Rational Approximation of Semigroups without Scaling and Squaring

Project Orchid

Part II: Proof of the Main Result

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Rational Approximation with Scaling and Squaring

Theorem (Hersh-Kato, Brenner-Thomée 1979)

Let r be an $\mathcal{A}$-stable rational approximation of order m, $(\mathrm{e}^{tA})_{t\geq 0}$ a C_0-semigroup of type $(M, 0)$. Then

$$\left\| r\left(\frac{t}{n}A\right)^n x - \mathrm{e}^{tA}x \right\| \leq C(r)M\frac{t^{m+1}}{n^m}\left\|A^{m+1}x\right\|$$

for $n \in \mathbb{N}$, $t \geq 0$, $x \in \mathrm{D}(A^{m+1})$.

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- $r\left(\frac{t}{n}A\right)^{n}$ contains higher powers of resolvents of A if $n > 1$.
- Restricting to $n = 1$ requires information on the size of C.
- "Compute" $C(r)$ for $n = 1$, refine by increasing m.

Approximation Without Scaling and Squaring

Let

- $r = \frac{P}{Q}$ the subdiagonal Padé approximation of order $m = 2p + 1$, i.e. $\deg(P) = p$, $\deg(Q) = p + 1$

$$|r(z) - \mathrm{e}^z| \leq C|z|^{m+1} \quad \text{for } |z| \leq \delta.$$

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- $\lambda_j \in \mathbb{C}_+$ the pairwise distinct roots of Q. Hence

$$r(z) = b_1\Big(\lambda_1 - z\Big)^{-1} + \ldots + b_{p+1}\Big(\lambda_{p+1} - z\Big)^{-1}.$$

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$$r(tA) = b_1\left(\lambda_1 - tA\right)^{-1} + \ldots + b_{p+1}\left(\lambda_{p+1} - tA\right)^{-1}.$$

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Establish

$$\left\|r(tA)x - \mathrm{e}^{tA}x\right\| \leq C(m)Mt^{m+1}\left\|A^{m+1}x\right\|, \quad C(m) \overset{m\to\infty}{\longrightarrow} 0 \text{ rapidly.}$$

Properties of Subdiagonal Padé Approximations

Lemma

Let $r = \frac{P}{Q}$ be the subdiagonal Padé approximation of order $m = 2p + 1$. Then

$$P(z) = \sum_{j=0}^{p} \frac{(m-j)!p!}{m!j!(p-j)!} z^j = \sum_{j=0}^{m} \gamma^{(m-j)}(1) z^j,$$

$$Q(z) = \sum_{j=0}^{p+1} \frac{(m-j)!(p+1)!}{m!j!(p+1-j)!} z^j = \sum_{j=0}^{m} \gamma^{(m-j)}(0) z^j,$$

where

$$\gamma(z) = \frac{(-1)^{p+1}}{m!} z^p (1-z)^{p+1}.$$

The Main Result

Theorem (Neubrander, Özer, Sandmaier 2012)

Let r_m be the subdiagonal Padé approximation of order $m = 2p + 1$ with singularities $\lambda_{1,p}, \ldots, \lambda_{p+1,p}$. Then

$$\left\| r_m(tA)x - \mathrm{e}^{tA}x \right\| \leq \frac{M(p+1)!}{(2p+2)!\,\mathrm{Re}(\lambda_{1,p}) \cdots \mathrm{Re}(\lambda_{p+1,p})} \|A^{m+1}x\|$$

for $t \geq 0$, $x \in \mathrm{D}(A^{m+1})$. Moreover,

$$\prod_{j=1}^{p+1} (\mathrm{Re}(\lambda_{j,p}))^{-1} \leq \begin{cases} p^{-p-1} & 1 \leq p \leq 15 \\ (p-1)^{-p-1} & 16 \leq p \leq 28 \end{cases}.$$

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- Similar estimates if $\deg(Q) - 4 \leq \deg P \leq \deg(Q)$ for $x \in \mathrm{D}(A^k)$, $\deg(Q) \leq k \leq m$.
- $r_m(tA)x \overset{m\to\infty}{\longrightarrow} \mathrm{e}^{tA}x$ locally uniformly in t for all $x \in \mathrm{D}(A)$?