On an elliptic mixed boundary value problem

Moritz Egert

TECHNISCHE UNIVERSITÄT DARMSTADT

Delft, November 13, 2014

The setup

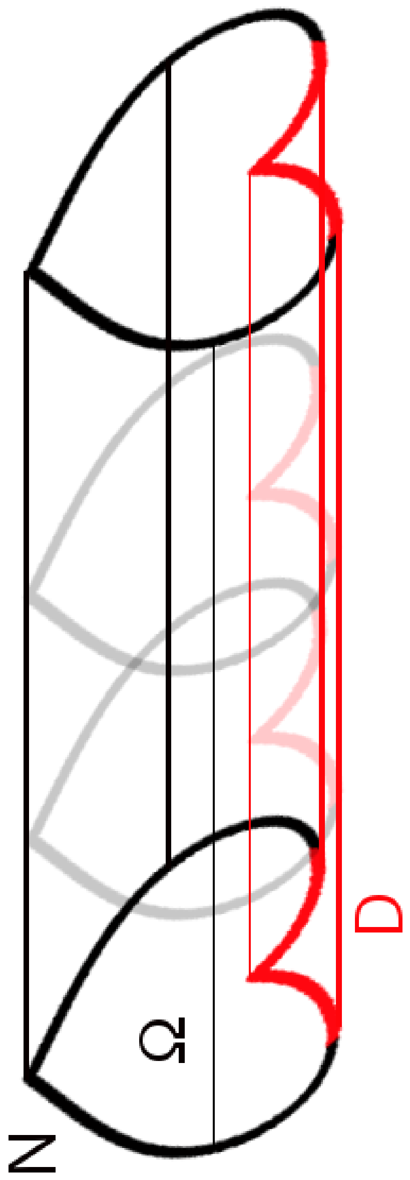
TECHNISCHE UNIVERSITÄT DARMSTADT

Elliptic mixed BVP

$$-\mathrm{div}_{t,x} A(x) \nabla_{t,x} U = 0 \qquad (\mathbb{R}^+ \times \Omega)$$
$$U = 0 \qquad (\mathbb{R}^+ \times D)$$
$$\partial_{\nu_A} U = 0 \qquad (\mathbb{R}^+ \times N)$$
$$\partial_{\nu_A} U = g \qquad (\{0\} \times \Omega)$$

Assumptions

- $\Omega \subseteq \mathbb{R}^d$, $D \subseteq \partial\Omega$ closed
- $A \in \mathrm{L}^\infty$ pointwise elliptic
- $\mathcal{V}$ = closure of $\mathrm{C}_c^\infty(\mathbb{R}^d \setminus D)$ in $\mathrm{H}^1(\Omega)$
- Poincaré $\|\nabla u\|_2 \simeq \|u\|_{\mathrm{H}^1}$ on $\mathcal{V}$



Lax-Milgram approach

- $\mathcal{E} = \mathrm{L}^2(\mathcal{V}) \cap \mathrm{H}^1(\mathrm{L}^2)$ energy space with norm $\|\nabla_{t,x} \cdot \|_2$
- $\mathcal{V}_{1/2} = [\mathrm{L}^2, \mathcal{V}]_{\frac{1}{2}}$ its trace space

Formal computation

$$0 = \int_0^\infty \int_\Omega \mathrm{div}_{t,x} A \nabla_{t,x} U \cdot \overline{V}$$

Lax-Milgram approach

- $\mathcal{E} = \mathrm{L}^2(\mathcal{V}) \cap \mathrm{H}^1(\mathrm{L}^2)$ energy space with norm $\|\nabla_{t,x} \cdot \|_2$
- $\mathcal{V}_{1/2} = [\mathrm{L}^2, \mathcal{V}]_{\frac{1}{2}}$ its trace space

Formal computation

$$\begin{aligned} 0 &= \int_0^\infty \int_\Omega \mathrm{div}_{t,x} A \nabla_{t,x} U \cdot \overline{V} \\ &= \int_\Omega \int_0^\infty \partial_t (A \nabla_{t,x} U)_\perp \cdot \overline{V} + \int_0^\infty \int_\Omega \nabla_x (A \nabla_{t,x} U)_\parallel \cdot \overline{V} \end{aligned}$$

Lax-Milgram approach

- $\mathcal{E} = \mathrm{L}^2(\mathcal{V}) \cap \mathrm{H}^1(\mathrm{L}^2)$ energy space with norm $\|\nabla_{t,x} \cdot \|_2$
- $\mathcal{V}_{1/2} = [\mathrm{L}^2, \mathcal{V}]_{\frac{1}{2}}$ its trace space

Formal computation

$$\begin{aligned} 0 &= \int_0^\infty \int_\Omega \operatorname{div}_{t,x} A \nabla_{t,x} U \cdot \overline{V} \\ &= \int_\Omega \int_0^\infty \partial_t (A \nabla_{t,x} U)_\perp \cdot \overline{V} + \int_0^\infty \int_\Omega \nabla_x (A \nabla_{t,x} U)_\parallel \cdot \overline{V} \\ &= -\int_0^\infty \int_\Omega A \nabla_{t,x} U \cdot \nabla_{t,x} \overline{V} + \int_\Omega g \cdot \overline{V}|_{t=0} \end{aligned}$$

Lax-Milgram approach

TECHNISCHE UNIVERSITÄT DARMSTADT

- $\mathcal{E} = \mathrm{L}^2(\mathcal{V}) \cap \mathrm{H}^1(\mathrm{L}^2)$ energy space with norm $\|\nabla_{t,x} \cdot \|_2$
- $\mathcal{V}_{1/2} = [\mathrm{L}^2, \mathcal{V}]_{\frac{1}{2}}$ its trace space

Formal computation

$$
\begin{aligned}
0 &= \int_0^\infty \int_\Omega \operatorname{div}_{t,x} A \nabla_{t,x} U \cdot \overline{V} \\
&= \int_\Omega \int_0^\infty \partial_t (A \nabla_{t,x} U)_\perp \cdot \overline{V} + \int_0^\infty \int_\Omega \nabla_x (A \nabla_{t,x} U)_\parallel \cdot \overline{V} \\
&= \underbrace{- \int_0^\infty \int_\Omega A \nabla_{t,x} U \cdot \nabla_{t,x} \overline{V}}_{=: a(U,V)} + \underbrace{\int_\Omega g \cdot \overline{V}|_{t=0}}_{=: \phi_g (V|_{t=0})}
\end{aligned}
$$

Lax-Milgram approach

- $\mathcal{E} = \mathrm{L}^2(\mathcal{V}) \cap \mathrm{H}^1(\mathrm{L}^2)$ energy space with norm $\|\nabla_{t,x} \cdot \|_2$
- $\mathcal{V}_{1/2} = [\mathrm{L}^2, \mathcal{V}]_{\frac{1}{2}}$ its trace space

Formal computation

$$
\begin{aligned}
0 &= \int_0^\infty \int_\Omega \mathrm{div}_{t,x} A \nabla_{t,x} U \cdot \overline{V} \\
&= \int_\Omega \int_0^\infty \partial_t (A\nabla_{t,x} U)_\perp \cdot \overline{V} + \int_0^\infty \int_\Omega \nabla_x (A \nabla_{t,x} U)_\parallel \cdot \overline{V} \\
&= \underbrace{- \int_0^\infty \int_\Omega A \nabla_{t,x} U \cdot \nabla_{t,x} \overline{V}}_{=: a(U,V)} + \underbrace{\int_\Omega g \cdot \overline{V}|_{t=0}}_{=: \phi_g(V|_{t=0})}
\end{aligned}
$$

Lemma

For each $g \in \mathcal{V}_{1/2}^*$ there exists a unique weak solution $U \in \mathcal{E}$.

A hidden semigroup structure

$$\big\langle (A\nabla_{t,x}U)_\perp|_{t=0}, V|_{t=0}\big\rangle = -\int_0^\infty \int_\Omega A\nabla_{t,x}U \cdot \nabla_{t,x}\overline{V} \qquad (V \in \mathcal{E})$$

Re-interpretation

$$(A\nabla_{t,x}U)_\perp|_{t=0} \sim v \mapsto -\int_0^\infty \int_\Omega A(x)\nabla_{t,x}U(0+t,x) \cdot \nabla_{t,x}\overline{V(t,x)}\,\mathrm{d}x\,\mathrm{d}t$$

where $V \in \mathcal{E}$ is any extension of $v \in \mathcal{V}_{1/2}$.

A hidden semigroup structure

TECHNISCHE UNIVERSITÄT DARMSTADT

$$\big\langle (A\nabla_{t,x}U)_\perp|_{t=0}, V|_{t=0}\big\rangle = -\int_0^\infty \int_\Omega A\nabla_{t,x}U \cdot \nabla_{t,x}\overline{V} \qquad (V \in \mathcal{E})$$

Re-interpretation

$$(A\nabla_{t,x}U)_\perp|_{t=s} \sim v \mapsto -\int_0^\infty \int_\Omega A(x)\nabla_{t,x}U(s+t,x) \cdot \nabla_{t,x}\overline{V(t,x)}\,\mathrm{d}x\,\mathrm{d}t$$

where $V \in \mathcal{E}$ is any extension of $v \in \mathcal{V}_{1/2}$.

A hidden semigroup structure

TECHNISCHE UNIVERSITÄT DARMSTADT

$$\big\langle (A\nabla_{t,x}U)_\perp|_{t=0}, V|_{t=0}\big\rangle = -\int_0^\infty \int_\Omega A\nabla_{t,x}U \cdot \nabla_{t,x}\overline{V} \qquad (V \in \mathcal{E})$$

Re-interpretation

$$(A\nabla_{t,x}U)_\perp|_{t=s} \sim v \mapsto -\int_0^\infty \int_\Omega A(x)\nabla_{t,x}U(s+t,x) \cdot \nabla_{t,x}\overline{V(t,x)}\,\mathrm{d}x\,\mathrm{d}t$$

where $V \in \mathcal{E}$ is any extension of $v \in \mathcal{V}_{1/2}$.

Obtain

- Natural semigroup flow $(A\nabla_{t,x}U)_\perp|_{t=s} = T(s)\big((A\nabla_{t,x}U)_\perp|_{t=0}\big)$
- T a C_0-smg. on $\mathcal{V}_{1/2}^*$.
- Is semigroup orbit a representative for $(A\nabla_{t,x}U)_\perp \in \mathrm{L}^2(\mathrm{L}^2)$?

Refined setup

TECHNISCHE UNIVERSITÄT DARMSTADT

Assumptions

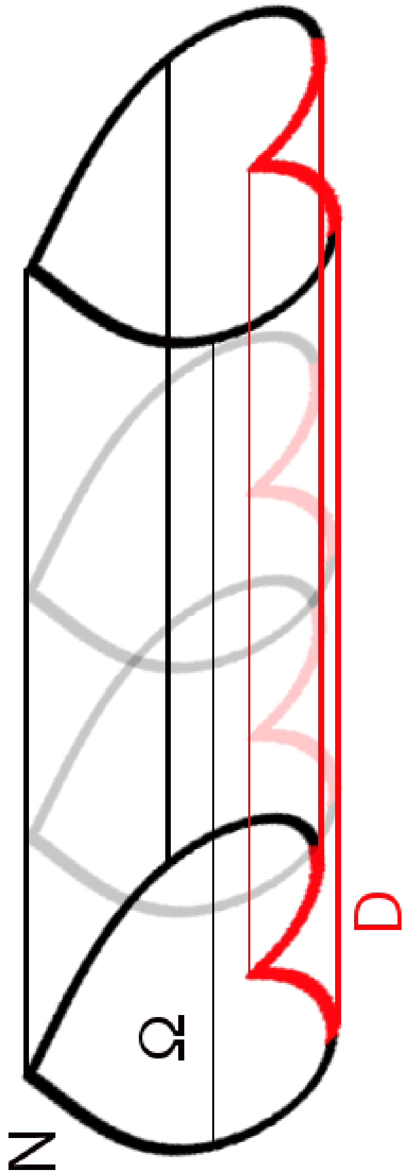
- $\Omega \subseteq \mathbb{R}^d$ bounded, $D \subseteq \partial\Omega$ closed
- $A \in \mathrm{L}^\infty$ pointwise elliptic
- Ω is a d-set, i.e.

$$|B(x,r) \cap \Omega| \simeq r^d \qquad (x \in \Omega,\ r \leq 1)$$

- D is a $(d-1)$-set, i.e.

$$\mathcal{H}_{d-1}(B(x,r) \cap D) \simeq r^{d-1} \qquad (x \in D,\ r \leq 1)$$

- Lipschitz charts around $\overline{N}$.



Second order equation

$$-\mathrm{div}_{t,x} A(x)\nabla_{t,x} U = 0 \qquad (\mathbb{R}^+ \times \Omega)$$

$$U = 0 \qquad (\mathbb{R}^+ \times D)$$

$$\partial_{\nu_A} U = 0 \qquad (\mathbb{R}^+ \times N)$$

Weak solutions

- $U \in \mathrm{L}^2_{\mathrm{loc}}(\mathcal{V}) \cap \mathrm{H}^1_{\mathrm{loc}}(\mathrm{L}^2)$
- $\displaystyle\int_0^\infty \int_\Omega A\nabla_{t,x} U \cdot \nabla_{t,x} \overline{V} = 0$

  for all $V \in \mathrm{C}^\infty_c(\mathcal{V})$

Second order equation

$$\begin{aligned} -\mathrm{div}_{t,x} A(x)\nabla_{t,x} U &= 0 \qquad (\mathbb{R}^+ \times \Omega) \\ U &= 0 \qquad (\mathbb{R}^+ \times D) \\ \partial_{\nu_A} U &= 0 \qquad (\mathbb{R}^+ \times N) \end{aligned}$$

Weak solutions

- $U \in \mathrm{L}^2_{\mathrm{loc}}(\mathcal{V}) \cap \mathrm{H}^1_{\mathrm{loc}}(\mathrm{L}^2)$
- $\displaystyle\int_0^\infty\int_\Omega A\nabla_{t,x} U \cdot \nabla_{t,x}\overline{V} = 0$

  for all $V \in \mathrm{C}^\infty_c(\mathcal{V})$

A formal computation

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{div}_{t,x} A(x)\nabla_{t,x} U \\ 0 \end{bmatrix} = \begin{bmatrix} \partial_t (A\nabla_{t,x} U)_\perp + \mathrm{div}_x (A\nabla_{t,x} U)_\parallel \\ \partial_t (\nabla_{t,x} U)_\parallel - \nabla_x (\nabla_{t,x} U)_\perp \end{bmatrix}$$

Second order equation

$$\begin{aligned} -\mathrm{div}_{t,x} A(x) \nabla_{t,x} U &= 0 \qquad (\mathbb{R}^+ \times \Omega) \\ U &= 0 \qquad (\mathbb{R}^+ \times D) \\ \partial_{\nu_A} U &= 0 \qquad (\mathbb{R}^+ \times N) \end{aligned}$$

Weak solutions

- $U \in \mathrm{L}^2_{\mathrm{loc}}(\mathcal{V}) \cap \mathrm{H}^1_{\mathrm{loc}}(\mathrm{L}^2)$
- $\displaystyle\int_0^\infty \int_\Omega A \nabla_{t,x} U \cdot \nabla_{t,x} \overline{V} = 0$

  for all $V \in \mathrm{C}_c^\infty(\mathcal{V})$

A formal computation

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{div}_{t,x} A(x) \nabla_{t,x} U \\ 0 \end{bmatrix} = \begin{bmatrix} \partial_t (A\nabla_{t,x} U)_\perp + \mathrm{div}_x (A \nabla_{t,x} U)_\parallel \\ \partial_t (\nabla_{t,x} U)_\parallel - \nabla_x (\nabla_{t,x} U)_\perp \end{bmatrix}$$

leads to the first order equation

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \partial_t F + \underbrace{\begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix}}_{=:\mathrm{D}} \mathrm{B} F, \quad \text{for} \quad F = \begin{bmatrix} (A\nabla_{t,x} U)_\perp \\ (\nabla_{t,x} U)_\parallel \end{bmatrix}$$

where B transfers A from the $\parallel$-part to the $\perp$-part

The first order formalism

Rigorously

- $\mathrm{D} = \begin{bmatrix} 0 & (-\nabla_{\mathcal{V}})^* \\ -\nabla_{\mathcal{V}} & 0 \end{bmatrix}$, where $\nabla_{\mathcal{V}} : \mathcal{V} \to (\mathrm{L}^2)^d$
- Study 1st order equation $\partial_t F + \mathrm{DB} F = 0$ through weak solutions $F \in \mathrm{L}^2_{\mathrm{loc}}(\overline{\mathcal{R}(\mathrm{DB})})$ defined by

$$\int_0^\infty \int_\Omega F \cdot \partial_t \overline{G} = \int_0^\infty \int_\Omega \mathrm{B} F \cdot \overline{\mathrm{D} G} \qquad (G \in \mathrm{C}_c^\infty(\mathcal{D}(\mathrm{D})))$$

Proposition

Weak solutions to 2nd order equation and 1st order equation are in one-to-one correspondence

$$U \sim \left[(A \nabla_{t,x} U)_\perp \quad \nabla_x U \right]^\top .$$

The DB-theorem

Theorem (E., Haller-Dintelmann, Tolksdorf '13)

Let $\mathrm{B} \in \mathrm{L}^\infty$ *be accretive on* $\mathcal{H} := \overline{\mathcal{R}(\mathrm{D})}$ *in the sense*

$$(\mathrm{BD}u \mid \mathrm{D}u) \gtrsim \|u\|^2 \qquad (u \in \mathcal{H}).$$

Then DB *is bi-sectorial on* L^2*, has range* $\mathcal{R}(\mathrm{DB}) = \mathrm{L}^2 \oplus \mathcal{R}(\nabla_{\mathcal{V}})$ *and satisfies quadratic estimates*

$$\int_0^\infty \|t\mathrm{DB}(1 + t^2(\mathrm{DB})^2)^{-1}u\|^2 \, \frac{\mathrm{d}t}{t} \simeq \|u\|^2 \qquad (u \in \mathcal{H}).$$

The DB-theorem

Theorem (E., Haller-Dintelmann, Tolksdorf '13)

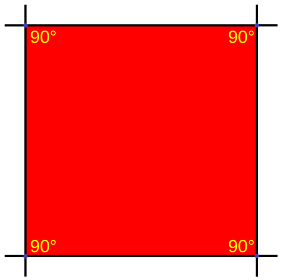
Let $\mathrm{B} \in \mathrm{L}^\infty$ *be accretive on* $\mathcal{H} := \overline{\mathcal{R}(\mathrm{D})}$ *in the sense*

$$(\mathrm{BD}u \mid \mathrm{D}u) \gtrsim \|u\|^2 \qquad (u \in \mathcal{H}).$$

Then DB *is bi-sectorial on* L^2*, has range* $\mathcal{R}(\mathrm{DB}) = \mathrm{L}^2 \oplus \mathcal{R}(\nabla_{\mathcal{V}})$ *and satisfies quadratic estimates*

$$\int_0^\infty \|t\mathrm{DB}(1 + t^2(\mathrm{DB})^2)^{-1}u\|^2 \, \frac{\mathrm{d}t}{t} \simeq \|u\|^2 \qquad (u \in \mathcal{H}).$$

Kato



Square

Root

Problem

Semigroup solutions

$$\partial_t F + \mathrm{DB} F = 0, \qquad F \in \mathrm{L}^2(\mathrm{L}^2)$$

Note

- $\mathbf{1}_{\mathbb{C}^+}(\mathrm{DB}) : \mathcal{H} \to \mathcal{H}^+$ projection
- DB sectorial on spectral subspace $\mathcal{H}^+$

Solve by $F(t) = \mathrm{e}^{-t[\mathrm{DB}]} F_0$, where $F_0 \in \mathcal{H}^+$.

Semigroup solutions

$$\partial_t F + \mathrm{DB} F = 0, \qquad F \in \mathrm{L}^2(\mathrm{L}^2)$$

Note

- $\mathbf{1}_{\mathbb{C}^+}(\mathrm{DB}) : \mathcal{H} \to \mathcal{H}^+$ projection
- DB sectorial on spectral subspace $\mathcal{H}^+$

Solve by $F(t) = \mathrm{e}^{-t[\mathrm{DB}]} F_0$, where $F_0 \in \mathcal{H}^+$? Not quite!

Semigroup solutions

$$\boxed{\partial_t F + \mathrm{DB} F = 0, \qquad F \in \mathrm{L}^2(\mathrm{L}^2)}$$

Note

- $\mathbf{1}_{\mathbb{C}^+}(\mathrm{DB}) : \mathcal{H} \to \mathcal{H}^+$ projection
- DB sectorial on spectral subspace $\mathcal{H}^+$

Solve by $F(t) = \mathrm{e}^{-t[\mathrm{DB}]} F_0$, where $F_0 \in \mathcal{H}^+$? Not quite!

$$\int_0^\infty \| \sqrt{t[\mathrm{DB}]}\, \mathrm{e}^{-t[\mathrm{DB}]} F_0 \|^2 \, \frac{\mathrm{d}t}{t} \simeq \| F_0 \|^2$$

Semigroup solutions

$$\boxed{\partial_t F + \mathrm{DB} F = 0, \qquad F \in \mathrm{L}^2(\mathrm{L}^2)}$$

Note

- $\mathbf{1}_{\mathbb{C}^+}(\mathrm{DB}) : \mathcal{H} \to \mathcal{H}^+$ projection
- DB sectorial on spectral subspace $\mathcal{H}^+$

Solve by $F(t) = \mathrm{e}^{-t[\mathrm{DB}]} F_0$, where $F_0 \in \mathcal{H}^+$? Not quite!

$$\int_0^\infty \| \sqrt{t[\mathrm{DB}]}\mathrm{e}^{-t[\mathrm{DB}]} F_0 \|^2 \, \frac{\mathrm{d}t}{t} \simeq \|F_0\|^2$$

Theorem

The weak solutions $F \in \mathrm{L}^2(\mathrm{L}^2)$ are precisely the functions

$$\left\{ \sqrt{[\mathrm{DB}]}\mathrm{e}^{-\bullet[\mathrm{DB}]} F_0 ;\; F_0 \in \mathcal{H}^+ \right\}.$$

Semigroup solutions

$$\boxed{\partial_t F + \mathrm{DB} F = 0, \qquad F \in \mathrm{L}^2(\mathrm{L}^2)}$$

Note

- $\mathbf{1}_{\mathbb{C}^+}(\mathrm{DB}) : \mathcal{H} \to \mathcal{H}^+$ projection
- DB sectorial on spectral subspace $\mathcal{H}^+$

Solve by $F(t) = \mathrm{e}^{-t[\mathrm{DB}]} F_0$, where $F_0 \in \mathcal{H}^+$? Not quite!

$$\int_0^\infty \|\sqrt{t[\mathrm{DB}]}\mathrm{e}^{-t[\mathrm{DB}]} F_0\|^2 \,\frac{\mathrm{d}t}{t} \simeq \|F_0\|^2$$

Theorem

The weak solutions $F \in \mathrm{L}^2(\mathrm{L}^2)$ are precisely the functions

$$\left\{ \sqrt{[\mathrm{DB}]}\mathrm{e}^{-\bullet[\mathrm{DB}]} F_0;\ F_0 \in \mathcal{H}^+ \right\}.$$

First extrapolation space for DB

$$\mathcal{H}^{-1} = \overline{\left(\mathcal{R}(\mathrm{DB}), \|(\mathrm{DB})^{-1} \cdot \|\right)} = \overline{\left(\mathcal{R}(\mathrm{D}), \|\mathrm{D}^{-1} \cdot \|\right)}$$

Then

- Functional calculus extrapolates to $\mathcal{H}^{-1}$
- $\sqrt{[\mathrm{DB}]}$ extends to isomorphism $\mathcal{H} \to \left[\mathcal{H}, \mathcal{H}^{-1}\right]_{1/2} =: \mathcal{H}^{-1/2}$
- In $\mathcal{H}^{-1/2}$ we have

$$\sqrt{[\mathrm{DB}]}\mathrm{e}^{-t[\mathrm{DB}]} F_0 = \mathrm{e}^{-t[\mathrm{DB}]} \sqrt{[\mathrm{DB}]} F_0 \qquad (F_0 \in \mathcal{H}^{+})$$

First extrapolation space for DB

$$\mathcal{H}^{-1} = \overline{\big(\mathcal{R}(\mathrm{DB}), \|(\mathrm{DB})^{-1}\cdot\|\big)} = \overline{\big(\mathcal{R}(\mathrm{D}), \|\mathrm{D}^{-1}\cdot\|\big)}$$

Then

- Functional calculus extrapolates to $\mathcal{H}^{-1}$
- $\sqrt{[\mathrm{DB}]}$ extends to isomorphism $\mathcal{H} \to \big[\mathcal{H}, \mathcal{H}^{-1}\big]_{1/2} =: \mathcal{H}^{-1/2}$
- In $\mathcal{H}^{-1/2}$ we have

$$\sqrt{[\mathrm{DB}]}\mathrm{e}^{-t[\mathrm{DB}]}F_0 = \mathrm{e}^{-t[\mathrm{DB}]}\sqrt{[\mathrm{DB}]}F_0 \qquad (F_0 \in \mathcal{H}^+)$$

First extrapolation space for DB

$$\mathcal{H}^{-1} = \overline{\big(\mathcal{R}(\mathrm{DB}), \|(\mathrm{DB})^{-1}\cdot\|\big)} = \overline{\big(\mathcal{R}(\mathrm{D}), \|\mathrm{D}^{-1}\cdot\|\big)}$$

Then

- Functional calculus extrapolates to $\mathcal{H}^{-1}$
- $\sqrt{[\mathrm{DB}]}$ extends to isomorphism $\mathcal{H} \to \big[\mathcal{H}, \mathcal{H}^{-1}\big]_{1/2} =: \mathcal{H}^{-1/2}$
- In $\mathcal{H}^{-1/2}$ we have

$$\sqrt{[\mathrm{DB}]}\mathrm{e}^{-t[\mathrm{DB}]}F_0 = \mathrm{e}^{-t[\mathrm{DB}]}\sqrt{[\mathrm{DB}]}F_0 \qquad (F_0 \in \mathcal{H}^{+})$$

Let $v \in \mathcal{R}(\mathrm{D})_{\perp}$

- Write $v = D\begin{bmatrix}0 & u\end{bmatrix}^{\top} = (-\nabla_{\mathcal{V}})^{*}u \quad$ with $\quad u \in \mathcal{H}_{\|} = \mathcal{R}(\nabla_{\mathcal{V}})$.

$$\|v\|_{\mathcal{V}^*} = \sup_{\substack{w\in\mathcal{V} \\ \|\nabla w\|_2=1}} \Big|\Big((-\nabla_{\mathcal{V}})^{*}u \;\Big|\; w\Big)_2\Big| = \sup_{\substack{w\in\mathcal{V} \\ \|\nabla w\|_2=1}} \Big|\Big(u \;\Big|\; \nabla_{\mathcal{V}}w\Big)_2\Big| = \|u\|_2$$

First extrapolation space for DB

$$\mathcal{H}^{-1} = \overline{\big(\mathcal{R}(\mathrm{DB}), \|(\mathrm{DB})^{-1}\cdot\|\big)} = \overline{\big(\mathcal{R}(\mathrm{D}), \|\mathrm{D}^{-1}\cdot\|\big)}$$

Then

- Functional calculus extrapolates to $\mathcal{H}^{-1}$
- $\sqrt{[\mathrm{DB}]}$ extends to isomorphism $\mathcal{H} \to \big[\mathcal{H}, \mathcal{H}^{-1}\big]_{1/2} =: \mathcal{H}^{-1/2}$
- In $\mathcal{H}^{-1/2}$ we have

$$\sqrt{[\mathrm{DB}]}\mathrm{e}^{-t[\mathrm{DB}]}F_0 = \mathrm{e}^{-t[\mathrm{DB}]}\sqrt{[\mathrm{DB}]}F_0 \qquad (F_0 \in \mathcal{H}^+)$$

Let $v \in \mathcal{R}(\mathrm{D})_\perp$

- Write $v = D\begin{bmatrix}0 & u\end{bmatrix}^\top = (-\nabla_{\mathcal{V}})^* u \quad$ with $\quad u \in \mathcal{H}_\| = \mathcal{R}(\nabla_{\mathcal{V}})$.

$$\|v\|_{\mathcal{V}^*} = \sup_{\substack{w\in\mathcal{V} \\ \|\nabla w\|_2=1}} \Big|\big((-\nabla_{\mathcal{V}})^* u \;\big|\; w\big)_2\Big| = \sup_{\substack{w\in\mathcal{V} \\ \|\nabla w\|_2=1}} \Big|\big(u \;\big|\; \nabla_{\mathcal{V}} w\big)_2\Big| = \|u\|_2$$

$\Longrightarrow$ $\mathcal{H}_\perp^{-1} = \mathcal{V}^*$ and $\mathcal{H}_\perp^{-1/2} = \mathcal{V}^*_{1/2}$ the space of Lax-Milgram semigroup

Back to the Lax-Milgram semigroup

Let $U \in \mathcal{E}$ be the weak solution obtained by Lax-Milgram

Back to the Lax-Milgram semigroup

Let $U \in \mathcal{E}$ be the weak solution obtained by Lax-Milgram

1. $\begin{bmatrix} (A\nabla_{t,x} U)_\perp \\ \nabla_x \end{bmatrix} \in \mathrm{L}^2(\mathrm{L}^2)$ weak solution of 1st order equation

Back to the Lax-Milgram semigroup

TECHNISCHE UNIVERSITÄT DARMSTADT

Let $U \in \mathcal{E}$ be the weak solution obtained by Lax-Milgram

1. $\begin{bmatrix} (A\nabla_{t,x} U)_\perp \\ \nabla_x \end{bmatrix} \in \mathrm{L}^2(\mathrm{L}^2)$ weak solution of 1st order equation
2. Semigroup representation in $\mathcal{H}^{-1/2}$:

$$\exists! F_0 \in \mathcal{H}^+ : \quad \begin{bmatrix} (A\nabla_{t,x} U)_\perp \\ \nabla_x U \end{bmatrix} = \mathrm{e}^{-\bullet[\mathrm{DB}]} \sqrt{[\mathrm{DB}]} F_0$$

Back to the Lax-Milgram semigroup

TECHNISCHE UNIVERSITÄT DARMSTADT

Let $U \in \mathcal{E}$ be the weak solution obtained by Lax-Milgram

1. $\begin{bmatrix} (A\nabla_{t,x}U)_\perp \\ \nabla_x \end{bmatrix} \in \mathrm{L}^2(\mathrm{L}^2)$ weak solution of 1st order equation
2. Semigroup representation in $\mathcal{H}^{-1/2}$:

$$\exists! F_0 \in \mathcal{H}^+ : \quad \begin{bmatrix} (A\nabla_{t,x}U)_\perp \\ \nabla_x U \end{bmatrix} = \mathrm{e}^{-\bullet[\mathrm{DB}]}\sqrt{[\mathrm{DB}]}F_0$$

3. Reconstruction of the Lax-Milgram flow

$$(A\nabla_{t,x}U)_\perp = \left(\mathrm{e}^{-\bullet[\mathrm{DB}]}\sqrt{[\mathrm{DB}]}F_0\right)_\perp \overset{!}{=} (A\nabla_{t,x}U)_\perp|_{t=\bullet} = T(\bullet)(A\nabla_{t,x}U)_\perp|_{t=0}.$$

Back to the Lax-Milgram semigroup

TECHNISCHE UNIVERSITÄT DARMSTADT

Let $U \in \mathcal{E}$ be the weak solution obtained by Lax-Milgram

1. $\begin{bmatrix} (A\nabla_{t,x}U)_\perp \\ \nabla_x \end{bmatrix} \in \mathrm{L}^2(\mathrm{L}^2)$ weak solution of 1st order equation
2. Semigroup representation in $\mathcal{H}^{-1/2}$:

$$\exists! F_0 \in \mathcal{H}^+ : \quad \begin{bmatrix} (A\nabla_{t,x}U)_\perp \\ \nabla_x U \end{bmatrix} = \mathrm{e}^{-\bullet[\mathrm{DB}]}\sqrt{[\mathrm{DB}]}F_0$$

3. Reconstruction of the Lax-Milgram flow

$$(A\nabla_{t,x}U)_\perp = \left(\mathrm{e}^{-\bullet[\mathrm{DB}]}\sqrt{[\mathrm{DB}]}F_0\right)_\perp \overset{!}{=} (A\nabla_{t,x}U)_\perp|_{t=\bullet} = T(\bullet)(A\nabla_{t,x}U)_\perp|_{t=0}.$$

Corollary

There is a Neumann-to-Dirichlet map $\mathcal{V}^*_{1/2} \to \mathcal{H}_{\|}^{-1/2}$ given by

$$\partial_{\nu_A} U|_{t=0} \longrightarrow U \longrightarrow F \longrightarrow (F_{\|})(0) \longrightarrow \nabla_x U|_{t=0}$$

Thank you for your attention!