The Porous Medium Equation

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Physical motivation

Flow of an ideal gas through a homogeneous porous medium can be described by

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\begin{cases}
\varepsilon \partial_t \rho + \mathrm{div}(\rho \mathbf{v}) = 0 & \text{mass balance} \\
\mu \mathbf{v} = -k \, \nabla \mathrm{p} & \text{Darcy's law} \\
\mathrm{p} = \mathrm{p}_0 \, \rho^\gamma & \text{state equation}
\end{cases}
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\varepsilon \partial_t \rho = -\mathrm{div}(\rho \mathbf{v}) = \frac{k}{\mu} \mathrm{div}(\rho \nabla \mathrm{p}) = \frac{k \mathrm{p}_0}{\mu} \mathrm{div}(\rho \nabla \rho^\gamma)
$$

- ρ : density
- p : pressure
- $\mathbf{v}$: velocity
- ε, k, $\mu > 0$: material constants
- $\gamma \geq 1$: polytropic exponent
- p_0 : reference pressure

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Definition

The **porous medium equation** (PME) is

$$\partial_t u(t,x) = \Delta_x u^m(t,x), \; u \geq 0, \; m > 1 \quad (t,x) \in (0,\infty) \times \mathbb{R}^n$$

$$\varepsilon \partial_t \rho = -\operatorname{div}(\rho \mathbf{v}) = \frac{k}{\mu} \operatorname{div}(\rho \nabla \mathrm{p}) = \frac{k \gamma \mathrm{p}_0}{\mu(\gamma+1)} \Delta(\rho^{\gamma+1})$$

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Scaling properties

Let

- u a classical solution of the PME in $(0, \infty) \times \mathbb{R}^n$
- $\alpha, \beta > 0$ constants with $\alpha(m-1) + 2\beta = 1$

Define for $\lambda > 0$

- $u_\lambda(t, x) := \lambda^\alpha u(\lambda t, \lambda^\beta x) \;\Rightarrow\; \partial_t u_\lambda - \Delta u_\lambda^m = 0$

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Idea

Scaling $u\to u_\lambda$ maps solutions of the PME to other solutions. Find a **scaling invariant** solution, that is $u_\lambda=u$ for all $\lambda>0$.

Ansatz

- $u(t,x)=t^{-\alpha}u(1,t^{-\beta}x)=:t^{-\alpha}v(t^{-\beta}x),\quad v:\mathbb{R}^n\to\mathbb{R}$

Reduction to one space variable

- $0=-t^{\alpha+1}(\partial_t u-\Delta u^m)=\alpha v(t^{-\beta}x)+\beta Dv(t^{-\beta}x)\cdot t^{-\beta}x+\Delta v^m(t^{-\beta}x)$

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Derivation of a special solution

Ansatz

- v is radial, *i.e.* $v(y) = w(|y|) := w(r), \quad w : \mathbb{R} \to \mathbb{R}$

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$$\partial_r w^{m-1} = -\frac{m-1}{m}\beta r \;\Rightarrow\; w^{m-1} = C - \frac{m-1}{2m}\beta r^2, \; C > 0$$

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u(t,x) = t^{-\alpha}\left(\left(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}\right)^+\right)^{\frac{1}{m-1}}
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Barenblatt's solution

Definition

Let $\alpha = \frac{n}{n(m-1)+2}$, $\beta = \frac{\alpha}{n}$, $C > 0$. **Barenblatt's solution** to the PME is

$$U_m(t,x;C) := t^{-\alpha}\left(\left(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}\right)^+\right)^{\frac{1}{m-1}}$$

It is also known as **ZKB solution** in literature.

Remarks

- U_m is a smooth solution where $U_m > 0$
- Finite propagation speed
- Non-smoothness on $|x| = t^\beta \left(\frac{C}{\beta}\frac{2m}{(m-1)}\right)^{\frac{1}{2}} =: r(t)$ for $m \geq 2$
- Scaling invariance
- Which role does C play?

Elimination of the free parameter

Lemma (Mass conservation)

Fix $C > 0$. As a map $(0, \infty) \to L^1(\mathbb{R}^n)$, U_m is mass preserving, i.e. $M := \|U_m(t, \cdot\, ; C)\|_{L^1(\mathbb{R}^n)}$ is independent of t and is called ***mass*** *of U_m.*

Proof

- Let $t_1, t_2 > 0$ and $\lambda = \frac{t_1}{t_2}$
- Scaling invariance: $U_m(t_1, x; C) = \lambda^\alpha U_m(t_2, \lambda^\beta x; C)$
- $\|U_m(t_1, \cdot\, ; C)\|_1 = \lambda^\alpha \lambda^{-n\beta} \|U_m(t_2, \cdot\, ; C)\|_1 = \|U_m(t_2, \cdot\, ; C)\|_1$

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Definition (Mass as parameter)

Let $\gamma = \frac{1}{m-1} + \frac{n}{2}$. The mass M and the free parameter C are related by

$$M = a(m,n) \cdot C^\gamma$$

Write $U_m(t, x; M)$ for Barenblatt's solution with mass M.

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$$M = a(m,n) \cdot C^\gamma = \pi^{\frac{n}{2}} \cdot \left(\frac{m\alpha - m}{2mn}\right)^{-\frac{n}{2}} \cdot \frac{\Gamma(\frac{m}{m-1})}{\Gamma(\frac{m}{m-1} + \frac{n}{2})} \cdot C^\gamma$$

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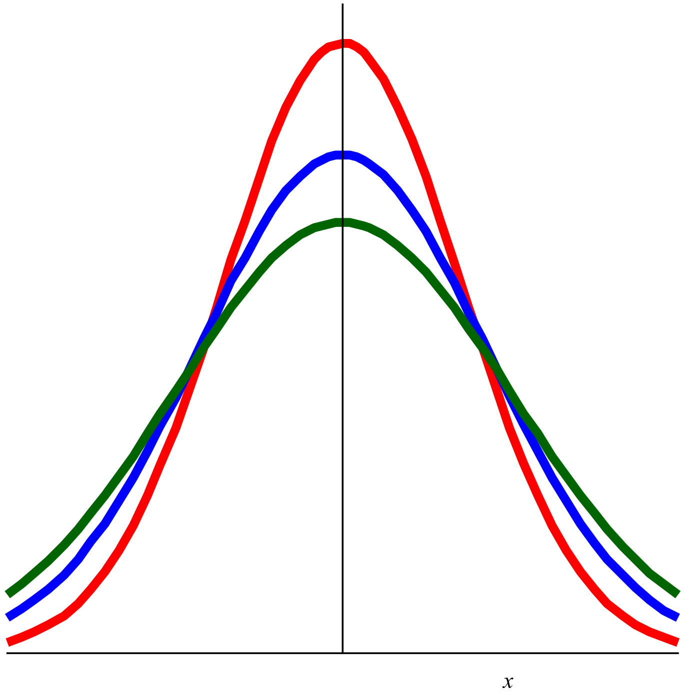
Comparison to the heat equation

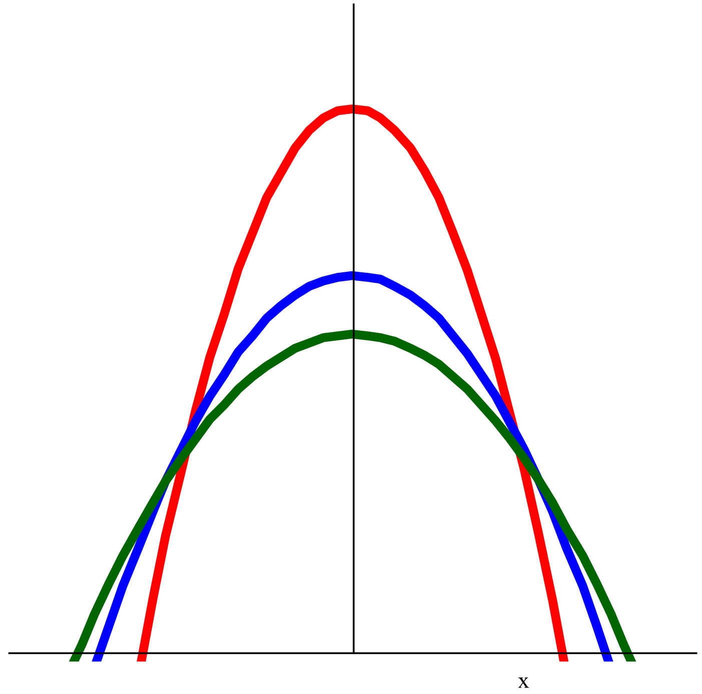
Note

- For $m = 1$ the PME becomes the heat equation

$$\partial_t u - \Delta u = 0 \quad \text{(HE)}$$

- Fundamental solution for the HE given by the Gaussian kernel $G(t,x) = (4\pi t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{4t})$





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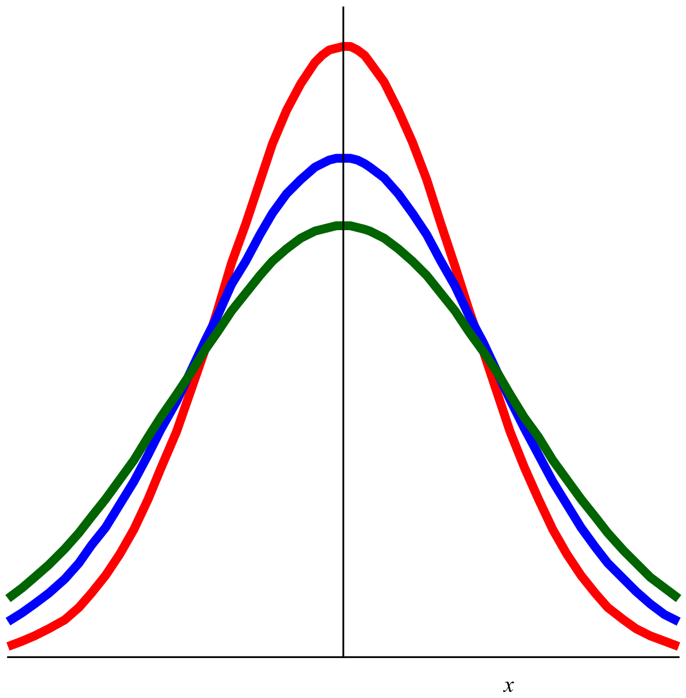

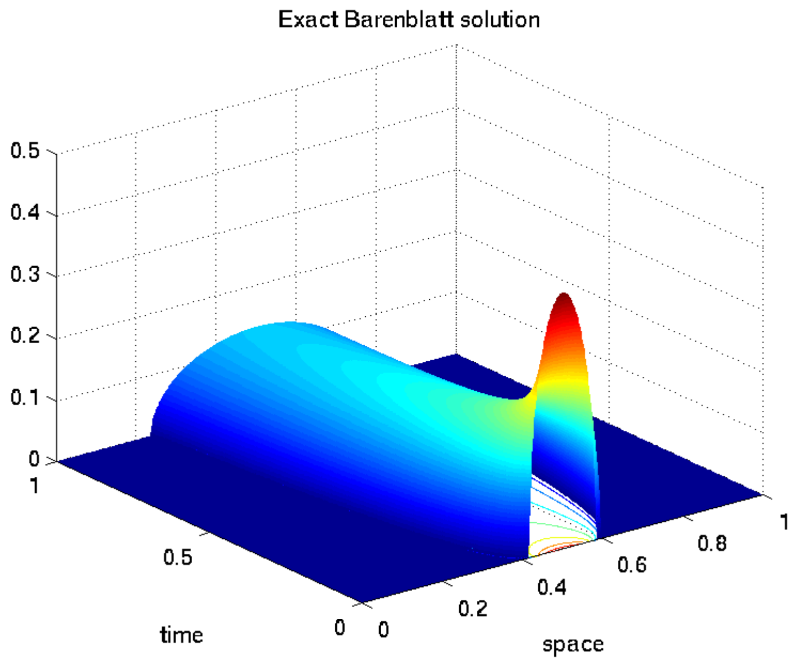

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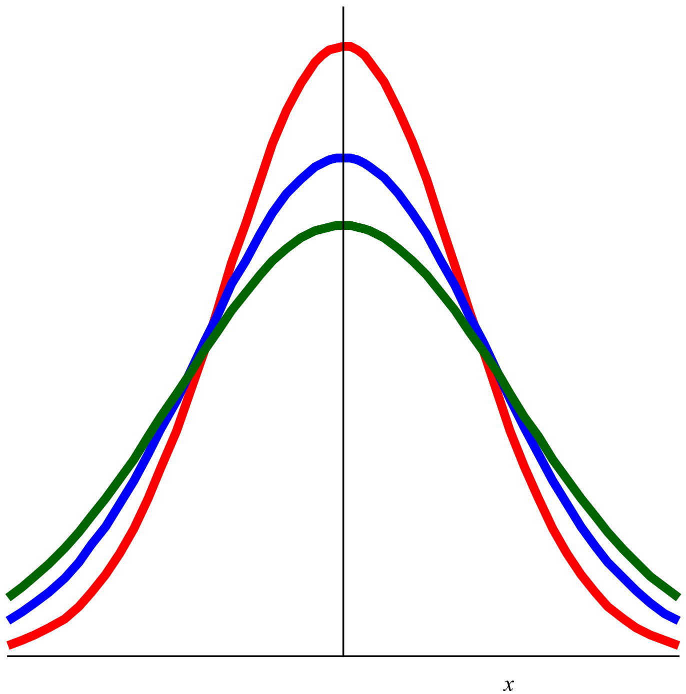

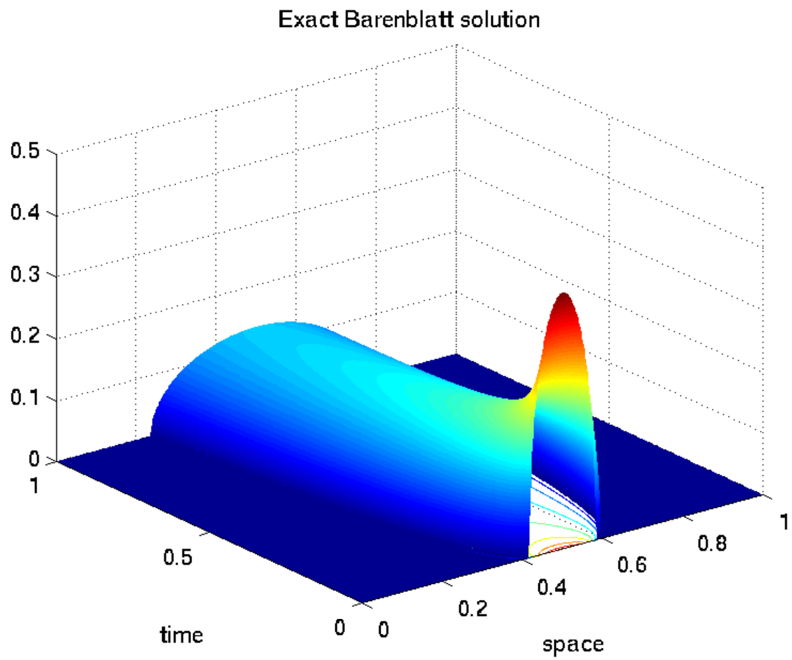

What happens to Barenblatt's solution in the limit $m \to 1$?

Asymptotics of Barenblatt's solution

Theorem

Let $U_m(t, x; M)$ Barenblatt's solution with mass M. We have the limits

$$\begin{aligned} \lim_{t \to 0} U_m(t, \cdot\ M) &= M\delta_0 && \textit{in the sense of distributions} \\ \lim_{m \to 1} U_m(t, x; M) &= MG(t, x) && \textit{pointwise on } (0, \infty) \times \mathbb{R}^n \end{aligned}$$

Proof

- supp $U_m(t, \cdot\ ; M) \subseteq B\left(0, t^\beta \left(\frac{C}{\beta} \frac{2m}{(m-1)}\right)^{\frac{1}{2}}\right)$
- By mass preservation: $\lim_{t \to 0} U_m(t, \cdot\ M) = M\delta_0$

Asymptotics of Barenblatt's solution

Theorem

Let $U_m(t,x;M)$ Barenblatt's solution with mass M. We have the limits

$$\begin{aligned} \lim_{t\to 0} U_m(t,\cdot\, M) &= M\delta_0 && \textit{in the sense of distributions} \\ \lim_{m\to 1} U_m(t,x;M) &= MG(t,x) && \textit{pointwise on } (0,\infty)\times\mathbb{R}^n \end{aligned}$$

Proof

- $\operatorname{supp} U_m(t,\cdot\,;M) \subseteq B\left(0, t^{\beta}\left(\frac{C}{\beta}\frac{2m}{(m-1)}\right)^{\frac{1}{2}}\right)$
- By mass preservation: $\lim_{t\to 0} U_m(t,\cdot\, M) = M\delta_0$
- Limit for $m \to 1$ is a truly marvelous calculation but this margin is to narrow to contain it

The Cauchy Dirichlet problem (CDP)

Let

- $\Omega \subseteq \mathbb{R}^n$ bounded with $\partial\Omega$ smooth, $T \in (0, \infty]$
- $Q := \mathbb{R}_+ \times \Omega$, $Q_T := (0, T) \times \Omega$
- $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$
- $\Phi \in C(\mathbb{R})$ strictly increasing with $\Phi(\pm\infty) = \pm\infty$, $\Phi(0) = 0$

Consider

$$(CDP)\begin{cases} \partial_t u - \Delta(\Phi(u)) = f & \text{in } Q_T \\ u(0, x) = u_0(x) & \text{in } \Omega \\ u(t, x) = 0 & \text{on } [0, T) \times \partial\Omega \end{cases}$$

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- Choose $\Phi(u) = |u|^{m-1} u$ and $f = 0$ for the PME

Weak solutions for the CDP

Definition

A weak solution of CDP in Q_T is a function $u \in L^1(Q_T)$ s.t.

1. $w := \Phi(u) \in L^1(0, T; W_0^{1,1}(\Omega))$
2. $$\iint_{Q_T} (\nabla w \cdot \nabla \eta - u \partial_t \eta)\, dx\, dt = \int_\Omega u_0(x)\eta(0, x)\, dx + \iint_{Q_T} f\eta\, dx\, dt$$

holds for any $\eta \in C^1(\overline{Q_T})$ which vanishes on $[0, T) \times \partial\Omega$ and for $t = T$

Remarks

- Integration by parts shows: smooth solutions are weak solutions
- What about initial data...?

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- satisfied in the sense that for any $\varphi \in C^1(\overline{\Omega})$ with $\varphi = 0$ on $\partial\Omega$

$$\lim_{t \to 0} \int\limits_{\Omega} u(t)\varphi\, dx = \int\limits_{\Omega} u_0 \varphi\, dx$$

A well-known weak solution

Modify Barenblatt's solution

- Take $x_0 \in \Omega$, $\tau > 0$
- Set $v(t,x) := U_m(t+\tau, x - x_0; M)$
- Let $T > 0$ be small enough so that $v = 0$ on $[0, T) \times \partial\Omega$

Theorem

Define $v(t,x)$ as above. Then v is a weak solution of the CDP for the PME in Q_T. If $m \geq 2$, then v is not a classical solution of that problem.

Proof

- v has the stated regularity
- Let $P := \{(t,x) \in Q_T \mid v(t,x) > 0\}$
- v is smooth solution within P and v^m is C^1 up to $|x| = r(t)$
- Integration by parts yields the integral equality (2)