On abstract variational inequalities in viscoplasticity with frictional contact

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abstract

In this paper, we study quasistatic abstract variational inequalities with time dependent constraints. We prove existence results and we present an approximation method valid for non smooth constraints. We then apply our results to the approximation (in view of a numerical study) of a quasistatic evolution of an elastic body in bilateral contact with a rigid foundation. The contact involves viscous friction of Tresca or Coulomb type. We prove existence results for approximate problems and we give the full asymptotic analysis.

Keywords. Quasistatic frictional contact, regularization, variational inequalities, asymptotic analysis.

1. Introduction

Modelling bilateral contact with Coulomb friction law problems between an elastic body and a rigid obstacle leads to variational inequalities of the form : find $u \in H^1(0,T;V)$ with

$$(I1) \qquad \forall v \in V, \quad a(u(t), v - \dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle,$$

where V is a Banach space, $\dot{u}$ denotes the time derivative of the quantity $u(t,x)$ and $j(v_1, v_2)$ is a non smooth functional defined on $V \times V$: for example, in case of a Coulomb friction law, $j(v_1, v_2)$ involves $|v_2|$.

The paper is motivated both by the well-posedness and the approximation of such inequalities when one regularizes a non smooth functional j by smooth functionals j_ρ. Constructing auxiliar smooth problems (for which variational inequalities become variational equalities), we prove existence results and give the asymptotic analysis when parameter ρ tends to 0. Notice that, in the whole literature on that subject, no uniqueness result is known.

In order to precise the scientific context of this work, we now detail the main parts of the paper and mention known works on the subject.

In the context of frictional contact, existence results have been obtained by several authors : M. Cocou, E. Pratt and M. Raous proved ([9]) existence of a solution for an elastic body in case of bilateral or unilateral contact and a non local Coulomb friction law in 1996; their results have been generalized by A. Amassad and C. Fabre in the case of an elastic-viscoplastic constitutive relationship in ([4]) in 2002. Related problems have also been studied for viscoelastic body : we refer to [17] in case of a

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bilateral contact and to [5] in case of an unilateral contact. All these papers involve in an essential way a functional j which is supposed to be a seminorm with respect to its second argument. This is a very restrictive assumption when one is interested in general frictional law and their numerical implementations.

We present in Theorem 2.4 an existence result making rid of this assumption on j. Our method uses a fixed point theorem on the first argument arising in the functional j which leads us to consider variational inequalities of the form : find $u \in H^1(0,T;V)$ with

$$(I2) \qquad \forall v \in V, \quad a(u(t), v - \dot{u}(t)) + j(t,v) - j(t, \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle,$$

where j is a functional defined on $[0,T] \times V$.
Our second interest concerns such inequalities (notice that they include the modelling of bilateral contact whith Tresca friction law). We study both the well-posedness and the approximation with respect to j related to them.

Concerning the well-posedness, the following facts are already known : the first way to solve the problem is part of the theory of variational inequalities. This theory has been deeply developped after the seventies and one can see the works of H. Brezis, G. Duvaut, R. Glowinski, D. Kinderlehrer, J.L. Lions, P.D. Panagiotopoulos, G. Stampacchia and L. Tremollieres, ([7] , [12], [13],[15], [16]) for more details. The particular case of evolution (and quasistatic) problems with constraints on the time derivative of the solution has been studied by H. Brezis in ([7]) where he considered convex but not necessarly smooth functional j. In order to prove existence result, his method involved convex duality and subdifferential theory and is based on a very deep and abstract result due to T. Kato saying that solving an equation as $\dot{w}(t) \in Aw(t)$ with A being a multivoque map, can be seen as $\dot{w}(t) = (Aw(t))^{\circ}$ where C° is an element of minimum norm in the convex set C. Roughly speaking, in that result, the map A corresponds to $\partial j^*(t,w)$ and it is supposed to be independent of the time variable. This means that, in the case where $j(t,v) = j(v)$ does not depend on t, existence result of a solution to the above inequality (with smooth or non smooth functional j) is given by H. Brezis. However, notice that even in that case, his method does not yield to a constructive method (but that was not his purpose).

The second way that can be used consists in a discretization in time approximating u by piecewise linear functions. This method is now a classical one : its advantage is that it is a constructive method. Its inconvenience is that it does not apply to the situation where one wants to regularize j because it is only valid in the very restrictive case where the functional j satisfies $j(t, \lambda v) = |\lambda| j(t,v)$.

Let us also point that, when a displacement formulation is used, H. Ben Dhia has given a numerical formulation for the model which gives nice results (see [11]).

In this paper, we first present a constructive method, based on a smoothing effect of the whole inequality, in order to solve variational inequalities $(I2)$. Adding to natural assumptions on j (such as continuity and differentiability in some sense), our main original assumption is

$$\exists B \in L^1(0,T),\ B \geq 0, \quad \exists b > 0, \qquad \forall v \in V, \quad a(v,v) + \frac{\partial j}{\partial t}(t,v) \geq b|v|_V^2 - B(t).$$

We will then give the asymptotic analysis when one considers regular functional j_ρ ($\rho > 0$) converging to a functional j. Two specific cases will be precise for both Coulomb or Tresca friction law : the cases where j involves the modulus function which can be approximate by $\sqrt{x^2+\rho^2}$ or by $|x|^{1+\rho}$. We will prove the strong convergence results in the case $\sqrt{x^2+\rho^2}$ whereas only weak convergence results will be obtained in the other case.

We then end this paper with the study related to a non local Coulomb friction law : using previous results, we prove the existence of solution for the approximate problems. Notice that no uniqueness result is given. We then study the approximation (with respect to j) and the asymptotic analysis and we give convergence results.

In the next section, we precisely state the results which will be proved and the assumptions on the data that will be valid in future sections .

2. Notation and main results

We first introduce few notations that will be used in all the paper.

Let Ω be a bounded and regular open set of $\mathbb{R}^d$ with boundary Γ. We assume that Γ is divided into three disjoint parts $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, with Γ_1 being of non zero measure. We denote by S_d ($d = 2$ or 3) the space of symmetric tensors of order d on $\mathbb{R}^d$ and it is endowed with its natural scalar product. If ν is the unit exterior normal on the boundary Γ, and if v is a vector in $\mathbb{R}^d$, we write $v_\nu = v.\nu$ and $v_\tau = v - v_\nu \nu$ the normal and tangential decomposition of the vector v. In a same way, we write $\sigma_\nu = \sigma\nu.\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$ the normal and tangential components of the vector $\sigma\nu$ for a tensor σ.

We consider the following spaces (repeated indexes convention is used):

$$H = [L^2(\Omega)]^d,$$
$$\mathcal{H} = \{\ (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\ \} = [L^2(\Omega)]_s^{d^2},$$
$$H_1 = [H^1(\Omega)]^d,$$
$$\mathcal{H}_1 = H(Div, \Omega) = \{\ \sigma \in \mathcal{H} |\ (\sigma_{ij,j}) \in H\ \}.$$

All these spaces are endowed with their natural norms and scalar products :

$$(u,v)_H = \int_\Omega u_i v_i\ dx, \quad (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\ dx,$$
$$((u,v))_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}$$

and

$$((\sigma,\tau))_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\ \sigma, Div\ \tau)_H.$$

When no confusion can be made, we will omit the index in the writing of these scalar products and in any case, the notation $\langle\ ,\ \rangle_{X',X}$ will always denote the duality bracket between a space X and its dual X'.
We recall the Green formula (valid in regular cases) :

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\ \sigma, v)_H = \langle \sigma\nu, v\rangle_{H^{-1/2}(\Gamma)^d, H^{1/2}(\Gamma)^d} \quad \forall v \in H_1, \tag{2.1}$$

which allows to define $\sigma\nu \in H^{-1/2}(\Gamma)^d$ for $\sigma \in \mathcal{H}_1$ in order that Green formula still holds.

We define

$$V = \{\ v \in H_1 \mid v = 0 \ \text{ on } \ \Gamma_1\}$$

and V_0 any closed subspace of V. We consider a bilinear continuous form $a = a(u, v)$ on $H_1 \times H_1$ coercive on V.

The abstract variational inequality in which we are interested can be written : let $T > 0$, be fixed, find $u : [0, T] \to V_0$ solution of
Problem (P_j^e) :

$$\begin{cases} \forall v \in V_0, \ \text{a.e. } t \in (0, T), \\ \qquad a(u, v - \dot{u}(t)) + j(t, v) - j(t, \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle_{V',V} \\ u(0) = u^0, \end{cases} \tag{2.2}$$

In (2.2), $\dot{u}$ denotes the time derivative of u and $j = j(t, v)$ is any positive and convex functional defined on $[0, T] \times V$. For simplicity (but this is not restrictive), we will assume that

$$\forall t, \quad j(t, 0) = 0. \tag{2.3}$$

As we already said in the introduction, problem (P_j^e) is part of the theory of variational inequalities for which we refer to ([7] , [12], [13],[15], [16]).

We suppose that j satisfies

$$v \in L^2(0, T; V) \ \mapsto \ \int_0^T j(t, v(t))dt \ \text{ is weakly lower semicontinuous (l.s.c).} \tag{2.4}$$

when we are in the regular case, we suppose that $j(t, .)$ is differentiable on V for every t, and we denote by $j'(t, .)$ the derivative of $j(t, .)$ with respect to v.

$$\exists b : V \to \mathbb{R}^+, \quad \exists D \in L^2(0, T), \qquad \forall v \in V, \\ |j'(t + h, v) - j'(t, v)|_{V'} \leq b(v) \int_t^{t+h} D(s)ds, \qquad \forall t, t + h \in (0, T) \tag{2.5}$$

with $b(v(\ .\ )) \in L^\infty(0, T)$ if $v \in L^\infty(0, T; V)$.
We also suppose that for every $v \in V$, $j(., v)$ is differentiable with respect to time and that

$$\exists B = B(t) \in L^1(0,T),\ B \geq 0, \quad \exists b_0 > 0\ , \quad \forall v \in V, \\ a(v,v) + \frac{\partial j}{\partial t}(t,v) \geq b_0 |v|_V^2 - B(t) \quad a.e.\ t \in (0,T). \tag{2.6}$$

Notice that in the case where j does not depend on time, assertions 2.5 and 2.6 are fullfilled with $b = B = 0$.

We prove

Theorem 2.1.

Suppose that (2.3), (2.4), (2.5), *and* (2.6) *hold. If* $u_0 \in V_0$ *and* $f \in H^1(0,T;V')$ *satisfy*

$$\exists c_0 > 0,\ \ a(u_0,v) + j(0,v) \geq \langle f(0), v\rangle_{V',V} - c_0 \qquad \forall v \in V, \tag{2.7}$$

there exists a unique solution $u \in L^2(0,T;V_0)$ *with* $u \in H^1(0,T_0;V_0)$ *for every* $0 < T_0 < T$ *of problem* P_j^e.

Furthermore, the following estimates are valid

(i) there exists $C > 0$ *independent of* j *such that for every* $u_0 \in V_0$, $f \in H^1(0,T;V')$, *we have*

$$|u|_{L^2(0,T;V_0)} \leq C[|u_0|_{V_0} + |f|_{L^2(0,T;V')}], \tag{2.8}$$

(ii) for every $T_0 < T$, *there exists* $C(T_0) > 0$, *independent on* j, *such that for every* $u_0 \in V_0$, $f \in H^1(0,T;V')$, *we have*

$$|u|_{H^1(0,T_0;V_0)} \leq \frac{C(T_0)}{b_0}[|u_0|_{V_0} + |f|_{H^1(0,T;V')} + |B|^{1/2}_{L^1(0,T)}]. \tag{2.9}$$

We then consider a functional j with (2.4) and (2.6) such that there exists functionals j_ρ for $\rho > 0$ which satisfy (2.3), (2.4), (2.5), and (2.6) uniformly in ρ and converging to j in the (different) following senses :

$$\begin{cases} \forall v \in L^2(0,T;V_0),\ \ \lim\limits_{\rho\to 0} \displaystyle\int_0^T j_\rho(t,v(t))dt \int_0^T j(t,v(t))dt \\ \forall v_\rho \rightharpoonup v \text{ in } L^2(0,T;V_0) \text{ weak}, \quad \liminf\limits_{\rho\to 0} \displaystyle\int_0^T j_\rho(t,v_\rho(t))dt \geq \int_0^T j(t,v(t))dt \end{cases} \tag{2.10}$$

or

$$\exists c > 0,\ \ \forall v \in V, |j_\rho(t,v) - j(t,v)| \leq c\rho. \tag{2.11}$$

Remark that (2.11) implies (2.10).

The asymptotic analysis with respect to the parameter ρ is given in

Theorem 2.2.

Under hypotheses of Theorem 2.1 and (2.10), the solutions u_ρ *of problems* $P^e_{j_\rho}$ *weakly converge in* $L^2(0,T;V_0)$ *and in* $H^1_{loc}([0,T[;V_0)$ *to the solution* u *of problem* P^e_j.

If moreover, (2.11) is fullfilled then for every $T_0 < T$,

$$\lim_{\rho\to 0} u_\rho = u \ \text{ in } \ C([0,T_0];V_0),$$

$$\lim_{\rho\to 0} \sigma_\rho = \sigma \ \text{ in } \ C([0,T_0];\mathcal{H}_1).$$

We now turn to the application of these abstract results to several kinds of mechanical problems related to the (bilateral) contact theory with Tresca friction law.

The setting of our problem is as follows. An elastic-viscoplastic body occupies the domain Ω and is acted upon by given forces and tractions. The body is clamped on $(0,T)\times\Gamma_1$ and surface tractions φ_2 act on $(0,T)\times\Gamma_2$. Moreover, a volume force of density φ_1 acts on the body in $(0,T)\times\Omega$. The solid is in frictional contact with a rigid obstacle on $(0,T)\times\Gamma_3$. We first assume a quasistatic process and we consider bilateral contact with Tresca friction law involving a friction coefficient $\mu>0$ and a friction yield limit $g=g(t,x)$ which may depends on the time variable. The mechanical problem in which our interest lies is :

$$\begin{cases} \dot\sigma(t) = \mathcal{E}\varepsilon(\dot u)(t) + G(\sigma(t),\varepsilon(u)(t)) \ \text{ in } \ (0,T)\times\Omega, \\ Div\ \sigma(t) + \varphi_1(t) = 0 \ \text{ in } \ \Omega\times(0,T), \\ u(t) = 0 \ \text{ on } \ (0,T)\times\Gamma_1, \\ \sigma\nu(t) = \varphi_2(t) \ \text{ on } \ (0,T)\times\Gamma_2, \\ u_\nu(t) = 0, \quad |\sigma_\tau(t)| \le \mu|g(t)| \ \text{ on } \ (0,T)\times\Gamma_3, \\ \text{with } \begin{cases} |\sigma_\tau(t)| < \mu|g(t)| \Rightarrow \dot u_\tau(t) = 0, \\ |\sigma_\tau(t)| = \mu|g(t)| \Rightarrow \exists\lambda\ge 0 \text{ such that } \sigma_\tau(t) = -\lambda\dot u_\tau(t), \end{cases} \\ u(0) = u_0, \qquad \sigma(0) = \sigma_0 \ \text{ in } \ \Omega. \end{cases} \tag{2.12}$$

Existence and uniqueness of the solution $u\in H^1(0,T;V_0)$ and $\sigma\in H^1(0,T;\mathcal{H}_1)$ are classical results and we refer to [18], [6] and [3] and their bibliographies in case of interest. Solving this problem leads to an equivalent formulation involving a variational inequality in which only one term (the friction one) presents a lack of regularity with respect to the velocity $\dot u$ and involves the function $\psi_0(v)=|v|$. With

$$j_0(t,v) = \mu\int_{\Gamma_3} |g(t,a)||\psi_0(v(a))|da,$$

it can be written

Problem P_0 :

$$\begin{cases} \dot\sigma(t) = \mathcal{E}\varepsilon(\dot u(t)) + G(\sigma(t),\varepsilon(u(t))) \qquad \text{a.e. } t\in(0,T), \\ \forall v\in V_0, \ \text{a.e. } t\in(0,T), \\ \qquad (\sigma(t),\varepsilon(v)-\varepsilon(\dot u(t)))_{\mathcal{H}} + j_0(t,v) - j_0(t,\dot u(t)) \ge \langle f(t), v-\dot u(t)\rangle_{V',V} \\ u(0)=u_0, \qquad \sigma(0)=\sigma_0. \end{cases} \tag{2.13}$$

Notice that Tresca friction law is a simple way to modelize Coulomb friction law when the friction yield limit is known and homogeneous on the contact part of the boundary. Indeed, in the case of

Coulomb law, the friction yield limit depends on the solution $u(x,t)$ (and more precisely on the normal stress tensor).
It is known that one can not expect existence of a solution without the following compatibility condition between u_0, σ_0, and $f(0)$:

$$\begin{cases} u_0 \in V_0, \quad \sigma_0 \in \mathcal{H}_1 \\ \forall v \in V_0 \qquad (\sigma_0, \varepsilon(v))_{\mathcal{H}} + j_0(g(0)), v) \geq \langle f(0), v \rangle_{V',V}, \end{cases} \tag{2.14}$$

We will suppose it to be true in all the paper.
For numerical reasons, one wants to regularize the term ψ_0 introducing an approximation of ψ_0. The idea is then to replace the functional ψ_0 by functionals ψ_ρ which would be more regular. Unfortunately, the method of proof of existence of a solution to problem (2.12) uses in a crucial way the fact that ψ_0 is a semi-norm. Functions ψ_ρ will no more verify this and that makes the main tools of the proof fail.

Applying the previous results, we first present the analysis when one considers functionals ψ_ρ instead of ψ_0 and then the asymptotic analysis when ρ tends to zero. Before stating precisely our results, we need to introduce more notations and all hypotheses on the data that will be used in what follows and we start with the hypotheses concerning the constitutive law.

Hypotheses on the tensor $\mathcal{E}$:

$$\mathcal{E} : \Omega \times S_d \to S_d \ \text{ is a symmetric definite positive tensor which means}$$

$$\begin{cases} (a)\ \mathcal{E}_{ijkl} \in L^\infty(\Omega) \quad \forall i,j,k,l = 1, \cdots, d \\ (b)\ \mathcal{E}\sigma \cdot \tau = \sigma \cdot \mathcal{E}\tau \quad \forall \sigma, \tau \in S_d, \text{ a.e. in } \Omega \\ (c) \text{ there exists } \quad \alpha > 0 \text{ such that } \mathcal{E}\sigma \cdot \sigma \geq \alpha |\sigma|^2 \quad \forall \sigma \in S_d, \end{cases} \tag{2.15}$$

Let V_0 be the closed subspace in V of admissible displacements :

$$V_0 = V \cap \{v_\nu = 0 \ \text{ on } \Gamma_3\}.$$

We recall that Korn inequality leads to the following statement on V (see [12]) :

$$|\varepsilon(u)|_{\mathcal{H}} \geq C||u||_V \qquad \forall u \in V$$

which, with (2.15), implies that $v \to \sqrt{a(v,v)}$ with

$$a(v,v) = \int_\Omega \mathcal{E}\varepsilon(v)(x).\varepsilon(v)(x)dx,$$

defines an equivalent norm on V.

Hypotheses on the nonlinearity G :

$$G : \Omega \times S_d \times S_d \to S_d$$

with

$$\begin{cases} (a) \text{ there exists } \quad L > 0 \text{ such that} \\ \quad |G(x,\sigma_1,\varepsilon_1) - G(x,\sigma_2,\varepsilon_2)| \leq L(|\sigma_1-\sigma_2| + |\varepsilon_1 - \varepsilon_2|) \\ \qquad \forall\, \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2 \in S_d, \text{a.e. in } \Omega \\ (b)\ x \mapsto G(x,\sigma,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \text{for all } \sigma,\varepsilon \in S_d \\ (c)\ x \mapsto G(x,0,0) \in \mathcal{H}, \end{cases} \tag{2.16}$$

This means in particular that G is globally Lipschitz with respect to the two last arguments.

We now turn to hypotheses on forces and the approximate framework.
We consider a function $g = g(t,a)$ in $W^{1,\infty}(0,T;L^2(\Gamma_3))$ or in $H^1(0,T;L^2(\Gamma_3))$, a real $\mu > 0$, and we introduce the functional defined on V,

$$j_0(g(t),v) = \mu \int_{\Gamma_3} |g(t,a)||v_\tau(a)|da,$$

where da is the surface measure on Γ_3.

For every $\rho > 0$, we consider a positive convex and $C^1(\mathbb{R}^d)$ function $\psi_\rho : \mathbb{R}^d \longrightarrow \mathbb{R}$ with

$$\psi_\rho(0) = 0. \tag{2.17}$$

We suppose

$$\begin{array}{ll} \text{If } d=2, & \exists 0 \leq r < 2, |\psi_\rho(x)| = \mathcal{O}(|x|^r + 1) \quad \text{and } |\psi'_\rho(x)| = \mathcal{O}(|x|+1). \\ \text{If } d>2, & \exists 0 \leq r < \frac{d-1}{d-2}, |\psi_\rho(x)|\mathcal{O}(|x|^r+1) \quad \text{and } |\psi'_\rho(x)| = \mathcal{O}(|x|^{\frac{1}{d-2}}+1). \end{array} \tag{2.18}$$

Remark that for $d = 2$ or 3, the exponants in (2.18) are the same ($\frac{3-1}{3-2} = 2$ and $\frac{1}{3-2} = 1$). There are several possible choice of the function ψ_ρ. A classical one is to use the well known approximation of the modulus function :

$$\psi_\rho(x) = \sqrt{\rho^2 + |x|^2} - \rho \quad \forall x \in \mathbb{R}^d.$$

One can easily see that such a choice of ψ_ρ satisfies (2.17)-(2.18) with $r = 1$ since $|\psi_\rho(x)| = \mathcal{O}(|x|)$ and $\psi'_\rho(x) = x/\sqrt{\rho^2+|x|^2} = \mathcal{O}(1)$.

Another possible choice is to consider the sub-linear functions defined by

$$\psi_\rho(x) = |x|^{1+\rho}$$

with $0 < \rho < 1$ in two space dimension or with $0 \leq \rho < \frac{1}{d-2}$ in others space dimensions.

In the study of the asymptotic analysis when ρ tends to zero, we will consider two kinds of hypotheses illustrating examples given above.
The first kind of hypotheses will be

$$\exists L_0 > 0, \;\; \forall \rho > 0, \;\; \forall x \in (\mathbb{R}^d), \;\; |\psi_\rho(x) - |x|| \leq L_0 \rho, \tag{2.19}$$

and will lead to a strong converge results concerning displacements and stresses.
The second kind will only lead to weak convergence results and its hypothesis is

$$\left\{\begin{array}{l}\lim_{\rho\to 0}\psi_\rho = |\ \ | \quad \text{in} \ \ C_c(\mathbb{R}^d),\\ \lim_{\rho\to 0}\psi'_\rho(x) = S(x) \quad \text{for every} \quad x\in\mathbb{R}^d.\end{array}\right. \tag{2.20}$$

In (2.20), $S(x) = x/|x|$.
We denote

$$j_\rho(g(t),v) = \mu\int_{\Gamma_3}|g(t,a)|\psi_\rho(v_\tau(a))da.$$

The functional $j_\rho(g(t),.)$ is convex differentiable and

$$j_\rho(g(t),0) = 0. \tag{2.21}$$

If we denote by $j'_\rho(t,u)$ the derivative of j_ρ with respect to v, we get :

$$\forall u\in V \quad j'_\rho(u)\in V' \quad \text{and} \quad \langle j'_\rho(t,u),v\rangle_{V',V} = \int_{\Gamma_3} g(t,a)\ \psi'_\rho(u_\tau)\cdot v_\tau\ da.$$

In order to write the variational formulation related to the mechanical problem, we suppose in all the paper that

$$\varphi_1\in H^1(0,T;H), \quad \varphi_2\in H^1(0,T;L^2(\Gamma_2)^d), \tag{2.22}$$

and we define $f\in H^1(0,T;V')$ with ($\gamma_0 v$ is the trace over Γ of the vector v) by

$$\langle f(t),v\rangle_{V',V} = \int_\Omega \varphi_1(t)vdx + (\varphi_2(t),\gamma_0 v)_{L^2(\Gamma_2)^d}$$

that we write $f(t) = \varphi_1 + \varphi_2\delta_{\Gamma_2}$.

We now introduce the variational formulation for the original problem and its approximation : for $\rho\geq 0$, we write

Problem P_ρ : Find a displacement field $u_\rho : [0,T]\longrightarrow V_0$ and a stress tensor $\sigma_\rho : [0,T]\longrightarrow \mathcal{H}_1$ with

$$\left\{\begin{array}{l}\dot\sigma_\rho(t) = \mathcal{E}\varepsilon(\dot u_\rho(t)) + G(\sigma_\rho(t),\varepsilon(u_\rho(t))) \qquad \text{a.e.} \ \ t\in(0,T),\\ \forall v\in V_0, \ \text{a.e.} \ t\in(0,T),\\ \qquad (\sigma_\rho(t),\varepsilon(v)-\varepsilon(\dot u_\rho(t)))_{\mathcal{H}} + j_\rho(g(t),v) - j_\rho(g(t),\dot u_\rho(t)) \geq \langle f(t), v-\dot u_\rho(t)\rangle_{V',V}\\ u_\rho(0) = u_0, \qquad \sigma_\rho(0) = \sigma_0.\end{array}\right. \tag{2.23}$$

We prove

Theorem 2.3. *Suppose that (2.14), (2.15), (2.16), and (2.22) hold.*
(i) Suppose that ψ_ρ satisfies (2.17) and (2.18), then for every $\rho > 0$, there exists a unique solution $u_\rho \in H^1(0,T;V_0)$ and $\sigma_\rho \in L^2(0,T;\mathcal{H}_1)$ of problem P_ρ.
(ii) Furthermore, if (2.19) is satisfied, there exits a constant $C_T > 0$ (depending on T) with

$$|u_\rho - u|_{C([0,T];V_0)} + |\sigma_\rho - \sigma|_{C([0,T];\mathcal{H}_1)} \le C_T\sqrt{\rho},$$

and $\dot u_\rho$ (respectively $\dot\sigma_\rho$) weakly converges to $\dot u$ in $L^2(0,T;V_0)$ (respectively to $\dot\sigma$ in $L^2(0,T;\mathcal{H}_1)$). We recall that $u \in\in H^1(0,T;V_0)$ and $\sigma \in L^2(0,T;\mathcal{H}_1)$ is the unique solution of problem P_0.
(iii) If (2.20) is fulfilled and in the case where a Hooke law is valid ($G = 0$), then

$$\lim_{\rho\to 0} u_\rho = u \text{ weak in } H^1([0,T];V_0),$$
$$\lim_{\rho\to 0} \sigma_\rho = \sigma \text{ weak in } H^1([0,T];\mathcal{H}_1),$$

Remarks

(i) For the existence part of a solution of problem P_ρ, we could have taken a more general condition than (2.14). Indeed, (2.14), where j is replaced by j_ρ is sufficient.

(ii) Notice that the regularity in time enounced in Theorem 2.3 is global on $(0,T)$ whereas it is a local one in Theorem 2.1.

(iii) Finally, notice that $\psi_\rho(x) = \sqrt{|x|^2+\rho^2} - \rho$ satisfies (2.19) and that $\psi_\rho(x) = |x|^{1+\rho}$ satisfies (2.20). This results explains why numerical implementations fit better with $\psi_\rho(x) = \sqrt{|x|^2+\rho^2} - \rho$ than with $|x|^{1+\rho}$.

We now turn to the case of Coulomb friction law : we suppose that the friction coefficient μ satisfies

$$\mu \in L^\infty(\Gamma_3) \quad \text{and} \quad \mu(x) \ge 0 \quad a.e. \text{ on } \Gamma_3. \tag{2.24}$$

In the sequel R will represent a normal regularization operator that is a linear and continuous operator $R : H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma)$. We shall need it to regularize the trace of the stress tensor on Γ. Notice that we do not make any hypothesis on compactness property of the operator R.

The mathematical formulation of the problem in which we are interested is :
Problem P_c : find a displacement field $u : \Omega\times[0,T] \to \mathbb{R}^d$ and a stress tensor $\sigma : \Omega\times[0,T] \to S_d$ with

$$\begin{cases} \dot\sigma(t) = \mathcal{E}\varepsilon(\dot u)(t) + G(\sigma(t),\varepsilon(u)(t)) \text{ sur } (0,T)\times\Omega, \\ Div\ \sigma(t) + \varphi_1(t) = 0 \text{ in } \Omega\times(0,T), \\ u(t) = 0 \text{ on } (0,T)\times\Gamma_1, \\ \sigma\nu(t) = \varphi_2(t) \text{ on } (0,T)\times\Gamma_2, \\ u_\nu(t) = 0, \quad |\sigma_\tau(t)| \le \mu|R\sigma_\nu| \text{ on } (0,T)\times\Gamma_3, \\ \text{with } \begin{cases} |\sigma_\tau(t)| < \mu|R\sigma_\nu| \Rightarrow \dot u_\tau(t) = 0, \\ |\sigma_\tau(t)| = \mu|R\sigma_\nu| \Rightarrow \exists\lambda\ge 0 \text{ such that } \sigma_\tau(t) = -\lambda\dot u_\tau(t), \end{cases} \\ u(0) = u_0, \qquad \sigma(0) = \sigma_0 \text{ in } \Omega. \end{cases} \tag{2.25}$$

The variational formulation of problem P_c is

$$\begin{cases} \textbf{Problem } P_{cv} : \text{find } u : [0,T] \to V_0 \text{ and } \sigma : [0,T] \to \mathcal{H}_1 \text{ with :} \\ \dot{\sigma}(t) = \mathcal{E}\varepsilon(\dot{u}(t)) + G(\sigma(t), \varepsilon(u)(t)) \quad \text{a.e. } t \in (0,T), \\ \forall v \in V_0, \text{ a.e. } t \in (0,T), \\ \qquad (\sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathcal{H}} + j(R\sigma_\nu, v) - j(R\sigma_\nu, \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle_{V',V} \\ u(0) = u_0, \qquad \sigma(0) = \sigma_0, \end{cases} \tag{2.26}$$

where

$$j(R\sigma_\nu, v) = \int_{\Gamma_3} \mu(a)|R\sigma_\nu(a)||v_\tau(a)|da. \tag{2.27}$$

Our aim is twice : we want to obtain an existence result when one replaces the modulus functional $|\cdot|$ by the smoothing term ψ_ρ satisfying (2.19) and we want to give the asymptotic analysis when ρ tends to 0. In this case, the Coulomb friction term becomes

$$j_\rho(R\sigma_\nu, v) = \int_{\Gamma_3} \mu(a)|R\sigma_{\rho\nu}(a)|\psi_\rho(v_\tau(a))da, \tag{2.28}$$

which leads to the variational formulation

$$\begin{cases} \textbf{Problem } P_{rcv} : \text{find } u_\rho : [0,T] \to V_0 \text{ and } \sigma_\rho : [0,T] \to \mathcal{H}_1 \text{ with :} \\ \dot{\sigma}_\rho(t) = \mathcal{E}\varepsilon(\dot{u}_\rho(t)) + G(\sigma_\rho(t), \varepsilon(u_\rho)(t)) \quad \text{a.e. } t \in (0,T), \\ \forall v \in V_0, \text{ a.e. } t \in (0,T), \\ \qquad (\sigma_\rho(t), \varepsilon(v) - \varepsilon(\dot{u}_\rho(t)))_{\mathcal{H}} + j_\rho(R\sigma_{\rho\nu}, v) - j_\rho(R\sigma_{\rho\nu}, \dot{u}_\rho(t)) \geq \langle f(t), v - \dot{u}_\rho(t)\rangle_{V',V} \\ u_\rho(0) = u_0, \qquad \sigma_\rho(0) = \sigma_0. \end{cases} \tag{2.29}$$

Finally, we denote by B_M the close ball centered at 0 with radius M in $H^1(0,T;V_0) \times H^1(0,T;\mathcal{H}_1)$. We prove

Theorem 2.4.
(i) If ψ_ρ satisfies (2.19), there then exists $\mu_0 > 0$ and $M > 0$ independent on ρ such that for every $\rho \in]0,1[$ and μ with $|\mu|_{L^\infty(\Gamma_3)} \leq \mu_0$, problem (2.29) possesses (at least) a solution (u_ρ, σ_ρ) in B_M.
(ii) Every weak limit point (in $H^1(0,T;V_0) \times H^1(0,T;\mathcal{H}_1)$) of (u_ρ, σ_ρ) is also a strong limit point in $C(0,T;V_0) \times C(0,T;\mathcal{H}_1)$) and is solution of the Coulomb problem P_{cv}.

Remark 2.5. (i) Uniqueness of the solution(s) of (2.29) is still an open problem.
(ii) We could not obtain a similar result when one supposes that ψ_ρ satisfies hypotheses (2.20) instead of (2.19).

We now detail our plan. Sections 3 and 4 concern the simplest case modelling the contact with Tresca friction law. In section 3, we prove Theorem 2.1 (existence and uniqueness result for problem P_j^e) and Theorem 2.2 (asymptotic analysis with respect to j). We will also state consequences that will be used in next sections. In section 4, we are interested in elastic-viscoplastic constitutive relationship and we prove Theorem 2.3. In section 5, we consider Coulomb friction law and prove Theorem 2.4.

Finally, notice that c, C will denote any positive constant that might depend only on Ω, T and that may change line to line.

3. Existence and continuity for abstract variational inequality.

Since V_0 is a closed subspace of V, we will suppose in Theorem 2.1's proof that $V_0 = V$ without loosing any generality. We are interested in the analysis *in a constructive way* of the following problem

$$a(u(t), v - \dot{u}(t)) + j(t, v) - j(t, \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t) \rangle_{V',V} \qquad (3.1)$$
$$\forall v \in V, \text{ a.e. } t \in (0, T),$$

$$u(0) = u_0. \qquad (3.2)$$

Since $j(t, .)$ is convex and differentiable, (3.1) is equivalent to

$$a(u(t), v) + \langle j'(t, \dot{u}(t)), v \rangle_{V',V} = \langle f(t), v \rangle_{V',V} \quad \forall v \in V, \text{ a.e. } t \in (0, T). \qquad (3.3)$$

We introduce an auxiliary problem that will approximate (3.1). For $n \in \mathbb{N}^*$, we consider
Problem $\widetilde{P}_n$: find $u_n : [0, T] \longrightarrow V$ with $u_n(0) = u_0$ such that
$\forall v \in V$, a.e. $t \in (0, T)$,

$$\frac{1}{n} (\dot{u}_n(t), v - \dot{u}_n(t))_V + a(u_n(t), v - \dot{u}_n(t)) + j(t, v) - j(t, \dot{u}_n(t)) \geq \langle f(t), v - \dot{u}_n(t) \rangle_{V',V} \qquad (3.4)$$

or equivalently

$$\frac{1}{n} (\dot{u}_n(t), v)_V + a(u_n(t), v) + \langle j'(t, \dot{u}_n(t)), v \rangle_{V',V} = \langle f(t), v \rangle_{V',V} \qquad (3.5)$$

In the case where $j(t, v) = j(v)$ does not depend on the time variable, we proved in [2] that for $u_0 \in V$ and $f \in H^2(0, T; V')$, problem $\widetilde{P}_n$ has a unique solution $u_n \in H^1(0, T; V)$. Following the idea of [2], we generalize here this result with

Lemma 3.1. *Suppose that j satisfies (2.3), (2.5), (2.6) and that $u_0 \in V$ and $f \in H^1(0, T; V')$, then problem $\widetilde{P}_n$ has a unique solution $u_n \in H^2(0, T; V)$.*

Proof of Lemma 3.1

The proof is carried out in two steps: the first entails the construction of an auxiliary problem in which a displacement like function is assumed to be known. We establish the existence of a unique solution of this auxiliary problem. In the second step we use a fixed point argument.
For this, for any $\alpha \in H^1(0, T; V)$ (representing a displacement) we consider the following abstract problem :
Problem $P_{n\alpha}$: Find a displacement field $v_{n\alpha} \in H^1(0, T; V)$ such that

$$\frac{1}{n}(v_{n\alpha}(t), v)_V + \langle j'(t, v_{n\alpha}(t)), v \rangle_{V',V} = \langle f(t), v \rangle_{V',V} - a(\alpha(t), v) \qquad \forall v \in V, \ \forall\, t \in [0, T]. \qquad (3.6)$$

(i) First step. Let us prove that there exists a unique solution to problem $P_{n\alpha}$ such that $v_{n\alpha} \in H^1(0,T;V)$.

The problem (3.6) is equivalent to the following minimization problem

$$\text{Find } v_{n\alpha}(t) \in V, \;\; \mathcal{I}_{n\alpha}(t, v_{n\alpha}(t)) = \inf_{v\in V} \mathcal{I}_{n\alpha}(t,v) \quad \forall\, t \in (0,T), \tag{3.7}$$

where

$$\mathcal{I}_{n\alpha}(t,v) = \frac{1}{2n}|v|_V^2 + j(t,v) - \langle f(t), v\rangle_{V',V} + a(\alpha(t), v).$$

The functional $\mathcal{I}_{n\alpha}(t,\cdot)$ is proper, continuous, strictly convex and coercive on V. Therefore, the problem (3.7) has a unique solution $v_{n\alpha}(t) \in V$, a.e. $t \in (0,T)$.

Now, we shall show that $v_{n\alpha} \in H^1(0,T;V)$. Let $t \in [0,T]$ and taking $v = v_{n\alpha}(t)$ in (3.6). Since $\mathcal{I}_{n\alpha}(t, v_{n\alpha}(t)) \leq \mathcal{I}_{n\alpha}(t,0)$ and $j(t,0) = 0$, we have for almost every t,

$$|v_{n\alpha}(t)|_V^2 \leq 2n\big(|f(t)|_{V'}|v_{n\alpha}(t)|_V\big),$$

and we deduce that $v_{n\alpha} \in L^\infty(0,T;V)$. If we take $t = t+h$ in (3.6) we obtain

$$\frac{1}{n}\langle v_{n\alpha}(t+h), v\rangle_V + \langle j'(t+h, v_{n\alpha}(t+h)), v\rangle_{V',V}\langle f(t+h), v\rangle_{V',V} - a(\alpha(t+h), v) \qquad \forall v \in V. \tag{3.8}$$

Let us set $v = v_{n\alpha}(t+h) - v_{n\alpha}(t)$ in (3.8) and $v = v_{n\alpha}(t) - v_{n\alpha}(t+h)$ in (3.6) and subtract these equations, we obtain

$$\begin{aligned} \frac{1}{n}\langle v_{n\alpha}(t+h) - v_{n\alpha}(t), \;\; & v_{n\alpha}(t+h) - v_{n\alpha}(t)\rangle_V \\ & + \langle j'(t+h, v_{n\alpha}(t+h)) - j'(t, v_{n\alpha}(t)), v_{n\alpha}(t+h) - v_{n\alpha}(t)\rangle_{V',V} \\ & = \langle f(t+h) - f(t), v_{n\alpha}(t+h) - v_{n\alpha}(t)\rangle_{V',V} \\ & - a(\alpha(t+h) - \alpha(t), v_{n\alpha}(t+h) - v_{n\alpha}(t)). \end{aligned} \tag{3.9}$$

We write

$$j'(t+h, v_{n\alpha}(t+h)) - j'(t, v_{n\alpha}(t)) = j'(t+h, v_{n\alpha}(t+h)) - j'(t+h, v_{n\alpha}(t)) + j'(t+h, v_{n\alpha}(t)) - j'(t, v_{n\alpha}(t)).$$

We have $\langle j'(t+h, v_{n\alpha}(t+h)) - j'(t+h, v_{n\alpha}(t)), v_{n\alpha}(t+h) - v_{n\alpha}(t)\rangle_{V',V} \geq 0$, and using (2.5),

$$|\langle j'(t+h, v_{n\alpha}(t)) - j'(t, v_{n\alpha}(t)), v_{n\alpha}(t+h) - v_{n\alpha}(t)\rangle_{V',V}| \leq b(v_{n\alpha}(t))|v_{n\alpha}(t+h) - v_{n\alpha}(t)|_V \int_t^{t+h} D(s)ds.$$

From Korn's inequality and (2.5), we get

$$\begin{aligned} |v_{n\alpha}(t+h) - v_{n\alpha}(t)|_V \leq C(n)(|f(t+h) - f(t)|_{V'} \;\; & + b(v_{n\alpha}(t)) \int_t^{t+h} D(s)ds \\ & + |\alpha(t+h) - \alpha(t)|_V), \end{aligned} \tag{3.10}$$

with (since $v_{n\alpha} \in L^\infty(0,T;V)$), $b(v_{n\alpha}) \in L^\infty(0,T)$.

The regularity property $v_{n\alpha} \in H^1(0,T;V)$ follows from (3.10), since $b(v_{n\alpha}) \in L^\infty(0,T)$, $f \in H^1(0,T;V)$, $\alpha \in H^1(0,T;V)$, $D \in L^2(0,T)$ and theorem 1.4.40 p23 of [8].

Let then $u_{n\alpha} : [0,T] \to V$ be the function given by

$$u_{n\alpha}(t) = \int_0^t v_{n\alpha}(s)ds + u_0, \tag{3.11}$$

we have $u_{n\alpha} \in H^2(0,T;V)$ and $u_{n\alpha}(0) = u_0$.
Next, we consider the map $\Lambda_n : H^1(0,T;V) \to H^1(0,T;V)$ defined by

$$\Lambda_n(\alpha)(t) = u_{n\alpha}(t) \qquad \forall \alpha \in H^1(0,T;V),\ t \in [0,T]. \tag{3.12}$$

(ii) Second step. Let us prove that Λ_n has a unique fixed point $\alpha^\star$. We shall prove that for p large enough, one iterate Λ_n^p is a contraction. For this, let $\alpha_1,\ \alpha_2 \in H^1(0,T;V)$ and $t \in [0,T]$. Let us denote $v_i = v_{n\alpha_i}$ and $u_i = u_{n\alpha_i}$, where $v_{n\alpha_i}$ is a solution of the problem $P_{n\alpha_i}$ and $u_{n\alpha_i}$ is defined by (3.11) for $i = 1,2$. Using (3.11)-(3.12), we have

$$\begin{aligned} |\dot\Lambda_n(\alpha_1)(t) - \dot\Lambda_n(\alpha_2)(t)|_V^2 + \ & |\Lambda_n(\alpha_1)(t) - \Lambda_n(\alpha_2)(t)|_V^2 = |\dot u_1(t) - \dot u_2(t)|_V^2 + |u_1(t) - u_2(t)|_V^2 \\ & \leq |v_1(t) - v_2(t)|_V^2 + \int_0^t |v_1(s) - v_2(s)|_V^2 ds. \end{aligned} \tag{3.13}$$

If we choose, for $i = 1,2$, $v = v_1(t) - v_2(t)$ in (3.6), we obtain

$$\begin{aligned} \frac{1}{n}\langle v_1(t), v_1(t) - v_2(t)\rangle_V + \langle j'(t, v_1(t)), v_1(t) - v_2(t)\rangle_{V',V} = \langle f(t) \ & , v_1(t) - v_2(t)\rangle_{V',V} \\ & -a(\alpha_1(t), v_1(t) - v_2(t)) \\ \frac{1}{n}\langle v_2(t), v_1(t) - v_2(t)\rangle_V + \langle j'(t, v_2(t)), v_1(t) - v_2(t)\rangle_{V',V} = \langle f(t) \ & , v_1(t) - v_2(t)\rangle_{V',V} \\ & -a(\alpha_2(t), v_1(t) - v_2(t)). \end{aligned}$$

Subtracting these equalities, since $\langle j'(t, v_1(t)) - j'(t, v_2(t)), v_1(t) - v_2(t)\rangle_{V',V} \geq 0$, we get

$$|v_1(t) - v_2(t)|_V \leq C(n)|\alpha_1(t) - \alpha_2(t)|_V. \tag{3.14}$$

From (3.13) and (3.14), it follows that

$$\begin{aligned} |\dot\Lambda_n(\alpha_1)(t) \ & -\dot\Lambda_n(\alpha_2)(t)|_V^2 + |\Lambda_n(\alpha_1)(t) - \Lambda_n(\alpha_2)(t)|_V^2 \leq C(n)(|\alpha_1(t) - \alpha_2(t)|_V^2 \\ & + \int_0^t |\alpha_1(s) - \alpha_2(s)|_V^2 ds) \leq C(n)\int_0^t (|\dot\alpha_1(s) - \dot\alpha_2(s)|_V^2 + |\alpha_1(s) - \alpha_2(s)|_V^2)ds. \end{aligned} \tag{3.15}$$

Reiterating this inequality p times and integrating on t leads to

$$|\Lambda_n^p(\alpha_1) - \Lambda_n^p(\alpha_2)|_{H^1(0,T;V)} \leq \sqrt{\frac{(C(n)T)^{p+1}}{(p+1)!}}|\alpha_1 - \alpha_2|_{H^1(0,T;V)}, \tag{3.16}$$

which implies that for n sufficiently large, a power Λ_n^p is a contraction on $L^2(0,T;V)$. Thus, there exists a unique $\alpha^\star \in L^2(0,T;V)$ such that $\Lambda_n^p\alpha^\star = \alpha^\star$ and $\alpha^\star$ is also the unique fixed point of Λ_n.

Let $\alpha^\star \in L^2(0,T;V)$ be the fixed point of Λ_n and let $u_{n\alpha^\star}$ the function given by (3.11)-(3.12) for $\alpha = \alpha^\star$. We proved that $u_{n\alpha^\star}$ is a solution of problem $\widetilde{P}_n$. The uniqueness part can be proved directly from (3.6) with Gronwall inequality and that ends Lemma 3.1's proof.

We now prove

Lemma 3.2. *(i) There exists $C > 0$ independent on j, such that*
$\forall n \in \mathbb{N}$,

$$|u_n|_{\mathcal{C}([0,T],V)} + \frac{1}{\sqrt{n}}|\dot{u}_n|_{L^2(0,T;V)} + \int_0^T j(t, \dot{u}_n(t))dt \leq C(|u_0|_V + |f|_{H^1(0,T;V')}) \tag{3.17}$$

where $(u_n)_n$ is the sequence of solutions of $(\widetilde{P}_n)$. We then have

$$(u_n) \text{ is bounded in } \mathcal{C}([0,T];V), \tag{3.18}$$

$$(\tfrac{1}{\sqrt{n}}(\dot{u}_n)) \text{ is bounded in } L^2(0,T;V), \tag{3.19}$$

$$(j(\dot{u}_n)) \text{ is bounded in } L^1(0,T;V). \tag{3.20}$$

(ii) If moreover the initial data satisfies the compatibility condition (2.7) then for every $T_0 < T$, the sequence $(\dot{u}_n)$ is is bounded in $L^2(0,T_0;V)$ with

$$|\dot{u}_n|_{L^2(0,T_0;V)} \leq \frac{c(T_0)}{b_0}(|u_0|_V + |f|_{H^1(0,T;V')} + \int_0^{T_0} B(t)dt). \tag{3.21}$$

Proof of Lemma 3.2
Let input $v = \dot{u}_n(t)$ in (3.5). After integration with respect to the time variable between 0 and t, we get

$$\frac{1}{n}\int_0^t (\dot{u}_n(s), \dot{u}_n(s))_V ds + \int_0^t a(u_n(s), \dot{u}_n(s))ds + \int_0^t \langle j'(s, \dot{u}_n(s)), \dot{u}_n(s)\rangle_{V',V} ds = \int_0^t \langle f(s), \dot{u}_n(s)\rangle_{V',V} ds.$$

But, on one hand, we have,

$$\int_0^t \langle f(s), \dot{u}_n(s)\rangle_{V',V} ds = \langle f(t), u_n(t)\rangle - \langle f(0), u_n(0)\rangle - \int_0^t \langle \dot{f}(s), u_n(s)\rangle_{V',V} ds,$$

and on the other hand,

$$\int_0^t a(u_n(s), \dot{u}_n(s))ds = \frac{1}{2}a(u_n(t), u_n(t)) - \frac{1}{2}a(u_0, u_0),$$

thus, we get

$$\begin{aligned}\frac{1}{n}\int_0^t |\dot{u}_n(s)|_V^2 ds + \frac{1}{2}a(u_n(t), u_n(t)) &- \frac{1}{2}a(u_0, u_0) + \int_0^t \langle j'(s, \dot{u}_n(s)), \dot{u}_n(s)\rangle_{V',V} ds \\ &= \langle f(t), u_n(t)\rangle_{V',V} - \langle f(0), u_n(0)\rangle_{V',V} - \int_0^t \langle \dot{f}(s), u_n(s)\rangle_{V',V} ds.\end{aligned}$$

Since $j(t,.)$ is convex and differentiable, we may write for almost every s,

$$j(s,u) - j(s,v) \geq \langle j'(s,v), u - v\rangle_{V',V} \ \forall u, v$$

and since $j(s,0)=0$, we obtain for $u=0$,

$$j(s,v) \leq \langle j'(s,v), v\rangle_{V',V} \ \forall v \in V,$$

and thus

$$\int_0^t \langle j'(s,\dot{u}_n(s)), \dot{u}_n(s)\rangle_{V',V} ds \geq \int_0^t j(\dot{u}_n(t))ds.$$

We then get

$$\begin{aligned}\frac{1}{n}\int_0^t |\dot{u}_n(s)|_V^2 ds &+ |u_n(t)|_V^2 + \int_0^t j(s,\dot{u}_n(s))ds \\ &\leq C\left(|f(t)|_{V'}|u_n(t)|_V + |f(0)|_{V'}|u_0|_V + |u_0|_V^2 + \int_0^t |\dot{f}(s)|_{V'}|u_n(s)|_V ds\right),\end{aligned}$$

which easily leads to (3.17) and by the way, to part (i) of Lemma 2.2.

Let us now turn to part (ii) which is of course the main point. We have to prove that

$$\forall \delta > 0, \quad \exists C_\delta > 0, \ \forall n, \ |\dot{u}_n|^2_{L^2(0,T-\delta;V)} \leq C_\delta. \tag{3.22}$$

For this, we consider a function $\varphi \in W^{1,\infty}(0,T)$ with $\varphi \geq 0$ and $\varphi(T)=0$ and we take $v = \ddot{u_n}(t)\varphi(t)$ in (3.5). After integration in time between 0 and T, we get

$$\begin{aligned}\frac{1}{n}\int_0^T \varphi(t)(\dot{u}_n(t), \ddot{u_n}(t))_V dt + \int_0^T \varphi(t)a(u_n(t), \ddot{u_n}(t))dt &+ \int_0^T \varphi(t)\langle j'(t,\dot{u}_n(t)), \ddot{u_n}(t)\rangle_{V',V} dt \\ &= \int_0^T \varphi(t)\langle f(t), \ddot{u_n}(t)\rangle_{V',V} dt,\end{aligned} \tag{3.23}$$

inequality in which we now study the different terms.
Since $\varphi(T)=0$, we have :

$$\begin{aligned}\frac{1}{n}\int_0^T \varphi(t)(\dot{u}_n(t), \ddot{u_n}(t))_V dt &= \frac{1}{2n}\int_0^T \frac{d}{dt}(\varphi(t)(\dot{u}_n(t), \dot{u}_n(t))_V)dt - \frac{1}{2n}\int_0^T \dot{\varphi}(t)(\dot{u}_n(t), \dot{u}_n(t))_V dt \\ &= \frac{-1}{2n}\left[\varphi(0)(\dot{u}_n(0), \dot{u}_n(0))_V + \int_0^T \dot{\varphi}(t)(\dot{u}_n(t), \dot{u}_n(t))_V dt\right].\end{aligned} \tag{3.24}$$

For the second term, we integrate by parts,

$$
\begin{aligned}
\int_0^T \varphi(t)a(u_n(t),\ddot{u_n}(t))dt \quad & -\int_0^T \varphi(t)a(\dot{u}_n(t),\dot{u}_n(t))dt - \int_0^T \dot{\varphi}(t)a(u_n(t),\dot{u}_n(t))dt \\
& \qquad + [\varphi(t)a(u_n(t),\dot{u}_n(t))]_0^T , \\
& = -\int_0^T \varphi(t)a(\dot{u}_n(t),\dot{u}_n(t))dt - \int_0^T \dot{\varphi}(t)a(u_n(t),\dot{u}_n(t))dt \\
& \qquad -\varphi(0)a(u_n(0),\dot{u}_n(0)).
\end{aligned}
\tag{3.25}
$$

For the third one,

$$
\begin{aligned}
\int_0^T \varphi(t)\langle j'(t,\dot{u}_n(t)),\ddot{u_n}(t)\rangle_{V',V}dt &= \int_0^T \varphi(t)\frac{\partial}{\partial t}\left(j(t,\dot{u}_n(t))\right)dt - \int_0^T \varphi(t)\frac{\partial j}{\partial t}(t,\dot{u}_n(t))dt \\
&= -\int_0^T j(t,\dot{u}_n(t))\dot{\varphi}(t)dt + [j(t,\dot{u}_n(t))\varphi(t)]_0^T - \int_0^T \varphi(t)\frac{\partial j}{\partial t}(t,\dot{u}_n(t))dt \\
&= -\int_0^T j(t,\dot{u}_n(t))\dot{\varphi}(t)dt - j(t,\dot{u}_n(0))\varphi(0) - \int_0^T \varphi(t)\frac{\partial j}{\partial t}(t,\dot{u}_n(t))dt.
\end{aligned}
\tag{3.26}
$$

And finally, for the last one,

$$
\begin{aligned}
\int_0^T \varphi(t)\langle f(t),\ddot{u_n}(t)\rangle_{V',V}dt \quad &= \quad -\int_0^T \varphi(t)\langle \dot{f}(t),\dot{u}_n(t)\rangle_{V',V}dt \\
& - \quad \int_0^T \dot{\varphi}(t)\langle f(t),\dot{u}_n(t)\rangle_{V',V}dt + \left[\varphi(t)\langle f(t),\dot{u}_n(t)\rangle_{V',V}\right]_0^T ,
\end{aligned}
$$

hence

$$
\begin{aligned}
\int_0^T \varphi(t)\langle f(t),\ddot{u_n}(t)\rangle_{V',V}dt \quad &= \quad -\int_0^T \varphi(t)\langle \dot{f}(t),\dot{u}_n(t)\rangle_{V',V}dt \\
& - \quad \int_0^T \dot{\varphi}(t)\langle f(t),\dot{u}_n(t)\rangle_{V',V}dt - \varphi(0)\langle f(0),\dot{u}_n(0)\rangle_{V',V}.
\end{aligned}
\tag{3.27}
$$

Taking into account (3.24), (3.25), (3.26) and (3.27) in (3.23), we obtain

$$
\begin{aligned}
-\frac{1}{2n}\varphi(0)|\dot{u}_n(0)|_V^2 - \frac{1}{2n}\int_0^T \dot{\varphi}(t)|\dot{u}_n(t)|_V^2 dt - \int_0^T \varphi(t)a(\dot{u}_n(t),\dot{u}_n(t))dt - \int_0^T \dot{\varphi}(t)a(u_n(t),\dot{u}_n(t))dt \\
-\varphi(0)a(u_n(0),\dot{u}_n(0)) - \int_0^T j(t,\dot{u}_n(t))\dot{\varphi}(t)dt - j(0,\dot{u}_n(0))\varphi(0) - \int_0^T \varphi(t)\frac{\partial j}{\partial t}(t,\dot{u}_n(t))dt \\
= -\int_0^T \varphi(t)\langle \dot{f}(t),\dot{u}_n(t)\rangle_{V',V}dt - \int_0^T \dot{\varphi}(t)\langle f(t),\dot{u}_n(t)\rangle_{V',V}dt - \varphi(0)\langle f(0),\dot{u}_n(0)\rangle_{V',V},
\end{aligned}
$$

or

$$
\int_0^T \varphi(t)a(\dot{u}_n(t),\dot{u}_n(t))dt + \int_0^T \varphi(t)\frac{\partial j}{\partial t}(t,\dot{u}_n(t))dt = -\frac{1}{2n}\varphi(0)|\dot{u}_n(0)|_V^2 - \frac{1}{2n}\int_0^T \dot{\varphi}(t)|\dot{u}_n(t)|_V^2 dt
$$

$$-\int_0^T j(t,\dot u_n(t))\dot\varphi(t)dt-\int_0^T \dot\varphi(t)a(u_n(t),\dot u_n(t))dt+\int_0^T \varphi(t)\langle \dot f(t),\dot u_n(t)\rangle_{V',V}dt$$

$$+\int_0^T \dot\varphi(t)\langle f(t),\dot u_n(t)\rangle_{V',V}dt+\varphi(0)[\langle f(0),\dot u_n(0)\rangle_{V',V}-a(u_n(0),\dot u_n(0))-j(0,\dot u_n(0))]. \tag{3.28}$$

In order to get an upper bound, we can remark that the first term in the right hand side of (3.28) is negative and using estimate (3.17) of Lemma 3.2, we get

$$\frac{1}{2n}\int_0^T |\dot\varphi(t)||\dot u_n(t)|_V^2 dt\le C \quad \text{and} \quad \int_0^T j(t,\dot u_n(t))|\dot\varphi(t)|dt\le C, \tag{3.29}$$

where C does not depend on j.
Using (2.6) and (2.7), we get

$$\begin{aligned} b_0\int_0^T \varphi(t)|\dot u_n(t)|_V^2 dt \;&\le\; C(1+\int_0^T |\dot\varphi(t)||u_n(t)|_V|\dot u_n(t)|_V dt+\int_0^T \varphi(t)|\dot f(t)|_{V'}|\dot u_n(t)|_V dt \\ &+\int_0^T |\dot\varphi(t)||f(t)|_{V'}|\dot u_n(t)|_V dt+\int_0^T B(t)\varphi(t)dt). \end{aligned} \tag{3.30}$$

- We now chose φ such that $\dfrac{|\dot\varphi|}{\sqrt{\varphi}}\in L^\infty(0,T)$ (notice that $\varphi(t)=(T-t)^2$ realizes it). We then obtain

$$\int_0^T |\dot\varphi(t)||u_n(t)|_V|\dot u_n(t)|_V dt\le\int_0^T \sqrt{|\varphi(t)|}|\dot u_n(t)|_V|u_n(t)|_V\frac{|\dot\varphi(t)|}{\sqrt{|\varphi(t)|}}dt,$$

  and then

$$\int_0^T |\dot\varphi(t)||u_n(t)|_V|\dot u_n(t)|_V dt\le\frac{1}{\theta}\int_0^T |\varphi(t)||\dot u_n(t)|_V^2 dt+C\theta\int_0^T |u_n(t)|_V^2 dt. \tag{3.31}$$

- Using analogous argument, we can write

$$\int_0^T |\dot\varphi(t)||f(t)|_{V'}|\dot u_n(t)|_V dt\le\frac{1}{\theta}\int_0^T |\varphi(t)||\dot u_n(t)|_V^2 dt+C\theta\int_0^T |f(t)|_V^2 dt. \tag{3.32}$$

- Finally, with Young's inequality, we have

$$\int_0^T \varphi(t)|\dot f(t)|_{V'}|\dot u_n(t)|_V dt\le\theta\int_0^T \varphi(t)|\dot f(t)|_{V'}^2 dt+\frac{1}{\theta}\int_0^T \varphi(t)|\dot u_n(t)|_V^2 dt. \tag{3.33}$$

With (3.31), (3.32) and (3.33) for a large θ, (3.30) becomes

$$b_0\int_0^T \varphi(t)|\dot u_n(t)|_V^2 dt\le C\left(1+|u_n|^2_{L^2(0,T;V)}+|\dot f|^2_{L^2(0,T;V')}+|f|^2_{L^2(0,T;V')}+|B|_{L^1(0,T)}\right).$$

Since we can chose φ with

$$\inf_{s\in[0,T-\delta]}\varphi(s)>0,\quad \forall t\in[0,T-\delta],$$

for a fixed $\delta > 0$, we deduce (using part (i)) that there exists $C_\delta > 0$ independent on j such that

$$|\dot{u}_n|^2_{L^2(0,T-\delta;V)} \leq C_\delta \tag{3.34}$$

which ends Lemma 3.2's proof.

We now end the proof of Theorem 2.1 in case of a linear elastic constitutive relationship. Using Lemma 3.2, there exists an element $u \in L^\infty(0,T;V)$ and in $H^1(0,T_0;V)$ for $T_0 < T$ such that, after extraction of a subsequence still denoted by (u_n), we have for every $T_0 < T$,

$$u_n \rightharpoonup u \text{ weak } * \text{ in } L^\infty(0,T;V), \tag{3.35}$$

$$\dot{u}_n \rightharpoonup \dot{u} \text{ weak in } L^2(0,T_0;V). \tag{3.36}$$

Lemma 3.3. *The sequence $(u_n)_n$ has a unique weak limit point u and we have*

$$u_n \longrightarrow u \text{ strongly in } \mathcal{C}_c([0,T[,V). \tag{3.37}$$

Furthermore, u solves P^e_j.

Proof. We prove that for every $T_0 < T$, $(u_n)_n$ is a Cauchy sequence in $\mathcal{C}([0,T_0],V)$. Using (3.5) at rank n with $v = \dot{u}_n(t) - \dot{u}_p(t)$ and at rank p with $v = \dot{u}_p(t) - \dot{u}_n(t)$ with $p > n$, we get with $u_{np}(t) = u_p(t) - u_n(t)$,

$$\frac{1}{n}\left(\dot{u}_n(t), -\dot{u}_{np}(t)\right)_V + a(u_n(t), -\dot{u}_{np}(t)) + \langle j'(t,\dot{u}_n(t)), -\dot{u}_{np}(t)\rangle_{V',V} = \langle f(t), -\dot{u}_{np}(t)\rangle_{V',V},$$

$$\frac{1}{p}\left(\dot{u}_p(t), \dot{u}_{np}(t)\right)_V + a(u_p(t), \dot{u}_{np}(t)) + \langle j'(t,\dot{u}_p(t)), \dot{u}_{np}(t)\rangle_{V',V} = \langle f(t), \dot{u}_{np}(t)\rangle_{V',V}.$$

Adding these two inequalities and since

$$\langle j'(t,\dot{u}_n(t)) - j'(t,\dot{u}_p(t)), \dot{u}_n(t) - \dot{u}_p(t)\rangle_{V',V} \geq 0,$$

we obtain

$$\left(\frac{1}{n}\dot{u}_n(t) - \frac{1}{p}\dot{u}_p(t), \dot{u}_n(t) - \dot{u}_p(t)\right)_V + a(u_n(t) - u_p(t), \dot{u}_n(t) - \dot{u}_p(t)) \leq 0,$$

then

$$\frac{1}{2}\int_0^t \frac{d}{ds}(a(u_n(s) - u_p(s), u_n(s) - u_p(s)))ds \leq 2\int_0^t \sup\left(\frac{1}{n},\frac{1}{p}\right)(|\dot{u}_n(s)|^2_V + |\dot{u}_p(s)|^2_V)ds,$$

hence, for $0 \leq t \leq T_0$,

$$a(u_n(t) - u_p(t)), u_n(t) - u_p(t)) - 0 \leq \frac{2}{n}(|\dot{u}_n|^2_{L^2(0,T_0;V)} + |\dot{u}_p|^2_{L^2(0,T_0;V)}).$$

Using part (ii) of Lemma 3.2, this ends the proof of Lemma 3.3.

We still have to prove that u solves P^e_j. Notice that we already have for every $T_0 < T$, $u \in L^2(0,T;V) \cap H^1(0,T_0;V)$.

We consider in (3.5) test function $v = v(t)$ with $v \in L^2(0,T;V_0)$ and we integrate between 0 and T_0 for a $T_0 < T$. We get

$$\frac{1}{n}\int_0^{T_0} (\dot{u}_n(t), v(t) - \dot{u}_n(t))_V dt + \int_0^{T_0} a(u_n(t), v(t) - \dot{u}_n(t))dt + \int_0^{T_0} (j(t, v(t)) - j(t, \dot{u}_n(t)))dt \geq \int_0^{T_0} \langle f(t), v(t) - \dot{u}_n(t)\rangle_{V',V} dt. \tag{3.38}$$

With Lemma 3.2, we have

$$\int_0^{T_0} (\dot{u}_n(t), v(t))_V dt \to \int_0^{T_0} (\dot{u}(t), v(t))_V dt,$$

and

$$\frac{1}{n}\int_0^{T_0} (\dot{u}_n(t), v(t))_V dt \to 0, \tag{3.39}$$

$$\frac{1}{n}\int_0^{T_0} (\dot{u}_n(t), \dot{u}_n(t))_V dt \to 0, \tag{3.40}$$

and

$$\int_0^{T_0} \langle f(t), v - \dot{u}_n(t)\rangle_{V',V} dt \to \int_0^{T_0} \langle f(t), v - \dot{u}(t)\rangle_{V',V} dt, \tag{3.41}$$

Using now both Lemma 3.2 and Lemma 3.3, we get

$$\int_0^{T_0} a(u_n(t), v - \dot{u}_n(t))dt \to \int_0^{T_0} a(u(t), v - \dot{u}(t))dt, \tag{3.42}$$

since (u_n) strongly converge to u in $L^2(0,T_0,V)$.
Finally, (2.4) implies that

$$\liminf_{n\to+\infty} \int_0^{T_0} j(t, \dot{u}_n(t))dt \geq \int_0^{T_0} j(t, \dot{u}(t))dt. \tag{3.43}$$

Taking into account (3.39)-(3.43) in (3.38), we get

$$\int_0^{T_0} a(u(t), v - \dot{u}(t))dt + \int_0^{T_0} (j(v) - j(\dot{u}(t)))dt \geq \int_0^{T_0} \langle f(t), v - \dot{u}(t)\rangle_{V',V} dt,$$

which ends the existence part of Theorem 2.1. Assertions (2.8) and (2.9) follow essentially from (3.34) and weak convergence of $\dot{u}_n$ to $\dot{u}$ in $L^2(0,T_0;V_0)$.
In order to prove the uniqueness of a solution, we consider two solutions u_1 and u_2. We use (3.1) (twice) with $(u_1, v = \dot{u}_1(t) - \dot{u}_2(t))$ and $(u_2, v = \dot{u}_2(t) - \dot{u}_1(t))$, and we get

$$a(u_1(t), \dot{u}_1(t) - \dot{u}_2(t)) + \langle j'(t, \dot{u}_1(t)), \dot{u}_1(t) - \dot{u}_2(t)\rangle_{V',V} = \langle f(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle_{V',V},$$

$$a(u_2(t), \dot{u}_2(t) - \dot{u}_1(t)) + \langle j'(t, \dot{u}_2(t)), \dot{u}_2(t) - \dot{u}_1(t)\rangle_{V',V} = \langle f(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle_{V',V}.$$

Subtracting them and since $\langle j'(t, \dot{u}_1(t)) - j'(t, \dot{u}_2(t)), \dot{u}_1(t) - \dot{u}_2(t)\rangle_{V',V} \geq 0$, we can write

$$a(u_1(t) - u_2(t), \dot{u}_1(t) - \dot{u}_2(t)) \leq 0 \ \text{ a.e } t \in (0,T),$$

which leads to

$$\frac{1}{2}\int_0^t \frac{d}{ds}(a(u_1(t)-u_2(t),\dot{u}_1(t)-\dot{u}_2(t)))ds \le 0$$

and then ends the proof of the uniqueness part in Theorem 2.1.

The following (easy) consequence of Theorem 2.1 will be used in future sections of the paper.

Corollary 3.4.

Suppose that (2.3), (2.4), (2.5), *and* (2.6) *hold. If* $u_0 \in V_0$ *and* $f \in H^1(0,T;V')$ *satisfy*

$$\exists c_0 > 0, \;\; (\sigma_0, \varepsilon(v))_{\mathcal{H}} + j(0,v) \ge \langle f(0), v\rangle_{V',V} - c_0 \qquad \forall v \in V, \tag{3.44}$$

there exists a unique solution $u \in L^2(0,T;V_0)$ *and* $\sigma \in L^2(0,T;\mathcal{H}_1)$ *with* $u \in H^1(0,T_0;V_0)$ *and* $\sigma \in H^1(0,T_0;\mathcal{H}_1)$ *for every* $0 < T_0 < T$ *of problem* P_j.
Furthermore, there exists $C > 0$ *independent of* j *such that for every* $u^0 \in V$, $f \in H^1(0,T;V')$, *we have*

$$|u|_{L^2(0,T;V_0)} + |\sigma|_{L^2(0,T;\mathcal{H}_1)} \le C[|u_0|_{V_0} + |f|_{L^2(0,T;V')}], \tag{3.45}$$

and for every $T_0 < T$, *there exists* $C(T_0) > 0$, *such that for every* $u^0 \in V$, $f \in H^1(0,T;V')$, *we have*

$$|u|_{H^1(0,T_0;V_0)} + |\sigma|_{H^1(0,T_0;\mathcal{H}_1)} \le \frac{C(T_0)}{b_0}[|u_0|_{V_0} + |f|_{H^1(0,T;V')} + |G(0,0)|_{\mathcal{H}} + |B|^{1/2}_{L^1(0,T)}]. \tag{3.46}$$

Remarks

Applying Brezis's result, M. Shillor and M. Sofonéa proved in [18] existence for (2.13) when $j(t,v) = j(v)$.

Proof of Corollary 3.4

The proof of Corollary 3.4 is now straightforward with the use a Banach fixed point. For $\eta \in L^2(0,T;\mathcal{H})$, we write

$$z_\eta(t) = \sigma_0 - \mathcal{E}\varepsilon(u_0) + \int_0^t \eta(s).$$

Applying Theorem 2.1 with second right hand side $\tilde{f}$ defined by

$$\langle \tilde{f}, v\rangle_{V',V} = \langle f, v\rangle_{V',V} - (z_\eta, \varepsilon(v))_{\mathcal{H}},$$

there exists a unique solution (u_η, σ_η) to problem P_j^e.
We then consider the map

$$\Lambda : \eta \in L^2(0,T;\mathcal{H}) \mapsto G(\sigma_\eta, u_\eta) \in L^2(0,T;\mathcal{H}).$$

It is easy to prove that

$$|\Lambda(\eta_1)(t) - \Lambda(\eta_2)(t)|^2_{\mathcal{H}} \le c\int_0^t |\eta_1(s) - \eta_2(s)|^2_{\mathcal{H}} ds.$$

We deduce that Λ possesses a contractive iterate and then a unique fixed point. The corresponding solution solves problem P_j. Assertions (3.45) and (3.46) are easy consequences of such inequalities in the elastic case, the Lipschitz character of G and Gronwall's Lemma.

The uniqueness part in Corollary 3.4 is an easy consequence of Gronwall's Lemma.

We now turn to the

Proof of Theorem 2.2.

Recall that u_ρ is solution of

$$\begin{cases} \forall v \in V_0, \text{ a.e. } t \in (0,T), \\ \qquad a(u_\rho(t), v - \dot{u}_\rho(t))_{\mathcal{H}} + j_\rho(t,v) - j_\rho(t, \dot{u}_\rho(t)) \geq \langle f(t), v - \dot{u}_\rho(t)\rangle_{V',V} \\ u_\rho(0) = u_0. \end{cases} \tag{3.47}$$

Theorem 2.1's estimates (2.8) and (2.9) imply that there exists $\tilde{u} \in L^2(0,T,V_0) \cap H^1(0,T_0,V_0)$ (for every $T_0 < T$) such that (after extraction of a subsequence), we have

$$u_\rho \rightharpoonup \tilde{u} \text{ weak in } \ H^1(0,T_0,V_0).$$

Remark that this implies (by continuous and linear injection of $H^1(0,T_0,X)$ in $C([0,T_0],X)$) that for every $t \in [0,T[$, $u_\rho(t) \rightharpoonup \tilde{u}(t)$ weakly in V_0.

Let $v \in C^\infty([0,T_0];V_0)$. We have for every $s < T_0 < T$,

$$\begin{aligned} \int_0^s a(u_\rho(t), v(t))dt + \int_0^s j_\rho(t,v(t))dt \geq \frac{1}{2}a(u_\rho(s),u_\rho(s)) - \frac{1}{2}a(u^0,u^0) + \int_0^s j_\rho(t,\dot{u}_\rho(t))dt \\ + \int_0^s \langle f(t), v(t) - \dot{u}_\rho(t)\rangle_{V',V} dt. \end{aligned} \tag{3.48}$$

Using (2.10), we have on one hand,

$$\lim_{\rho\to 0} \int_0^s a(u_\rho(t),v(t))dt + \int_0^s j_\rho(t,v(t))dt = \int_0^s a(\tilde{u}(t),v(t))dt + \int_0^s j(t,v(t))dt,$$

$$\lim_{\rho\to 0} \int_0^s \langle f(t), v(t) - \dot{u}_\rho(t)\rangle_{V',V} dt = \int_0^s \langle f(t), v(t) - \dot{\tilde{u}}(t)\rangle_{V',V} dt;$$

and on the other hand,

$$\liminf_{\rho\to 0} \frac{1}{2}a(u_\rho(s),u_\rho(s)) \geq \frac{1}{2}a(\tilde{u}(s),\tilde{u}(s))$$

and

$$\liminf_{\rho\to 0} \int_0^s j_\rho(t,\dot{u}_\rho(t))dt \geq \int_0^s j(t,\dot{\tilde{u}}(t))dt.$$

It is now sufficient to take the infimum limit in (3.48) in order to prove the convergence of problems $P^e_{j_\rho}$ to P^e_j. This proves existence of a solution of $P^e_{j_\rho}$. Uniqueness of it is an easy and classical result. First part of Theorem 2.2 is proved.
We now turn to the case where (2.11) is fullfilled. Let us input $v = \dot{u}_\rho(t)$ in P^e_j, $v = \dot{u}(t)$ in $P^e_{j_\rho}$ and add the two inequalities to obtain

$$a(u_\rho(t) - u(t), \dot{u}_\rho(t) - \dot{u}(t)) \leq [j(t, \dot{u}_\rho(t)) - j_\rho(t, \dot{u}_\rho(t))] + [j_\rho(t, \dot{u}(t)) - j(t, \dot{u}(t))]] \leq c\rho$$

which ends Theorem 2.2's proof by an integration in time.

4. Asymptotic analysis when $\rho \to 0$

In this section, we prove Theorem 2.3 applying Corollary 3.4 and Theorem 2.2. We start with the existence part of Theorem 2.3.

Lemma 4.1. *Under hypotheses (2.17) and (2.18), functionals j_ρ satisfy (2.3), (2.5), (2.6) and (2.4). If moreover, ψ_ρ satisfies (2.19), then hypothesis $g \in H^1(0, T; L^2(\Gamma_3))$ is sufficient in order to ensure existence and uniqueness of the solution of P_ρ. More over, there exists $C_0 > 0$ independent on ρ such that*

$$|u_\rho|_{L^2(0,T,V_0)} + |\sigma_\rho|_{L^2(0,T;\mathcal{H}_1)} \leq C_0[|f|_{H^1(0,T,V')} + |u_0|_{V_0} + |g|_{H^1(0,T,L^2(\Gamma_3))} + |G(0, 0)|_{\mathcal{H}}]$$

and, for every $0 < T_0 < T$,

$$|u_\rho|_{H^1(0,T_0,V_0)} + |\sigma_\rho|_{H^1(0,T_0;\mathcal{H}_1)} \leq C_0[|f|_{H^1(0,T,V')} + |u_0|_{V_0} + |G(0, 0)|_{\mathcal{H}} + |g|_{H^1(0,T,L^2(\Gamma_3))} + \rho].$$

Proof of Lemma 4.1
Recall that

$$j_\rho(t, v) = \mu \int_{\Gamma_3} |g(t, a)|\psi_\rho(v_\tau(a))da,$$

where $g \in W^{1,\infty}(0, T; L^2(\Gamma_3))$.
Using (2.18) and since the trace map is linear continuous from V into $L^{m_d}(\Gamma_3)$ with $m_d \geq 2$ if $d = 2$ and $m_d = 2(d-1)/(d-2)$ if $d > 2$ (see [1]), one can prove that $\psi_\rho(v) \in L^2(\Gamma_3)$ for $v \in V$ and j_ρ is well defined. Of course, (2.3) is valid.

(i) **Proof of (2.4)**. We denote by

$$F(v) = \int_0^T \int_{\Gamma_3} |g(t, a)|\psi_\rho(v(t, a))dtda.$$

It is clear that F is defined and convex on $L^2(0, T; V)$. Since every convex and lower semicontinous function on $L^2(0, T; V)$ is weakly lower semicontinous, (2.4) will be proved with the l.s.c property of F. Let then v_n be a sequence in $L^2(0, T; V)$ which converges in that space to a function v. Let v_{n_k} be a subsequence of v_n such that

$$\liminf_{n\to+\infty} F(v_n) = \lim_{k\to+\infty} F(v_{n_k}).$$

Using Tonelli's theorem, we deduce that, after extraction of a subsequence still denoted by v_{n_k} we have

$$v_{n_k} \to v \quad a.e.\ \ on\ \ (0,T) \times \Gamma_3,$$

and, since ψ_ρ is continuous,

$$|g|\psi_\rho(v_{n_k}) \to |g|\psi_\rho(v) \quad a.e.\ \ on\ \ (0,T) \times \Gamma_3.$$

Fatou's Lemma implies that

$$\liminf_{k\to+\infty} F(v_{n_k}) \geq \int_0^T \int_{\Gamma_3} |g(t,a)|\psi_\rho(v(t,a))dtda,$$

which is exactly the l.s.c character of F.

(ii) **Proof of (2.5)**. We write

$$\langle j'(t+h,v) - j'(t,v), u\rangle_{V',V} = \int_{\Gamma_3} (|g(t+h,a)| - |g(t,a)|)\psi'_\rho(v(a)).u(a)da,$$

hence

$$|\langle j'(t+h,v) - j'(t,v), u\rangle_{V',V}| \leq |g(t+h) - g(t)|_{L^2(\Gamma_3)} |\psi'_\rho(v).u|_{L^2(\Gamma_3)}. \tag{4.1}$$

We have, since the trace map is linear continuous from V in $L^{2(d-1)/(d-2)}(\Gamma_3)$, (see [1])

$$\begin{aligned} \int_{\Gamma_3} (\psi'_\rho(v(a)).u(a))^2 da \ &\leq \int_{\Gamma_3} |\psi'_\rho(v)|^2|u|^2 da \\ &\leq |u^2|_{L^{(d-1)/(d-2)}(\Gamma_3)} |\psi'_\rho(v)^2|_{L^{d-1}(\Gamma_3)} \\ &\leq |u|^2_{L^{2(d-1)/(d-2)}(\Gamma_3)} |\psi'_\rho(v)|^2_{L^{2(d-1)}(\Gamma_3)}. \end{aligned} \tag{4.2}$$

Using (2.18), we can write

$$\int_{\Gamma_3} |\psi'_\rho(v)|^{2(d-1)} \leq c[|v|_V^{2(d-1)/(d-2)} + 1], \tag{4.3}$$

hence

$$|\psi'_\rho(v)|^2_{L^{2(d-1)}(\Gamma_3)} \leq c[|v|_V^{2/(d-2)} + 1]. \tag{4.4}$$

Taking into account (4.3) in (4.1), we get

$$|j'(t+h,v) - j'(t,v)|_{V'} \leq ch[|v|_V^{2/(d-2)} + 1]^{1/2}, \tag{4.5}$$

which proves (2.5).

(iii) **Proof of (2.6)**.

We have

$$\frac{\partial j}{\partial t}(t,v) = \mu \int_{\Gamma_3} \frac{\partial}{\partial t}(|g|)(t,a)\psi_\rho(v(a))dtda.$$

We then get for a constant $c > 0$,

$$|\frac{\partial}{\partial t}(|g|)(t,a)|\psi_\rho(v_\tau(a)) \leq |\frac{\partial}{\partial t}(|g|)(t,a)|(|v_\tau(a)|^r + c)$$

Using (2.18), we can consider $1 < p \leq 2/r$. We write for every $\eta > 0$, with $p' = p/(p-1)$ and Young's inequality,

$$\begin{aligned}\int_{\Gamma_3} \frac{\partial}{\partial t}(|g|)(t,a)|\psi_\rho(v_\tau(a))da &\leq |\tfrac{\partial}{\partial t}(|g|)(t)|_{L^2(\Gamma_3)}(||v_\tau|^r|_{L^2(\Gamma_3)} + c) \\ &\leq |\tfrac{\partial}{\partial t}(|g|)(t)|_{L^2(\Gamma_3)}(|v_\tau(t)|^r_{L^{2r}(\Gamma_3)} + c) \\ &\leq \eta|v_\tau|^{pr}_{L^{2r}(\Gamma_3)} + C_\eta|\tfrac{\partial}{\partial t}(|g|)(t)|^{p'}_{L^2(\Gamma_3)} + c\end{aligned}$$

Recall that the trace map is linear continuous from V into $L^{2r}(\Gamma_3)$ hence we deduce

$$a(v,v) + \tfrac{\partial j}{\partial t}(t,v) \geq a(v,v) - \eta||v||^{pr}_V - C_\eta|\tfrac{\partial}{\partial t}(|g|)(t)|^{p'}_{L^2(\Gamma_3)} - c, \tag{4.6}$$

which (for η small enough) proves assertion (2.6).

Suppose now that ψ_ρ satisfies (2.19), then $\psi_\rho(x) = \mathcal{O}(|x| + \rho)$ and $p = p' = 2$ is a valid choice in (4.6) which easily (taking care of parameter ρ) leads to

$$a(v,v) + \tfrac{\partial j}{\partial t}(t,v) \geq c||v||^2_V - C_\eta|\tfrac{\partial}{\partial t}(|g|)(t)|^2_{L^2(\Gamma_3)} - c\rho^2.$$

which, with the previous section, ends Lemma 4.1's proof. □

Using Corollary 3.4, there exists a unique solution (u_ρ, σ_ρ) to problem P_ρ with the regularity

$$\forall T_0 < T, \quad u_\rho \in L^2(0,T;V_0) \cap H^1(0,T_0;V_0),$$

and

$$\forall T_0 < T, \quad \sigma_\rho \in L^2(0,T;\mathcal{H}_1) \cap H^1(0,T_0;\mathcal{H}_1).$$

We prove

Lemma 4.2. *Problems P_ρ possesses a unique solution $u_\rho \in H^1(0,T;V_0)$ and $\sigma_\rho \in H^1(0,T;\mathcal{H}_1)$, and there exists $C_0 > 0$, independent on ρ, such for every $\rho > 0$, we have*

$$|u_\rho|_{H^1(0,T,V_0)} + |\sigma_\rho|_{H^1(0,T;\mathcal{H}_1)} \leq C_0[|f|_{H^1(0,T,V')} + |u_0|_{V_0} + |g|_{W^{1,\infty}(0,T,L^2(\Gamma_3))} + 1].$$

If moreover ψ_ρ satisfies (2.19), then we can chose C_0 such that

$$|u_\rho|_{H^1(0,T,V_0)} + |\sigma_\rho|_{H^1(0,T;\mathcal{H}_1)} \leq C_0[|f|_{H^1(0,T,V')} + |u_0|_{V_0} + |g|_{H^1(0,T,L^2(\Gamma_3))} + \rho].$$

Proof of Lemma 4.2

We consider continuous extensions $\bar{f}$ of f in $H^1(0,T+1,V')$ and $\bar{g}$ of g in $W^{1,\infty}(0,T+1;L^2(\Gamma_3))$, the solution $\bar{u}$ associated to $\bar{f}$ on the interval $(0,T+1)$ satisfies

$$\bar{u} = u \text{ sur } [0,T].$$

We apply Corollary 3.4 to $\bar{u}$ on $[0,T+1-\delta]$ with $\delta = 1$ to conclude the proof of Lemma 4.2.

Proof of assertion (ii) of Theorem 2.3

We suppose here that ψ_ρ satisfies (2.19). We follow the proof of Theorem 2.2 concerning the case of an elastic constitutive relationship.

Input $v = \dot{u_\rho}(t)$ in P_0 and $v = \dot{u}(t)$ in P_ρ. We obtain

$$\begin{cases} (\sigma(t), \dot{u_\rho}(t) - \dot{u}(t))_{\mathcal{H}} + j_0(t, \dot{u_\rho}(t)) - j_0(\dot{u}(t)) \geq < f(t), \dot{u_\rho}(t) - \dot{u}(t) >_{V',V} \\ (\sigma_\rho(t), \dot{u}(t) - \dot{u_\rho}(t))_{\mathcal{H}} + j_\rho(t, \dot{u}(t)) - j_\rho(\dot{u_\rho}(t)) \geq < f(t), \dot{u}(t) - \dot{u_\rho}(t) >_{V',V} \end{cases}$$

Adding these two inequalities, we get

$$(\sigma_\rho(t) - \sigma(t), \dot{u_\rho}(t) - \dot{u}(t))_{\mathcal{H}} \leq [j_0(t, \dot{u_\rho}(t)) - j_\rho(t, \dot{u_\rho}(t))] + [j_\rho(t, \dot{u}(t)) - j_0(t, \dot{u}(t))] \qquad (4.7)$$

Using the constitutive relationship, the Lipschitz character of the plasticity map (see (2.16), Gronwall's Lemma and hypothesis (2.19), it is easy to prove that (4.7) leads to

$$\sup_{t\in[0,T]} [|u_\rho(t) - u(t)|_V^2 + |\sigma_\rho(t) - \sigma(t)|_{\mathcal{H}_1}^2] \leq C\rho,$$

which proves assertion (ii) of Theorem 2.3.

Proof of assertion (iii) of Theorem 2.3

Since $G = 0$, the constitutive relationship is $\sigma(t) = \mathcal{E}\varepsilon(u(t)) + z^0$, where $z^0 = \sigma_0 - \mathcal{E}\varepsilon(u_0)$. It is then a linear elastic relationship and assertion (iii) will be a consequence of Theorem 2.2.

We still have to prove that functionals j_ρ satisfy (2.10).

Let first $v \in L^2(0,T;V_0)$. We have

$$|g(t,a)|\psi_\rho(v_\tau(t,a)) \to |g(t,a)|\psi_0(v_\tau(t,a)) \ \ a.e. \text{ on } \ (0,T)\times\Gamma_3.$$

Furthermore, (2.18) gives

$$|g(t,a)|\psi_\rho(v_\tau(t,a)) \leq c|g(t,a)|(|v_\tau(t,a)|^{m_d} + 1),$$

with $m_2 = 2$ and $m_d = (d-1)/(d-2)$ for $d > 2$. We have

$$|g(t)|_{L^2(\Gamma_3)}||v_\tau(t)|^{m_d}|_{L^2(\Gamma_3)} \leq c|g(t)|_{L^2(\Gamma_3)}(|v_\tau(t)|^{m_d}_{L^{2m_d}(\Gamma_3)} + 1).$$

hence

$$
\begin{aligned}
\int_0^T |g(t)|_{L^2(\Gamma_3)} ||v_\tau(t)|^{m_d}|_{L^2(\Gamma_3)} dt & \leq c|g|_{L^\infty(0,T;L^2(\Gamma_3))} \int_0^T (|v_\tau(t)|^{m_d}_{L^{2m_d}(\Gamma_3)} + 1)dt \\
& \leq c|g|_{L^\infty(0,T;L^2(\Gamma_3))} \int_0^T (|v(t)|^{m_d}_V + 1)dt;
\end{aligned}
\tag{4.8}
$$

Since, in any case, $m_d \leq 2$, this proves that $|g(t,a)||v_\tau(t,a)|^{m_d} \in L^1(0,T;L^2(\Gamma_3))$. We deduce the first assertion of (2.20) with Lebesgue's Theorem.

We still have to prove that if v_ρ weakly converge to v in $L^2(0,T;V_0)$, then

$$
\liminf_\rho \int_0^T j_\rho(t, v_\rho(t))dt \geq \int_0^T j(t, v(t))dt.
$$

We write

$$
j_\rho(t, v_\rho(t)) = j_\rho(t, v_\rho(t)) - j_\rho(t, v(t)) + j_\rho(t, v(t)). \tag{4.9}
$$

On one hand, since $v \in L^2(0,T;V_0)$, assertion (2.10) implies

$$
\lim_{\rho \to 0} \int_0^T j_\rho(t, v(t))dt \int_0^T j(t, v(t))dt; \tag{4.10}
$$

and on the other hand,

$$
j_\rho(t, v_\rho(t)) - j_\rho(t, v(t)) \geq \langle j'_\rho(t, v(t)), v_\rho(t) - v(t) \rangle_{V',V}.
$$

Since $\psi'_\rho(x) \to S(x)$ on $\mathbb{R}^d$, we deduce that $\psi'_\rho(v_\tau(t,x)) \to S(v_\tau(t,x))$ for almost every (t,x) in $(0,T) \times \Gamma_3$. Furthermore, by (2.18), we have (with $m_2 = 1$ and $m_d = 1/(d-2) \leq 1$ if $d > 2$), for almost everywhere on $(0,T) \times \Gamma_3$,

$$
|g(t,a)||\psi'_\rho(v_\tau(t,a))| \leq |g(t,a)|(|v_\tau(t,a)^{m_d}| + c)
$$

with

$$
\begin{aligned}
\int_0^T \int_{\Gamma_3} |g(t,a)||v_\tau(t,a)^{m_d}| dtda & \leq \int_0^T |g(t)|_{L^2(\Gamma_3)} |v_\tau(t)|^{m_d}_{L^{2m_d}(\Gamma_3)} dt \\
& \leq |g|_{L^\infty(0,T;L^2(\Gamma_3))} \int_0^T |v(t)|^{m_d}_V dt < +\infty.
\end{aligned}
\tag{4.11}
$$

Using Lebesgue's Theorem, we have proved that $j'_\rho(t, v(t))$ strongly converge in $L^2(0,T;V')$. Since, $v_\rho - v$ weakly converge to zero in $L^2(0,T;V)$, we obtain that

$$
\int_0^T \langle j'_\rho(t, v(t)), v_\rho(t) - v(t) \rangle_{V',V} dt \quad \to 0,
$$

which easily leads to

$$
\liminf_{\rho \to 0} \int_0^T [j_\rho(t, v_\rho(t)) - j_\rho(t, v(t))]dt \geq 0. \tag{4.12}
$$

Using (4.9) and (4.10), we deduce the second assertion of (2.20). Assertion (iii) of Theorem 2.3 is now a consequence of Theorem 2.2.

5. The case of a Coulomb friction law

In this section, we suppose that that the friction coefficient μ satisfies (2.24) and we assume that ψ_ρ satisfies (2.19). We recall that this case is valid with

$$\psi_\rho(x) = \sqrt{|x|^2 + \rho^2} - \rho.$$

In order to prove the existence part of Theorem 2.4, we introduce the map Λ_ρ defined by

$$\begin{array}{rccc} \Lambda_\rho : & H^1(0,T;L^2(\Gamma_3)) & \rightarrow & H^1(0,T;L^2(\Gamma_3)) \\ & g & \rightarrow & R\sigma_{\rho\nu} \end{array} \tag{5.1}$$

where σ_ρ is the solution of problem P_ρ. For sake of simplicity and only in the first part of this section where the parameter ρ is fixed, we write Λ instead of Λ_ρ and (u,σ) instead of (u_ρ,σ_ρ) the solution of problem P_ρ. Notice that we will take care about the dependence of different constants with respect to the parameter ρ.
According to Lemma 4.2 (last part), one can remark that Λ is well defined. In what follows, we are going to prove that Λ has a fixed point using the second fixed point Schauder Theorem that we now recall (see [20]):

Theorem 5.1. (Second Schauder's fixed point theorem)
Suppose that :
(i) X is a reflexive and separable Banach space,
(ii) M is a bounded, closed, convex and non empty subset of X,
(iii) The map $T : M \subseteq X \rightarrow M$ is sequentially weakly continuous (which means that if $x_n \rightharpoonup x$ when $n \rightarrow +\infty$, then $T(x_n) \rightharpoonup T(x)$),
then , T has a fixed point.

We will set here $X = H^1(0,T;L^2(\Gamma_3))$ and we prove

Lemma 5.2. *There exists $\mu_0 > 0$ independent on $\rho \in]0,1[$, such that for every μ with $|\mu|_{L^\infty(\Gamma_3)} \leq \mu_0$, there exists $R_1 > 0$ independent on $\rho \in]0,1[$, such that $\Lambda(\overline{B(0,R_1)}) \subset \overline{B(0,R_1)}$ ($\overline{B(0,R_1)}$ denotes the close unit ball of de $H^1(0,T;L^2(\Gamma_3))$).*

Proof of Lemma 5.2.
From the last part of Lemma 4.2, with g replaced by μg, we get

$$\begin{aligned} |u|_{H^1(0,T;V_0)} + |\sigma|_{H^1(0,T;\mathcal{H}_1)} \leq\ & C_0[|u_0|_V + |\sigma_0|_{\mathcal{H}} + |G(0,0)|_{\mathcal{H}} \\ & + |f|_{H^1(0,T;V')} + |\mu|_{L^\infty(\Gamma_3)}|g|_{H^1(0,T;L^2(\Gamma_3))} + \rho], \end{aligned} \tag{5.2}$$

where C_0 does not depend on ρ. Moreover, we have (C_R is given by the continuity of R)

$$|R\sigma_\nu|_{H^1(0,T;L^2(\Gamma))} \leq C_R|\sigma_\nu|_{H^1(0,T;H^{-1/2}(\Gamma))},$$

and there exists $C_1 > 0$ independent on ρ such that

$$|R\sigma_\nu|_{H^1(0,T;L^2(\Gamma))} \leq C_1 |\sigma|_{H^1(0,T;\mathcal{H}_1)}.$$

We deduce from (5.2) that for $\rho < 1$,

$$\begin{aligned} |R\sigma_\nu|_{H^1(0,T;L^2(\Gamma))} &\leq C_0C_1[|u_0|_V + |\sigma_0|_{\mathcal{H}} + |G(0,0)|_{\mathcal{H}} \\ &\quad + |f|_{H^1(0,T;V')} + |\mu|_{L^\infty(\Gamma_3)}|g|_{H^1(0,T;L^2(\Gamma_3))} + 1], \end{aligned} \tag{5.3}$$

Let us introduce

$$D = |u_0|_V + |f|_{H^1(0,T;V')} + |\sigma_0|_{\mathcal{H}} + |G(0,0)|_{\mathcal{H}} + 1.$$

Using (5.2), we get

$$|R\sigma_\nu|_{H^1(0,T;L^2(\Gamma))} \leq C_0C_1[D + |\mu|_{L^\infty(\Gamma_3)}|g|_{H^1(0,T;L^2(\Gamma_3))}]. \tag{5.4}$$

Let us prove that we can choose R_1 with

$$C_0C_1D + C_0C_1|\mu|_{L^\infty(\Gamma_3)}R_1 \leq R_1.$$

This inequality can be written

$$R_1(1 - C_0C_1|\mu|_{L^\infty(\Gamma_3)}) \geq C_0C_1D. \tag{5.5}$$

Under the hypothesis

$$|\mu|_{L^\infty(\Gamma_3)} < \frac{1}{C_0C_1},$$

(5.5) leads to

$$R_1 \geq \frac{D}{\frac{1}{C_0C_1} - |\mu|_{L^\infty(\Gamma_3)}} \tag{5.6}$$

It is then sufficient to choose $\mu_0 = \frac{1}{C_0C_1}$ and R_1 with (5.6) both independent on ρ in order to ensure 5.2 being satisfied. $\square$

Lemma 5.3. *The map $\Lambda : \overline{B(0,R_1)} \to \overline{B(0,R_1)}$ is sequentially weakly continuous.*

Proof of Lemma 5.3.

Let us consider a sequence g_n of element of $\overline{B(0,R_1)}$ with $g_n \rightharpoonup g$ for a g in $\overline{B(0,R_1)}$.

(i) First step : let (u,σ) and (u_n,σ_n) be the solutions of P_ρ with data g et g_n.

Since $|g_n|_{H^1(0,T;L^2(\Gamma_3)} \leq R_1$, estimate (5.2) ensure that u_n and σ_n are bounded respectively in $H^1(0,T;V_0)$ and in $H^1(0,T;\mathcal{H}_1)$ and there then exists $\tilde{u}$ in $H^1(0,T;V_0)$ and $\tilde{\sigma}$ in $H^1(0,T;\mathcal{H}_1)$, such that after extraction of a subsequence still denote by (u_n,σ_n)

$$(u_n,\sigma_n) \rightharpoonup (\tilde{u},\tilde{\sigma}) \quad \text{in} \quad H^1(0,T;V_0) \times H^1(0,T;\mathcal{H}_1)\text{weak}, \tag{5.7}$$

Let us first prove with Cauchy's criterium that

$$(u_n, \sigma_n) \to (\tilde{u}, \tilde{\sigma}) \quad \text{in} \quad C([0,T]; V_0) \times C([0,T]; \mathcal{H}_1). \tag{5.8}$$

Using the variational inequalities defining u_n and u_p for $p > n$, and input as test functions respectively $v = \dot{u}_p(t)$ and $v = \dot{u}_n(t)$, we get

$$\begin{cases} a(u_p(t), \dot{u}_n(t) - \dot{u}_p(t)) & + \quad (z_p(t), \varepsilon(\dot{u}_n(t) - \dot{u}_p(t)))_{\mathcal{H}} \\ & + \quad j_p(t, \dot{u}_n(t)) - j_p(t, \dot{u}_p(t)) \geq \langle f(t), \dot{u}_n(t) - \dot{u}_p(t) \rangle_{V',V}, \\ a(u_n(t), \dot{u}_p(t) - \dot{u}_n(t)) & + \quad (z_n(t), \varepsilon(\dot{u}_p(t) - \dot{u}_n(t)))_{\mathcal{H}} \\ & + \quad j_n(t, \dot{u}_p(t)) - j_n(t, \dot{u}_n(t)) \geq \langle f(t), \dot{u}_p(t) - \dot{u}_n(t) \rangle_{V',V}, \end{cases}$$

where we have written

$$\forall v \in V, \quad j_n(t,v) = \int_{\Gamma_3} \mu(a) |g_n(t,a)| \psi_\rho(a) da, \quad \text{and} \quad z_n(t) = \sigma_0 - \mathcal{E}\varepsilon(u_0) + \int_0^t G(\sigma_n(s), \varepsilon(u_n(s))) ds.$$

Adding, we obtain

$$\begin{aligned} a(u_p(t) - u_n(t), \dot{u}_p(t) - \dot{u}_n(t)) \quad & \leq \quad (z_p(t) - z_n(t), \varepsilon(\dot{u}_n(t) - \dot{u}_p(t)))_{\mathcal{H}} \\ & + \quad (j_n(t, \dot{u}_p(t)) - j_p(t, \dot{u}_p(t))) + (j_p(t, \dot{u}_n(t)) - j_n(t, \dot{u}_n(t))), \end{aligned}$$

which leads, after integrating with respect to time variable

$$\begin{aligned} &\frac{1}{2} a(u_p(t) - u_n(t), u_p(t) - u_n(t)) \leq (z_p(t) - z_n(t), \varepsilon(u_n(t) - u_p(t)))_{\mathcal{H}} \\ &\quad - \int_0^t (\dot{z}_p(s) - \dot{z}_n(s), \varepsilon(u_n(s) - u_p(s)))_{\mathcal{H}} ds + \int_0^t J_{n,p}(s, \dot{u}_n(s), \dot{u}_p(s)) ds, \end{aligned} \tag{5.9}$$

where

$$J_{n,p}(t, \dot{u}_n(t), \dot{u}_p(t)) = (j_n(t, \dot{u}_p(t)) - j_p(t, \dot{u}_p(t))) + (j_p(t, \dot{u}_n(t)) - j_n(t, \dot{u}_n(t))).$$

For every $\theta > 0$, we have

$$|(z_p(t) - z_n(t), \varepsilon(u_n(t) - u_p(t)))_{\mathcal{H}}| \leq C_\theta |z_p(t) - z_n(t)|_{\mathcal{H}}^2 + C\theta |u_n(t) - u_p(t)|_V^2,$$

therefore for θ small enough,

$$\begin{aligned} |u_n(t) - u_p(t)|_V^2 &\leq C(|z_p(t) - z_n(t)|_{\mathcal{H}}^2 + \int_0^t |\dot{z}_p(s) - \dot{z}_n(s)|_{\mathcal{H}}^2 ds \\ &+ \int_0^t |u_n(s) - u_p(s)|_V^2 ds + \int_0^t J_{n,p}(s, \dot{u}_n(s), \dot{u}_p(s)) ds). \end{aligned} \tag{5.10}$$

On another hand, we have

$$|\dot{z}_p(t) - \dot{z}_n(t)|_{\mathcal{H}} = |G(\sigma_p(t), \varepsilon(u_p(t))) - G(\sigma_n(t), \varepsilon(u_n(t)))|_{\mathcal{H}},$$

and since the map G is Lipschitz, we get

$$\int_0^t |\dot{z}_p(t) - \dot{z}_n(t)|_{\mathcal{H}}^2 ds \leq L \left(\int_0^t |\sigma_p(s) - \sigma_n(s)|_{\mathcal{H}}^2 ds + \int_0^t |u_n(s) - u_p(s)|_V^2 ds \right). \tag{5.11}$$

Using again the constitutive relationship, we have

$$|z_p(t) - z_n(t)|_{\mathcal{H}} = |\int_0^t G(\sigma_p(s), \varepsilon(u_p(s))) - G(\sigma_n(s), \varepsilon(u_n(s)))ds|_{\mathcal{H}},$$

and thus

$$|z_p(t) - z_n(t)|_{\mathcal{H}}^2 \leq L\left(\int_0^t |\sigma_p(s) - \sigma_n(s)|_{\mathcal{H}}^2 ds + \int_0^t |u_n(s) - u_p(s)|_V^2 ds\right). \tag{5.12}$$

Using (5.12), (5.11), (5.10), we have

$$|u_n(t) - u_p(t)|_V^2 + |\sigma_p(t) - \sigma_n(t)|_{\mathcal{H}}^2 \leq C(\int_0^t |\sigma_p(s) - \sigma_n(s)|_{\mathcal{H}}^2 ds$$

$$+ \int_0^t |u_n(s) - u_p(s)|_V^2 ds + \int_0^t J_{n,p}(s, \dot{u}_n(s), \dot{u}_p(s))ds).$$

We now use Gronwall Lemma to deduce that there exists $C_2 > 0$ independent on ρ such that

$$|u_n(t) - u_p(t)|_V^2 + |\sigma_p(t) - \sigma_n(t)|_{\mathcal{H}}^2 \leq C_2 \int_0^T J_{n,p}(s, \dot{u}_n(s), \dot{u}_p(s))ds. \tag{5.13}$$

We still have to prove that the last term in (5.13) tends to 0. We have

$$j_n(t, \dot{u}_p(t)) - j_p(t, \dot{u}_p(t)) = \int_{\Gamma_3} \mu(a)(|g_n(t,a)| - |g_p(t,a)|)\psi_\rho(\dot{u}_{p\tau}(a))da.$$

After extraction of a subsequence, we get

$$g_n \rightharpoonup g \quad \text{in} \quad H^1(0,T;L^2(\Gamma_3))\text{weak} \Rightarrow g_n \to g \quad \text{in} \quad L^2(0,T;H^{-1/2}(\Gamma_3)) \ \text{ strong},$$

thus $(g_n)_n$ converges and satifies Cauchy's criterium in $L^2(0,T;H^{-1/2}(\Gamma_3))$.
Furthermore, for $v \in L^2(0,T;V)$, we have $\psi_\rho(v) \in L^2(0,T;V)$ and since $\gamma(\psi_\rho(v)) = \psi_\rho(\gamma v)$, we can write that

$$\dot{u}_p \quad \text{is bounded in} \quad L^2(0,T;V_0) \Rightarrow \psi_\rho(\dot{u}_{p\tau}) \quad \text{is bounded in} \quad L^2(0,T;H^{1/2}(\Gamma)).$$

As

$$\int_0^T (j_n(t, \dot{u}_p(t)) - j_p(t, \dot{u}_p(t)))dt = \langle |g_p| - |g_n|, \ \mu\psi_\rho(\dot{u}_{p\tau})\rangle_{L^2(0,T;H^{-1/2}(\Gamma_3)),L^2(0,T;H^{1/2}(\Gamma_3))},$$

we conclude that this last term also satisfies Cauchy's criterium.
Finally, we can treated the term $J_{n,p}$ in an analogous way.
We have proved the existence of a sub-sequence satisfying Cauchy's criterium. With (5.7) and the uniqueness property of the limit point, assertion (5.8) follows.

(ii) Second step : using (5.7) and (5.8), we prove that $(\tilde{u}, \tilde{\sigma})$ is solution of the contact with Tresca friction problem with g as friction yield limit.

First, we recall that the subsequence (u_n, σ_n) satisfies

$$\begin{cases} \sigma_n(t) = \mathcal{E}\varepsilon(u_n(t)) + \sigma_0 - \mathcal{E}\varepsilon(u_0) + \displaystyle\int_0^t G(\sigma_n(s), \varepsilon(u_n)(s))ds \quad \text{ a.e. } t \in (0,T), \\ \forall v \in V_0, \text{ a.e. } t \in (0,T), \\ \qquad (\sigma_n(t), \varepsilon(v) - \varepsilon(\dot{u}_n(t)))_{\mathcal{H}} + j_n(R\sigma_{n\nu}, v) - j_n(R\sigma_{n\nu}, \dot{u}_n(t)) \geq \langle f(t), v - \dot{u}_n(t)\rangle_{V',V} \\ u_n(0) = u_0, \qquad \sigma_n(0) = \sigma_0 \end{cases}$$

With (5.8), we get

$$\tilde{\sigma}(t) = \mathcal{E}\varepsilon(\tilde{u}(t)) + \sigma_0 - \mathcal{E}\varepsilon(u_0) + \int_0^t G(\tilde{\sigma}(s), \varepsilon(\tilde{u})(s))ds. \tag{5.14}$$

We now prove that one can pass to the limit in this inequality. For $t \leq T$ and v in V, we have

$$\int_0^t (\sigma_n(s), v - \dot{u}_n(s))ds + \int_0^t j_n(s, v)ds - \int_0^t j_n(s, \dot{u}_n(s))ds \geq \int_0^t \langle f(t), v - \dot{u}_n(t)\rangle_{V',V} ds. \tag{5.15}$$

With (5.7), we deduce

$$\int_0^t \langle f(t), v - \dot{u}_n(t)\rangle_{V',V} ds \to \int_0^t \langle f(t), v - \dot{\tilde{u}}(t)\rangle_{V',V} ds.$$

Using (5.7) and (5.8), we get

$$\int_0^t (\sigma_n(s), v - \dot{u}_n(s))ds \to \int_0^t (\tilde{\sigma}(s), v - \dot{\tilde{u}}(s))ds.$$

Let us write

$$\int_0^t j_n(s, \dot{u}_n(s))ds = \int_0^t j_n(s, \dot{u}_n(s))ds - \int_0^t j(s, \dot{u}_n(s))ds + \int_0^t j(s, \dot{u}_n(s))ds.$$

One can easily prove (same proof as in the study of $J_{n,p}$) that

$$\int_0^t j_n(s, \dot{u}_n(s))ds - \int_0^t j(s, \dot{u}_n(s))ds \to 0.$$

With (5.7), the convexity and s.l.c properties of j, we obtain

$$\liminf_{n \to +\infty} \int_0^t j(s, \dot{u}_n(s))ds \geq \int_0^t j(s, \dot{\tilde{u}}(s))ds,$$

which leads to

$$\int_0^t (\tilde{\sigma}(s), v - \dot{\tilde{u}}(s))ds + \int_0^t j(s, v)ds - \int_0^t j(s, \dot{\tilde{u}}(s))ds \geq \int_0^t \langle f(t), v - \dot{\tilde{u}}(t)\rangle_{V',V} ds.$$

We deduce that $(\tilde{u}, \tilde{\sigma})$ is a solution of the Tresca friction contact problem with g as friction bound, where g is the weak limit of g_n in $H^1(0,T; L^2(\Gamma_3))$. On another hand, we know the existence and uniqueness of the solution of this last problem, solution that we denote by note (u, σ). This ensure that $(u, \sigma) = (\tilde{u}, \tilde{\sigma})$ and that the whole sequence (u_n, σ_n) converges to (u, σ) in both senses (5.7) and

(5.8).
Since the trace map and R, being linear continuous, are weakly continuous, we get

$$R\sigma_{n\nu} \rightharpoonup R\sigma_\nu \quad \text{in} \quad H^1(0,T;L^2(\Gamma_3)).$$

This ends the proof of Lemma 5.3. □

Proof Theorem 2.4, part (i).

Theorem 2.4 is now a direct consequence of existence result in Lemma 4.2, and of Theorem 5.1 applying to the map Λ defined by (5.1). Moreover, using (5.2), we deduce that for $\rho < 1$,

$$\begin{aligned} |u_\rho|_{H^1(0,T;V_0)} + |\sigma_\rho|_{H^1(0,T;\mathcal{H}_1)} \;&\leq\; C_0[|u_0|_V + |\sigma_0|_{\mathcal{H}} + |G(0,0)|_{\mathcal{H}} \\ &\quad + |f|_{H^1(0,T;V')} + \mu_0 R_1 + 1], \end{aligned} \tag{5.16}$$

The choice of

$$M = C_0[|u_0|_V + |\sigma_0|_{\mathcal{H}} + |G(0,0)|_{\mathcal{H}} + |f|_{H^1(0,T;V')} + \mu_0 R_1 + 1]$$

ends the proof of the existence part of Theorem 2.4. □

Remark 5.4.
Uniqueness of solution(s) is an open problem.

We end by the
Proof Theorem 2.4, part (ii).
Let us now write (u_ρ, σ_ρ) the solutions of P_{rcv} that we have just built. Estimate (5.16) proves that they are uniformly bouded in $H^1(0,T;V_0) \times H^1(0,T;\mathcal{H}_1)$ with respect to $\rho \in]0,1[$. Let then denote by (u,σ) a weak limit point in that space.
We first prove the strong convergence in $C([0,T];V_0)\times C([0,T];\mathcal{H}_1)$. We use the variational inequalities defining u_ρ and $u_{\rho'}$, input as test functions respectively $v = \dot{u}_{\rho'}(t)$ and $v = \dot{u}_\rho(t)$, and we obtain

$$\left\{ \begin{array}{lll} a(u_\rho(t), \dot{u}_{\rho'}(t) - \dot{u}_\rho(t)) & + & (z_\rho(t), \varepsilon(\dot{u}_{\rho'}(t) - \dot{u}_\rho(t)))_{\mathcal{H}} \\ & + & j_\rho(u_\rho, \dot{u}_{\rho'}(t)) - j_\rho(u_\rho, \dot{u}_\rho(t)) \geq \langle f(t), \dot{u}_{\rho'}(t) - \dot{u}_\rho(t)\rangle_{V',V}, \\ a(u_{\rho'}(t), \dot{u}_\rho(t) - \dot{u}_{\rho'}(t)) & + & (z_{\rho'}(t), \varepsilon(\dot{u}_\rho(t) - \dot{u}_{\rho'}(t)))_{\mathcal{H}} \\ & + & j_{\rho'}(u_{\rho'}, \dot{u}_\rho(t)) - j_{\rho'}(u_{\rho'}, \dot{u}_{\rho'}(t)) \geq \langle f(t), \dot{u}_\rho(t) - \dot{u}_{\rho'}(t)\rangle_{V',V}, \end{array} \right.$$

where we have written $g_\rho = R\sigma_{\rho\nu}$ and

$$\forall v \in V, \quad j_\rho(u_\rho, v) = \int_{\Gamma_3} \mu(a)|g_\rho(t,a)|\psi_\rho(v_\tau(a))da, \;\text{ and }\quad z_\rho(t) = \sigma_0 - \mathcal{E}\varepsilon(u_0) + \int_0^t G(\sigma_\rho(s), \varepsilon(u_\rho(s)))ds.$$

Adding, we get

$$\begin{aligned} a(u_\rho(t) - u_{\rho'}(t), \dot{u}_\rho(t) - \dot{u}_{\rho'}(t)) &\leq (z_\rho(t) - z_{\rho'}(t), \varepsilon(\dot{u}_{\rho'}(t) - \dot{u}_\rho(t)))_{\mathcal{H}} \\ &\quad + j_\rho(u_\rho(t), \dot{u}_{\rho'}(t)) - j_\rho(u_\rho(t), \dot{u}_\rho(t)) + j_{\rho'}(u_{\rho'}(t), \dot{u}_\rho(t)) - j_{\rho'}(u_{\rho'}(t), \dot{u}_{\rho'}(t)), \end{aligned}$$

which leads, after integrating with respect to time variable

$$\frac{1}{2}a(u_\rho(t)-u_{\rho'}(t),u_\rho(t)-u_{\rho'}(t)) \leq (z_\rho(t)-z_{\rho'}(t),\varepsilon(u_{\rho'}(t)-u_\rho(t)))_{\mathcal{H}} \\ -\int_0^t(\dot{z}_\rho(s)-\dot{z}_{\rho'}(s),\varepsilon(u_{\rho'}(s)-u_\rho(s)))_{\mathcal{H}}ds+\int_0^t J_{\rho,\rho'}(u_\rho,u_{\rho'},\dot{u}_\rho,\dot{u}_{\rho'})ds, \tag{5.17}$$

where

$$J_{\rho,\rho'}(u_\rho,u_{\rho'},\dot{u}_\rho,\dot{u}_{\rho'})=j_\rho(u_\rho,\dot{u}_{\rho'})-j_\rho(u_\rho,\dot{u}_\rho)+j_{\rho'}(u_{\rho'},\dot{u}_\rho)-j_{\rho'}(u_{\rho'},\dot{u}_{\rho'}).$$

Following Lemma 5.3's proof, assertions (5.10)-(5.13), we deduce from (5.17) that there exists $C_2>0$ independent on ρ with

$$|u_\rho(t)-u_{\rho'}(t)|_V^2+|\sigma_\rho(t)-\sigma_{\rho'}(t)|_{\mathcal{H}}^2 \leq C_2\int_0^t J_{\rho,\rho'}(u_\rho(s),u_{\rho'}(s),\dot{u}_\rho(s),\dot{u}_{\rho'}(s))ds. \tag{5.18}$$

Using (2.19), we get

$$|u_\rho(t)-u_{\rho'}(t)|_V^2+|\sigma_\rho(t)-\sigma_{\rho'}(t)|_{\mathcal{H}}^2 \leq 2C_2L_0\mu_0R_1(\rho+\rho') \\ +\int_0^t\int_{\Gamma_3}\mu(a)[|g_\rho(s)||\dot{u}_{\rho'\tau}(s)|-|g_\rho(s)||\dot{u}_{\rho\tau}(s)|+|g_{\rho'}(s)|\dot{u}_{\rho\tau}(s)|-|g_{\rho'}(s)||\dot{u}_{\rho'\tau}(s)|]dads \tag{5.19}$$

and hence

$$|u_\rho(t)-u_{\rho'}(t)|_V^2+|\sigma_\rho(t)-\sigma_{\rho'}(t)|_{\mathcal{H}}^2 \leq 2C_2L_0\mu_0R_1(\rho+\rho') \\ +2\mu_0(\sup_{\rho>0}|\dot{u}_\rho|_{L^2(0,T;V)})\big||g_\rho|-|g_{\rho'}|\big|_{L^2(0,T;H^{-1/2}(\Gamma_3))} \tag{5.20}$$

Since $g_\rho=R\sigma_{\rho\nu}$, they weakly converge to $R\sigma_\nu$ in $H^1(0,T;L^2(\Gamma_3))$. Up to a subsequence, they strongly converge in $L^2(0,T;H^{-1/2}(\Gamma_3))$ and then satisfy Cauchy's criterium. Since the sequence $(\dot{u}_\rho)_{\rho>0}$ is bounded in $L^2(0,T;V)$, we deduce that a subsequence of (u_ρ,σ_ρ) satisfies Cauchy criterium and hence strongly converge in $C([0,T];V)\times C([0,T;\mathcal{H}_1)$ to (u,σ). We finally conclude that the whole sequence converges according to the uniqueness of the limit point and the known weak convergence.

We still have to prove that the weak limit point is solution of problem P_{cv}. This is done in two steps considering first the constitutive relationship and secondly the variational inequality. Using the strong convergence that we stated above, there is no difficulty in proving that the limit constitutive law is the one given in (2.26) and we now turn to the variational inequality that we first integrate in time (as usual) for a test function $v\in L^2(0,T;V)$. With the strong convergence result above, there is just one term that makes a difficulty in the asymptotic study of the variational inequelity in (2.26). This term is $j_\rho(R\sigma_{\rho\nu},\dot{u}_\rho)$.
We then write

$$\int_0^T j_\rho(R\sigma_{\rho\nu},\dot{u}_\rho)dt=\int_0^T\int_{\Gamma_3}\mu[(|R\sigma_{\rho\nu}|-|R\sigma_\nu|)\psi_\rho(\dot{u}_{\rho\tau})+|R\sigma_\nu|(\psi_\rho(\dot{u}_{\rho\tau})-|\dot{u}_{\rho\tau}|)+|R\sigma_\nu||\dot{u}_{\rho\tau}|]dadt.$$

We end as in Lemma 5.1's proof using the convergence of the two first terms and the inf-limit of the last one : we then get

$$\liminf_{\rho \to 0} \int_0^T j_\rho(R\sigma_{\rho\nu}, \dot{u}_\rho) dt \geq \int_0^T \int_{\Gamma_3} \mu |R\sigma_\nu| |\dot{u}_\tau|] dadt,$$

which is sufficient in order to end Theorem 2.4's proof.

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