1. Thermodynamics formalism

Let \((X, \mathcal{T})\) be a dynamical system with \(X\) a compact space and \(\mathcal{T}: X \to X\) a continuous function.

**Definition:** A potential is a continuous function \(V: X \to \mathbb{R}_+\).

A potential defines a pressure function in the following way:

\[
p\beta(V) = \sup_{\mu \in \mathcal{M}_\mu} \left( \int V \, d\mu - \beta \right)
\]

where the sup is taken on all \(\beta\)-invariant probability measures and \(h_\mu\) is the Kolmogorov-Sinai entropy.

2. Some properties of the pressure function

The pressure function is decreasing and convex.

**Definition:** A point \(x\) such that \(h\) is not analytic in \(x\) is called a phase transition.

**Theorem:** [Rue04] Let \((X, \mathcal{T})\) be a subshift of finite type on a finite alphabet. Then, \(\mathcal{A}^0\) is the set of unilateral infinite sequences on this alphabet (an element of \(\mathcal{A}^0\) is a configuration). This set is endowed with the product topology which makes it a Cantor set.

**Definition:** A substitution \(\sigma: \mathcal{A}^0 \to \mathcal{A}^0\) is a non-erasing word morphism.

Examples:

- **k-bonacci substitution:**
  \[
  \begin{array}{c|c}
  0 & 0 \rightarrow 0 \\
  1 & 0 \rightarrow 1 \\
  1 & 1 \rightarrow 0 \\
  k & 1 \rightarrow 0
  \end{array}
  \]

- **Thue-Morse substitution:**
  \[
  \begin{array}{c|c}
  0 & 0 \rightarrow 0 \\
  1 & 1 \rightarrow 0 \\
  1 & 0 \rightarrow 1 \\
  0 & 1 \rightarrow 0
  \end{array}
  \]

- **k-bonacci substitution:**
  \[
  \begin{array}{c|c}
  0 & 0 \rightarrow 0 \\
  1 & 0 \rightarrow 1 \\
  1 & 1 \rightarrow 0 \\
  k & 1 \rightarrow 0
  \end{array}
  \]

Remark that \(x_2\) is the Fibonacci substitution.

**Theorem:** [E] Let \(k \geq 2\), denote \(x_k\) the \(k\)-bonacci substitution, there exists a potential \(U \in C^0(\mathcal{A}^1, \mathbb{R}_+)\) such that \(H_U = U\) given by:

\[
U(x) = 1 + \sum_{n=1}^{\infty} \sum_{y \in \mathcal{A}^n} |x| y_0 \eta(x, y_0) v_0
\]

where \(\lambda\) is the dominant root of the polynomial

\[
X^k - \sum_{j=0}^{k-1} \lambda^j X^j = 0
\]

Moreover, if \(V: \mathcal{A}^1 \to \mathbb{R}_+\) is of the form:

\[
V(x) = \frac{v(x)}{\lambda_1 v_0} - \frac{\lambda_0}{\lambda_1} v_0
\]

with \(v\) a positive continuous function, \(\lambda_0\) being \(0\) on \(x_0\), and continuous and \(\alpha > 0\), then, for any \(x\) in \(\mathcal{A}^2\):

\[
\lim_{n \to \infty} R^* V(x) = 0
\]

Remark for any integer \(n\), for any configuration \(x:\)

\[
R^n V(x) = \sum_{j=1}^{n-1} V(x \sigma^j\).
\]

**Lemma:** For any configuration \(x\) and for any integer \(n\), the maximal prefix of \(x^n\) is \(L_0\) if:

\[
x^n \in L_\alpha
\]

Remark that we could expect, for any \(j < n\), to have the relation:

\[
\delta(\sigma^{n-j} x^n) = \delta(\sigma^{-j} x^n) - j.
\]

But there might be some integer \(j\), such that

\[
\delta(\sigma^{n-j} x^n) - \delta(\sigma^{-j} x^n) = 1
\]

Such points are called accidental.

**Figure 1:** Illustration of an accident at point \(x_2\).

**References**


