which solve approximate problems and we prove an estimate on it. The end of proof of our mean result (Theorem 2.1) is given in Section 5.

2. Main Results

In this short section, we list the hypotheses on the data involved in the constitutive law and the forces; we state our main existence result.

Hypotheses on the Viscosity Tensor $\mathscr{A}$. $\mathscr{A}:\Omega \times S_d \to S_d$ is a symmetric definite-positive tensor, which means that

$$\mathscr{A}_{ijkl} \in L^\infty(\Omega), \qquad \forall i, j, k, l = 1, \cdots, d, \tag{8a}$$

$$\mathscr{A}\sigma \cdot \tau = \sigma \cdot \mathscr{A}\tau, \qquad \forall \sigma, \tau \in S_d, \text{ a.e. in } \Omega, \tag{8b}$$

$$\text{there exists } \alpha > 0 \text{ such that } \mathscr{A}\sigma \cdot \sigma \geq \alpha|\sigma|^2, \qquad \forall \sigma \in S_d. \tag{8c}$$

Hypotheses on the Nonlinearity Elasticity Map G. $G: \Omega \times S_d \to S_d$, with

there exists $L_1 > 0$ such that

$$|G(x, \epsilon_1) - G(x, \epsilon_2)| \leq L_1 |\epsilon_1 - \epsilon_2|, \qquad \forall \epsilon_1, \epsilon_2 \in S_d, \text{ a.e. in } \Omega, \tag{9a}$$

$$x \mapsto G(x, \epsilon) \text{ is Lebesgue measurable on } \Omega, \qquad \forall \epsilon \in S_d, \tag{9b}$$

$$x \mapsto G(x, 0) \in \mathscr{H}. \tag{9c}$$

Let us notice that we do not need the existence of $\alpha > 0$ such that

$$(G(x, \epsilon), \epsilon) \geq \alpha|\epsilon|^2.$$

Hypotheses on Forces. Here,

$$\varphi_1 \in L^\infty(0, T; H), \qquad \varphi_2 \in L^\infty(0, T; L^2(\Gamma_2)^d). \tag{10}$$

Now, we can write the variational formulation of problem (3).

Problem P. Find a displacement field $u\colon [0,T]\to K$ and a stress field $\sigma\colon [0,T]\to \mathscr{H}_1$ such that

$$\sigma = \mathscr{A}(\epsilon(\dot u)) + G(\epsilon(u)), \qquad \text{a.e. } t\in(0,T), \tag{11}$$

$$\langle \sigma(t), \epsilon(v) - \epsilon(\dot u(t))\rangle_{\mathscr{H}} + j(\sigma(t), v) - j(\sigma(t), \dot u(t))$$
$$\geq \langle f(t), v - \dot u(t)\rangle_{V',V} + \langle \sigma_\nu(t), v_\nu - \dot u_\nu(t)\rangle, \qquad \forall v\in V, \text{ a.e. } t\in(0,T), \tag{12}$$

$$\langle \sigma_\nu(t), w_\nu - u_\nu(t)\rangle \geq 0, \qquad \forall w\in K, \quad \forall t\in[0,T], \tag{13}$$

$$u(0) = u_0. \tag{14}$$

Now, we can state our result.

Theorem 2.1. Assume that $u_0\in K$ and that (8a)–(8c), (9a)–(9c), and (10) hold. There exists $\mu_0 > 0$, which depends only on Ω, Γ, $\mathscr{A}$ such that, if $|\mu|_{L^\infty(\Gamma_3)} \leq \mu_0$, then there exists a solution (u, σ) of the Problem P satisfying

$$u\in W^{1,\infty}(0,T;V)\cap C([0,T];K), \qquad \sigma\in L^\infty(0,T;\mathscr{H}_1).$$

Remark 2.1.

(i) Let us precise that, in the case of bilateral contact (which means that condition $u_\nu = 0$ on Γ_3 replaces the Signorini contact conditions that we consider), the variational formulation does not contain anymore inequality (13): then, it leads to a well-posed optimization problem with respect to the velocity $\dot u$ on the space $V_0 = V\cap[\dot u_\nu = 0 \textit{ on } \Gamma_3]$.

(ii) We could have taken data φ_1 and φ_2 with the regularity L^2 in the time variable; then, we would obtain the existence of $u\in H^1(0,T;K)$ and $\sigma\in L^2(0,T;\mathscr{H}_1)$.

(iii) The uniqueness problem of a solution is an open problem.

3. Auxiliary Problems

In the sequel, we need the following notations. We denote by $a\colon V\times V\to\mathbb{R}$ the bilinear form defined by

$$a(u,v) = \langle \mathscr{A}(\epsilon(u)), \epsilon(v)\rangle_{\mathscr{H}},$$

and we denote by $b\colon V\times V\to\mathbb{R}$ the map defined by

$$b(u,v) = \langle G(\epsilon(u)), \epsilon(v)\rangle_{\mathscr{H}}.$$

Notice that b is only linear with respect to the second argument.

With these notations, Problem P can be written as follows.

Problem P. Find a displacement field $u\colon [0,T] \to K$ and a stress field $\sigma\colon [0,T] \to \mathscr{H}_1$ such that

$$\sigma = \mathscr{A}(\epsilon(\dot{u})) + G(\epsilon(u)), \qquad \text{a.e. } t \in (0, T), \tag{15}$$

$$a(\dot{u}(t), v - \dot{u}(t)) + b\,(u(t), v - \dot{u}(t)) + j(\sigma(t), v) - j(\sigma(t), \dot{u}(t))$$
$$\geq \langle f(t), v - \dot{u}(t)\rangle_{V', V} + \langle \sigma_\nu(t), v_\nu - \dot{u}_\nu(t)\rangle, \qquad \forall v \in V, \text{ a.e. } t \in (0, T), \tag{16}$$

$$\langle \sigma_\nu(t), w_\nu - u_\nu(t)\rangle \geq 0, \qquad \forall w \in K, \forall t \in [0, T], \tag{17}$$

$$u(0) = u_0. \tag{18}$$

We introduce now auxiliary problems that we derive from the discretization. For this, let $N \in \mathbb{N}$; we write

$$h = T/N \text{ and } t_n = nh, \qquad \text{for } 0 \leq n \leq N.$$

We consider a sequence of elements $f^n \in V'$, $1 \leq n \leq N$, such that

$$\sum_k f^{k+1} 1_{]t_k, t_{k+1}]} \to f, \qquad \text{strongly in } L^2(0, T; V'). \tag{19}$$

Then, we turn to the study of the following sequence of variational formulations.

Problem P^{n+1}. For $0 \leq n \leq N-1$, find a displacement field $u^{n+1} \in K$ such that, writing

$$\delta u^{n+1} = (u^{n+1} - u^n)/h, \tag{20a}$$

$$\sigma^{n+1} = \mathscr{A}(\epsilon(\delta u^{n+1})) + G(\epsilon(u^n)), \tag{20b}$$

we obtain

$$a(\delta u^{n+1}, v - \delta u^{n+1}) + b\,(u^n, v - \delta u^{n+1}) + j(\sigma^{n+1}, v) - j(\sigma^{n+1}, \delta u^{n+1})$$
$$\geq \langle f^{n+1}, v - \delta u^{n+1}\rangle_{V', V} + \langle \sigma_\nu^{n+1}, v_\nu - \delta u_\nu^{n+1}\rangle, \qquad \forall v \in V, \tag{21}$$

$$\langle \sigma_\nu^{n+1}, w_\nu - u_\nu^{n+1}\rangle \geq 0, \qquad \forall w \in K, \tag{22}$$

$$u^0 = u_0. \tag{23}$$

We prove the following lemma.

Lemma 3.1. There exists $\mu_0 > 0$ such that, for $|\mu|_{L^\infty(\Gamma_3)} \leq \mu_0$ and for every $N \geq 0$, Problem P^{n+1} has a unique solution $u^{n+1} \in K$ for $0 \leq n \leq N-1$.

Proof. In order to solve Problem P^{n+1}, we write

$$K_h^{n+1} = (K - u^n)/h, \qquad \text{for all } n = 0, \dots, N-1.$$

The sets K_h^{n+1} are closed in V, convex, and it is important to notice that

$$\delta u^{n+1} + \mathscr{D}(\Omega)^d \subset K_h^{n+1}, \tag{24}$$

where $\mathscr{D}(\Omega)$ is the space of $C^\infty(\Omega)$ functions with compact support in Ω.

On the one hand, assertion (24) is the main point in order to see that inequalities (20)–(23) are equivalent to the following systems (in the sense that they represent their variational formulation):

$$(S^{n+1}): \quad \sigma^{n+1} = \mathscr{A}(\epsilon(\delta u^{n+1})) + G(\epsilon(u^n)), \qquad \text{in } \Omega, \tag{25a}$$

$$\operatorname{Div} \sigma^{n+1} + \varphi_1^{n+1} = 0, \qquad \text{in } \Omega, \tag{25b}$$

$$u^{n+1} = 0, \qquad \text{on } \Gamma_1, \tag{25c}$$

$$\sigma^{n+1}\nu = \varphi_2^{n+1}, \qquad \text{on } \Gamma_2, \tag{25d}$$

$$u_\nu^{n+1} \le 0, \quad \sigma_\nu^{n+1} \le 0, \quad \sigma_\nu^{n+1} u_\nu^{n+1} = 0, \quad |\sigma_\tau^{n+1}| \le \mu |R\sigma_\nu^{n+1}|, \tag{25e}$$

with on Γ_3,

$$|\sigma_\tau^{n+1}| < \mu |R\,\sigma_\nu^{n+1}| \Rightarrow u_\tau^{n+1} = u_\tau^n, \tag{25f}$$

$$|\sigma_\tau^{n+1}| = \mu |R\,\sigma_\nu^{n+1}| \Rightarrow \exists \lambda \ge 0, \qquad \text{with } \sigma_\tau^{n+1} = -\lambda(\delta u_\tau^{n+1}). \tag{25g}$$

On the other hand, it is easy to see that (22) is equivalent to

$$\langle \sigma_\nu^{n+1}, w_\nu - \delta u_\nu^{n+1} \rangle \ge 0, \qquad \forall w \in K_h^{n+1}. \tag{26}$$

In order to solve (20)–(23), we use the following iterative method involving a fixed-point argument at each step.

Step 1. First, we recall that $u^0 \in K$ is known. For $g \in \mathscr{H}_1$, we consider the following optimization problem:

$$J^1(\delta u^1) = \min_{v \in K_h{}^1} J^1(v), \tag{27a}$$

$$J^1(v) = (1/2)\, a(v, v) + b\,(u^0, v) + j(g, v) - \langle f^1, v \rangle_{V', V}. \tag{27b}$$

The functional J^1 is proper, continuous, strictly convex, and coercive on the closed and convex set K_h^1. Therefore, the problem (27) has a unique solution $\delta u^1 \in K_h^1$. Then, we define $u^1 = u^1(g) \in V$ and $\sigma^1 = \sigma^1(g)$ by

$$u^1 = u^0 + h\delta u^1,$$

$$\sigma^1 = \mathscr{A}(\epsilon(\delta u^1)) + G(\epsilon(u^0)).$$

Since

$$K = u^0 + hK_h^1,$$

we deduce that $u^1 \in K$.

Writing the optimality condition of the minimization problem (27), we obtain that

$$a(\delta u^1, w - \delta u^1) + b(u^0, w - \delta u^1) + j(g, w) - j(g, \delta u^1)$$
$$\geq \langle f^1, w - \delta u^1 \rangle_{V', V}, \qquad \forall w \in K_h^1. \tag{28}$$

Let us prove that

$$\langle \sigma_\nu^1, w_\nu - u_\nu^1 \rangle \geq 0, \qquad \forall w \in K. \tag{29}$$

Using (24), for every $\varphi \in \mathscr{D}(\Omega)^d$, we see that the function $w = \delta u^1 + \varphi$ is in K_h^1; hence, inequality (28) leads easily to

$$\mathrm{Div}(\sigma^1) + \varphi_1^1 = 0, \qquad \text{in } \Omega,$$

in the distribution sense. This allows us to define the normal stress tensor trace $\sigma^1 \nu \in H^{-1/2}(\Gamma)^d$.

For any test function $\varphi \in K$, we write

$$w = (\varphi - u^0)/h.$$

We have $w \in K_h^1$ and

$$w - \delta u^1 = (\varphi - u^1)/h.$$

Using the equation

$$\mathrm{Div}(\sigma^1) + \varphi_1^1 = 0,$$

assertion (28), and the Green formula, one computes easily that, for every $\varphi \in K$ with $\varphi = 0$ on Γ_2,

$$\langle \sigma^1 \nu, \varphi - u^1 \rangle + j(g, \varphi - u^0) - j(g, u^1 - u^0) \geq 0.$$

In order to obtain (29) and the boundary condition on Γ_3 given in the system S^1, we split the normal and tangential part in this last inequality.

We now prove that, for $|\mu|_{L^\infty(\Gamma_3)}$ small enough, the map $\Lambda: g \in \mathscr{H}_1 \to \sigma^1 \in \mathscr{H}_1$ has a unique fixed point.

For g_i in $\mathscr{H}_1$, with $i = 1, 2$, we write

$$\Lambda(g_i) = \sigma_i^1,$$

δu_i^1 the solution of the minimization problem (27) with respect to g_i, and

$$u_i^1 = u^0 + h\delta u_i^1.$$

Taking as test function $v = \delta u_j^1$ in the inequality (28) satisfied by u_i^1, $i \neq j$, and adding them, we obtain

$$a(\delta u_1^1 - \delta u_2^1, \delta u_1^1 - \delta u_2^1) - \langle \sigma_{2\nu}^1 - \sigma_{1\nu}^1, \delta u_{2\nu}^1 - \delta u_{1\nu}^1 \rangle$$
$$\leq j(g_1, \delta u_2^1) - j(g_1, \delta u_1^1) + j(g_2, \delta u_1^1) - j(g_2, \delta u_2^1). \tag{30}$$

Moreover, using condition (29) with

$$w = \delta u_{j\nu}^1 \in K_h^1$$

in the inequality satisfied by σ_i^1, $i \neq j$, one gets easily

$$\langle \sigma_{2\nu}^1 - \sigma_{1\nu}^1, \delta u_{2\nu}^1 - \delta u_{1\nu}^1 \rangle \leq 0. \tag{31}$$

Combining (30), (31), and (7), there exists a constant $c > 0$, which depends on R and $\mathscr{A}$, such that, for every $\mu > 0$, we obtain

$$a(\delta u_1^1 - \delta u_2^1, \delta u_1^1 - \delta u_2^1) \leq c\mu |g_1 - g_2|_{\mathscr{H}_1} |\delta u_2^1 - \delta u_1^1|_{L^2(\Gamma_3)^d}, \tag{32}$$

which leads easily to the existence of a constant $\mu_0 > 0$ such that, for every $0 < |\mu|_{L^\infty(\Gamma_3)} \leq \mu_0$, we can write

$$|\Lambda(g_1) - \Lambda(g_2)|_{\mathscr{H}_1} \leq k |g_1 - g_2|_{\mathscr{H}_1}, \tag{33}$$

with $0 < k < 1$. Then, we end with the Banach fixed-point theorem: the map Λ possesses a unique fixed point denoted by σ^1. It is straightforward to see that the corresponding solution solves

$$a(\delta u^1, w - \delta u^1) + b(u^0, w - \delta u^1) + j(\sigma^1, w) - j(\sigma^1, \delta u^1)$$
$$\geq \langle f^1, w - \delta u^1 \rangle_{V', V}, \qquad \forall w \in K_h^1, \tag{34}$$

$$\langle \sigma_\nu^1, w_\nu - u_\nu^1 \rangle \geq 0, \qquad \forall w \in K. \tag{35}$$

In order to see that this solution solves problem P^1, we use the equivalence between Problem P^1, the system S^1, and (34), (35).

Step n + 1. In this step, we assume that we have constructed u^1, u^2, $\ldots$, $u^n = u^{n-1} + h\delta u^n$ elements of K, σ^1, $\sigma^2, \ldots,$ $\sigma^n = \mathscr{A}(\epsilon(\delta u^n)) + G(\epsilon(u^{n-1}))$ elements of $\mathscr{H}_1$, and we consider Problem P^{n+1}.

Following the first step, we introduce for $g \in \mathscr{H}_1$ the solution

$$\delta u^{n+1} = \delta u^{n+1}(g) \in K_h^{n+1}$$

of

$$J^{n+1}(\delta u^{n+1}) = \inf_{v \in K_h^{n+1}} J^{n+1}(v), \tag{36a}$$

where

$$J^{n+1}(v) = (1/2)\, a(v, v) + b(u^n, v) + j(g, v) - \langle f^{n+1}, v \rangle_{V', V}. \tag{36b}$$

The functional J^{n+1} is proper, continuous, strictly convex, and coercive on the closed convex set K_h^{n+1}. Therefore, the problem (36) has a unique solution

$$\delta u^{n+1} \in K_h^{n+1}.$$

Then, we write

$$u^{n+1} = u^n + h\delta u^{n+1},$$

$$\sigma^{n+1} = \mathscr{A}(\epsilon(\delta u^{n+1})) + G(\epsilon(u^n)).$$

By definition of K_h^{n+1}, we get $u^{n+1} \in K$. Furthermore, since

$$\delta u^{n+1} + \mathscr{D}(\Omega)^d \subset K_h^{n+1},$$

one can prove easily that

$$\mathrm{Div}(\sigma^{n+1}) + \varphi_1^{n+1} = 0, \qquad \text{in } \Omega,$$

is satisfied, and thus,

$$\sigma^{n+1} \in \mathscr{H}_1.$$

We then argue as in the first step in order to prove that the map $g \to \sigma^{n+1}$ possesses a unique fixed point. The equivalence between the different formulations can then be proved, and we obtain that u^{n+1} and σ^{n+1} solve Problem P^{n+1}. This ends the proof of Lemma 3.1. □

4. Asymptotic Analysis

In Section 3, we have constructed for all $n = 0, \ldots, N-1$ a unique pair of functions $(u^{n+1}, \sigma^{n+1}) \in K \times \mathscr{H}_1$ satisfying Problem P^{n+1}. In order to study the convergence of (u^{n+1}, σ^{n+1}) for all $n = 0, \ldots, N-1$ when $N \to \infty$, we introduce the following functions and tensor:

$$u^N(t) = [(t - t_n)/h]\,(u^{n+1} - u^n) + u^n, \qquad \text{on } [t_n, t_{n+1}], \tag{37a}$$

$$\tilde{u}^N(t) = u^{n+1}, \qquad \tilde{\sigma}^N(t) = \sigma^{n+1}, \qquad \tilde{f}^N(t) = f^{n+1}, \tag{37b}$$

$$\bar{u}^N(t) = u^n, \qquad \forall t \in]t_n, t_{n+1}]. \tag{37c}$$

The vector function u^N is in $W^{1,\infty}(0, T; V) \cap C([0, T]; K)$, both $\bar{u}^N(t)$ and $\tilde{u}^N(t)$ are in $L^\infty(0, T; K)$, and finally, using Section 3, $\tilde{\sigma}^N \in L^\infty(0, T; \mathscr{H}_1)$.

We prove the following lemma.

Lemma 4.1. There exists a pair of functions $(u,\sigma)\in (W^{1,\infty}(0, T; V)\cap C([0, T]; K))\times L^{\infty}(0, T; \mathscr{H}_1)$ such that, after extraction of a subsequence still denoted by $(u^N, \tilde{\sigma}^N)$, we have

$$u^N \rightharpoonup u, \qquad \text{weak* in } L^{\infty}(0, T; K), \tag{38}$$

$$\dot{u}^N \rightharpoonup \dot{u}, \qquad \text{weak* in } L^{\infty}(0, T; V), \tag{39}$$

$$\tilde{\sigma}^N \rightharpoonup \sigma, \qquad \text{weak* in } L^{\infty}(0, T; \mathscr{H}_1), \tag{40}$$

$$u^N \to u, \qquad \text{strongly in } C([0, T]; K). \tag{41}$$

Proof. Using (20)–(23) and (37), we obtain, for almost every t,

$$\tilde{\sigma}^N(t) = \mathscr{A}(\epsilon(\dot{u}^N(t))) + G(\epsilon(\bar{u}^N(t))), \tag{42}$$

$$a(\dot{u}^N(t), v - \dot{u}^N(t)) + b(\bar{u}^N(t), v - \dot{u}^N(t)) + j(\tilde{\sigma}^N(t), v) - j(\tilde{\sigma}^N(t), \dot{u}^N(t))$$

$$\geq \langle \tilde{f}^N(t), v - \dot{u}^N(t)\rangle_{V', V} + \langle \tilde{\sigma}_\nu^N(t), v_\nu - \dot{u}_\nu^N(t)\rangle, \qquad \forall v\in V, \tag{43}$$

$$\langle \tilde{\sigma}_\nu^N(t), w_\nu - \tilde{u}_\nu^N(t)\rangle \geq 0, \qquad \forall w\in K, \quad \forall t\in]t_n, t_{n+1}]. \tag{44}$$

A Priori Estimate I. As $u^n\in K$, we have $0\in K_h^{n+1}$; thus, taking $w=0$ as test function in (26) and using (37a)–(37b), we obtain

$$\langle \tilde{\sigma}_\nu^N(t), \dot{u}_\nu^N(t)\rangle \leq 0, \qquad \text{a.e. } t\in (t_n, t_{n+1}). \tag{45}$$

Taking now as test function $v=0$ in inequality (43), and since $j(\tilde{\sigma}^N(t), \dot{u}^N(t))\geq 0$, we get, for $t_n < t < t_{n+1}$,

$$a(\dot{u}^N(t), \dot{u}^N(t)) \leq \langle \tilde{f}^N(t), \dot{u}^N(t)\rangle_{V', V} - b(\bar{u}^N(t), \dot{u}^N(t)). \tag{46}$$

Using the V-ellipticity of $a(\cdot, \cdot)$, (37a)–(37b), and the hypotheses on the map G(which is supposed to be Lipschitz), we obtain

$$|\dot{u}^N(t)|_V \leq C\,(|\tilde{f}^N(t)|_{V'} + |\bar{u}^N(t)|_V + |G(0)|_{\mathscr{H}}),$$

$$\leq C\,(|f|_{L^\infty(0, T; V')} + |u_0|_V + \int_0^t |\dot{u}^N(s)|_V\, ds + |G(0)|_{\mathscr{H}}). \tag{47}$$

Therefore, by the Gronwall lemma, it follows that, almost everywhere in time,

$$|\dot{u}^N(t)|_V \leq C\,(|f|_{L^\infty(0, T; V')} + |u_0|_V + |G(0)|_{\mathscr{H}}). \tag{48}$$

Then, we deduce that

$$|u^N(t)|_V \leq \int_0^t |\dot{u}^N(s)|_V\, ds + |u_0|_V \leq C\,(|f|_{L^\infty(0, T; V')} + |u_0|_V + |G(0)|_{\mathscr{H}}). \tag{49}$$

From (48), (49), and (42), it results that

$$(u^N) \text{ is a bounded sequence in } L^\infty(0, T; K), \tag{50}$$

$$(\dot{u}^N) \text{ is a bounded sequence in } L^\infty(0, T; V), \tag{51}$$

$$(\tilde{\sigma}^N) \text{ is a bounded sequence in } L^\infty(0, T; \mathscr{H}_1). \tag{52}$$

A Priori Estimate II. In order to show the strong convergence of $(u^N)_N$ stated in (41), first we notice that (43) implies that, taking $v = 2\dot{u}^N(t)$,

$$\begin{aligned} &a(\dot{u}^N(t), v) + b(\bar{u}^N(t), v) + j(\tilde{\sigma}^N(t), v) \\ &\geq \langle \tilde{f}^N(t), v\rangle_{V', V} + \langle \tilde{\sigma}^N_\nu(t), v_\nu\rangle, \qquad \forall v \in V. \end{aligned}$$

We take

$$v = u^N(t) - u^{N+p}(t)$$

in this inequality at the order $N+p$ and

$$v = u^{N+p}(t) - u^N(t)$$

at the order N in the same inequality. Adding them, we obtain, for almost every $t \in (0, T)$,

$$\begin{aligned} &a(\dot{u}^{N+p}(t) - \dot{u}^N(t), u^{N+p}(t) - u^N(t)) + b(\bar{u}^{N+p}(t), u^{N+p}(t) - u^N(t)) \\ &- b(\bar{u}^N(t), u^{N+p}(t) - u^N(t)) \\ &\leq j(\tilde{\sigma}^{N+p}(t), u^N(t) - u^{N+p}(t)) + j(\tilde{\sigma}^N(t), u^{N+p}(t) - u^N(t)) \\ &+ \langle \tilde{f}^{N+p}(t) - \tilde{f}^N(t), u^{N+p}(t) - u^N(t)\rangle_{V', V} \\ &+ \langle \tilde{\sigma}^{N+p}_\nu(t) - \tilde{\sigma}^N_\nu(t), u^{N+p}_\nu(t) - u^N_\nu(t)\rangle. \end{aligned} \tag{53}$$

Integrating (53) with respect to time on the interval $[0, t]$, and after some computations, we obtain

$$\begin{aligned} &(1/2)\, a(u^{N+p}(t) - u^N(t), u^{N+p}(t) - u^N(t)) \\ &\leq |\mu|_{L^\infty(\Gamma_3)} \sup_N |R\,\tilde{\sigma}^N_\nu|_{L^1(0, T; L^2(\Gamma_3))} |u^{N+p}_\tau - u^N_\tau|_{L^2(0, T; L^2(\Gamma_3)^d)} \\ &\quad + CL_2 \int_0^t |\bar{u}^{N+p}(s) - \bar{u}^N(s)|_V |u^{N+p}(s) - u^N(s)|_V\, ds \\ &\quad + (\gamma/2) \int_0^t |\tilde{f}^{N+p}(s) - \tilde{f}^N(s)|^2_{V'}\, ds \\ &\quad + (1/2\gamma) \int_0^t |u^{N+p}(s) - u^N(s)|^2_V\, ds \\ &\quad + 2 \sup_N |R\,\tilde{\sigma}^N_\nu|_{L^2(0, T, L^2(\Gamma_3))} |u^{N+p}_\nu - u^N_\nu|_{L^2(0, T; L^2(\Gamma_3))}. \end{aligned} \tag{54}$$

From (50)–(52) and using the compact imbedding of $H^{1/2}(\Gamma)$ in $L^2(\Gamma)$, after extraction of a suitable subsequence, it results that, $\forall \epsilon > 0, \exists N_\epsilon, \forall N \geq N_\epsilon, \forall p \in \mathrm{N}, \forall t$, we have

$$|u^{N+p}(t) - u^N(t)|_{L^2(\Gamma_3)^d} \leq \epsilon. \tag{55}$$

On the other hand, since for $t_n \leq t \leq t_{n+1}$ we have

$$|u^N(t) - \bar{u}^N(t)|_V \leq \frac{T}{N} |\dot{u}^N(t)|_V,$$

with the estimate (48), we obtain

$$\begin{aligned} &|\bar{u}^{N+p}(t) - \bar{u}^N(t)|_V \\ &\leq |\bar{u}^{N+p}(t) - u^{N+p}(t)|_V + |u^{N+p}(t) - u^N(t)|_V + |u^N(t) - \bar{u}^N(t)|_V \\ &\leq (C/N)\,(|f|_{L^\infty(0,\,T;\,V')} + |u_0|_V + |G(0)|_{\mathscr{H}}) + |u^{N+p}(t) - u^N(t)|_V. \end{aligned} \tag{56}$$

Assertions (54)–(56) imply that

$$\begin{aligned} &|u^{N+p}(t) - u^N(t)|_V^2 \\ &\leq \epsilon C\,(|\mu|_{L^\infty(\Gamma_3)} + 2) \sup_N |R\,\tilde{\sigma}_\nu^N|_{L^\infty(0,\,T;\,L^2(\Gamma_3))} \\ &\quad + (C/N)\,L_2(f, u_0, G(0)) \int_0^t |u^{N+p}(s) - u^N(s)|_V\, ds \\ &\quad + CL_1 \int_0^t |u^{N+p}(s) - u^N(s)|_V^2\, ds + C\,|\tilde{f}^{N+p} - \tilde{f}^N|^2_{L^2(0,\,T;\,V')}, \end{aligned} \tag{57}$$

with

$$L_2(f, u_0, G(0)) = L_1\,(|f|_{L^2(0,\,T;\,V')} + |u_0|_V + |G(0)|_{\mathscr{H}}).$$

Therefore, using the Gronwall inequality, we obtain

$$|u^{N+p}(t) - u^N(t)|_V \leq CL_3(\sqrt{\epsilon} + T/N + C|\tilde{f}^{N+p} - \tilde{f}^N|_{L^2(0,\,T;\,V')}). \tag{58}$$

From the last inequality and the strong convergence of $\tilde{f}^N$ in $L^2(0, T; V')$, one has (41), and Lemma 4.1 is proved. □

5. Proof of Theorem 2.1

We start this section with the following lemma.

Lemma 5.1. Every pair (u,σ) weak* limit point in $W^{1,\infty}(0, T; V) \times L^\infty(0, T; \mathscr{H}_1)$ of the sequence $(u^N, \tilde{\sigma}^N)_N$ is solution of the variational formulation P.

Proof. With Lemma 4.1, we can suppose that, for such a limit point, the displacement u, which is in $C([0, T]; K)$ since K is convex and closed in V, is also a strong limit point of a subsequence of $(u^N)_N$ in $C([0, T]; K)$.

We now analyze assertions (42)–(44). We begin with (42) and (44). Using the strong convergence in $C([0, T]; V)$ of $(u^N)_N$ to u and

$$|u^N - \bar{u}^N|_{L^2(0, T; K)} \to 0,$$

we deduce that

$$|u - \bar{u}^N|_{L^2(0, T; K)} \to 0.$$

Now, with (9a)–(9b), this leads to

$$|G(\epsilon(u)) - G(\epsilon(\bar{u}^N))|_{L^2(0, T; \mathscr{H})} \to 0. \tag{59}$$

We recall that

$$\tilde{\sigma}^N(t) = \mathscr{A}(\epsilon(\dot{u}^N(t))) + G(\epsilon(\bar{u}^N(t)))$$

hence, one can deduce easily that $\tilde{\sigma}^N$ weakly converges in $L^2(0, T; \mathscr{H})$ to $\mathscr{A}(\epsilon(\dot{u})) + G(\epsilon(u))$, and thus,

$$\sigma(t) = \mathscr{A}(\epsilon(\dot{u}(t))) + G(\epsilon(u(t))), \qquad \text{a.e. } t \in (0, T).$$

From (38)–(41), and taking the limit of inequality (44) when N tends to infinity, we obtain that

$$\langle \sigma_\nu(t), w_\nu(t) - u_\nu(t) \rangle \geq 0, \qquad \forall w \in K. \tag{60}$$

Then, we deduce that a limit point satisfies

$$\langle \sigma_\nu(t), u_\nu(t) \rangle = \langle \sigma_\nu(t), \dot{u}_\nu(t) \rangle = 0. \tag{61}$$

We now study (43). Bearing in mind (19), we obtain

$$\tilde{f}^N \to f, \qquad \text{in } L^2(0, T; V'). \tag{62}$$

We split inequality (43) in two parts and we use (45) for the second part.

Part (i). For almost every $t \in (0, T)$, for every $v \in V$,

$$a(\dot{u}^N(t), v) + b\,(\bar{u}^N(t), v) + j(\tilde{\sigma}^N(t), v) \geq \langle \tilde{f}^N(t), v \rangle_{V', V} + \langle \tilde{\sigma}_\nu^N(t), v_\nu \rangle.$$

Part (ii). For almost every $t \in (0, T)$,

$$a(\dot{u}^N(t), \dot{u}^N(t)) + b\,(\bar{u}^N(t), \dot{u}^N(t)) + j(\tilde{\sigma}^N(t), \dot{u}^N(t)) \leq \langle \tilde{f}^N(t), \dot{u}^N(t) \rangle_{V', V}.$$

In order to take the limit in Part (i), we consider $v \in L^2(0, T; V)$, we use again (38)–(41), and since $|R\tilde{\sigma}_\nu^N| = -R\tilde{\sigma}_\nu^N$ on Γ_3, we get

$$\int_0^T (a(\dot{u}(t), v(t)) + b(u(t), v(t)) + j(\sigma(t), v(t)))\, dt$$

$$\geq \int_0^T \langle f(t), v(t)\rangle_{V', V} dt + \int_0^T \langle \sigma_\nu(t), v_\nu(t)\rangle dt, \tag{63}$$

and after a classical application of Lebesgue points for L^1, we obtain

$$a(\dot{u}(t), v(t)) + b(u(t), v(t)) + j(\sigma(t), v(t))$$

$$\geq \langle f(t), v(t)\rangle_{V', V} + \langle \sigma_\nu(t), v_\nu(t)\rangle, \qquad \forall v \in V. \tag{64}$$

In the study of Part (ii), we introduce the nonnegative function

$$F(\dot{u}^N(t)) = a(\dot{u}^N(t), \dot{u}^N(t)) + j(\tilde{\sigma}^N(t), \dot{u}^N(t)).$$

We have

$$F(\dot{u}^N(t)) + b(\bar{u}^N(t), \dot{u}^N(t)) \leq \langle \tilde{f}^N(t), \dot{u}^N(t)\rangle_{V', V}, \tag{65}$$

and using (61)–(63), it is sufficient to prove that

$$F(\dot{u}(t)) + b(u(t), \dot{u}(t)) \leq \langle f(t), \dot{u}(t)\rangle_{V', V}$$

in order to ensure (12). This will follow from the lemma below.

Lemma 5.2. There exists $\mu_0 > 0$ such that, if $|\mu|_{L^\infty(\Gamma_3)} \leq \mu_0$, then

$$F(\dot{u}^N(t)) \rightharpoonup F(\dot{u}(t)) \qquad \text{weakly in } L^2(0, T). \tag{66}$$

Proof. Let $\psi \in L^2(0, T)$ such that

$$\psi(t) \geq 0, \qquad \text{for almost every } t \in [0, T].$$

Using (65), we get

$$\int_0^T \psi(t) F(\dot{u}^N(t)) dt \leq \int_0^T \psi(t)[\langle \tilde{f}^N(t), \dot{u}^N(t)\rangle_{V', V} - b(\bar{u}^N(t), \dot{u}^N(t))]\, dt.$$

With Lemma 4.1 and the strong convergence of $\bar{u}^N$ in $L^2(0, T; V)$, we get

$$\limsup_{N \to +\infty} \int_0^T \psi(t) F(\dot{u}^N(t))\, dt \leq \int_0^T \psi(t)[\langle f(t), \dot{u}(t)\rangle_{V', V} - b(u(t), \dot{u}(t))]\, dt.$$

Using now (64) with $v=\dot u$ and (61), we deduce that

$$\int_0^T \psi(t)[\langle f(t), \dot u(t)\rangle_{V',V} - b\,(u(t), \dot u(t))]\,dt \le \int_0^T \psi(t)F(\dot u(t))\,dt$$

hence,

$$\limsup_{N\to+\infty} \int_0^T \psi(t)F(\dot u^N(t))\,dt \le \int_0^T \psi(t)F(\dot u(t))\,dt. \tag{67}$$

On the other hand, we write

$$\begin{aligned} F(\dot u^N(t)) &= a(\dot u^N(t), \dot u(t)) + a(\dot u^N(t)-\dot u(t), \dot u^N(t)-\dot u(t)) + a(\dot u(t), \dot u^N(t)-\dot u(t)) \\ &\quad + \int_{\Gamma_3} \mu|R(\tilde\sigma_\nu^N(t))||\dot u_\tau(t)|da \\ &\quad + \int_{\Gamma_3} \mu(|R(\tilde\sigma_\nu^N(t))| - |R(\sigma_\nu(t))|)\,(|\dot u^N(t)| - |\dot u_\tau(t)|)\,da \\ &\quad + \int_{\Gamma_3} \mu|R(\sigma_\nu(t))|(|\dot u_\tau^N(t)| - |\dot u_\tau(t)|)\,da. \end{aligned}$$

It is easy to see that, for $|\mu|_{L^\infty(\Gamma_3)}$ small enough, we have

$$\begin{aligned} &a(\dot u^N(t)-\dot u(t), \dot u^N(t)-\dot u(t)) + \int_{\Gamma_3} \mu(|R(\tilde\sigma_\nu^N(t))| - |R(\sigma_\nu(t))|)(|\dot u_\tau^N(t)| - |\dot u_\tau(t)|)\,da \\ &\ge a(\dot u^N(t)-\dot u(t), \dot u^N(t)-\dot u(t)) - c|\mu|_{L^\infty(\Gamma_3)}|\tilde\sigma^N(t)-\sigma(t)|_{\mathscr{H}_1}|\dot u^N(t)-\dot u(t)|_V \\ &\ge (c_\alpha - c|\mu|_{L^\infty(\Gamma_3)})|\dot u^N(t)-\dot u(t)|_V^2 - c|\mu|_{L^\infty(\Gamma_3)}|G(\epsilon(\bar u^N(t))) \\ &\quad - G(\epsilon(u(t)))|_{\mathscr{H}}\,|\dot u^N(t)-\dot u(t)|_V \\ &\ge -c|\mu|_{L^\infty(\Gamma_3)}|G(\epsilon(\bar u^N(t))) - G(\epsilon(u(t)))|_{\mathscr{H}}\,|\dot u^N(t)-\dot u(t)|_V, \end{aligned}$$

which proves, thanks to the strong convergence of $\tilde u^N$ and $\bar u^N$ to u in $L^2(0, T; V)$, that

$$\begin{aligned} \liminf_{N\to+\infty} \int_0^T \psi(t)[&a(\dot u^N(t)-\dot u(t), \dot u^N(t)-\dot u(t)) \\ &+ \int_{\Gamma_3} \mu(|R(\tilde\sigma_\nu^N(t))| - |R(\sigma_\nu(t))|)\,(|\dot u_\tau^N(t)| - |\dot u_\tau(t)|)\,da]\,dt \ge 0. \end{aligned}$$

This proves that

$$\liminf_{N\to+\infty} \int_0^T \psi(t)F(\dot u^N(t))\,dt \ge \int_0^T \psi(t)F(\dot u(t))\,dt. \tag{68}$$

Combining (67) and (68), we deduce that, for $\psi \geq 0$ and $\psi \in L^2(0, T)$, we get

$$\lim_{N \to +\infty} \int_0^T \psi(t) F(\dot{u}^N(t))\, dt = \int_0^T \psi(t) F(\dot{u}(t))\, dt,$$

which is sufficient in order to end the proof of Lemma 5.2 (writing $\psi = \psi^+ - \psi^-$ in the general case). □

We have proved that, in the weak* sense of Lemma 5.1, each limit point (u, σ) of the sequence $(u^N, \tilde{\sigma}^N)$ is a solution of Problem P. Since the set of such points is not empty (by Lemma 4.1), this proves Theorem 2.1.

□

References

1. Kikuchi, N., and Oden, J. T., *Contact Problems in Elasticity*, SIAM, Philadelphia, Pennsylvania, 1988.
2. Shillor, M., and Sofonea, M., *A Quasistatic Viscoelastic Contact Problem with Friction*, International Journal of Engineering Sciences, Vol 38, pp. 212–236, 2000.
3. Hlavacek, I., and Necas, J., *Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction*, Elsevier, Amsterdam, Holland, 1981.
4. Cocu, M., Pratt, E., and Raous, M., *Formulation and Approximation of Quasistatic Frictional Contact*, International Journal of Engineering Sciences, Vol 34, pp. 783–798. 1996.
5. Andrews, K. T., Kuttler, K. L., and Shillor, M., *On the Dynamic Behavior of a Thermoviscoelastic Body in Frictional Contact with a Rigid Obstacle*, European Journal of Applied Mathematics, Vol 8, pp. 417–436, 1997.
6. Shillor, M., and Rochdi, M., *A Dynamic Thermoviscoelastic Frictional Contact Problem with Damped Response*, Preprint, 1998.
7. Shillor, M., Editor, *Recent Advances in Contact Mechanics*, Mathematical Computational Modelling, Vol 28, pp. 4–8, 1998.
8. Amassad, A., and Sofonea, M., *Analysis of a Quasistatic Viscoplastic Problem Involving the Tresca Friction Law*, Discrete and Continuous Dynamical Systems, Vol 4, pp. 55–72, 1998.
9. Amassad, A., Shillor, M., and Sofonea, M., *A Quasistatic Contact Problem with Slip Dependent Coefficient of Friction*, Mathematical Methods in the Applied Sciences, Vol 22, pp. 267–284, 1999.
10. Duvaut, G., and Lions, J. L., *Les Inéquations en Mécanique et en Physique*, Dunod, Paris, France, 1972.
11. Glowinski, R., Lions, J. L., and Tremolieres, R., *Analyse Numérique des Inéquations Variationnelles*, Collection des Méthodes Mathématiques de l'Informatique, 1981.