

The Kato Square Root Problem for Mixed Boundary Conditions

Moritz Egert

joint work with R. Haller-Dintelmann and P. Tolksdorf



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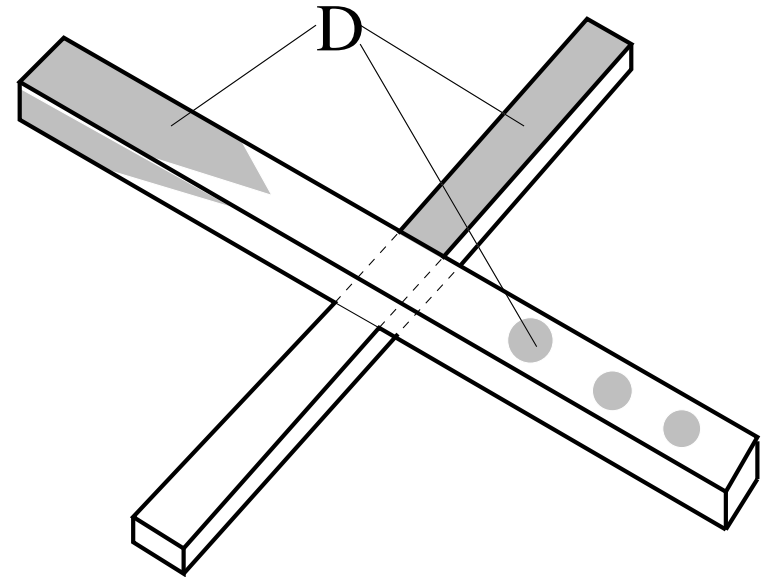
Setup



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Let

- ▶ $\Omega \subseteq \mathbb{R}^d$ bdd. ‘rough’ domain,
- ▶ $D \subseteq \partial\Omega$ closed,
- ▶ $A_D u \sim \sum_{\alpha, \beta=1}^d -\partial_\alpha (a_{\alpha, \beta} \partial_\beta u)$,
- ▶ $a_{\alpha, \beta} \in L^\infty(\Omega)$.



Realize A_D on $L^2(\Omega)$ via form method

$$\begin{cases} a_D(u, v) & := \sum_{\alpha, \beta=1}^d \int_{\Omega} a_{\alpha, \beta} \partial_\beta u \cdot \partial_\alpha \bar{v}, \\ \text{dom}(a_D) & := H_D^1(\Omega) := \overline{\{u|_{\Omega} : u \in C_D^\infty(\mathbb{R}^d)\}}^{H^1(\Omega)}. \end{cases}$$

Assume Gårding Inequality

$$\exists \lambda > 0 : \quad \Re(a_D(u, u)) \geq \lambda \|\nabla u\|_{L^2(\Omega)}^2 \quad (u \in H_D^1(\Omega)).$$

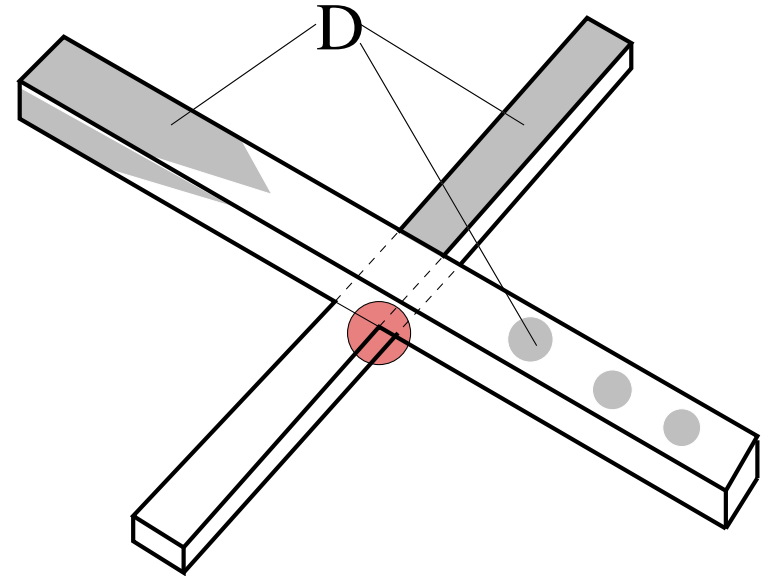
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Properties of A_D



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- ▶ $-A_D \sim$ analytic contraction semigroup on $L^2(\Omega)$.
- ▶ $H_D^1(\Omega) \subseteq \text{dom}(A_D) \not\subseteq H_D^1(\Omega) \cap H^2(\Omega)$ in general.
- ▶ $\text{dom}(\sqrt{A_D}) = \text{dom}(a_D) = H_D^1(\Omega)$ if A_D self-adjoint.

The Kato Square Root Problem

Does $\text{dom}(\sqrt{A_D}) = \text{dom}(a_D) = H_D^1(\Omega)$ always hold?

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Reduction to $-\Delta_D$



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Minimal smoothness assumptions

- ▶ Existence of a bounded extension operator $H^1(\Omega) \rightarrow H^1(\mathbb{R}^d)$,
- ▶ d -set property:

$$|B(x, r) \cap \Omega| \sim r^d \quad (x \in \Omega, r \in (0, r_0)).$$

- ▶ No assumptions on D .
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If $\text{dom}((-\Delta_D)^{\frac{1+\alpha}{2}}) \hookrightarrow H^{1+\alpha}(\Omega)$ for an $\alpha > 0$, then $H_D^1(\Omega) = \text{dom}(\sqrt{A_D})$.

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geometry \iff operator theory

Our Geometric Setting

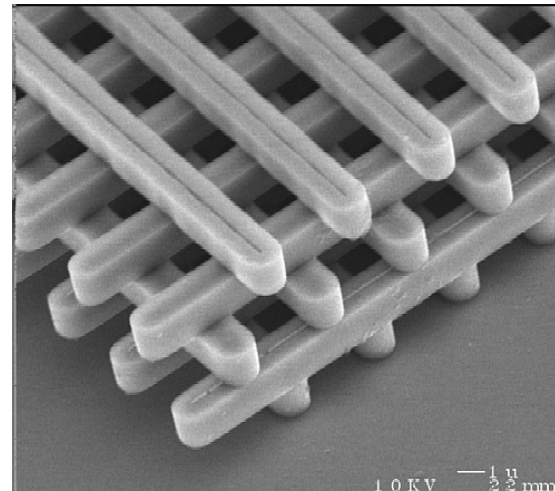
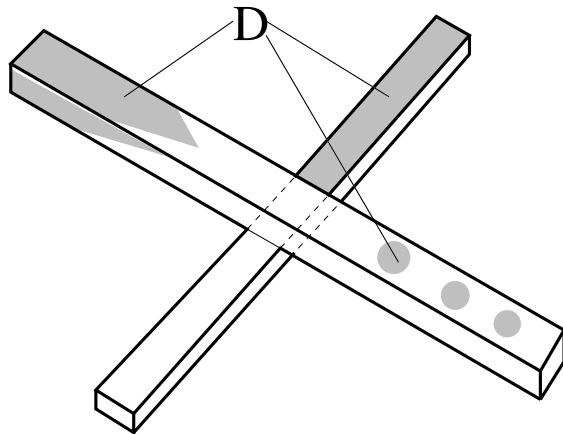


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Let

- ▶ Ω bounded Lipschitz domain,
- ▶ D a $(d - 1)$ -set:

$$|B(x, r) \cap D| \sim r^{d-1} \quad (x \in D, r \in (0, r_0)).$$



Introduce

- ▶ $H_D^{1+\alpha}(\Omega) := \left\{ u \in H^{1+\alpha}(\Omega) : u|_D = 0 \text{ } \mathcal{H}^{d-1}\text{-a.e.} \right\} \quad (|\alpha| < \frac{1}{2}).$
- ▶ $\{H_D^\theta(\Omega)\}_{\frac{1}{2} < \theta < \frac{3}{2}}$ complex interpolation scale.

From Extrapolation to Interpolation



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Goal

(♠) $\text{dom}\left((-\Delta_D)^{\frac{1+\alpha}{2}}\right) \hookrightarrow H_D^{1+\alpha}(\Omega)$ for small $\alpha \in (0, \frac{1}{2})$.

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$$(\spadesuit) \quad \text{dom}((-\Delta_D)^{\frac{1+\alpha}{2}}) \hookrightarrow H_D^{1+\alpha}(\Omega) \text{ for small } \alpha \in (0, \frac{1}{2}).$$

Lemma

One gets (\spadesuit) if $H_D^{1-\alpha}(\Omega) \hookrightarrow \text{dom}((-\Delta_D)^{\frac{1-\alpha}{2}})$ holds for $\alpha \in (0, \frac{1}{2})$.

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- ▶ Reduces extrapolation result to interpolation result.

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- ▶ Main ingredient for interpolation result: Fractional Hardy Inequality

$$\int_{\Omega} \frac{|u(x)|^2}{\text{dist}_D(x)^{2(1-\alpha)}} dx \lesssim \|u\|_{H_D^{1-\alpha}(\Omega)}^2 \quad (u \in H_D^{1-\alpha}(\Omega)).$$

The Main Result



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Theorem

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$$\text{dom}(\sqrt{A_D}) = H_D^1(\Omega)$$

and moreover,

$$\text{dom}((A_D)^{\frac{\alpha}{2}}) = \begin{cases} H_D^\alpha(\Omega) & \alpha \in (\frac{1}{2}, 1] \\ H^\alpha(\Omega) & \alpha \in (0, \frac{1}{2}) \end{cases}.$$

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