Polynomial decay of bounded C_0-semigroups on Hilbert spaces

Project M

Moritz Egert, Stefan Kunkel, Sándor Kelemen

Coordinator: Roland Schnaubelt

Blaubeuren, 10 June 2013

Assumptions

- X is a complex Banach space,
- A generates C_0-semigroup $(T(t))_{t\geq 0}$ on X of type $(M, 0)$.

Problem

- Study long-time behavior of solutions to

$$\dot{u}(t) - Au(t) = 0 \quad (t > 0)$$
$$u(0) = u_0.$$

Assumptions

- X is a complex Banach space,
- A generates C_0-semigroup $(T(t))_{t\geq 0}$ on X of type $(M,0)$.

Problem

- Study long-time behavior of solutions to

$$\dot{u}(t) - Au(t) = 0 \quad (t > 0)$$
$$u(0) = u_0.$$

- When does $m : \mathbb{R}_+ \to \mathbb{R}_+$ with $m(t) \overset{t\to\infty}{\longrightarrow} 0$ exist s.t.

$$\|T(t)u_0\| \leq m(t)\|u_0\| \quad (u_0 \in X,\ t > 0)\ ?$$

Assumptions

- X is a complex Banach space,
- A generates C_0-semigroup $(T(t))_{t\geq 0}$ on X of type $(M, 0)$.

Problem

- Study long-time behavior of solutions to

$$\dot{u}(t) - Au(t) = 0 \quad (t > 0)$$
$$u(0) = u_0.$$

- When does $m : \mathbb{R}_+ \to \mathbb{R}_+$ with $m(t) \stackrel{t\to\infty}{\longrightarrow} 0$ exist s.t.

$$\|T(t)u_0\| \leq m(t)\|u_0\| \quad (u_0 \in X,\ t > 0)\ ?$$

Answer

- Easy.

Assumptions

- X is a complex Banach space,
- A generates C_0-semigroup $(T(t))_{t\geq 0}$ on X of type $(M, 0)$.

Problem

- Study long-time behavior of solutions to

$$\dot{u}(t) - Au(t) = 0 \quad (t > 0)$$
$$u(0) = u_0.$$

- When does $m : \mathbb{R}_+ \to \mathbb{R}_+$ with $m(t) \stackrel{t\to\infty}{\longrightarrow} 0$ exist s.t.

$$\|T(t)u_0\| \leq m(t)\|u_0\| \quad (u_0 \in X,\ t > 0)\ ?$$

Answer

- Easy. If and only if $\omega_0(T(\cdot)) < 0$. In this case $m(t) \leq C\mathrm{e}^{-\omega t}$.

Assumptions

- X is a complex Banach space,
- A generates C_0-semigroup $(T(t))_{t\geq 0}$ on X of type $(M, 0)$.

Problem

- Study long-time behavior of solutions to

$$\dot{u}(t) - Au(t) = 0 \quad (t > 0)$$
$$u(0) = u_0.$$

- When does $m : \mathbb{R}_+ \to \mathbb{R}_+$ with $m(t) \stackrel{t\to\infty}{\longrightarrow} 0$ exist s.t.

$$\|T(t)u_0\| \leq m(t)\|u_0\|_{D(A)} \quad (u_0 \in D(A),\ t > 0)\ ?$$

Answer

- Easy. If and only if $\omega_0(T(\cdot)) < 0$. In this case $m(t) \leq C\mathrm{e}^{-\omega t}$.
- New phenomena if one restricts to classical solutions.

Assumptions

- X is a complex Banach space,
- A generates C_0-semigroup $(T(t))_{t\geq 0}$ on X of type $(M, 0)$.

Problem

- Study long-time behavior of solutions to

$$\dot{u}(t) - Au(t) = 0 \quad (t > 0)$$
$$u(0) = u_0.$$

- When does $m : \mathbb{R}_+ \to \mathbb{R}_+$ with $m(t) \stackrel{t\to\infty}{\longrightarrow} 0$ exist s.t.

$$\|T(t)u_0\| \leq Ct^{-\frac{1}{\alpha}}\|u_0\|_{\mathrm{D}(A)} \quad (u_0 \in \mathrm{D}(A),\ t > 0)\ ?$$

Answer

- Easy. If and only if $\omega_0(T(\cdot)) < 0$. In this case $m(t) \leq C\mathrm{e}^{-\omega t}$.
- New phenomena if one restricts to classical solutions.

Polynomial stability

Definition

Let $\alpha > 0$. Call $(T(t))_{t\geq 0}$ *polynomially stable* with rate $t^{-\frac{1}{\alpha}}$ if

$$\|T(t)u_0\| \leq Ct^{-\frac{1}{\alpha}}\|u_0\|_{\mathrm{D}(A)} \qquad (t \geq 1,\ u_0 \in \mathrm{D}(A))$$

for some $C > 0$ or, equivalently,

$$\|T(t)(I-A)^{-1}\| \leq Ct^{-\frac{1}{\alpha}} \qquad (t \geq 1).$$

Polynomial stability

Definition

Let $\alpha > 0$. Call $(T(t))_{t\geq 0}$ *polynomially stable* with rate $t^{-\frac{1}{\alpha}}$ if

$$\|T(t)u_0\| \leq Ct^{-\frac{1}{\alpha}}\|u_0\|_{\mathrm{D}(A)} \qquad (t \geq 1,\ u_0 \in \mathrm{D}(A))$$

for some $C > 0$ or, equivalently,

$$\|T(t)(I-A)^{-1}\| \leq Ct^{-\frac{1}{\alpha}} \qquad (t \geq 1).$$

Observation

If $(T(t))_{t\geq 0}$ is polynomially stable, then it is strongly stable, i.e.

$$T(t)u_0 \overset{t\to\infty}{\longrightarrow} 0 \qquad (u_0 \in X)$$

but without uniform rate in general.

Polynomial stability

Definition

Let $\alpha > 0$. Call $(T(t))_{t\geq 0}$ *polynomially stable* with rate $t^{-\frac{1}{\alpha}}$ if

$$\|T(t)u_0\| \leq Ct^{-\frac{1}{\alpha}}\|u_0\|_{\mathrm{D}(A)} \qquad (t \geq 1,\ u_0 \in \mathrm{D}(A))$$

for some $C > 0$ or, equivalently,

$$\|T(t)(I - A)^{-1}\| \leq Ct^{-\frac{1}{\alpha}} \qquad (t \geq 1).$$

Observation

If $(T(t))_{t\geq 0}$ is polynomially stable, then it is strongly stable, i.e.

$$T(t)u_0 \overset{t\to\infty}{\longrightarrow} 0 \qquad (u_0 \in X)$$

but without uniform rate in general.

The general philosophy

Semigroup decay $\leftrightsquigarrow$ resolvent growth on $\mathrm{i}\mathbb{R}$

Theorem (Gearhart)

If X is a Hilbert space then

$\Updownarrow$

1. $(T(t))_{t\geq 0}$ *is exponentially stable.*
2. $i\mathbb{R} \subseteq \rho(A)$ *and* $\|(i\tau - A)^{-1}\| \leq C \quad (\tau \in \mathbb{R})$.

The general philosophy

Semigroup decay $\leftrightsquigarrow$ resolvent growth on $\mathrm{i}\mathbb{R}$

Theorem (Gearhart)

If X is a Hilbert space then

$\Updownarrow$

1. $(T(t))_{t\geq 0}$ *is exponentially stable.*
2. $i\mathbb{R} \subseteq \rho(A)$ *and* $\|(i\tau - A)^{-1}\| \leq C \quad (\tau \in \mathbb{R})$.

The general philosophy

Semigroup decay $\leftrightsquigarrow$ resolvent growth on $\mathrm{i}\mathbb{R}$

Theorem (Gearhart)

If X is a Hilbert space then

$\Updownarrow$

1. $(T(t))_{t\geq 0}$ *is exponentially stable.*
2. $i\mathbb{R} \subseteq \rho(A)$ *and* $\|(i\tau - A)^{-1}\| \leq C \quad (\tau \in \mathbb{R})$.

Theorem (Batty-Duyckaerts '08)

For $\alpha > 0$ it holds

$\Downarrow$

1. $(T(t))_{t\geq 0}$ *is polynomially stable with rate* $t^{-\frac{1}{\alpha}}$.
2. $i\mathbb{R} \subseteq \rho(A)$ *and* $\left\|(\mathrm{i}\tau - A)^{-1}\right\| \leq C(1+|\tau|)^{\alpha} \quad (\tau \in \mathbb{R})$.

The general philosophy

Semigroup decay $\leftrightsquigarrow$ resolvent growth on $\mathrm{i}\mathbb{R}$

Theorem (Gearhart)

If X is a Hilbert space then

$\Updownarrow$

1. $(T(t))_{t\geq 0}$ *is exponentially stable.*
2. $i\mathbb{R} \subseteq \rho(A)$ *and* $\|(i\tau - A)^{-1}\| \leq C \quad (\tau \in \mathbb{R})$.

Theorem (Batty-Duyckaerts '08)

For $\alpha > 0$ it holds

$\Downarrow$

1. $(T(t))_{t\geq 0}$ *is polynomially stable with rate* $t^{-\frac{1}{\alpha}}$.
2. $i\mathbb{R} \subseteq \rho(A)$ *and* $\left\|(\mathrm{i}\tau - A)^{-1}\right\| \leq C(1 + |\tau|)^{\alpha} \quad (\tau \in \mathbb{R})$.

A necessary spectral condition

Lemma (Bátkai-Engel-Prüss-Schnaubelt '06)

Assume $\mathrm{i}\mathbb{R} \cap \sigma(A) = \emptyset$ *and fix* $\delta > 0$ *s.t.* $[-\delta, 0] \subseteq \rho(A)$. *If*

$$\|(\mathrm{i}\tau - A)^{-1}\| \leq C(1 + |\tau|)^{\alpha} \quad (\tau \in \mathbb{R})$$

for $C, \alpha > 0$, *then*

$$|\mathrm{Im}\,\lambda| \geq C'(-\mathrm{Re}\,\lambda)^{-\frac{1}{\alpha}} \quad (\lambda \in \sigma(A) \cap \{\mathrm{Re}\,z \geq -\delta\}).$$

Remarks

- Purely spectral condition not sufficient in general.
- Exception: e.g. X Hilbert space, A normal, and $\sigma(A) \subseteq \mathbb{C}_-$.

Lemma (Bátkai-Engel-Prüss-Schnaubelt '06)

If A is invertible, then for $\alpha > 0$:

$\Updownarrow$

1. $\|T(t)(1-A)^{-1}\| \le Ct^{-\frac{1}{\alpha}} \quad (t \ge 1)$,
2. $\|T(t)(-A)^{-1}\| \le Ct^{-\frac{1}{\alpha}} \quad (t \ge 1)$.
3. $\|T(t)(-A)^{-\alpha}\| \le Ct^{-1} \qquad (t \ge 1)$.

Note

- $(-A)^{-\alpha} \in \mathcal{L}(X)$ is a "fractional power" of $-A$ defined by

$$(-A)^{-\alpha} := \frac{1}{2\pi \mathrm{i}} \int_\gamma (-z)^\alpha (z-A)^{-1} \, \mathrm{d}z.$$

- Specialize to decay rate t^{-1}, pay the price to deal with $u_0 \in \mathrm{Rg}((-A)^{-\alpha})$.

Lemma (Bátkai-Engel-Prüss-Schnaubelt '06)

If A is invertible, then for $\alpha > 0$:

$\Updownarrow$

1. $\|T(t)(1-A)^{-1}\| \leq Ct^{-\frac{1}{\alpha}} \quad (t \geq 1)$,
2. $\|T(t)(-A)^{-1}\| \leq Ct^{-\frac{1}{\alpha}} \quad (t \geq 1)$.
3. $\|T(t)(-A)^{-\alpha}\| \leq Ct^{-1} \qquad (t \geq 1)$.

Note

- $(-A)^{-\alpha} \in \mathcal{L}(X)$ is a "fractional power" of $-A$ defined by

$$(-A)^{-\alpha} := \frac{1}{2\pi \mathrm{i}} \int_{\gamma} (-z)^{\alpha} (z - A)^{-1} \, \mathrm{d}z.$$

- Specialize to decay rate t^{-1}, pay the price to deal with $u_0 \in \mathrm{Rg}((-A)^{-\alpha})$.

Lemma (Bátkai-Engel-Prüss-Schnaubelt '06)

If A is invertible, then for $\alpha > 0$:

$\Updownarrow$

1. $\|T(t)(1-A)^{-1}\| \leq Ct^{-\frac{1}{\alpha}} \quad (t \geq 1)$,
2. $\|T(t)(-A)^{-1}\| \leq Ct^{-\frac{1}{\alpha}} \quad (t \geq 1)$.
3. $\|T(t)(-A)^{-\alpha}\| \leq Ct^{-1} \qquad (t \geq 1)$.

Note

- $(-A)^{-\alpha} \in \mathcal{L}(X)$ is a "fractional power" of $-A$ defined by

$$(-A)^{-\alpha} := \frac{1}{2\pi \mathrm{i}} \int_\gamma (-z)^\alpha (z-A)^{-1} \, \mathrm{d}z.$$

- Specialize to decay rate t^{-1}, pay the price to deal with $u_0 \in \mathrm{Rg}((-A)^{-\alpha})$.