Square Roots of Elliptic Systems

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October 3, 2012

What is a Square Root of an Operator?

Definition

A square root of $A : \mathrm{D}(A) \subseteq X \to X$ is an operator $A^{\frac{1}{2}}$ such that

$$A^{\frac{1}{2}} A^{\frac{1}{2}} x = Ax, \quad \text{for all } x \in \mathrm{D}(A).$$

Example

- Important-Constants-Matrix on $X = \mathbb{C}^2$

$$A = \begin{pmatrix} \pi & 0 \\ 0 & \mathrm{e} \end{pmatrix}, \ \mathrm{D}(A) = \mathbb{C}^2, \quad A^{\frac{1}{2}} = \begin{pmatrix} \sqrt{\pi} & 0 \\ 0 & \sqrt{\mathrm{e}} \end{pmatrix}, \ \mathrm{D}(A^{\frac{1}{2}}) = \mathbb{C}^2.$$

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$$-\Delta u = \mathcal{F}^{-1}(\xi \mapsto |\xi|^2 \mathcal{F}u(\xi)), \ \mathrm{D}(-\Delta) = H^2(\mathbb{R}^d)$$
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Outline

1. Elliptic systems and their square root
2. Maximal parabolic regularity on distribution spaces

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Guideline

"In non-smooth situations, the square root behaves much better than the operator itself."

(Saying in applied analysis)

Setup

Let

- A an $N \times N$ system

$$Au = \begin{pmatrix} -\sum_{\alpha,\beta=1}^{d} \sum_{k=1}^{N} \partial_\alpha (a_{\alpha,\beta}^{1,k} \partial_\beta u_k) \\ \vdots \\ -\sum_{\alpha,\beta=1}^{d} \sum_{k=1}^{N} \partial_\alpha (a_{\alpha,\beta}^{N,k} \partial_\beta u_k) \end{pmatrix} = -\sum_{\alpha,\beta=1}^{d} \partial_\alpha (a_{\alpha,\beta} \partial_\beta u)$$

  of 2nd-order differential equations in divergence form.
- $a_{\alpha,\beta}^{m,k} \in L^\infty(\mathbb{R}^d; \mathbb{C})$, $a_{\alpha,\beta} = (a_{\alpha,\beta}^{m,k})_{1 \leq m,k \leq N} \in L^\infty(\mathbb{R}^d; \mathbb{C}^{N \times N})$.

We will

- realize A as an m-accretive operator on $L^2(\mathbb{R}^d; \mathbb{C}^N)$.
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Properties of A

Theorem (Kato)

$-A$ generates a bounded analytic contraction C_0-semigroup $(\mathrm{e}^{-tA})_{t\geq 0}$.

Consequences

- A has a unique m-accretive square root $A^{\frac{1}{2}}$ and

$$A^{\frac{1}{2}}u = \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t}\mathrm{e}^{-tA}Au \, \frac{\mathrm{d}t}{t}, \quad u \in \mathrm{D}(A).$$

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What can be said about $\mathrm{D}(A^{\frac{1}{2}})$?

The Kato Square Root Problem

Theorem (Auscher, Hofmann, Lacey, McIntosh,Tchamitchian '02)

Kato was right, i.e. $\mathrm{D}(A^{\frac{1}{2}}) = \mathrm{D}(a) = W^{1,2}(\mathbb{R}^d; \mathbb{C}^N)$ *and moreover,*

$$\|A^{\frac{1}{2}}u\|_{L^2} \sim \|\nabla u\|_{L^2}, \quad u \in \mathrm{D}(A^{\frac{1}{2}}).$$

Note that

- in general $\mathrm{D}(A) \neq W^{2,2}(\mathbb{R}^d; \mathbb{C}^N)$, i.e. $A^{\frac{1}{2}}$ has the expected regularity properties and A has not.
- the Riesz transform

$$\nabla A^{-\frac{1}{2}} : L^2(\mathbb{R}^d; \mathbb{C}^N) \to L^2(\mathbb{R}^d; \mathbb{C}^{dN})$$

  is bounded.

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What happens for integrability exponents $p \neq 2$?

The Riesz Transform

Observe that

- for $u \in \mathrm{D}(A) \cap \mathrm{Rg}(A)$

$$\nabla A^{-\frac{1}{2}} u = \nabla A^{-\frac{1}{2}} A v = \nabla A^{\frac{1}{2}} v = \nabla \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t} \mathrm{e}^{-tA} A v \, \frac{\mathrm{d}t}{t}$$
$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t} \nabla \mathrm{e}^{-tA} u \, \frac{\mathrm{d}t}{t}$$

- $(\sqrt{t}\nabla \mathrm{e}^{-tA})_{t>0} \subseteq \mathcal{L}(L^2(\mathbb{R}^d;\mathbb{C}^N), L^2(\mathbb{R}^d;\mathbb{C}^{dN}))$ is bounded, thus the above integral singular.

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Obtain boundedness on L^p-spaces

- via Calderón-Zygmund-Type results due to Blunck-Kunstmann ($p < 2$) and Auscher, Coulhon, Duong, Hofmann ($p > 2$)
- if $(\sqrt{t}\nabla \mathrm{e}^{-tA})_{t>0}$ behaves “sufficiently regular” on L^p.

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The Main Result

Theorem (Auscher '07)

The sets

$$\mathcal{I}(A) := \Big\{ p \in (1,\infty) \mid \nabla A^{-\frac{1}{2}} \text{ is } L^p\text{-bounded} \Big\},$$
$$\mathcal{N}(A) := \Big\{ p \in [1,\infty] \mid (\sqrt{t}\nabla \mathrm{e}^{-tA})_{t>0} \text{ is uniformly } L^p\text{-bounded} \Big\}$$

share the same interior. Moreover,

$$\|A^{\frac{1}{2}}u\|_{L^p} \sim \|\nabla u\|_{L^p}, \quad u \in C_c^\infty(\mathbb{R}^d;\mathbb{C}^N)$$

if, and only if $p \in \mathcal{I}(A)$.

Note that

- this is the L^p-analogon of the solution to the Square Root Problem.
- $[\frac{2d}{d+2}, 2] \Subset \mathcal{I}(A)$ but $\mathcal{I}(A) \neq (1,\infty)$ in general.

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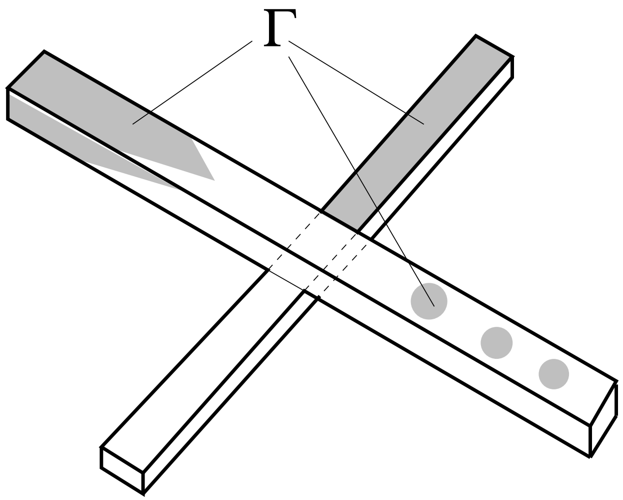
Elliptic Differential Equations on Rough Domains

Let

- Ω a bounded Lipschitz domain
- Γ an open subset of $\partial\Omega$
- $\mu \in \mathbb{R}^{d \times d}$ symmetric
- $J = (T_0, T)$ bounded.



Consider

$$(PB) \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t)(x) - (\nabla \cdot \mu \nabla) u(t)(x) & = f(t)(x), \quad t \in J, x \in \Omega \\ u(t)(x) & = 0, \quad t \in J, x \in \partial\Omega \setminus \Gamma \\ \nu^t \mu \nabla u(t)(x) & = 0, \quad t \in J, x \in \Gamma \\ u(T_0)(x) & = 0, \quad x \in \Omega. \end{cases}$$

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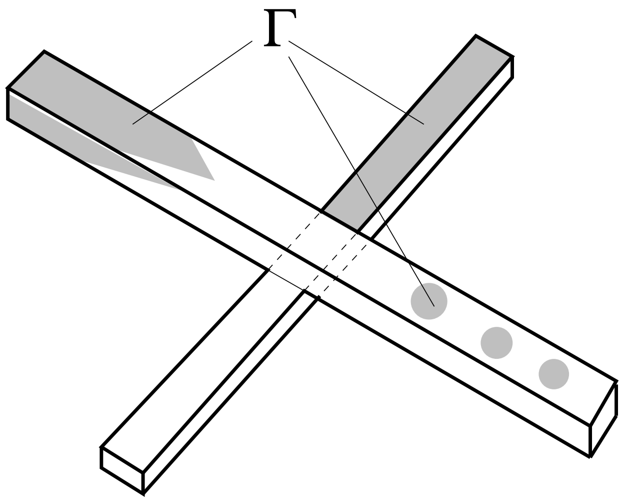

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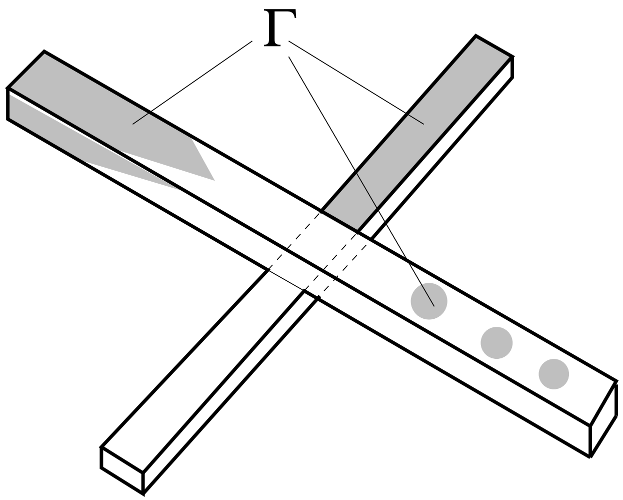

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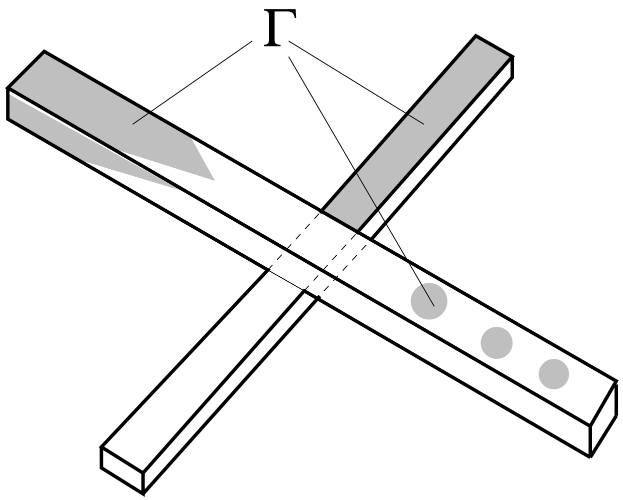

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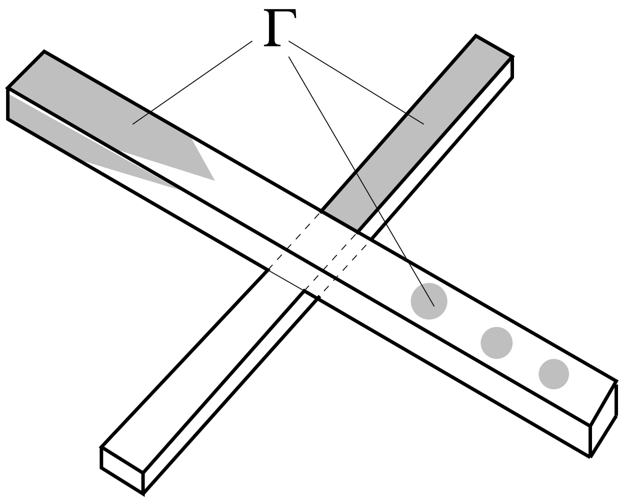

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- Incorporate spatial boundary conditions into definition of $A_{W^{1,p}_\Gamma} := -(\nabla\cdot\mu\nabla):\ \mathrm{D}(A_{W^{1,p}_\Gamma}) \subseteq W^{1,p}_\Gamma(\Omega) \to L^p(\Omega)$.

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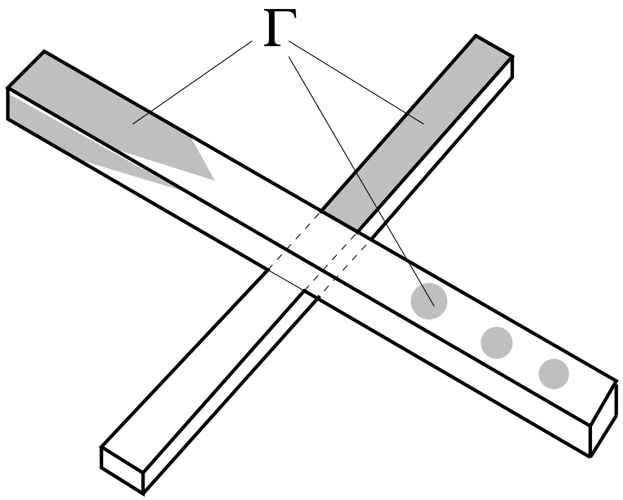

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Maximal Parabolic Regularity on L^p-spaces

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Parabolic solution operator

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} + A_{W_\Gamma^{1,p}}\Big)^{-1} \colon L^s(J; L^p(\Omega)) \to W^{1,s}(J; L^p(\Omega)) \cap L^s(J; \mathrm{D}(A_{W_\Gamma^{1,p}}))$$

- for $p, s \in (1, \infty)$ well-defined and continuous.

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$$(PB) \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) + A_{W_\Gamma^{1,p}} u(t) & = f(t), \quad t \in J \\ u(T_0) & = 0 \end{cases}$$

Parabolic solution operator

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + A_{W_\Gamma^{1,p}}\right)^{-1} : L^s(J; L^p(\Omega)) \to W^{1,s}(J; L^p(\Omega)) \cap L^s(J; \mathrm{D}(A_{W_\Gamma^{1,p}}))$$

- for $p, s \in (1, \infty)$ well-defined and continuous.

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- in real-world applications: $f \in L^s(J; W^{-1,p}_\Gamma(\Omega))$, e.g. to model electrical surface densities.

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Can we carry over max. reg. from L^p- to $W_\Gamma^{-1,p}$-spaces?

Maximal Parabolic Regularity on Distribution Spaces

Idea

- Find isomorphism $B : L^p(\Omega) \to W_\Gamma^{-1,p}(\Omega)$ that "suits" $(\frac{\mathrm{d}}{\mathrm{d}t} + A_{W_\Gamma^{1,p}})^{-1}$
- Carry over maximal regularity via $B(\frac{\mathrm{d}}{\mathrm{d}t} + A_{W_\Gamma^{1,p}})^{-1}B^{-1}$

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Upcoming project: no self-adjointness, complex coefficients, $N > 1$

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