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Optimal control of an elastic contact problem involving Tresca friction law

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1. Introduction

In this paper we establish existence results and necessary optimality conditions for a class of optimal control problems where the state is described by elliptic variational inequalities modelling the frictional contact between an elastic body and a rigid foundation. We use the *Tresca's law* of dry friction, in which the friction bound is prescribed for the contact condition. We assume that slowly varying time-dependent volume forces and surface tractions act on the body so that the acceleration in the system will be neglected.

Existence and uniqueness results for initial and boundary value problems involving Tresca friction law were obtained in [2,6,10] in the case of an elastic body, in [4] in the case of elasto-viscoplastic models and in [3] in the case of perfectly plastic solids.

The aim of this paper is to study optimal control problems for mechanical models describing the quasistatic process of bilateral frictional contact between an elastic body and a rigid foundation. The state of this system is governed by a variational inequality

$$u(t) \in V,\ a(u(t), v - \dot{u}(t)) + j(v) - j(\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle_{V',V} \quad \forall v \in V, \quad \text{a.e. } t, \tag{1.1}$$

where $u = (u_i)$ is the displacement field, $\dot{u} = \partial u/\partial t$, a is a bilinear form related to the elastic part of the stress, f denotes applied forces, j represents the friction functional

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and V is a Hilbert space. The well-posedness of this problem has been proved in [2,4] using a time discretization method and standard arguments of elliptic variational inequalities.

The optimal control problem for a system governed by elliptic variational inequalities is proposed by Lions [11] and studied by Mignot [12], Mignot and Puel [13], Barbu [5], Sokolowski and Zolesio [14], Haslinger and Neittaanmäki [8], White and Oden [15] and Yvon [16]. From the point of view of optimal control theory, the most characteristic property of variational inequalities is that their solution does not depend smoothly, in general, on the control.

The paper is organized as follows. In Section 2, we introduce some notations, recall the state equation modelling the contact problem with friction and its variational formulation, which is an inequation. We also recall the assumptions on the data and the main existence and uniqueness result (Theorem 2.1). Then, we formulate the optimal control problem governed by the variational inequality (1.1) and establish the existence of an optimal control β (Theorem 2.3). In Section 3, we use a regularization method in order to obtain optimality conditions that can be exploited numerically and set conditions allowing us to pass to the limit with respect to the regularization parameter. We regularize both the frictional function j and the material's law in order to get smoothing effects. We give the analysis (existence and uniqueness of a solution) and control results for the perturbed problem. Finally, we make the asymptotic analysis when the parameter of regularization tends to zero proving that a solution of the initial optimal control problem can be obtained as a limit of smoother's one.

2. A frictional contact problem

2.1. Preliminaries and notations

Let us introduce, in this section, various notations and spaces which will be used in the formulation and analysis of the mechanical problem. For further details on this preliminary material we refer the reader to [1,9].

Let Ω be an open and regular subset of $\mathbb{R}^M$ ($M = 2, 3$ in the applications) with a boundary Γ and let Γ_1 be a measurable part of Γ such that $\text{meas}(\Gamma_1) > 0$. Since the boundary is Lipschitz continuous, the unit outward normal vector v is defined a.e. on Γ.

Let $\mathscr{S}_M$ represent the space of second-order symmetric tensors on $\mathbb{R}^M$, or equivalently, the space of the symmetric matrices of order M. We define the inner products and the corresponding norms on $\mathbb{R}^M$ and $\mathscr{S}_M$ by

$$(u, v) = u_i v_i, \quad |v| = (v, v)^{1/2} \quad \forall u,\ v \in \mathbb{R}^M,$$

$$\sigma{:}\ \tau = \sigma_{ij}\tau_{ij}, \quad |\tau| = (\tau : \tau)^{1/2} \quad \forall \sigma,\ \tau \in \mathscr{S}_M.$$

Here and below, $i, j = 1, 2, \ldots, M$, and the summation convention over repeated indices is adopted. Moreover in the sequel, the index that follows a comma indicates a partial derivative, e.g., $u_{i,j} = \partial u_i / \partial x_j$.

Now we introduce some functional spaces

$$H=\{u=(u_i)\,|\,u_i\in L^2(\Omega),\ i=1,\dots,M\},$$

$$\mathscr{H}=\{\sigma=(\sigma_{ij})\,|\,\sigma_{ij}=\sigma_{ji}\in L^2(\Omega),\ i,j=1,\dots,M\},$$

$$H_1=\{u=(u_i)\,|\,u_i\in H^1(\Omega),\ i=1,\dots,M\},$$

$$\mathscr{H}_1=\{\sigma\in\mathscr{H}\,|\,\sigma_{ij,j}\in H,\ i,j=1,\dots,M\}.$$

The spaces $H,\mathscr{H}$ and H_1 are real Hilbert spaces endowed with the inner products given by

$$\langle u,v\rangle_H=\int_\Omega(u,v)\,\mathrm{d}x=\int_\Omega u_iv_i\,\mathrm{d}x,$$

$$\langle\sigma,\tau\rangle_{\mathscr{H}}=\int_\Omega(\sigma:\tau)\,\mathrm{d}x=\int_\Omega\sigma_{ij}\tau_{ij}\,\mathrm{d}x,$$

$$\langle u,v\rangle_{H_1}=\langle u,v\rangle_H+\langle\varepsilon(u),\varepsilon(v)\rangle_{\mathscr{H}},$$

respectively, where $\varepsilon:H_1\to\mathscr{H}$ is the *deformation*, defined by

$$\varepsilon(v)=(\varepsilon_{ij}(v)),\quad \varepsilon_{ij}(v)=\tfrac{1}{2}(v_{i,j}+v_{j,i}).$$

The associated norms on the spaces $H,\mathscr{H}$, H_1 and $\mathscr{H}_1$ are denoted by $|\cdot|_H,|\cdot|_{\mathscr{H}},|\cdot|_{H_1}$ and $|\cdot|_{\mathscr{H}_1}$, respectively.

Let $H_\Gamma=[H^{1/2}(\Gamma)]^M$ and let $\gamma:H_1\to H_\Gamma$ be the trace map. For the sake of simplicity, we write v for the trace γv of $v\in H_1$, when no confusion is likely, and we denote by v_ν and v_τ, respectively, the normal and the tangential traces of v given by

$$v_\nu=(v,\nu),\quad v_\tau=v-v_\nu\nu. \tag{2.1}$$

Let $H'_\Gamma=[H^{-1/2}(\Gamma)]^M$ be the dual of H_Γ. From the Green formula, we know that if $\sigma\in\mathscr{H}_1$, there exists an element $\sigma\nu\in H'_\Gamma$ such that

$$\langle\sigma\nu,\gamma v\rangle_{H'_\Gamma,H_\Gamma}=\langle\sigma,\varepsilon(v)\rangle_{\mathscr{H}}+\langle Div\,\sigma,v\rangle_H\quad\forall v\in H_1, \tag{2.2}$$

where $Div:\mathscr{H}_1\to H$ is the *divergence* operators, defined by

$$Div\,\sigma=(\sigma_{ij,j}).$$

Moreover, if σ is a regular (say $\mathscr{C}^1$) function, then

$$\langle\sigma\nu,\gamma v\rangle_{H'_\Gamma,H_\Gamma}=\int_\Gamma(\sigma\nu,v)\,\mathrm{d}a\quad\forall v\in H_1, \tag{2.3}$$

where $\mathrm{d}a$ is the surface measure element. In this case the normal and the tangential components of σ are given by

$$\sigma_\nu=((\sigma\nu),\nu),\quad \sigma_\tau=\sigma\nu-\sigma_\nu\nu. \tag{2.4}$$

Let us finally recall that, for any normed space Y, $|\cdot|_Y$ denotes the norm on Y. Also, let X be any Banach space and X' his dual, we denote $\langle\cdot,\cdot\rangle_{X',X}$ the duality pairing between X' and X.

2.2. The mechanical problem and the variational formulation

In this section we describe a model for the contact process, present its variational formulation, list the assumptions imposed on the problem data and recall the existence and uniqueness result obtained in [2].

We consider an elastic body whose material particles occupy a bounded domain Ω with a lipschitz boundary Γ, partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}(\Gamma_1) > 0$. Let $T > 0$ be given, the time interval of interest is $[0,T]$. We assume that the body is held fixed on $\Gamma_1 \times (0,T)$, i.e., the displacement field vanishes. It is acted upon by a volume force of density φ_1 in $\Omega \times (0,T)$, and by a surface traction of density φ_2 on $\Gamma_2 \times (0,T)$. The solid is in bilateral frictional contact with a rigid fixed foundation on $\Gamma_3 \times (0,T)$ and it is submitted to friction forces. We assume that the applied forces and tractions vary slowly in time and consequently neglect the acceleration in the system. We assume that the material has a linear elastic constitutive law. The contact conditions we choose are Tresca's law of dry friction with prescribed friction bound g (hence $g \geq 0$) and the nonseparation condition $u_\nu = 0$. The classical formulation of the mechanical problem is

Problem $\mathscr{P}$. Find the displacement field $u\colon \Omega \times [0,T] \to \mathbb{R}^M$ such that

$$\sigma = \mathscr{R}\varepsilon(u) \quad \text{in } \Omega \times (0,T), \tag{2.5}$$

$$Div\,\sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.6}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.7}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.8}$$

$$u_\nu = 0, \quad |\sigma_\tau| \leq g \quad \text{on } \Gamma_3 \times (0,T), \tag{2.9}$$

$$|\sigma_\tau| < g \Rightarrow \dot{u}_\tau = 0,$$

$$|\sigma_\tau| = g \Rightarrow \text{ there exists } \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau,$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{2.10}$$

In order to write a weak formulation for the problem $\mathscr{P}$ we begin with the introduction of the functional space V defined by

$$V = \{v \in H_1 \mid v = 0 \textit{ on } \Gamma_1, \quad v_\nu = 0 \textit{ on } \Gamma_3\}.$$

It is a closed subspace of H_1. Since $\text{meas}(\Gamma_1) > 0$, the following Korn's inequality holds

$$|\varepsilon(u)|_{\mathscr{H}} \geq C|u|_{H_1}, \quad \forall u \in V \tag{2.11}$$

(see for instance [9]). Here C denotes a strictly positive constant which depends only on Ω and Γ_1.

On V we consider the inner product $\langle\cdot,\cdot\rangle_V$ given by

$$\langle u,v\rangle_V = \langle \varepsilon(u),\varepsilon(v)\rangle_{\mathscr{H}}$$

and $|\cdot|_V$ the associated norm. It follows from Korn's inequality (2.11) that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V,|\cdot|_V)$ is a Hilbert space.

We introduce the bilinear form $a: V\times V\to\mathbb{R}$ defined by

$$a(u,v) = \langle \mathscr{R}\varepsilon(u),\varepsilon(v)\rangle_{\mathscr{H}} \tag{2.12}$$

and we denote by $j: V\to\mathbb{R}$ the continuous seminorm on V, defined by

$$j(v) = \int_{\Gamma_3} g|v_\tau|\,\mathrm{d}a = \int_{\Gamma_3} g(x)|v_\tau(x)|\,\mathrm{d}a_x \tag{2.13}$$

where $\mathrm{d}a = \mathrm{d}a_x$ denotes the surface measure on Γ_3.

In the study of the mechanical problem $\mathscr{P}$ we make the following assumptions on the data:

$\mathscr{R}: \Omega\times\mathscr{S}_M\to\mathscr{S}_M$ is a symmetric and positive definite fourth-order tensor, i.e.

$$\begin{aligned}
&\text{(a) } \mathscr{R}_{ijkh}\in L^\infty(\Omega)\quad \forall i,j,k,h=\overline{1,M},\\
&\text{(b) } \mathscr{R}\sigma:\tau=\sigma:\mathscr{R}\tau\quad \forall\sigma,\tau\in\mathscr{S}_M,\quad \text{a.e. in } \Omega,\\
&\text{(c) there exists } \alpha>0 \text{ such that } \mathscr{R}\sigma:\sigma\ge\alpha|\sigma|^2\quad \forall\sigma\in\mathscr{S}_M.
\end{aligned} \tag{2.14}$$

We also suppose that the forces and the tractions have the regularity

$$\varphi_1\in H^1(0,T;H),\quad \varphi_2\in H^1(0,T;L^2(\Gamma_2)^M) \tag{2.15}$$

and, we denote by $f(t)$ the element of V' given by

$$\langle f(t),v\rangle_{V',V} = \langle\varphi_1(t),v\rangle_H + \langle\varphi_2(t),\gamma v\rangle_{L^2(\Gamma_2)^M}\quad \forall v\in V,\ t\in[0,T]. \tag{2.16}$$

Using (2.15) and (2.16) it follows that $f\in H^1(0,T;V')$. For the initial data, we assume that

$$u_0\in V, \tag{2.17}$$

$$a(u_0,v)+j(v)\ge\langle f(0),v\rangle_{V',V}\quad \forall v\in V. \tag{2.18}$$

The relation (2.18) represents a compatibility condition at the initial time. Finally, the friction bound on Γ_3 satisfies

$$g\in L^\infty(\Gamma_3). \tag{2.19}$$

We are now ready to state the following weak formulation of the problem $\mathscr{P}$.

Problem P. Find a displacement field $u:[0,T]\to V$ such that

$$a(u(t),v-\dot u(t))+j(v)-j(\dot u(t))\ge\langle f(t),v-\dot u(t)\rangle_{V',V}$$
$$\forall v\in V,\quad \text{a.e. } t\in(0,T), \tag{2.20}$$

$$u(0)=u_0. \tag{2.21}$$

The well-posedness of the problem P is studied in [2] where the following result is proved.

Theorem 2.1. *Let* (2.14)–(2.19) *hold. Then there exists a unique solution of problem* P *having the regularity* $u \in H^1(0,T;V)$. *Moreover, this solution satisfies* (2.5)–(2.10).

Remark 2.2. Using the arguments used in [2] it can be shown that the norm $|\dot{u}|_{L^2(0,T;V)}$ can be estimated with respect to $\dot{f}$. Indeed, we have

$$|\dot{u}|_{L^2(0,T;V)} \leq C|\dot{f}|_{L^2(0,T;V')}. \tag{2.22}$$

Here and below, C represents positive constants which may depend on Ω, Γ, $\mathscr{R}$, g and T but do not depend on the input data φ_1, φ_2 or u_0, and whose value may change from line to line.

2.3. Setting of the optimal control problem: existence result

We have seen in the previous section that for given loading functions (φ_1, φ_2), the problem P has a unique solution $u \in H^1(0,T;V)$.

Now, we would like to act on Γ so that the resulting stress σ be as close as possible to a given target. We choose a formulation corresponding to a mechanical situation:

- the quality of the stress is measured through the deviator (see [7]) which is $\sigma^D := \sigma - 1/M(tr\,\sigma)I$ where $tr\,\sigma$ is the trace of σ and I is the identity matrix.
- we search how to act in the best way through the boundary forces φ_2. As a matter of fact, we suppose that φ_2, which belongs to $H^1(0,T;L^2(\Gamma_2)^M)$ is decoupled in

$$\varphi_2(t,x) = \beta(t)\theta(x) \quad \forall t \in [0,T], \text{ a.e. } x \in \Gamma_2. \tag{2.23}$$

The function $\theta \in L^2(\Gamma_2)^M$ is given, we want to act through a good choice of β. The solution of (2.20) and (2.21), which of course depends on φ_2, is now a function of $\beta \in H^1(0,T)$. In the following we denote it by $u(\beta)$. The corresponding stress $\sigma(u(\beta)) \in H^1(0,T;\mathscr{H})$ will be denoted by $\sigma(\beta)$. So, let σ_g be a target that we suppose to be in $L^2(0,T;\mathscr{H})$. We want to minimize in $H^1(0,T)$ the functional

$$J(\beta) = \frac{1}{2}\int_0^T \int_\Omega \rho(t,x)|\sigma^D(\beta)(t,x) - \sigma_g(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}|\beta|^2_{H^1(0,T)}, \tag{2.24}$$

where $\rho(t,x)$ is a smooth weight function which will be described below, and γ is a positive constant. Specifically, we are interested in the case where the support of ρ lies a neighbourhood of the contact region $\Gamma_3 \times (0,T)$ where it is likely that σ^D is large.

Remark 2.3. (1) Of course, we could as well have considered a control acting in a distributed way. We would have then writen

$$\varphi_1(t,x) = \beta(t)\theta(x)$$

and take β as the control.

(2) We could as well have considered various examples of cost functions in which we can obtain analogous results without additional difficulty in the proofs. For instance, we could have considered the minimization in $L^2(\Omega)$ between the displacement $u(T)$ at a fixed time T and a desired target u_g.

The minimization problem (2.24) can be formulated as an optimal control one, using the functional $\mathscr{J}\colon H^1(0,T)\times H^1(0,T;V)\to\mathbb{R}$ defined by

$$\mathscr{J}(\beta,u)=\frac{1}{2}\int_0^T\int_\Omega \rho(t,x)|\sigma^D(u)(t,x)-\sigma_g(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{\gamma}{2}|\beta|^2_{H^1(0,T)}. \tag{2.25}$$

We have

$$J(\beta)=\mathscr{J}(\beta,u(\beta)). \tag{2.26}$$

The aim of this paragraph is to prove an existence result for the following abstract optimal control problem:

$$\begin{array}{l}\text{For a given } \sigma_g\in L^2(0,T;\mathscr{H}) \text{ find } \beta_0\in H^1(0,T) \text{ such that}\\ J(\beta_0)=\text{infimum}\{J(\beta);\beta\in H^1(0,T)\}.\end{array} \tag{2.27}$$

It is clear that the mapping $\beta\mapsto\varphi_2=\beta\theta: H^1(0,T)\to H^1(0,T;L^2(\Gamma_2)^M)$ is linear continuous for the strong topologies, so it is also continuous for the weak topologies. The only thing we need to know is that the mapping $\varphi_2\mapsto u(\varphi_2): H^1(0,T;L^2(\Gamma_2)^M)\to H^1(0,T;V)$ (where u is the solution of (2.20) and (2.21)) is continuous for the weak topologies. This gives the continuity of $\varphi_2\mapsto\sigma^D(u(\varphi_2)): H^1(0,T;L^2(\Gamma_2)^M)\to H^1(0,T;\mathscr{H})$ by linearity arguments.

As a matter of fact, we are going to prove the following continuity result:

Theorem 2.4. *Let* (2.14)–(2.19) *hold and let* $(\varphi_{1k})_k$, $(\varphi_{2k})_k$ *and* $(u_{0k})_k$ *be sequences in* $H^1(0,T;H)$, $H^1(0,T;L^2(\Gamma_2)^M)$ *and* V, *respectively*, *such that* (2.18) *holds and*

$$\varphi_{1k}\rightharpoonup\varphi_1 \text{ weakly in } H^1(0,T;H), \tag{2.28}$$

$$\varphi_{2k}\rightharpoonup\varphi_2 \text{ weakly in } H^1(0,T;L^2(\Gamma_2)^M). \tag{2.29}$$

Then u_{0k} *is a bounded sequence in* V *and if* u_0 *is a weak adherence value of this sequence*, *then there exists a subsequence still denoted by* u_{0k} *such that*

$$u_{0k}\to u_0 \text{ strongly in } V \tag{2.30}$$

and the corresponding sequence of solutions $u(\varphi_{1k},\varphi_{2k},u_{0k})$ *of* (2.20) *and* (2.21) *verifies*:

$$\begin{array}{l}(1)\ u(\varphi_{1k},\varphi_{2k},u_{0k})\rightharpoonup u(\varphi_1,\varphi_2,u_0) \text{ weakly in } H^1(0,T;V),\\ (2)\ u(\varphi_{1k},\varphi_{2k},u_{0k})\to u(\varphi_1,\varphi_2,u_0) \text{ strongly in } L^2(0,T;V),\\ (3)\ u(\varphi_{1k},\varphi_{2k},u_{0k})(t)\to u(\varphi_1,\varphi_2,u_0)(t) \text{ strongly in } V\quad \forall t\in[0,T],\end{array} \tag{2.31}$$

where $u(\varphi_1,\varphi_2,u_0)$ *is the solution of problem P with data* φ_1, φ_2 and u_0

Remark 2.5. (1) In the sequel, (2.31) (1) is sufficient for our need. Though, (2.31) (2) and (3) are also true. They come from the compatibility condition (2.18).

(2) If u_{0k} has a unique limit point u_0 then (2.31) holds for the whole sequence $u(\varphi_{1k},\varphi_{2k},u_{0k})$.

Proof. *Convergence of* u_{0k}: We first use the compatibility condition (2.18) in order to get (2.30). Let u_{0k} be the initial data corresponding to the forces (φ_{1k}) and the tractions (φ_{2k}). Taking $v = -v$, (2.18) is equivalent to

$$a(u_{0k}, v) - j(v) \leq \langle f_k(0), v\rangle_{V',V} \quad \forall v \in V. \tag{2.32}$$

Setting $v = u_{0k}$ in the previous inequality, we obtain

$$|u_{0k}|_V \leq C(1 + |f_k(0)|_{V'}) \leq C(1 + |\varphi_{1k}(0)|_H + |\varphi_{2k}(0)|_{L^2(\Gamma_2)^M}) \leq C$$

thus, there exists an element $u_0 \in V$ and a subsequence still denoted by u_{0k} such that

$$u_{0k} \rightharpoonup u_0 \quad \text{weakly in } V. \tag{2.33}$$

From (2.33) we have

$$a(u_{0k}, v) \to a(u_0, v) \quad \forall v \in V. \tag{2.34}$$

Using (2.16), the continuous imbedding of $H^1(0,T;V')$ in $C([0,T],V')$, we obtain

$$\langle f_k(0), v\rangle_{V',V} \to \langle f(0), v\rangle_{V',V} \quad \forall v \in V. \tag{2.35}$$

From (2.34) and (2.35) it follows that u_0 verifies the compatibility condition at the initial time

$$a(u_0, v) - j(v) \leq \langle f(0), v\rangle_{V',V} \quad \forall v \in V. \tag{2.36}$$

Taking $v = u_{0k} - u_0$ in (2.32) and $v = u_0 - u_{0k}$ in (2.36), we obtain

$$\begin{aligned} a(u_{0k}, u_{0k} - u_0) - j(u_{0k} - u_0) &\leq \langle f_k(0), u_{0k} - u_0\rangle_{V',V}, \\ a(u_0, u_0 - u_{0k}) - j(u_0 - u_{0k}) &\leq \langle f(0), u_0 - u_{0k}\rangle_{V',V} \end{aligned}$$

and, adding the previous inequalities, it follows that:

$$\begin{aligned} a(u_{0k} - u_0, u_{0k} - u_0) &\leq \langle f_k(0) - f(0), u_{0k} - u_0\rangle_{V',V} + 2j(u_{0k} - u_0) \\ &\leq \langle \varphi_{1k}(0) - \varphi_1(0), u_{0k} - u_0\rangle_H + \langle \varphi_{2k}(0) \\ &\quad -\varphi_2(0), u_{0k} - u_0\rangle_{L^2(\Gamma_2)^M} \\ &\quad + 2\int_{\Gamma_3} g(x)|u_{0k_\tau}(x) - u_{0\tau}(x)|\,\mathrm{d}a_x \\ &\leq \langle \varphi_{1k}(0) - \varphi_1(0), u_{0k} - u_0\rangle_{V',V} \\ &\quad + \langle \bar{\varphi}_{2k}(0) - \bar{\varphi}_2(0), u_{0k} - u_0\rangle_{H^{-1/2}(\Gamma)^M, H^{1/2}(\Gamma)^M} \\ &\quad + C|u_{0k} - u_0|_{L^2(\Gamma_3)^M}, \end{aligned} \tag{2.37}$$

where $\bar{\varphi}$ is the null extension of $\varphi \in L^2(\Gamma_2)$ on Γ. Using the compact imbeddings of $H^1(0,T;H)$ in $C([0,T];V')$, $H^1(0,T;L^2(\Gamma)^M)$ in $C([0,T];H^{-1/2}(\Gamma)^M)$, the compactness of the trace map from V into $L^2(\Gamma_3)^M$ and the weak convergence of u_{0k} to u_0, it follows that:

$$u_{0k} \to u_0 \text{ strongly in } V.$$

Now we prove (2.31), using standard a priori estimates and convergence.

A priori estimates: For sake of simplicity we denote $u(\varphi_{1k}, \varphi_{2k}) = u_k$, $\dot{u}(\varphi_{1k}, \varphi_{2k}) = \dot{u}_k$, where u_k is the solution of

$$\forall v \in V, \text{ a.e. } t \in (0,T),$$

$$a(u_k(t), v - \dot{u}_k(t)) + j(v) - j(\dot{u}_k(t)) \geq \langle f_k(t), v - \dot{u}_k(t)\rangle_{V',V} \tag{2.38}$$

$$u_k(0) = u_{0k}. \tag{2.39}$$

Setting $v = 0$ in (2.38) we obtain

$$a(u_k(t), \dot{u}_k(t)) \leq \langle f_k(t), \dot{u}_k(t)\rangle_{V',V}, \quad \text{a.e. } t \in (0,T). \tag{2.40}$$

We integrate by parts the previous inequality on $[0,t]$ and use the V-ellipticity of $a(\cdot,\cdot)$. We get

$$\forall t \in [0,T]$$

$$|u_k(t)|_V^2 \leq C\left(|u_{0k}|_V^2 + |f_k(t)|_{V'}^2 + |f_k(0)|_{V'}^2 + \int_0^t |\dot{f}_k(s)|_{V'}^2 \, ds + \int_0^t |u_k(s)|_V^2 \, ds\right) \tag{2.41}$$

Using Gronwall's inequality, we obtain for all $t \in [0,T]$

$$\begin{aligned} |u_k(t)|_V^2 &\leq C(1 + |f_k(0)|_{V'}^2 + |\dot{f}_k|_{L^2(0,T;V')}^2) \\ &\leq C(1 + |\varphi_{1k}(0)|_H^2 + |\varphi_{2k}(0)|_{L^2(\Gamma_2)^M}^2 + |\dot{\varphi}_{1k}|_{L^2(0,T;H)}^2 + |\dot{\varphi}_{2k}|_{L^2(0,T;L^2(\Gamma_2)^M)}^2) \\ &\leq C. \end{aligned} \tag{2.42}$$

Now, from (2.22) it follows:

$$\begin{aligned} \int_0^T |\dot{u}_k(t)|_V^2 \, dt &\leq C \int_0^T |\dot{f}_k(t)|_{V'}^2 \, dt \\ &\leq C\left(\int_0^T |\dot{\varphi}_{1k}(t)|_H^2 \, dt + \int_0^T |\dot{\varphi}_{2k}(t)|_{L^2(\Gamma_2)^M}^2 \, dt\right) \\ &\leq C. \end{aligned} \tag{2.43}$$

From (2.42) and (2.43), we obtain that

$$(u_k) \text{ is a bounded sequence in } L^\infty(0,T;V), \tag{2.44}$$

$$(\dot{u}_k) \text{ is a bounded sequence in } L^2(0,T;V). \tag{2.45}$$

Convergence results: We pass to the limit in (2.38) and (2.39). From (2.44) and (2.45), we deduce that there exists an element $\tilde{u} \in H^1(0,T;V)$ and a subsequence still denoted by (u_k) such that

$$u_k \rightharpoonup \tilde{u} \quad \text{in } L^\infty(0,T;V) \text{ weak}^*, \tag{2.46}$$

$$\dot{u}_k \rightharpoonup \dot{\tilde{u}} \quad \text{weakly in } L^2(0,T;V). \tag{2.47}$$

Next, we shall show that $\tilde{u}$ is a solution of problem (2.20) and (2.21) with data φ_1, φ_2 and u_0. First, we note that (2.46) and (2.47) imply

$$u_k(t) \rightharpoonup \tilde{u}(t) \text{ weakly in } V \quad \forall t \in [0,T]. \tag{2.48}$$

From (2.38), it is straightforward to show that for all $t \in [0,T]$, $u_k(t)$ satisfies

$$a(u_k(t),v) + j(v) \geq \langle f(t),v\rangle_{V',V} \quad \forall v \in V. \tag{2.49}$$

Now, using exactly the same arguments as in the proof of (2.30) but for a fixed time t, we have

$$u_k(t) \to \tilde{u}(t) \quad \text{strongly in } V \quad \forall t \in [0,T]$$

which, using *Lebesgue's Theorem*, proves that

$$u_k \to \tilde{u} \quad \text{strongly in } L^2(0,T;V). \tag{2.50}$$

Then, we obtain

$$\lim_k \int_0^T a(u_k(t),\dot{u}_k(t))\,\mathrm{d}t = \int_0^T a(\tilde{u}(t),\dot{\tilde{u}}(t))\,\mathrm{d}t. \tag{2.51}$$

Taking into account (2.47) and using standard lower semicontinuity arguments, we find that

$$\liminf_k \int_0^T j(\dot{u}_k(t))\,\mathrm{d}t \geq \int_0^T j(\dot{\tilde{u}}(t))\,\mathrm{d}t. \tag{2.52}$$

Let $v \in L^2(0,T;V)$, using (2.46), it follows that:

$$\int_0^T a(u_k(t),v(t))\,\mathrm{d}t \to \int_0^T a(\tilde{u}(t),v(t))\,\mathrm{d}t \tag{2.53}$$

and, having in mind (2.28) and (2.29), we obtain

$$\int_0^T \langle f_k(t),v(t)\rangle_{V',V}\,\mathrm{d}t \to \int_0^T \langle f(t),v(t)\rangle_{V',V}\,\mathrm{d}t. \tag{2.54}$$

Moreover, we have

$$\int_0^T \langle f_k(t),\dot{u}_k(t)\rangle_{V',V}\,\mathrm{d}t$$
$$= \langle f_k(T),u_k(T)\rangle_{V',V} - \langle f_k(0),u_k(0)\rangle_{V',V} - \int_0^T \langle \dot{f}_k(t),u_k(t)\rangle_{V',V}\,\mathrm{d}t, \tag{2.55}$$

from which we get, using (2.28) and (2.29), (2.50) and the strong convergence of $u_k(t)$ to $\tilde{u}(t)$ for all t

$$\int_0^T \langle f_k(t),\dot{u}_k(t)\rangle_{V',V}\,\mathrm{d}t \to \int_0^T \langle f(t),\dot{\tilde{u}}(t)\rangle_{V',V}\,\mathrm{d}t. \tag{2.56}$$

Now, using (2.48)–(2.54) and (2.56), we obtain

$$\int_0^T a(\tilde{u}(t), v(t) - \dot{\tilde{u}}(t))\,\mathrm{d}t + \int_0^T j(v(t))\,\mathrm{d}t - \int_0^T j(\dot{\tilde{u}}(t))\,\mathrm{d}t$$
$$\geq \int_0^T \langle f(t), v(t) - \dot{\tilde{u}}(t)\rangle_{V',V}\,\mathrm{d}t \tag{2.57}$$

for all $v \in L^2(0,T;V)$. Then, choosing any $t \in [0,T]$, setting (2.57) for the test function v defined by

$$v(s) = \begin{cases} w \in V & \text{for } s \in [t, t+h], \\ \dot{\tilde{u}}(s) & \text{otherwise} \end{cases}$$

and using Lebesgue points for an L^1 function, it is straightforward to see that the previous inequality is equivalent to the following inequality:

$$a(\tilde{u}(t), w - \dot{\tilde{u}}(t)) + j(w) - j(\dot{\tilde{u}}(t)) \geq \langle f(t), w - \dot{\tilde{u}}(t)\rangle_{V',V}$$
$$\forall w \in V, \quad \text{a.e. } t \in (0,T). \tag{2.58}$$

Thus, $\tilde{u}$ is the unique solution $u(\varphi_1, \varphi_2, u_0)$ of (2.58). From (2.46) and (2.47) we have (2.31)(1). From (2.49) and (2.50) we have (2.31)(2) and (2.31)(3). This completes the proof of Theorem 2.4. □

The main result of this section is the following:

Theorem 2.6. *Let* (2.14)–(2.19) *hold with* $u_0 \in V$ *defined by* $a(u_0, v) = \langle f(0), v\rangle_{V',V}$. *Then there exists a solution to problem* (2.27).

Proof. Using Theorem 2.4 and convexity arguments, it is straightforward to show that J is coercive on $H^1(0,T)$ and sequentially weakly lower semi-continuous. Therefore, the proof of Theorem 2.6 is given using standard arguments.

Now we want to get necessary optimality conditions for J. We notice that the mapping $\beta \to u(\beta)$ is not differentiable (the state is described by a variational inequality (2.20) and the friction functional j is nondifferentiable). To overcome this difficulty, we apply a regularization technique to problem P in order to obtain smooth problems. These problems have some interest on their own, and indeed may be taken as a basis for a convergent numerical algorithm for the original problem.

3. A regularized problem

3.1. Existence and uniqueness result

In this section we consider variational problems which are regularizations of (2.20) and (2.21). The idea is to approximate the nondifferentiable term by a sequence of differentiable ones. More precisely, we approximate the term $j(v)$ by

$$j_\varepsilon(v) = \int_{\Gamma_3} g(x)\psi_\varepsilon(v_\tau(x))\,\mathrm{d}a_x, \tag{3.1}$$

where $\varepsilon > 0$ is a small parameter, and for all $\varepsilon > 0$, $\psi_\varepsilon : \mathbb{R}^M \to \mathbb{R}$ is a non-negative and convex function satisfying

$$\psi_\varepsilon \in \mathscr{C}^2(\mathbb{R}^M), \quad \psi_\varepsilon(0) = 0 \tag{3.2}$$

and such that there exists positive constants C_0, C_1 and a positive function $C_2 : \mathbb{R} \to \mathbb{R}_+$ such that for every $\varepsilon > 0$ and $(x, y, z) \in (\mathbb{R}^M)^3$

$$\begin{aligned} &|\psi_\varepsilon(x) - |x|| \le C_0\varepsilon, \\ &|\psi_\varepsilon'(x) \cdot y| \le C_1|y|, \\ &|\psi_\varepsilon''(x)(y,z)| \le C_2(\varepsilon)|y||z|. \end{aligned} \tag{3.3}$$

There are many ways of constructing functions ψ_ε. For instance

$$\psi_\varepsilon(x) = \sqrt{\varepsilon^2 + |x|^2} - \varepsilon \quad \forall x \in \mathbb{R}^M$$

satisfies (3.2) and (3.3).

It is easy to show that, for every $\varepsilon > 0$, $j_\varepsilon : V \to \mathbb{R}_+$ is proper, convex and verifies

$$j_\varepsilon \in \mathscr{C}^2(V), \quad j_\varepsilon(0) = 0. \tag{3.4}$$

Denoting by $j_\varepsilon'(u)$ the differential of j_ε at the point u, we have

$$\forall u \in V \quad j_\varepsilon'(u) \in V' \quad \text{and} \quad \langle j_\varepsilon'(u), v\rangle_{V',V} = \int_{\Gamma_3} g\psi_\varepsilon'(u_\tau) \cdot v_\tau\,\mathrm{d}a$$

and also, for all $u \in V$, $j_\varepsilon''(u)$ is a bilinear and symmetric form on $V \times V$ given by

$$j_\varepsilon''(u)(v,w) = \int_{\Gamma_3} g\psi_\varepsilon''(u_\tau)(v_\tau, w_\tau)\,\mathrm{d}a \quad \forall v, w \in V.$$

From (3.3) it is straightforward to see that there exists

$$\begin{aligned} &C_0 > 0, \quad \forall v \in V \quad |j_\varepsilon(v) - j(v)| \le C_0\varepsilon, \\ &C_1 > 0, \quad \forall u \in V \quad |\langle j_\varepsilon'(u), v\rangle_{V',V}| \le C_1|v|_V \quad \forall v \in V, \\ &C_2(\varepsilon) > 0, \quad \forall u \in V \ |j_\varepsilon''(u)(v,w)| \le C_2(\varepsilon)|v|_V|w|_V \quad \forall v, w \in V. \end{aligned} \tag{3.5}$$

Thus, the regularized problem takes the following form:

Problem P_ε**.** Find a displacement field $u_\varepsilon : [0, T] \to V$ such that

$$\forall v \in V, \quad \text{a.e. } t \in (0, T),$$

$$\varepsilon\langle \dot u_\varepsilon(t), v - \dot u_\varepsilon(t)\rangle_V + a(u_\varepsilon(t), v - \dot u_\varepsilon(t)) + j_\varepsilon(v) - j_\varepsilon(\dot u_\varepsilon(t)) \ge \langle f(t), v - \dot u_\varepsilon(t)\rangle_{V',V} \tag{3.6}$$

$$u_\varepsilon(0) = u_0. \tag{3.7}$$

Since j_ε is differentiable, the problem (3.6) and (3.7) is clearly equivalent to the variational equation:

Find $u_\varepsilon : [0,T] \to V$ such that $u_\varepsilon(0) = u_0$ and

$\forall v \in V$, a.e. $t \in (0,T)$.

$$\varepsilon\langle \dot{u}_\varepsilon(t), v\rangle_V + a(u_\varepsilon(t), v) + \langle j'_\varepsilon(\dot{u}_\varepsilon(t)), v\rangle_{V',V} = \langle f(t), v\rangle_{V',V} \tag{3.8}$$

We will use one or the other of these two formulations according to our needs. Indeed, the use of (3.6) and (3.7) is particularly appropriate in weak topology arguments owing to the convexity of j_ε.

The existence and the uniqueness result for the variational problem P_ε is the following:

Theorem 3.1. *Under the assumptions* (2.14)–(2.19) *on the data and* (3.2)–(3.3), *problem* (3.6)–(3.7) *has a unique solution* $u_\varepsilon \in H^2(0,T;V)$.

Remark 3.2. As a matter of fact we do not need the compatibility condition (2.18) here.

The proof of Theorem 3.1 will be carried out in two steps: the first entails the construction of an auxiliary problem in which a displacement-like function is assumed to be known. We establish the existence of a unique solution of this auxiliary problem. In the second step we use a fixed point argument and establish Theorem 3.1.

For this, for any $\alpha \in H^1(0,T;V)$ (representing a displacement) we consider the following abstract problem:

Problem $P_{\varepsilon\alpha}$**.** Find a displacement field $v_{\varepsilon\alpha} \in H^1(0,T;V)$ such that

$\forall v \in V \quad \forall t \in [0,T]$

$$\varepsilon\langle v_{\varepsilon\alpha}(t), v\rangle_V + \langle j'_\varepsilon(v_{\varepsilon\alpha}(t)), v\rangle_{V',V} = \langle f(t), v\rangle_{V',V} - a(\alpha(t), v) \tag{3.9}$$

Lemma 3.3. *There exists a unique solution* $v_{\varepsilon\alpha}$ *to problem* $P_{\varepsilon\alpha}$ *in* $H^1(0,T;V)$.

Proof. Problem (3.9) is equivalent to the following minimization problem:

$$\text{Find } v_{\varepsilon\alpha}(t) \in V, \quad \mathscr{I}_{\varepsilon\alpha}(t, v_{\varepsilon\alpha}(t)) = \inf_{v\in V} \mathscr{I}_{\varepsilon\alpha}(t,v) \quad \forall t \in (0,T), \tag{3.10}$$

where

$$\mathscr{I}_{\varepsilon\alpha}(t,v) = \frac{\varepsilon}{2}|v|_V^2 + j_\varepsilon(v) - \langle f(t), v\rangle_{V',V} + a(\alpha(t), v).$$

The functional $\mathscr{I}_{\varepsilon\alpha}(t,\cdot)$ is proper, continuous, strictly convex and coercive on V. Therefore, problem (3.10) has a unique solution $v_{\varepsilon\alpha}(t) \in V$, a.e. $t \in (0,T)$.

Now, we show that $v_{\varepsilon\alpha} \in H^1(0,T;V)$. For any $t \in [0,T]$ we take $v = v_{\varepsilon\alpha}(t)$ in (3.9). Using the properties of j_ε and after some algebraic manipulations we obtain

$$|v_{\varepsilon\alpha}(t)|_V^2 \le C(\varepsilon)(|f(t)|_{V'}^2 + |\alpha(t)|_V^2),$$

for which we deduce that $v_{\varepsilon\alpha} \in L^2(0,T;V)$.

Let us now write (3.9) at $t+h$. We obtain

$$\varepsilon\langle v_{\varepsilon\alpha}(t+h), v\rangle_V + \langle j'_\varepsilon(v_{\varepsilon\alpha}(t+h)), v\rangle_{V',V} = \langle f(t+h), v\rangle_{V',V} - a(\alpha(t+h), v) \quad \forall v\in V. \tag{3.11}$$

Let us set $v = v_{\varepsilon\alpha}(t+h) - v_{\varepsilon\alpha}(t)$ in (3.11) and $v = v_{\varepsilon\alpha}(t) - v_{\varepsilon\alpha}(t+h)$ in (3.9) and subtract these equations, we obtain

$$\begin{aligned}\varepsilon\langle v_{\varepsilon\alpha}(t+h) - v_{\varepsilon\alpha}(t), v_{\varepsilon\alpha}(t+h) - v_{\varepsilon\alpha}(t)\rangle_V &+ \langle j'_\varepsilon(v_{\varepsilon\alpha}(t+h)) \\ &- j'_\varepsilon(v_{\varepsilon\alpha}(t)), v_{\varepsilon\alpha}(t+h) - v_{\varepsilon\alpha}(t)\rangle_{V',V} \\ &= \langle f(t+h) - f(t), v_{\varepsilon\alpha}(t+h) - v_{\varepsilon\alpha}(t)\rangle_{V',V} \\ &- a(\alpha(t+h) - \alpha(t), v_{\varepsilon\alpha}(t+h) - v_{\varepsilon\alpha}(t))\end{aligned} \tag{3.12}$$

and by using the properties of j_ε and Korn's inequality we get

$$\begin{aligned}\varepsilon|v_{\varepsilon\alpha}(t+h) - v_{\varepsilon\alpha}(t)|_V &\le C(|f(t+h) - f(t)|_{V'} + |\alpha(t+h) - \alpha(t)|_V) \\ &\le C\left(\int_t^{t+h} |\dot f(s)|_{V'}\,\mathrm{d}s + \int_t^{t+h} |\dot\alpha(s)|_V\,\mathrm{d}s\right).\end{aligned} \tag{3.13}$$

The regularity property $v_{\varepsilon\alpha} \in H^1(0,T;V)$ follows from (3.13), since $f \in H^1(0,T;V)$ and $\alpha \in H^1(0,T;V)$. This concludes the proof. □

Let $u_{\varepsilon\alpha} : [0,T] \to V$ be the function given by

$$u_{\varepsilon\alpha}(t) = \int_0^t v_{\varepsilon\alpha}(s)\,\mathrm{d}s + u_0 \tag{3.14}$$

we have $u_{\varepsilon\alpha} \in H^2(0,T;V)$ and $u_{\varepsilon\alpha}(0) = u_0$. Next, we consider the mapping $\Lambda_\varepsilon : H^1(0,T;V) \to H^1(0,T;V)$ defined by

$$\Lambda_\varepsilon(\alpha)(t) = u_{\varepsilon\alpha}(t) \quad \forall \alpha \in H^1(0,T;V),\ t\in[0,T]. \tag{3.15}$$

Lemma 3.4. *The operator Λ_ε has a unique fixed point α^*.*

Proof. We shall prove that for n large enough, Λ_ε^n is a contraction. For this, let $\alpha_1,\ \alpha_2 \in H^1(0,T;V)$ and $t\in[0,T]$. Let us denote $v_i = v_{\varepsilon\alpha_i}$ and $u_i = u_{\varepsilon\alpha_i}$, where $v_{\varepsilon\alpha_i}$ is a solution of the problem $P_{\varepsilon\alpha_i}$ and $u_{\varepsilon\alpha_i}$ is defined by (3.14), for $i=1,2$. Using (3.14) and (3.15), we have

$$\begin{aligned}&|\dot\Lambda_\varepsilon(\alpha_1)(t) - \dot\Lambda_\varepsilon(\alpha_2)(t)|_V^2 + |\Lambda_\varepsilon(\alpha_1)(t) - \Lambda_\varepsilon(\alpha_2)(t)|_V^2 \\ &\quad = |\dot u_1(t) - \dot u_2(t)|_V^2 + |u_1(t) - u_2(t)|_V^2 \\ &\quad \le |v_1(t) - v_2(t)|_V^2 + \int_0^t |v_1(s) - v_2(s)|_V^2\,\mathrm{d}s.\end{aligned} \tag{3.16}$$

If we choose, for $i = 1, 2$, $v = v_1(t) - v_2(t)$ in (3.9), we obtain

$$\varepsilon\langle v_1(t), v_1(t) - v_2(t)\rangle_V + \langle j'_\varepsilon(v_1(t)), v_1(t) - v_2(t)\rangle_{V',V}$$
$$= \langle f(t), v_1(t) - v_2(t)\rangle_{V',V} - a(\alpha_1(t), v_1(t) - v_2(t)),$$

$$\varepsilon\langle v_2(t), v_1(t) - v_2(t)\rangle_V + \langle j'_\varepsilon(v_2(t)), v_1(t) - v_2(t)\rangle_{V',V}$$
$$= \langle f(t), v_1(t) - v_2(t)\rangle_{V',V} - a(\alpha_2(t), v_1(t) - v_2(t))$$

subtracting these equalities, since $\langle j'_\varepsilon(v_1(t)) - j'_\varepsilon(v_2(t)), v_1(t) - v_2(t)\rangle_{V',V} \geq 0$, we get

$$|v_1(t) - v_2(t)|_V \leq C(\varepsilon)|\alpha_1(t) - \alpha_2(t)|_V. \tag{3.17}$$

From (3.16) and (3.17), it follows that:

$$|\dot{\Lambda}_\varepsilon(\alpha_1)(t) - \dot{\Lambda}_\varepsilon(\alpha_2)(t)|_V^2 + |\Lambda_\varepsilon(\alpha_1)(t) - \Lambda_\varepsilon(\alpha_2)(t)|_V^2$$
$$\leq C(\varepsilon)\left(|\alpha_1(t) - \alpha_2(t)|_V^2 + \int_0^t |\alpha_1(s) - \alpha_2(s)|_V^2\,\mathrm{d}s\right)$$
$$\leq C(\varepsilon)\int_0^t (|\dot{\alpha}_1(s) - \dot{\alpha}_2(s)|_V^2 + |\alpha_1(s) - \alpha_2(s)|_V^2)\,\mathrm{d}s. \tag{3.18}$$

Reiterating this inequality n times and integrating on t leads to

$$|\Lambda_\varepsilon^n(\alpha_1) - \Lambda_\varepsilon^n(\alpha_2)|_{H^1(0,T;V)} \leq \sqrt{\frac{(C(\varepsilon)T)^n H}{(nH)!}}|\alpha_1 - \alpha_2|_{H^1(0,T;V)} \tag{3.19}$$

which implies that for n sufficiently large a power Λ_ε^n is a contraction on $H^1(0,T;V)$. Thus, there exists a unique $\alpha^* \in H^1(0,T;V)$ such that $\Lambda_\varepsilon^n\alpha^* = \alpha^*$ and α^* is also the unique fixed point of Λ_ε. □

Proof of Theorem 3.1. *Existence.* Let $\alpha^* \in H^1(0,T;V)$ be the fixed point of Λ_ε and let $u_{\varepsilon\alpha^*}$ be the function given by (3.14) and (3.15) for $\alpha = \alpha^*$. We show that $u_{\varepsilon\alpha^*}$ is a solution of problem P_ε.

Uniqueness. The uniqueness part in Theorem 3.1 can be proved directly from (3.8) with Gronwall inequality and (3.3). □

3.2. Optimal control problem on the regularized equation

The control problem governed by these regularized equations is stated as follows:

For a given $\sigma_g \in L^2(0,T;\mathscr{H})$ find $\beta_\varepsilon \in H^1(0,T)$ such that

$$J_\varepsilon(\beta_\varepsilon) = \text{infimum}\{J_\varepsilon(\beta);\ \beta \in H^1(0,T)\}. \tag{3.20}$$

Let us recall that

$$J_\varepsilon(\beta) = \mathscr{J}(\beta, u_\varepsilon(\beta)) \tag{3.21}$$

with

$$\mathscr{J}(\beta, u_\varepsilon(\beta)) = \frac{1}{2}\int_0^T\int_\Omega \rho(t,x)|\sigma^D(u_\varepsilon(\beta))(t,x) - \sigma_g(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}|\beta|^2_{H^1(0,T)}.$$

We have the continuity result similar to Theorem 2.4.

Theorem 3.5. *Let* $\varepsilon > 0$ *be fixed and let* (2.14)–(2.19) *and* (3.2)–(3.3) *hold. Suppose that* (φ_{1k}), (φ_{2k}) *and* u_{0k} *are three sequences in* $H^1(0,T;H)$, $H^1(0,T;L^2(\Gamma_2)^M)$ *and* V, *respectively, such that*

$$\varphi_{1k} \rightharpoonup \varphi_1 \text{ weakly in } H^1(0,T;H), \tag{3.22}$$

$$\varphi_{2k} \rightharpoonup \varphi_2 \text{ weakly in } H^1(0,T;L^2(\Gamma_2)^M), \tag{3.23}$$

$$u_{0k} \rightharpoonup u_0 \text{ weakly in } V. \tag{3.24}$$

The corresponding sequence of solutions $u_\varepsilon(\varphi_{1k}, \varphi_{2k}, u_{0k})$ *of* (3.6)–(3.7) *verifies*

$$u_\varepsilon(\varphi_{1k}, \varphi_{2k}, u_{0k}) \rightharpoonup u_\varepsilon(\varphi_1, \varphi_2, u_0) \text{ weakly in } H^1(0,T;V), \tag{3.25}$$

where $u_\varepsilon(\varphi_1, \varphi_2, u_0)$ *is the solution of* P_ε *with data* φ_1, φ_2 *and* u_0.

Remark 3.6. If we compare this result to Theorem 2.4, (3.25) is similar to (2.31)(1). We do not have anything similar to (2.31)(2) and (3). This is because here we have no compatibility condition similar to (2.18).

Proof. *A priori estimates.* For sake of simplicity we denote by $u_{\varepsilon k} = u_\varepsilon(\varphi_{1k}, \varphi_{2k}, u_{0k})$), $\dot{u}_{\varepsilon k} = \dot{u}_\varepsilon(\varphi_{1k}, \varphi_{2k}, u_{0k})$. Setting $v = \dot{u}_{\varepsilon k}(t)$ in (3.8) we obtain

$$\begin{aligned}&\varepsilon|\dot{u}_{\varepsilon k}(t)|^2_V + a(u_{\varepsilon k}(t), \dot{u}_{\varepsilon k}(t)) + \langle j'_\varepsilon(\dot{u}_{\varepsilon k}(t)), \dot{u}_{\varepsilon k}(t)\rangle_{V',V}\\&\quad = \langle f_k(t), \dot{u}_{\varepsilon k}(t)\rangle_{V',V} \quad \text{a.e. } t \in (0,T).\end{aligned} \tag{3.26}$$

Since $j_\varepsilon(v) - j_\varepsilon(u) \geq \langle j'_\varepsilon(u), v-u\rangle_{V',V}$ $\forall u, v \in V$, and $j_\varepsilon(0) = 0$, we deduce that $\langle j'_\varepsilon(u), u\rangle_{V',V} \geq j_\varepsilon(u)$ $\forall u \in V$. As j_ε is a non-negative function, we get

$$\langle j'_\varepsilon(u), u\rangle_{V',V} \geq 0\ \forall u \in V.$$

Thus we obtain from (3.26) that

$$\varepsilon|\dot{u}_{\varepsilon k}(t)|^2_V + a(u_{\varepsilon k}(t), \dot{u}_{\varepsilon k}(t)) \leq \langle f_k(t), \dot{u}_{\varepsilon k}(t)\rangle_{V',V} \quad \text{a.e. } t \in (0,T). \tag{3.27}$$

We integrate the previous inequality on $[0,t]$ and use the V-ellipticity of $a(\cdot,\cdot)$, we get

$$\begin{aligned}&\varepsilon\int_0^t |\dot{u}_{\varepsilon k}(s)|^2_V\,\mathrm{d}s + \frac{1}{2}a(u_{\varepsilon k}(t), u_{\varepsilon k}(t))\\&\quad \leq \int_0^t \langle f_k(s), \dot{u}_{\varepsilon k}(s)\rangle_{V',V}\,\mathrm{d}s + \frac{1}{2}a(u_0, u_0)\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\varepsilon\int_0^t |\dot{u}_{\varepsilon k}(s)|^2_V\,\mathrm{d}s + \frac{\alpha C}{2}|u_{\varepsilon k}(t)|^2_V \leq\\&\quad C + \frac{\theta}{2}\int_0^t |f_k(s)|^2_{V'}\,\mathrm{d}s + \frac{1}{2\theta}\int_0^t |\dot{u}_{\varepsilon k}(s)|^2_V\,\mathrm{d}s\end{aligned} \tag{3.29}$$

which implies that

$$|u_{\varepsilon k}|_V^2 \le C(1 + |f_k|^2_{L^2(0,T;V')}), \tag{3.30}$$

$$|\dot u_{\varepsilon k}|^2_{L^2(0,T;V)} \le C(\varepsilon)(1 + |f_k|^2_{L^2(0,T;V')}). \tag{3.31}$$

We derive (3.8) with respect to t, take in the result $v = \ddot u_{\varepsilon k}(t)$ and use $j''(\dot u_{\varepsilon k})(\ddot u_{\varepsilon k}, \ddot u_{\varepsilon k}) \ge 0$, it follows:

$$\varepsilon|\ddot u_{\varepsilon k}(t)|_V^2 + a(\dot u_{\varepsilon k}(t), \ddot u_{\varepsilon k}(t)) \le \langle \dot f_k(t), \ddot u_{\varepsilon k}(t)\rangle_{V',V}. \tag{3.32}$$

Using Young's inequality, we can write

$$\varepsilon|\ddot u_{\varepsilon k}(t)|_V^2 \le \frac{C\theta}{2}|\dot u_{\varepsilon k}(t)|_V^2 + \frac{\theta}{2}|\dot f_k(t)|_{V'}^2 + \frac{C}{\theta}|\ddot u_{\varepsilon k}(t)|_V^2 \tag{3.33}$$

so that

$$|\ddot u_{\varepsilon k}(t)|_V^2 \le C(\varepsilon)(|\dot u_{\varepsilon k}(t)|_V^2 + |\dot f_k(t)|_{V'}^2). \tag{3.34}$$

From (3.30), (3.31) and (3.34), we obtain that

$$(u_{\varepsilon k}) \text{ is a bounded sequence in } L^\infty(0,T;V), \tag{3.35}$$

$$(\dot u_{\varepsilon k}) \text{ is a bounded sequence in } L^2(0,T;V), \tag{3.36}$$

$$(\ddot u_{\varepsilon k}) \text{ is a bounded sequence in } L^2(0,T;V). \tag{3.37}$$

Convergence results. From (3.35)–(3.37), we deduce that there exists an element $\tilde u_\varepsilon \in H^1(0,T;V)$ and a subsequence still denoted by $(u_{\varepsilon k})$ such that

$$u_{\varepsilon k} \rightharpoonup \tilde u_\varepsilon \quad \text{in } L^\infty(0,T;V) \text{ weak}^*, \tag{3.38}$$

$$\dot u_{\varepsilon k} \rightharpoonup \dot{\tilde u}_\varepsilon \quad \text{weakly in } H^1(0,T;V). \tag{3.39}$$

Next, we show that $\tilde u_\varepsilon$ is a solution of problem (3.6)–(3.7). First, we note that (3.38) and (3.39) imply

$$u_{\varepsilon k}(t) \rightharpoonup \tilde u_\varepsilon(t) \quad \text{weakly in } V \quad \forall t \in [0,T]. \tag{3.40}$$

Let us pass to the limit in the following equation which is equivalent to (3.8):

$$\varepsilon\int_0^T \langle \dot u_{\varepsilon k}(t), v(t)\rangle_V \,\mathrm{d}t + \int_0^T a(u_{\varepsilon k}(t), v(t))\,\mathrm{d}t + \int_0^T \langle j'_\varepsilon(\dot u_{\varepsilon k}(t)), v(t)\rangle_{V',V}\,\mathrm{d}t$$
$$= \int_0^T \langle f_k(t), v(t)\rangle_{V',V}\,\mathrm{d}t \;\forall v \in L^2(0,T;V).$$

From (3.38) and (3.39) it is clear that

$$\varepsilon\int_0^T \langle \dot u_{\varepsilon k}(t), v(t)\rangle_V \,\mathrm{d}t \to \varepsilon\int_0^T \langle \dot{\tilde u}_\varepsilon(t), v(t)\rangle_V\,\mathrm{d}t \tag{3.41}$$

and

$$\int_0^T a(u_{\varepsilon k}(t), v)\,\mathrm{d}t \to \int_0^T a(u_\varepsilon(t), v)\,\mathrm{d}t. \tag{3.42}$$

Moreover, using again (3.36) and (3.37) it follows that for $0<\delta<\frac{1}{2}$ (up to subsequences):

$$\dot{u}_{\varepsilon k} \to \dot{\tilde{u}}_{\varepsilon} \text{ strongly in } L^2(0,T;[H^{1-\delta}(\Omega)]^M) \tag{3.43}$$

and therefore

$$\dot{u}_{\varepsilon k|\Gamma_3} \to \dot{\tilde{u}}_{\varepsilon|\Gamma_3} \text{ strongly in } L^2([0,T]\times\Gamma_3). \tag{3.44}$$

So, up to a subsequence

$$\dot{u}_{\varepsilon k\tau}(t,x) \to \dot{\tilde{u}}_{\varepsilon\tau}(t,x), \quad \text{a.e. } (t,x)\in(0,T)\times\Gamma_3,$$

$$\psi_\varepsilon'(\dot{u}_{\varepsilon k\tau}(t,x)) \to \psi_\varepsilon'(\dot{\tilde{u}}_{\varepsilon\tau}(t,x)), \quad \text{a.e. } (t,x)\in[0,T]\times\Gamma_3.$$

Reminding that

$$\int_0^T \langle j_\varepsilon'(\dot{u}_{\varepsilon k}(t)), v(t)\rangle_{V',V}\,\mathrm{d}t = \int_0^T\int_{\Gamma_3} g\psi_\varepsilon'(\dot{u}_{\varepsilon k\tau}(t,x))v_\tau(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

and that from (3.3) $|\psi_\varepsilon'(\dot{u}_{\varepsilon k\tau}(t,x))v_\tau(t,x)| \le C_1|v_\tau(t,x)|$, the *Lebesgue-dominated convergence Theorem* gives

$$\lim_k \int_0^T \langle j_\varepsilon'(\dot{u}_{\varepsilon k}(t)), v(t)\rangle_{V',V}\,\mathrm{d}t = \int_0^T \langle j_\varepsilon'(\dot{\tilde{u}}_{\varepsilon}(t)), v(t)\rangle_{V',V}\,\mathrm{d}t. \tag{3.45}$$

Using now (3.22)–(3.23), (3.41), (3.42) and (3.45), we obtain

$$\varepsilon\int_0^T \langle \dot{\tilde{u}}_\varepsilon(t), v(t)\rangle_V\,\mathrm{d}t + \int_0^T a(\tilde{u}_\varepsilon(t), v(t))\,\mathrm{d}t + \int_0^T \langle j_\varepsilon'(\dot{\tilde{u}}_\varepsilon(t)), v(t)\rangle_{V',V}\,\mathrm{d}t$$
$$= \int_0^T \langle f(t), v(t)\rangle_{V',V}\,\mathrm{d}t.$$
$$\forall v\in L^2(0,T;V)$$

which implies

$$\varepsilon\langle\dot{\tilde{u}}_\varepsilon(t), v\rangle_V + a(\tilde{u}_\varepsilon(t), v) + \langle j_\varepsilon'(\dot{\tilde{u}}_\varepsilon(t)), v\rangle_{V',V} = \langle f(t), v\rangle_{V',V}$$
$$\forall v\in V, \text{ a.e. } t\in(0,T).$$

So $\tilde{u}_\varepsilon = u_\varepsilon$ which completes the proof. □

Using the same arguments as in Theorem 2.6, we deduce

Theorem 3.7. *Let* (2.14)–(2.19), (3.2) *and* (3.3) *hold. Then there exists a solution* β_ε *to problem* (3.20).

In the sequel we study the relation between solutions of P and P_ε when $\varepsilon\to 0$.

3.3. Convergence of the regularized problems, asymptotic analysis

We prove the convergence of u_ε to u and give an estimate of $|u_\varepsilon - u|_v$. Then we prove that the solution of the optimal regularized control converges to the solution of the initial control problem.

Theorem 3.8. *Let* (2.14)–(2.19) *and* (3.2)–(3.3) *hold and let* u_ε *be a solution of problem* (3.6)–(3.7) *corresponding to the parameter* ε *for the data* φ_1, φ_2 *and* u_0. *Let also* u *be the solution of* (2.20)–(2.21) *with the same data. Then*

$$\dot{u}_\varepsilon \rightharpoonup \dot{u} \text{ weakly in } L^2(0,T;V), \tag{3.46}$$

$$u_\varepsilon \to u \text{ strongly in } L^\infty(0,T;V) \tag{3.47}$$

and there exists $C > 0$ *independent of* ε *such that*

$$|u_\varepsilon - u|_{L^\infty(0,T;V)} \le C\sqrt{\varepsilon}(1 + |\dot{u}|^2_{L^2(0,T;V)}). \tag{3.48}$$

Proof. We take $v = \dot{u}_\varepsilon(t)$ in (2.20), $v = \dot{u}(t)$ in (3.6), add the two inequalities, and use (3.3) to obtain

$$\begin{aligned}
&\varepsilon|\dot{u}_\varepsilon(t)|^2_V + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a(u_\varepsilon(t) - u(t), u_\varepsilon(t) - u(t)) \\
&\quad = \varepsilon|\dot{u}_\varepsilon(t)|^2_V + a(u_\varepsilon(t) - u(t), \dot{u}_\varepsilon(t) - \dot{u}(t)) \\
&\quad \le [j(\dot{u}_\varepsilon(t)) - j_\varepsilon(\dot{u}_\varepsilon(t))] + [j_\varepsilon(\dot{u}(t)) - j(\dot{u}(t))] + \varepsilon\langle \dot{u}_\varepsilon(t), \dot{u}(t)\rangle_V \\
&\quad \le C\varepsilon + \frac{\varepsilon}{2\theta}|\dot{u}_\varepsilon(t)|^2_V + \frac{\varepsilon\theta}{2}|\dot{u}(t)|^2_V.
\end{aligned} \tag{3.49}$$

Integrating (3.49) over $[0,t]$ and using the V-ellipticity of a, we deduce that

$$|\dot{u}_\varepsilon|_{L^2(0,T;V)} \le C(1 + |\dot{u}|_{L^2(0,T;V)}),$$

$$|u_\varepsilon(t) - u(t)|^2_V \le C\varepsilon(1 + |\dot{u}|_{L^2(0,T;V)}) \tag{3.50}$$

which completes the proof of Theorem 3.8. □

Using the uniform estimate (3.48) and Theorem 2.4, one can easily deduce

Proposition 3.9. *Let* (2.14)–(2.19) *and* (3.2)–(3.3) *hold and suppose that*

$$\varphi_{1k} \rightharpoonup \varphi_1 \text{ weakly in } H^1(0,T;H),$$

$$\varphi_{2k} \rightharpoonup \varphi_2 \text{ weakly in } H^1(0,T;L^2(\Gamma_2)^M)$$

and $\varepsilon_k \to 0$ *as* $k \to \infty$. *Then*

$$u_{\varepsilon_k}(\varphi_{1k}, \varphi_{2k}) \to u(\varphi_1, \varphi_2) \text{ strongly in } L^2(0,T;V).$$

Now, we turn to the control problem and prove

Theorem 3.10. *Let* (2.14)–(2.19) *and* (3.2)–(3.3) *hold. Then the sequence* (β_ε) *defined in* (3.20) *is bounded in* $H^1(0,T)$ *and any weak* $H^1(0,T)$ *limit point* β^* *of the sequence* (β_ε) *as* $\varepsilon \to 0$ *is a solution of* (2.27) *and is a strong* $H^1(0,T)$ *limit point.*

Proof. We remind that

$$\mathscr{J}(\beta,u) = \frac{1}{2}\int_0^T\int_\Omega \rho(x,t)|\sigma^D(u)(x,t) - \sigma_g(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}|\beta|^2_{H^1(0,T)}$$

and

$$J(\beta) = \mathscr{J}(\beta,u(\beta))$$

where $u(\beta)$ is the solution of (2.20) and (2.21) for $\varphi_2(t,x) = \beta(t)\theta(x)$, and

$$J_\varepsilon(\beta) = \mathscr{J}(\beta,u_\varepsilon(\beta)),$$

where $u_\varepsilon(\beta)$ is the solution of (3.8).

Let us prove that the sequence β_ε defined by $J_\varepsilon(\beta_\varepsilon) \le J_\varepsilon(\beta)\ \forall\beta \in H^1(0,T)$ is bounded. From Theorem 3.8 and (3.47), it is straightforward to see that

$$\sigma^D(u_\varepsilon(\beta)) \to \sigma^D(u(\beta)) \text{ strongly in } L^2(0,T;\mathscr{H}) \tag{3.51}$$

for any $\beta \in H^1(0,T)$. So

$$J_\varepsilon(\beta) \to J(\beta) \quad \forall\beta \in H^1(0,T), \quad \text{as } \varepsilon \to 0 \tag{3.52}$$

and for any $\beta \in H^1(0,T)$, $J_\varepsilon(\beta)$ is bounded. As

$$\frac{\gamma}{2}|\beta_\varepsilon|^2_{H^1(0,T)} \le J_\varepsilon(\beta_\varepsilon) \le J_\varepsilon(\beta) \tag{3.53}$$

the sequence β_ε is bounded in $H^1(0,T)$. Let β^* be a weak limit of $\beta_\varepsilon \in H^1(0,T)$. Using Theorem 3.8 and Theorem 2.4, we can check that

$$\sigma^D(u_\varepsilon(\beta_\varepsilon)) \rightharpoonup \sigma^D(u(\beta^*)) \text{ weakly in } L^2(0,T;\mathscr{H}). \tag{3.54}$$

Then by convexity arguments, we have

$$J(\beta^*) \le \liminf_\varepsilon J_\varepsilon(\beta_\varepsilon). \tag{3.55}$$

Using (3.52)–(3.55), we obtain

$$J(\beta^*) \le J(\beta)$$

and β^* is a solution of (2.27). By the above inequality we have

$$J(\beta^*) \le \liminf_\varepsilon J_\varepsilon(\beta_\varepsilon) \le J(\beta^*)$$

then

$$J(\beta^*) = \liminf_\varepsilon J_\varepsilon(\beta_\varepsilon). \tag{3.56}$$

Now, we know that $|\beta^*|^2_{H^1(0,T)} \le \liminf_\varepsilon |\beta_\varepsilon|^2_{H^1(0,T)}$ and

$$\int_0^T\int_\Omega \rho(t,x)|\sigma^D(u(\beta^*))(t,x) - \sigma_g(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\le \liminf_\varepsilon \int_0^T\int_\Omega \rho(t,x)|\sigma^D(u_\varepsilon(\beta_\varepsilon))(t,x) - \sigma_g(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Thus, (3.56) proves that

$$\sqrt{\rho}\sigma^D(u_\varepsilon(\beta_\varepsilon)) \to \sqrt{\rho}\sigma^D(u(\beta^*)) \text{ strongly in } L^2(0,T;\mathscr{H})$$

and

$$\beta_\varepsilon \to \beta^* \text{ strongly in } H^1(0,T). \qquad \square$$

3.4. *Differentiation, optimality conditions for the regularized problems*

Having proved the existence of optimal controls for the regularized problems and their convergence properties as $\varepsilon \to 0$, we try now to give their regularity. Indeed, in this section, we consider the regularized problem (3.8) which has a unique solution $u_\varepsilon \in H^2(0,T;V)$. Our main interest lies in the differentiation of $J_\varepsilon(\beta) = \mathscr{J}(\beta, u_\varepsilon(\beta))$ defined by (3.21) and write that at the extremum the differential of J_ε must be null.

We start with a standard result that can be found in Lions [11] where we find the main ideas of the proof and we provide a sketch of it for the reader's convenience.

Lemma 3.11. *Let $\mathscr{B}$ be a Banach space, X and Y two reflexive Banach spaces. Let also be given two $\mathscr{C}^1$ functions*

$$F : \mathscr{B} \times X \to Y : (\beta, \tilde{u}) \mapsto F(\beta, \tilde{u}),$$

$$\tilde{\mathscr{J}} : \mathscr{B} \times X \to \mathbb{R} : (\beta, \tilde{u}) \mapsto \tilde{\mathscr{J}}(\beta, \tilde{u}).$$

We suppose that for all $\beta \in \mathscr{B}$

1. *There exists a unique $\tilde{u}(\beta)$ such that $F(\beta, \tilde{u}(\beta)) = 0$,*
2. *$\frac{\partial F}{\partial \tilde{u}}(\beta, \tilde{u}(\beta))$ is an isomorphism from X onto Y.*

Then $J(\beta) = \tilde{\mathscr{J}}(\beta, \tilde{u}(\beta))$ is differentiable and

$$\forall \delta\beta \in \mathscr{B} \quad \frac{\mathrm{d}J}{\mathrm{d}\beta}(\beta) \cdot \delta\beta = \frac{\partial \tilde{\mathscr{J}}}{\partial \beta}(\beta, \tilde{u}(\beta)) \cdot \delta\beta - \left\langle p(\beta), \frac{\partial F}{\partial \beta}(\beta, \tilde{u}(\beta)) \cdot \delta\beta \right\rangle_{Y',Y}, \tag{3.57}$$

where $p(\beta) \in Y'$ is the adjoint state, unique solution of

$$\left[\frac{\partial F}{\partial \tilde{u}}(\beta, \tilde{u}(\beta))\right]^* \cdot p(\beta) = \frac{\partial \tilde{\mathscr{J}}}{\partial \tilde{u}}(\beta, \tilde{u}(\beta)) \quad \text{in } X'. \tag{3.58}$$

Proof. The differentiability of J with respect to β follows from the *implicit function Theorem*, from which we know that the mapping $\beta \mapsto \tilde{u}(\beta)$ is $\mathscr{C}^1(\mathscr{B}, X)$.

For β fixed and for any $\delta\beta \in \mathscr{B}$, we denote

$$\tilde{u}' \cdot \delta\beta = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}\beta} \cdot \delta\beta, \quad A = \frac{\partial F}{\partial \tilde{u}}(\beta, \tilde{u}(\beta)).$$

The operator A is an isomorphism from X onto Y, his adjoint A^* is an isomorphism from Y' onto X'. By differentiation, we have

$$\frac{\partial F}{\partial \beta}(\beta, \tilde{u}(\beta)) \cdot \delta\beta + A \cdot (\tilde{u}' \cdot \delta\beta) = 0 \quad \text{in } Y, \tag{3.59}$$

$$\frac{\mathrm{d}J}{\mathrm{d}\beta}(\beta) \cdot \delta\beta = \frac{\partial \tilde{\mathscr{J}}}{\partial \beta}(\beta, \tilde{u}(\beta)) \cdot \delta\beta + \left\langle \frac{\partial \tilde{\mathscr{J}}}{\partial \tilde{u}}(\beta, \tilde{u}(\beta)), \tilde{u}' \cdot \delta\beta \right\rangle_{X',X}. \tag{3.60}$$

Eq. (3.59) is equivalent to

$$\langle l, A \cdot (\tilde{u}' \cdot \delta\beta)\rangle_{Y',Y} = -\left\langle l, \frac{\partial F}{\partial \beta}(\beta, \tilde{u}(\beta)) \cdot \delta\beta \right\rangle_{Y',Y} \qquad \forall l \in Y'.$$

Let $p(\beta) \in Y'$ be the unique solution of

$$A^* \cdot p(\beta) = \frac{\partial \tilde{\mathscr{J}}}{\partial \tilde{u}}(\beta, \tilde{u}(\beta)) \quad \text{in } X'.$$

By substitution in (3.60), we obtain

$$\frac{\mathrm{d}J}{\mathrm{d}\beta}(\beta) \cdot \delta\beta = \frac{\partial \tilde{\mathscr{J}}}{\partial \beta}(\beta, \tilde{u}(\beta)) \cdot \delta\beta - \left\langle p(\beta), \frac{\partial F}{\partial \beta}(\beta, \tilde{u}(\beta)) \cdot \delta\beta \right\rangle_{Y',Y}. \qquad \square$$

Now, we describe the functional spaces of our problem. In Eq. (3.8) the loading $f(\beta) \in V'$ is defined by

$$\langle f(\beta)(t), v\rangle_{V',V} = \langle \varphi_1(t), v\rangle_H + \beta(t)\langle \theta, \gamma v\rangle_{L^2(\Gamma_2)^M}.$$

Eq. (3.8), which has, for all v in V and for t almost everywhere in $[0, T]$, a unique solution $u_\varepsilon(\beta) \in H^2(0, T; V)$, is equivalent to

$$u_\varepsilon(\beta) \in H^2(0, T; V), \quad u_\varepsilon(\beta)(0) = u_0$$

$$\int_0^T [\varepsilon\langle \dot{u}_\varepsilon(\beta)(t), v(t)\rangle_V + a(u_\varepsilon(\beta)(t), v(t)) + \langle j_\varepsilon'(\dot{u}_\varepsilon(\beta)(t)), v(t)\rangle_{V',V}]\,\mathrm{d}t$$
$$= \int_0^T \langle f(\beta)(t), v(t)\rangle_{V',V}\,\mathrm{d}t \quad \forall v \in L^2(0, T; V). \tag{3.61}$$

In the next, we assume that $\varepsilon > 0$ is fixed. In order to simplify the notation, we suppress the ε subscript on $u(\beta)$. We can write

$$u(\beta) = u_0 + \tilde{u}(\beta),$$

where $\tilde{u}(\beta) \in X = \{\tilde{u} \in H^1(0, T; V) \,|\, \tilde{u}(0) = 0\}$.

We consider the function $F : H^1(0,T)\times X \to L^2(0,T;V') : (\beta,\tilde{u}) \mapsto F(\beta,\tilde{u})$ defined by

$$\begin{aligned}\langle F(\beta,\tilde{u}), v\rangle_{L^2(0,T;V'),L^2(0,T;V)} &= \int_0^T [\varepsilon\langle \dot{\tilde{u}}(t), v(t)\rangle_V + a(u_0+\tilde{u}(t), v(t))]\,\mathrm{d}t \\ &\quad + \int_0^T [\langle j'_\varepsilon(\dot{\tilde{u}}(t)), v(t)\rangle_{V',V} - \langle f(\beta)(t), v(t)\rangle_{V',V}]\,\mathrm{d}t, \quad \forall v\in L^2(0,T;V) \end{aligned} \tag{3.62}$$

and $\tilde{\mathscr{J}} : H^1(0,T)\times X \to \mathbb{R} : (\beta,\tilde{u}) \mapsto \tilde{\mathscr{J}}(\beta,\tilde{u})$ given by

$$\tilde{\mathscr{J}}(\beta,\tilde{u}) = \frac{1}{2}\int_0^T\int_\Omega \rho(x,t)|\sigma(u_0+\tilde{u})(x,t) - \sigma_g(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}|\beta|^2_{H^1(0,T)}. \tag{3.63}$$

The state equation (3.61) can be written as follows:

$$\text{find}\quad u(\beta) = u_0 + \tilde{u}(\beta) \in u_0 + X \quad \text{such that} \quad F(\beta,\tilde{u}(\beta)) = 0, \tag{3.64}$$

and we have $J(\beta) = \tilde{\mathscr{J}}(\beta,\tilde{u}(\beta))$.

We are exactly in the framework of Lemma 3.11, taking $\mathscr{B} = H^1(0,T)$, $X = \{u \in H^1(0,T;V)\,|\,u(0)=0\}$ and $Y = L^2(0,T;V')$. It is straightforward to show that F and $\tilde{\mathscr{J}}$ are of class $\mathscr{C}^1(\mathscr{B}\times X)$. In order to use Lemma 3.11, we may now show that

Lemma 3.12. *For all $\beta \in \mathscr{B}$, the operator $\frac{\partial F}{\partial \tilde{u}}(\beta,\tilde{u}(\beta))$ is an isomorphism from X onto Y.*

Proof. Let $h \in L^2(0,T;V')$. We have to prove that there exists a unique element $\xi \in X$ such that $\frac{\partial F}{\partial \tilde{u}}(\beta,\tilde{u}(\beta))\cdot\xi = h$, which is equivalent to solve the following parabolic system:

$$\begin{aligned} &\text{Find } \xi \in H^1(0,T;V) \quad \text{such that} \quad \xi(0) = 0 \quad \text{and} \\ &\varepsilon\langle\dot{\xi}(t), v\rangle_V + a(\xi(t), v) + j''_\varepsilon(\dot{u}(\beta)(t))(\dot{\xi}(t), v) = \langle h(t), v\rangle_{V',V} \\ &\quad \forall v\in V, \quad \text{a.e. } t\in(0,T). \end{aligned} \tag{3.65}$$

In this proof, $u(\beta) \in H^1(0,T;V)$ and $\varepsilon > 0$ are fixed. For sake of simplicity we denote $j''(\dot{u})$ for $j''_\varepsilon(\dot{u}(\beta)(t))$.

Existence: We use *a Galerkin method*. Let $\mathscr{B} = (w_i)_{i\geq 1}$ be a Hilbertien basis of V. We first prove that there exists $\xi_N(t) = \sum_{k=1}^N \lambda_k(t) w_k$ such that

$$\begin{aligned} &\forall k = 1\ldots N, \quad \lambda_k \in H^1(0,T), \quad \lambda_k(0) = 0 \quad \text{and} \\ &\varepsilon\langle\dot{\xi}_N(t), w_i\rangle_V + a(\xi_N(t), w_i) + j''(\dot{u})(\dot{\xi}_N(t), w_i) = \langle h(t), w_i\rangle_{V',V} \\ &\quad \forall i = 1..N, \quad \text{a.e. } t\in(0,T). \end{aligned} \tag{3.66}$$

The linear system (3.66) can be written as follows:

$$(\varepsilon\mathbf{I}+\mathbf{G}(t))\dot{\Lambda}(t)+\mathbf{A}\Lambda(t)=H(t),$$
$$\Lambda(0)=0, \tag{3.67}$$

where $\Lambda(t)$ and $H(t)$ are real vectors defined by

$$\Lambda(t)=(\lambda_k(t))_{1\le k\le N}, \quad H(t)=(\langle h(t),w_i\rangle_{V',V})_{1\le i\le N},$$

$\mathbf{I}$, $\mathbf{G}(t)$ and $\mathbf{A}$ are $(N\times N)$ real symmetric matrices defined by

$$\mathbf{I}=(\langle w_k,w_i\rangle_V)_{1\le i,k\le N},$$

$$\mathbf{G}(t)=(j''(\dot{u})(w_k,w_i))_{1\le i,k\le N},$$

$$\mathbf{A}=(a(w_k,w_i))_{1\le i,k\le N}.$$

Since $\langle\mathbf{G}(t)\Lambda,\Lambda\rangle=\int_{\Gamma_3}(\sum_{k=1}^{N}\sqrt{\psi_\varepsilon''(\dot{u}_\varepsilon(t))}\lambda_i\ w_i)^2\,\mathrm{d}a\ge 0$, $(\varepsilon\mathbf{I}+\mathbf{G}(t))$ is postive definite and (3.67) can be rewritten as

$$\dot{\Lambda}(t)+(\varepsilon\mathbf{I}+\mathbf{G}(t))^{-1}\mathbf{A}\Lambda(t)=(\varepsilon\mathbf{I}+\mathbf{G}(t))^{-1}H(t),$$
$$\Lambda(0)=0. \tag{3.68}$$

From the theory of systems of ordinary differential equations, it is known that (3.68) has a unique solution in $H^1(0,T)^N$. Now, in order to pass to the limit in N, we proceed to obtain an a priori *estimate* on the solution ξ_N.

Setting $v=\dot{\xi}_N(t)$ in (3.66) and using the convexity of j_ε, it follows that:

$$\varepsilon|\dot{\xi}_N(t)|_V^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a(\xi_N(t),\xi_N(t))\le\langle h(t),\dot{\xi}_N(t)\rangle_{V',V}. \tag{3.69}$$

Integrating (3.69) in time from 0 to t, using the V-ellipticity property of $a(\cdot,\cdot)$ and Young's inequality, we obtain

$$\varepsilon\int_0^t|\dot{\xi}_N(s)|_V^2\,\mathrm{d}s+\frac{\alpha}{2}|\xi_N(t)|_V^2\le\frac{\theta}{2}\int_0^t|h(s)|_{V'}^2\,\mathrm{d}s+\frac{1}{2\theta}\int_0^t|\dot{\xi}_N(s)|_V^2\,\mathrm{d}s, \tag{3.70}$$

so that

$$|\xi_N|_{H^1(0,T;V)}\le C. \tag{3.71}$$

So the sequence $(\xi_N)_N$ is weakly compact in $H^1(0,T;V)$. It is clear that any weak limit point is a solution of (3.65).

Uniqueness: We now prove that (3.65) has at most one solution. Let ξ_1 and ξ_2 be two solutions of problem (3.65), for the same data. Denote by ξ the difference $\xi_1-\xi_2$. Writing the variational statement (3.65) for ξ_1 and ξ_2 and substracting the equations we obtain

$$\varepsilon\langle\dot{\xi}(t),v\rangle_V+a(\xi(t),v)+j_\varepsilon''(\dot{u}_\varepsilon(t))(\dot{\xi}(t),v)=0\quad\forall v\in V,\quad\text{a.e. } t\in(0,T)$$
$$\xi(0)=0. \tag{3.72}$$

Setting $v=\dot{\xi}(t)$ in (3.72), using the convexity of j_ε and integrating in time, we easily get

$$\varepsilon\int_0^t |\dot{\xi}(s)|_{V^2}\,\mathrm{d}s+\frac{1}{2}a(\xi(t),\xi(t))\leq 0 \quad \forall t\in[0,T],$$
$$\xi(0)=0 \tag{3.73}$$

from which we conclude that $\xi(t)=0\ \forall t\in[0,T]$. □

We obtain the following theorem from Lemma 3.11.

Theorem 3.13. *J_ε is differentiable on $H^1(0,T)$ and for all $\delta\beta\in H^1(0,T)$*

$$\frac{\mathrm{d}J_\varepsilon}{\mathrm{d}\beta}(\beta)\cdot\delta\beta=\int_0^T\langle\theta,\,p_\varepsilon(\beta)(t)\rangle_{L^2(\Gamma_2)^M}\,\delta\beta(t)\,\mathrm{d}t+\gamma\langle\beta,\delta\beta\rangle_{H^1(0,T)}, \tag{3.74}$$

where the adjoint state $p_\varepsilon(\beta)$ is the unique solution of the problem

$$p_\varepsilon(\beta)\in L^2(0,T;V)$$
$$\int_0^T\varepsilon\langle\dot{w}(t),\,p_\varepsilon(\beta)(t)\rangle_V\,\mathrm{d}t+\int_0^T a(w(t),\,p_\varepsilon(\beta)(t))\,\mathrm{d}t$$
$$+\int_0^T j''_\varepsilon(\dot{u}_\varepsilon(\beta)(t))(\dot{w}(t),\,p_\varepsilon(\beta)(t))\,\mathrm{d}t$$
$$=\int_0^T\int_\Omega\rho(\sigma^D(u_\varepsilon(\beta)-\sigma_g:\sigma^D(w))(t,x)\,\mathrm{d}x\,\mathrm{d}t\quad\forall w\in X \tag{3.75}$$

and $u_\varepsilon(\beta)$ is the solution of the regularized problem (3.8).

Proof. We apply Lemma 3.11

1. *The adjoint state*: Noticing that $\dot{u}(\beta)=\dot{\tilde{u}}(\beta)$, we can write the differential of (3.62) with respect to $\tilde{u}$ as follows:

$$\left\langle\frac{\partial F}{\partial\tilde{u}}(\beta,\tilde{u}(\beta))\cdot w,v\right\rangle_{L^2(0,T;V'),L^2(0,T;V)}$$
$$=\left\langle w,\left[\frac{\partial F}{\partial\tilde{u}}(\beta,\tilde{u}(\beta))\right]^*\cdot v\right\rangle_{X,X'}$$
$$=\int_0^T[\varepsilon\langle\dot{w}(t),v(t)\rangle_V+a(w(t),v(t))+j''_\varepsilon(\dot{u}_\varepsilon(\beta)(t))(\dot{w}(t),v(t))]\,\mathrm{d}t\quad\forall w\in X. \tag{3.76}$$

Moreover

$$\frac{\partial\tilde{\mathscr{J}}}{\partial\tilde{u}}(\beta,\tilde{u}(\beta))\cdot w=\int_0^T\int_\Omega\rho(t,x)[(\sigma^D(u_\varepsilon(\beta))-\sigma_g:\sigma^D(w))](t,x)\,\mathrm{d}x\,\mathrm{d}t\quad\forall w\in X. \tag{3.77}$$

The adjoint state $p_\varepsilon(\beta)$ is the solution of

$$\left\langle p_\varepsilon(\beta), \frac{\partial F}{\partial \tilde{u}}(\beta, \tilde{u}(\beta)) \cdot w \right\rangle_{L^2(0,T;V),L^2(0,T;V')} = \frac{\partial \tilde{\mathscr{J}}}{\partial \tilde{u}}(\beta, \tilde{u}(\beta)) \cdot w \quad \forall w \in X. \tag{3.78}$$

From (3.76)–(3.78), we obtain (3.75).

2. *Differentiation of* J_ε: We differentiate (3.62) and (3.63) with respect to β

$$\frac{\partial \tilde{\mathscr{J}}}{\partial \beta}(\beta, \tilde{u}(\beta)) \cdot \delta\beta = \gamma\langle \beta, \delta\beta\rangle_{H^1(0,T)}, \tag{3.79}$$

$$\begin{aligned} &\left\langle p_\varepsilon(\beta), \frac{\partial F}{\partial \beta}(\beta, \tilde{u}(\beta)) \cdot \delta\beta \right\rangle_{L^2(0,T;V),L^2(0,T;V')} \\ &\quad = -\int_0^T \left\langle \frac{\mathrm{d}f}{\mathrm{d}\beta}(\beta)(t) \cdot \delta\beta(t), p_\varepsilon(\beta)(t) \right\rangle_{V',V} \mathrm{d}t \\ &\quad = -\int_0^T \langle \theta, p_\varepsilon(\beta)(t)\rangle_{L^2(\Gamma_2)^M} \delta\beta(t)\,\mathrm{d}t. \end{aligned} \tag{3.80}$$

Finally, from Lemma 3.11, it follows that:

$$\begin{aligned} &\frac{\mathrm{d}J_\varepsilon}{\mathrm{d}\beta}(\beta) \cdot \delta\beta = \int_0^T \langle \theta, p_\varepsilon(\beta)(t)\rangle_{L^2(\Gamma_2)^M} \delta\beta(t)\,\mathrm{d}t + \gamma\langle \beta, \delta\beta\rangle_{H^1(0,T)} \\ &\quad \forall \delta\beta \in H^1(0,T). \end{aligned} \tag{3.81}$$

Writing that at the optimum, the differential of J_ε must be zero, we derive the following optimality condition:

$$\gamma\langle \beta_\varepsilon, \delta\beta\rangle_{H^1(0,T)} = -\int_0^T \langle \theta, p_\varepsilon(\beta_\varepsilon)(t)\rangle_{L^2(\Gamma_2)^M} \delta\beta(t)\,\mathrm{d}t \quad \forall\ \delta\beta \in H^1(0,T). \tag{3.82}$$

Remark 3.14 (Coupled control on traction forces and elasticity tensor). We end this paper with a version of the control problem (3.20) in which the elasticity tensor $\mathscr{R}$ of the material depends on a control $\phi \in \mathscr{C}$ where $\mathscr{C}$ is an open subset of a given Banach space Z. For seak of completness, we give a sketch of proofs (which are now classical in optimal control theory). Therefore, we assume that

$$\mathscr{R} = \mathscr{R}(\phi), \quad \text{a.e. in } \Omega. \tag{3.83}$$

We suppose that

(a) $\mathscr{R}_{ijkh}(\phi) \in L^\infty(\Omega)$ for all $\phi \in \mathscr{C}$,

(b) $\mathscr{R}(\phi)\sigma : \tau = \sigma : \mathscr{R}(\phi)\tau$, a.e. in Ω, $\forall \phi \in \mathscr{C}$ $\forall \sigma,\ \tau \in \mathscr{S}_M$,

(c) $\exists \alpha > 0 \mid \forall \phi \in \mathscr{C}$, $\mathscr{R}(\phi)\sigma : \sigma \geq \alpha|\sigma|^2$ $\forall \sigma \in \mathscr{S}_M$

(d) $\forall ijkh$, $\mathscr{R}_{ijkh} : \mathscr{C} \to L^\infty(\Omega)$ is differentiable.

We denote $a(\phi, u, v) = \langle \mathscr{R}(\phi)\varepsilon(v) > \mathscr{H}$

Let us denote $\eta = (\beta, \phi) \in H^1(0,T) \times \mathscr{C}$. From Theorem 3.1 we deduce that Eq. (3.8) has a unique solution $u_\varepsilon(\eta) \in H^1(0,T;V)$.

We consider the following abstract optimal control problem:

For a given $\sigma_g \in L^2(0,T;\mathscr{H})$ find $\eta_\varepsilon = (\beta_\varepsilon, \phi_\varepsilon) \in H^1(0,T) \times \mathscr{C}$ such that

$$\bar{J}_\varepsilon(\eta_\varepsilon) = \text{infimum}\{\bar{J}_\varepsilon(\eta);\ \eta \in H^1(0,T) \times \mathscr{C}\}, \tag{3.84}$$

where

$$\bar{J}_\varepsilon(\eta) = \mathscr{J}(\beta, u_\varepsilon(\eta)). \tag{3.85}$$

In the next, we assume that there exists the solution $\eta_\varepsilon = (\beta_\varepsilon, \phi_\varepsilon)$ to problem (3.84). In order to apply Lemma 3.10 and obtain necessary optimality condition, we consider the functions $F : (H^1(0,T) \times \mathscr{C}) \times X \to L^2(0,T;V') : (\eta, \tilde{u}) \mapsto F(\eta, \tilde{u})$ given by

$$\begin{aligned}
&\langle F(\eta,\tilde{u}), v\rangle_{L^2(0,T;V'),L^2(0,T;V)} \\
&\quad = \int_0^T [\varepsilon\langle \dot{\tilde{u}}(t), v(t)\rangle_V + a(\phi(t), u_0 + \tilde{u}(t), v(t))]\,\mathrm{d}t \\
&\qquad + \int_0^T [\langle j'_\varepsilon(\dot{\tilde{u}}(t)), v(t)\rangle_{V',V} + \langle f(\beta)(t), v(t)\rangle_{V',V}]\mathrm{d}t \quad \forall v \in L^2(0,T;V),
\end{aligned} \tag{3.86}$$

where $X = \{\tilde{u} \in H^1(0,T;V) \mid \tilde{u}(0) = 0\}$, and $\tilde{\mathscr{J}} : (H^1(0,T) \times \mathscr{C}) \times X \to \mathbb{R} : (\eta, \tilde{u}) \mapsto \tilde{\mathscr{J}}(\eta, \tilde{u})$ given by

$$\tilde{\mathscr{J}}(\eta,\tilde{u}) = \frac{1}{2}\int_0^T\int_\Omega \rho(x,t)|\sigma^D(u_0+\tilde{u})(x,t) - \sigma_g(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}|\beta|^2_{H^1(0,T)}. \tag{3.87}$$

We have

Theorem 3.15. *$\bar{J}_\varepsilon$ is differentiable on $H^1(0,T) \times \mathscr{C}$ and*

$$\begin{aligned}
\frac{\partial \bar{J}_\varepsilon}{\partial \beta}(\eta)\cdot\delta\beta + \frac{\partial \bar{J}_\varepsilon}{\partial \phi}(\eta)\cdot\delta\phi = &-\int_0^T \frac{\partial a}{\partial \phi}(\phi(t), u_\varepsilon(\eta)(t), p_\varepsilon(\eta)(t))\cdot\delta\phi(t)\,\mathrm{d}t \\
&+ \int_0^T \langle \theta, p_\varepsilon(\eta)(t)\rangle_{L^2(\Gamma_2)^M}\delta\beta(t)\,\mathrm{d}t + \gamma\langle\beta,\delta\beta\rangle_{H^1(0,T)},
\end{aligned} \tag{3.88}$$

where the adjoint state $p_\varepsilon(\eta)$ is the unique solution of the problem

$$\begin{aligned}
&p_\varepsilon(\eta) \in L^2(0,T;V) \\
&\int_0^T \varepsilon\langle \dot{w}(t), p_\varepsilon(\eta)(t)\rangle_V\,\mathrm{d}t + \int_0^T a(\phi(t)w(t), p_\varepsilon(\eta)(t))\,\mathrm{d}t \\
&\quad + \int_0^T j''_\varepsilon(\dot{u}_\varepsilon(\eta)(t))(\dot{w}(t), p_\varepsilon(\eta)(t))\,\mathrm{d}t \\
&\quad = \int_0^T\int_\Omega \rho(t,x)(\sigma^D(u_\varepsilon(\eta)(t,x)) - \sigma_g(t,x)) : \sigma^D(w(t,x))\,\mathrm{d}x\,\mathrm{d}t \quad \forall w \in X
\end{aligned} \tag{3.89}$$

and $u_\varepsilon(\eta)$ is the solution of the regularized problem (3.8).

Proof. This proof is similar to Theorem 3.13. Since the modifications are straigtforward, we omit the details.

1. *The state equation*: The state equation can be written as follows:

$$u_\varepsilon(\eta) = u_0 + \tilde{u}(\eta) \quad \text{such that } F(\eta, \tilde{u}(\eta)) = 0$$
$$\text{and} \quad \bar{J}_\varepsilon(\eta) = \tilde{\mathscr{J}}(\eta, \tilde{u}(\eta)). \tag{3.90}$$

2. *The adjoint state*: The differential of (3.87) and (3.88) with respect to $\tilde{u}$ are the following:

$$\left\langle \frac{\partial F}{\partial \tilde{u}}(\beta, \tilde{u}(\beta)) \cdot w, v \right\rangle_{L^2(0,T;V'),L^2(0,T;V)}$$
$$= \left\langle w, \left[\frac{\partial F}{\partial \tilde{u}}(\eta,\ \tilde{u}(\eta))\right]^* \cdot v \right\rangle_{X,X'}$$
$$= \int_0^T [\varepsilon\langle \dot{w}(t), v(t)\rangle_V + a(\phi(t), w(t), v(t)) + j_\varepsilon''(\dot{u}_\varepsilon(\eta)(t))(\dot{w}(t), v(t))]\,\mathrm{d}t. \tag{3.91}$$

$$\frac{\partial \tilde{\mathscr{J}}}{\partial \tilde{u}}(\eta, \tilde{u}(\eta)) \cdot w = \int_0^T \int_\Omega \rho(t,x)(\sigma^D(u_\varepsilon(\eta)(t,x)) - \sigma_g(t,x)) : \sigma^D(w(t,x))\,\mathrm{d}x\,\mathrm{d}t. \tag{3.92}$$

The adjoint state $p_\varepsilon(\eta)$ is the solution of

$$\left\langle p_\varepsilon(\eta), \frac{\partial F}{\partial \tilde{u}}(\eta, \tilde{u}(\eta)) \cdot w \right\rangle_{L^2(0,T;V),L^2(0,T;V')} = \frac{\partial \tilde{\mathscr{J}}}{\partial \tilde{u}}(\eta, \tilde{u}(\eta)) \cdot w \quad \forall w \in X. \tag{3.93}$$

From (3.90)–(3.93), we obtain (3.89).

3. *Differentiation of* $\bar{J}_\varepsilon$: We differentiate (3.86) and (3.87) with respect to $\eta = (\beta, \phi)$ in the direction $\delta\eta = (\delta\beta, \delta\phi)$

$$\frac{\partial \tilde{\mathscr{J}}}{\partial \eta} \cdot \delta\eta = \frac{\partial \tilde{\mathscr{J}}}{\partial \beta}(\eta, \tilde{u}(\eta)) \cdot \delta\beta + \frac{\partial \tilde{\mathscr{J}}}{\partial \phi}(\eta, \tilde{u}(\eta)) \cdot \delta\phi = \gamma\langle \beta, \delta\beta\rangle_{H^1(0,T)}, \tag{3.94}$$

$$\langle p_\varepsilon(\eta), \frac{\partial F}{\partial \tilde{\eta}}(\eta, \tilde{u}(\eta)) \cdot \delta\eta\rangle_{L^2(0,T;V),L^2(0,T;V')}$$
$$= \int_0^T \frac{\partial}{\partial \phi} a(\phi, u_0 + \tilde{u}(\eta), p_\varepsilon(\eta)) \cdot \partial\phi\,\mathrm{d}t - \int_0^T \left\langle \frac{\mathrm{d}f}{\mathrm{d}\beta}(\beta)(t) \cdot \delta\beta, p_\varepsilon(\eta)(t) \right\rangle_{V',V} \mathrm{d}t$$
$$= \int_0^T \frac{\partial}{\partial \phi} a(\phi, u_0 + \tilde{u}(\eta), p_\varepsilon(\eta)) \cdot \partial\phi\,\mathrm{d}t - \int_0^T \langle \theta, p_\varepsilon(\eta)(t)\rangle_{L^2(\Gamma_2)^M} \delta\beta\,\mathrm{d}t.$$
$$\forall w \in L^2(0,T;V). \tag{3.95}$$

From Lemma 3.11, it follows that:

$$\frac{\partial \bar{J}_\varepsilon}{\partial \eta} \cdot \delta\eta = \frac{\partial \bar{J}_\varepsilon}{\partial \beta}(\eta)(t) \cdot \delta\beta + \frac{\partial \bar{J}_\varepsilon}{\partial \phi}(\eta)(t) \cdot \delta\phi = -\int_0^T \frac{\partial}{\partial \phi} a(\phi, u_\varepsilon(\eta), p_\varepsilon(\eta)) \cdot \delta\phi\,\mathrm{d}t$$
$$+ \int_0^T \langle \theta, p_\varepsilon(\eta)(t)\rangle_{L^2(\Gamma_2)^M} \delta\beta\,\mathrm{d}t + \gamma\langle \beta, \delta\beta\rangle_{H^1(0,T)} \quad \forall \delta\beta \in H^1(0,T). \tag{3.96}$$

4. *Optimality condition*

$$\gamma\langle\beta_\varepsilon,\delta\beta\rangle_{H^1(0,T)} = \int_0^T \frac{\partial}{\delta\phi}a(\phi,u_\varepsilon(\eta_\varepsilon),p_\varepsilon(\eta_\varepsilon))\cdot\delta\phi\mathrm{d}t - \int_0^T \delta\beta(t)\langle\theta,p_\varepsilon(\eta_\varepsilon)(t)\rangle_V\,\mathrm{d}t \quad \forall\delta\beta\in H^1(0,T). \tag{3.97}$$

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