On Kato’s Square Root Problem

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T. Kato 1960s: Non-autonomous parabolic evolution equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t)(x) + A(t)u(t)(x) = 0 & (t > 0,\ x \in \Omega), \\ u(0) = u_0 \in \mathrm{L}^2(\Omega). \end{cases}$$

- $A(t) \sim -\nabla_x \cdot \mu(t, x)\nabla_x$ via elliptic form $\mathfrak{a}(t) : \mathcal{V}(\Omega) \times \mathcal{V}(\Omega) \to \mathbb{C}$.
- $u(t)(x) = \mathrm{e}^{-tA} u_0(x)$ if $A(t) = A$ for all $t > 0$.

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Kato Square Root Problem (1961)

"We do not know whether or not $\mathcal{D}(A^{1/2}) = \mathcal{D}(A^{*1/2})$. This is perhaps not true in general. But the question is open even when A is regularly accretive. In this case it appears reasonable to suppose that both $\mathcal{D}(A^{1/2})$ and $\mathcal{D}(A^{*1/2})$ coincide with $\mathcal{D}(\mathfrak{a})$, where $\mathfrak{a}$ is the regular sesquilinear form which defines A."

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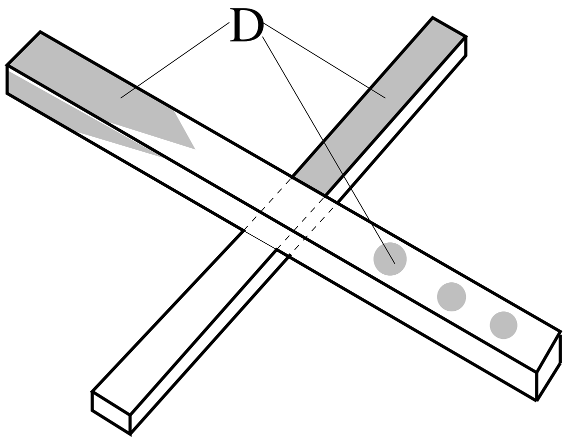
- Counterexamples by Lions 1962, M$^{\mathrm{c}}$Intosh 1982
- Specialize to divergence-form operators.

Setup

Let

- $\Omega \subseteq \mathbb{R}^d$ domain, $D \subseteq \partial\Omega$ closed, $\mu \in \mathrm{L}^\infty(\Omega)$
- $A \sim -\nabla \cdot \mu \nabla$ accretive operator on $\mathrm{L}^2(\Omega)$ associated with

$$\mathrm{W}^{1,2}_D(\Omega) \times \mathrm{W}^{1,2}_D(\Omega) \to \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \mu \nabla u \cdot \overline{\nabla v}.$$



- $A^{1/2}$ square root of A defined by e.g.

$$A^{1/2} u = \frac{1}{\pi} \int_0^\infty t^{-1/2} A (t + A)^{-1} \, \mathrm{d}t.$$

Kato conjecture

It holds $\mathcal{D}(A^{1/2}) = \mathrm{W}^{1,2}_D(\Omega)$ with equivalent norms.

Why do we care about the Kato conjecture?

Philosophy

- Elliptic non-regularity results $\mathcal{D}(A) \not\subseteq \mathrm{W}^{2,2}(\Omega)$.
- Kato Conjecture $\sim$ optimal Sobolev regularity for $A^{1/2}$.

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$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t)(x) + \nabla \cdot \mu(x) \nabla u(t,x) = 0 & (t > 0,\ x \in \mathbb{R}^d), \\ u(0,x) = u_0(x) \in \mathrm{W}^{1,2}(\mathbb{R}^d). & \end{cases}$$

- Solution $u(t,x) = \mathrm{e}^{-tA^{1/2}} u_0(x)$.
- Kato conjecture $\sim$ Rellich inequality "$\|\partial_t u|_{t=0}\|_2 \sim \|\nabla u_0\|_2$".

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Ex. 2: Maximal parabolic regularity (e.g. Haller-Dintelmann-Rehberg)

- In L^p-setting study parabolic equation

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- Goal: Transport Max. Reg. from $\mathrm{L}^p(\Omega)$ to $\mathrm{W}_D^{-1,p}(\Omega)$.

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Many further examples, e.g. Cauchy-Integral along Lipschitz curve, hyperbolic wave equations, ...

Positive answers

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Whole space $\Omega = \mathbb{R}^d$

- $d = 1$: Coifman - M^{c}Intosh - Meyer '82.
- $d \geq 2$: Auscher-Hofmann-Lacey-M^{c}Intosh-Tchamitchian '01, Axelsson-Keith-M^{c}Intosh '06.

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Bounded domains

- Ω Lipschitz, $D \in \{\emptyset, \partial\Omega\}$: Auscher-Tchamitchian '03, '01 ($p \neq 2$).
- Ω smooth, smooth $D \leftrightsquigarrow \partial\Omega \setminus D$ interface: Axelsson-Keith-M^{c}Intosh '06.
- Ω Lipschitz around $\overline{\partial\Omega \setminus D}$:
  Auscher-Badr-Haller-Dintelmann-Rehberg '12 ($p \neq 2$).

Kato for mixed boundary conditions

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Theorem (E.-Haller-Dintelmann-Tolksdorf '14)

Suppose

- $\Omega \subseteq \mathbb{R}^d$ *bounded* d*-Ahlfors regular domain,*
- $D \subseteq \partial\Omega$ *closed and* $(d-1)$*-Ahlfors regular,*
- Ω *Lipschitz around* $\overline{\partial\Omega \setminus D}$*.*

Then

$$\mathcal{D}(A^{1/2}) = \mathrm{W}^{1,2}_D(\Omega) \quad \textit{with} \quad \|A^{1/2}u\|_2 \sim \|\nabla u\|_2.$$

- First formulated by J.-L. Lions 1962.
- For rough (= L^∞) coefficients new even on Lipschitz domains.

Some ideas of the proof

1. First-order approach via perturbed Dirac operators á la AKM ’06, H^∞-functional calculus.

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Reduction-Theorem (E.-Haller-Dintelmann-Tolksdorf '14)

In essence, the following holds: If $\mathcal{D}\left((-\Delta_{\mathcal{V}})^s\right) \hookrightarrow \mathrm{H}^{2s,2}(\Omega)$ for some $s > \frac{1}{2}$, then $\mathcal{D}(A^{1/2}) = \mathrm{W}^{1,2}_D(\Omega)$.

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Extrapolate Kato for $-\Delta_{\mathcal{V}}$ $\Longrightarrow$ Kato property for general A

geometry, potential theory $\leftrightsquigarrow$ harmonic analysis

3. $\mathcal{D}((-\Delta_{\mathcal{V}})^{1/2}) = \mathrm{W}^{1,2}_D(\Omega)$ by self-adjointness. Extrapolate by Sneiberg's stability theorem and the following result.

Theorem (E.-Haller-Dintelmann-Tolksdorf '14)

Let $\theta \in (0,1)$ *and* $s_0, s_1 \in (\frac{1}{2}, \frac{3}{2})$. *Put* $s_\theta := (1-\theta)s_0 + \theta s_1$. *Then,*

- $$\mathrm{W}^{1,2}_D(\Omega) = \mathrm{H}^{1,2}_D(\Omega)$$
- $$\left[\mathrm{H}^{s_0,2}_D(\Omega), \mathrm{H}^{s_1,2}_D(\Omega)\right]_\theta = \mathrm{H}^{s_\theta,2}_D(\Omega).$$
- $$\left[\mathrm{L}^2(\Omega), \mathrm{H}^{1,2}_D(\Omega)\right]_\theta = \begin{cases} \mathrm{H}^{\theta,2}_D(\Omega), & \text{if } \theta > \frac{1}{2}, \\ \mathrm{H}^{\theta,2}(\Omega), & \text{if } \theta < \frac{1}{2}. \end{cases}$$

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4. In fact, $\mathcal{D}((-\Delta_{\mathcal{V}})^s) = \mathrm{H}_D^{2s,2}(\Omega)$ for $|\frac{1}{2} - s| < \varepsilon$.

Elliptic BVPs on cylindrical domains

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Elliptic mixed BVP

$$
\begin{aligned}
-\mathrm{div}_{t,x}\mu(x)\nabla_{t,x}U &= 0 && (\mathbb{R}^+ \times \Omega)\\
U &= 0 && (\mathbb{R}^+ \times D)\\
\partial_{\nu_\mu}U &= 0 && (\mathbb{R}^+ \times N)\\
\partial_{\nu_\mu}U = f &\in \mathrm{L}^2 && (\{0\} \times \Omega)
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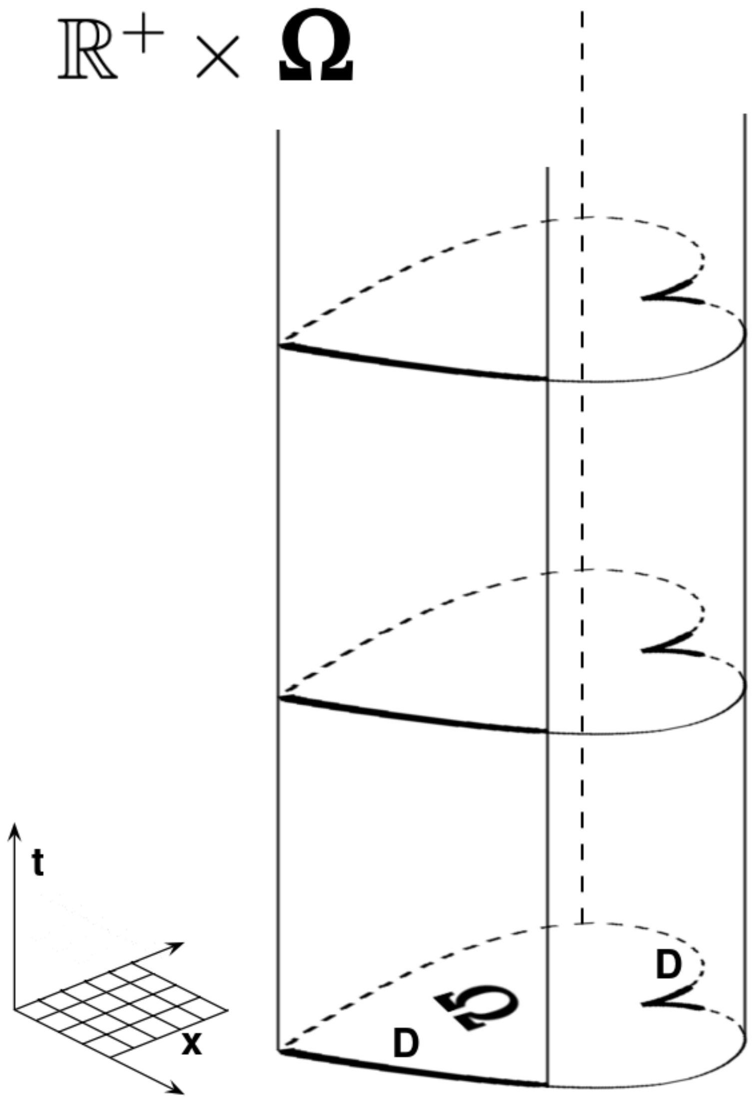

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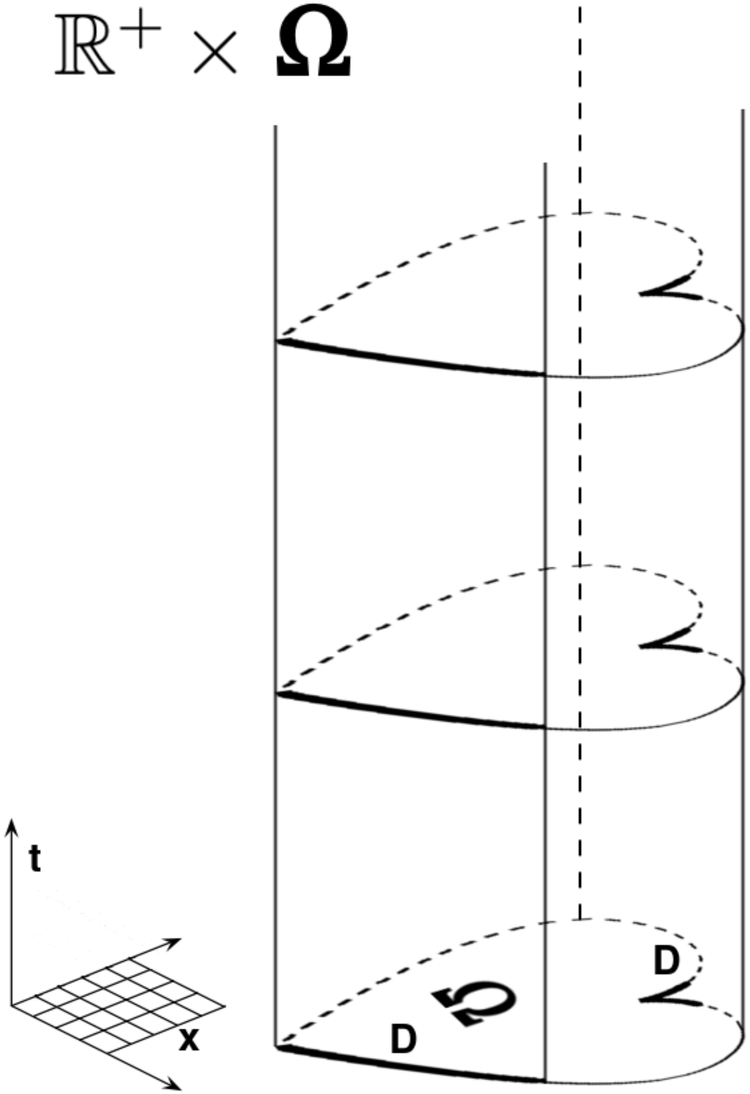
$$\Updownarrow \quad F \sim \begin{bmatrix} \partial_{\nu_\mu} U \\ \nabla_x U \end{bmatrix}$$

First order equation

$$\partial_t F + \underbrace{\begin{bmatrix} 0 & (-\nabla_{\mathcal{V}})^* \\ -\nabla_{\mathcal{V}} & 0 \end{bmatrix}}_{\mathbb{D}} \mathbb{B} F = 0 \quad (t > 0)$$

$$F(0)_\perp = f$$

$\mathbb{R}^+ \times \mathbf{\Omega}$



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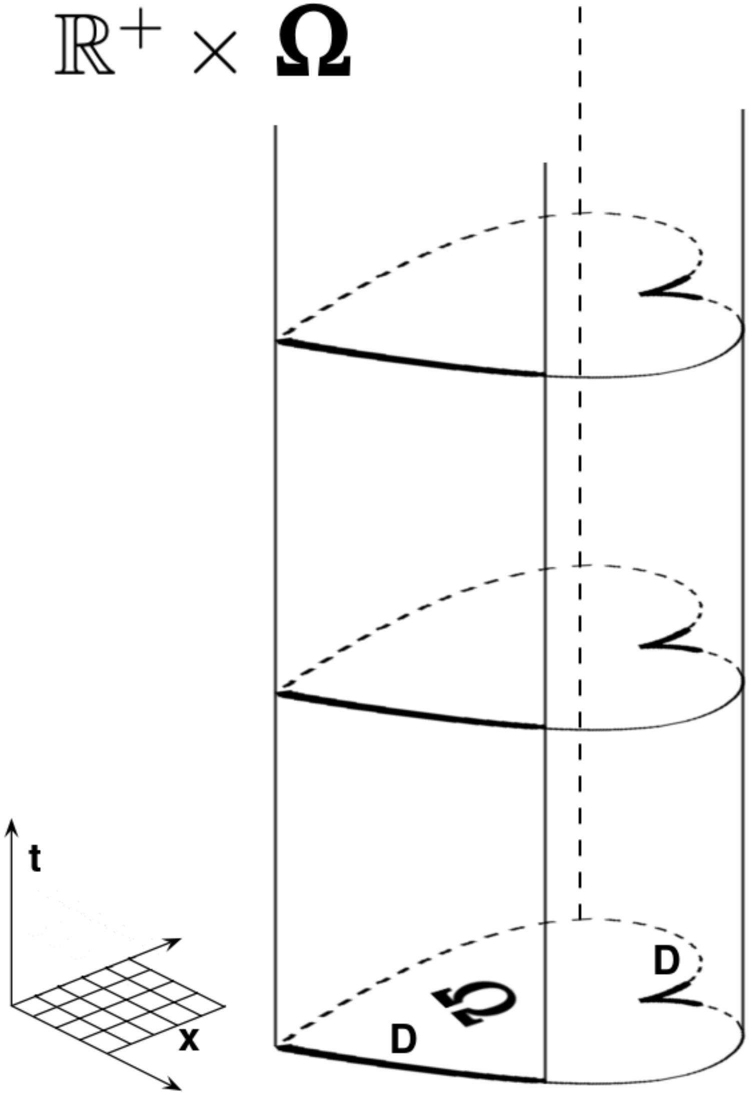

$\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\Omega))$ setting

Semigroup solutions via $\mathbb{DB}$-formalism

$\mathbb{DB}$ has bounded H^∞-calculus on $\mathcal{H} = \overline{\mathcal{R}(\mathbb{DB})}$ (Kato Technology).

Theorem (Auscher-E. '14)

1. *For every $F(0) \in \mathcal{H}^+ := \mathbf{1}_{\mathbb{C}^+}(\mathbb{DB})\mathcal{H}$ a solution to the first-order system is*

$$F(t) = \mathrm{e}^{-t\mathbb{DB}} F(0) \qquad (t \geq 0).$$

Via $F \sim \begin{bmatrix} \partial_{\nu_\mu} U \\ \nabla_x U \end{bmatrix}$ these functions are in one-to-one correspondence with weak solutions U such that

$$\widetilde{N}_*(|\nabla_{t,x} U|) \in \mathrm{L}^2(\mathbb{R}^+ \times \Omega).$$

2. *If μ is either block-diagonal or Hermitean, then for each $f \in \mathrm{L}^2(\Omega)$ there exists a unique such solution u.*

Thank you for your attention!

Kato

Square

Root

Problem

Literature

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