# Flexion algebra meets tree algebra.

As mathematical objects, finite trees would seem to be nearly as basic and ubiquitous as the natural integers, were it not for their apparent 'chemical inertness', by which we mean the paucity of natural operations (of any given arity) defined on them. One way to redress this state of affairs is by bringing trees into close relation with $\mathrm{Flex}(\mathfrak{E}^\bullet)$ — the flexion polyalgebra generated by a so-called flexion unit $\mathfrak{E}^\bullet$ — and by uploading the rich structure of that polyalgebra onto trees. The rapprochement also benefits $\mathrm{Flex}(\mathfrak{E}^\bullet)$, leading in particular
(i) to a neat filtration by depth and alternality codegree,
(ii) to exact formulae for the corresponding dimensions,
(iii) to remarkable expansions for all the main elements of $\mathrm{Flex}(\mathfrak{E}^\bullet)$.

The construction naturally leads to a notion of *pre-associative algebra*, parallel to that of pre-Lie algebra and potentially capable of rendering roughly the same services.

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**NB:** A *flexion unit* is a depth-1 bimould $\mathfrak{E}^\bullet$ that verifies the identities:

$$\mathfrak{E}^{\binom{-u_1}{-v_1}} \equiv -\mathfrak{E}^{\binom{u_1}{v_1}} \qquad ; \qquad \mathfrak{E}^{\binom{u_1}{v_1}}\,\mathfrak{E}^{\binom{u_2}{v_2}} \equiv \mathfrak{E}^{\binom{u_1+u_2}{v_1}}\,\mathfrak{E}^{\binom{u_2}{v_2-v_1}} + \mathfrak{E}^{\binom{u_1+u_2}{v_2}}\,\mathfrak{E}^{\binom{u_1}{v_1-v_2}}$$

Although there exist wildly different realisations of $\mathfrak{E}^\bullet$, all polyalgebras $\mathrm{Flex}(\mathfrak{E}^\bullet)$ are isomorphic and admit natural indexations by trees, whether of the *binary* or *ordered* sort.