Mixed Boundary Value Problems on Cylindrical Domains

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December 2, 2015, WIAS Berlin

(based on joint work with P. Auscher)

The Problem

Elliptic divergence-form equation

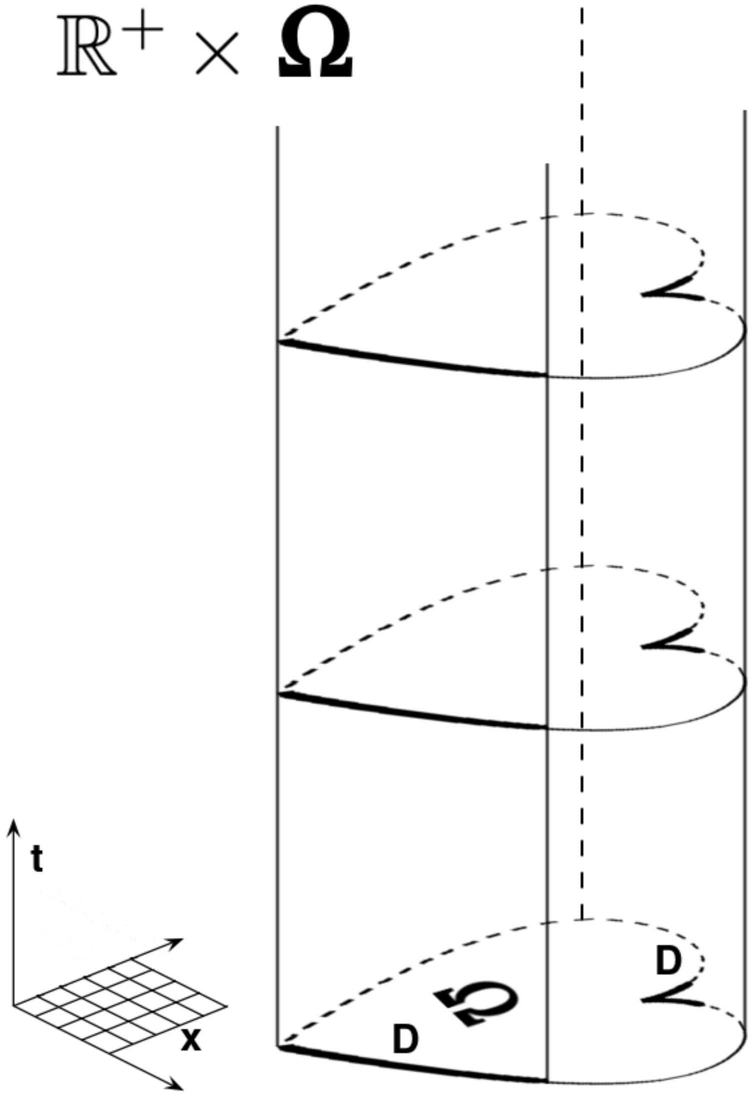
$$
\begin{aligned}
-\mathrm{div}_{t,x} A(t,x) \nabla_{t,x} u &= 0 && (\mathbb{R}^+ \times \Omega) \\
u &= 0 && (\mathbb{R}^+ \times \mathscr{D}) \\
\nu \cdot A \nabla_{t,x} u &= 0 && (\mathbb{R}^+ \times \mathscr{N})
\end{aligned}
$$

$+$ Boundary data in $L^2(\Omega)$ for either

u or $\nu \cdot A \nabla_{t,x} u$ or $\nabla_x u$

General assumptions

- $A : \mathbb{R}^+ \times \Omega \to \mathcal{L}(\mathbb{C}^{(d+1)})$ bdd. and elliptic
- $\Omega \subseteq \mathbb{R}^d$ bounded domain, $\mathscr{D} \subseteq \partial\Omega$ closed
- Ω Lipschitz near $\mathscr{N}$, Ahlfors-regular near $\mathscr{D}$



Second-order equation

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Weak solutions

- $u \in L^2_{loc}(W^{1,2}_{\mathscr{D}}) \cap W^{1,2}_{loc}(L^2)$
- $\displaystyle\int_0^\infty \int_\Omega A\nabla_{t,x} u \cdot \nabla_{t,x} \overline{v} \; dx \, dt = 0$

  for all $v \in \mathrm{C}^\infty_c(W^{1,2}_{\mathscr{D}})$

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$$\Updownarrow \qquad f \sim \begin{bmatrix} (A\nabla_{t,x} u)_\perp \\ \nabla_x u \end{bmatrix}$$

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First-order system

$$
\partial_t f_t + \underbrace{\begin{bmatrix} 0 & (-\nabla_{\mathscr{D}})^* \\ -\nabla_{\mathscr{D}} & 0 \end{bmatrix}}_{=:D} B_t f_t = 0
$$

Non-autonomous "evolution equation"

Weak solutions

- $f \in L^2_{loc}(\overline{\mathcal{R}(D)})$
- $\int_0^\infty (B_t f_t \mid D g_t)_{L^2(\Omega)} \, dt = 0$

  for all $g \in \mathrm{C}_c^\infty(\mathcal{D}(D))$

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$$\Updownarrow \qquad f \sim \begin{bmatrix} (A\nabla_{t,x} u)_\perp \\ \nabla_x u \end{bmatrix}$$ one-to-one correspondence

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Solving the first-order system: t-independent case

- Suppose A independent of t, i.e., $A(t,x) = A_0(x)$.
- "Evolution equation"

$$\partial_t f_t + DB_0 f_t = 0$$

Theorem (E.–Haller-Dintelmann–Tolksdorf 2013)

The operator DB_0 is bisectorial on $L^2(\Omega)^{1+d}$ and has a bounded H^∞-calculus on $\mathcal{H} := \overline{\mathcal{R}(D)}$.

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Consequences

- $E^\pm := \mathbf{1}_{\mathbb{C}^\pm}(DB_0) \in \mathcal{L}(\mathcal{H})$ define spectral projections
- For each $h^+ \in E_0^\pm \mathcal{H}$ we can construct solutions to first-order system:

$$f_t = e^{-tDB_0} h^+ \qquad \text{and} \qquad g_t = DB_0 e^{-tDB_0} h^+.$$

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- Is any "reasonable" solution of this type?

Interior controls

$$\widetilde{N}_*(f)(x) = \left(\sup_{t>0} \fint_{\frac{t}{2}}^{2t} \fint_{B(x,t)\cap\Omega} |f(s,y)|^2 \, dy \, ds \right)^{1/2}$$

$$S(f)(x) = \left(\int_0^\infty |f(x,t)|^2 \, \frac{dt}{t} \right)^{1/2}$$

Theorem

1. *f weak sol. with $\widetilde{N}_*(f) \in L^2(\Omega) \iff f = e^{-tDB_0} h^+$ for some $h^+ \in E_0^+ \mathcal{H}$.*
2. *f weak sol. with $S(f) \in L^2(\Omega) \iff f = DB_0 e^{-tDB_0} h^+$ for some $h^+ \in E_0^+ \mathcal{H}$. In this case*

$$u = (-B_0 e^{-tDB_0} h^+)_\perp$$

is weak sol. with $\widetilde{N}_*(u) \in L^2(\Omega)$ and $S(\nabla_{t,x} u) \in L^2(\Omega)$.

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Needs reverse Hölder estimate for $2-\varepsilon \le q \le p \le 2+\varepsilon$:

$$\left(\fint_{\frac{t}{2}}^{2t} \fint_{B(x,t)\cap\Omega} |f(s,y)|^p \, dy \, ds \right)^{1/p} \lesssim \left(\fint_{\frac{t}{2}}^{2t} \fint_{B(x,t)\cap\Omega} |f(s,y)|^q \, dy \, ds \right)^{1/q}$$

BVPs for t-independent equations

Showcase: Neumann problem

$$\begin{aligned} -\mathrm{div}_{t,x} A(t,x)\nabla_{t,x} u &= 0 \quad (\mathbb{R}^+ \times \Omega) \\ u &= 0 \quad (\mathbb{R}^+ \times \mathscr{D}) \\ \nu \cdot A\nabla_{t,x} u &= 0 \quad (\mathbb{R}^+ \times \mathscr{N}) \\ \lim_{t\to 0} (A\nabla_{t,x} u)_\perp &= \varphi \end{aligned}$$

Well-posedness

$\forall\ \varphi \in \mathrm{L}^2(\Omega)\ \exists!$ weak solution u with $\widetilde{N}(\nabla_{t,x} u) \in L^2(\Omega)$:

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Examples, when this can be checked

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A = \begin{bmatrix} A_{\perp\perp} & 0 \\ 0 & A_{\|\|} \end{bmatrix} \implies E_0^+ \mathcal{H} = \left\{ \begin{bmatrix} h_\perp \\ -\nabla_{\mathscr{D}} ((\nabla_{\mathscr{D}})^* A_{\|\|} \nabla_{\mathscr{D}})^{-1/2} h_\perp \end{bmatrix} ; h_\perp \in L^2(\Omega) \right\}
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$$A = A^* \implies \text{Rellich estimate } \|(h^+)_\perp\|_2 \simeq \|(h^+)_{\|}\|_2 \simeq \|h^+\|_2$$

Thank you for your attention!